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Rational solutions to the Johnson equation of order N depending on $2N - 2$ parameters.

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Abstract

We construct rational solutions of order N depending on $2N - 2$ parameters. They can be written as a quotient of 2 polynomials of degree $2N(N + 1)$ in x, t and $4N(N + 1)$ in y depending on $2N - 2$ parameters. We explicitly construct the expressions of the rational solutions of order 4 depending on 6 real parameters and we study the patterns of their modulus in the plane (x, y) and their evolution according to time and parameters $a_1, a_2, a_3, b_1, b_2, b_3$.

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1 Introduction

Johnson introduced a new equation in a paper written in 1980, [1] to describe waves surfaces in shallow incompressible fluids [2, 3]. This equation was later derived for internal waves in a stratified medium [4]. The Johnson equation is a dissipative equation and there is no soliton-like solution with a linear front localized along straight lines in the (x, y) plane. We consider the Johnson equation (J) in the following normalization

$$(u_t + 6uu_x + u_{xxx} + \frac{u}{2t})_x - 3\frac{u_{yy}}{t^2} = 0, \quad (1)$$

where as usual, subscripts x, y and t denote partial derivatives.

Johnson constructed the first solutions in 1980 [1]. Golinko, Dryuma, and Stepanyants found other types of solutions in 1984 [5]. A new approach to solve this equation was given in 1986 [6] by giving a connection between solutions of the Kadomtsev-Petviashvili (KP) [9] and solutions of the Johnson equation. Another types of solutions were obtained by using the Darboux transformation [7]. More recently, in 2013, extension to the elliptic case has been considered [8]. In the following, we recall that the solutions can be expressed in terms of Fredholm determinants of order $2N$ depending on $2N - 1$ parameters. They can also be given in terms of wronskians of order $2N$ with $2N - 1$ parameters. These representations allow to obtain an infinite hierarchy of solutions to the Johnson equation, depending on $2N - 1$ real parameters. We use these results to build rational solutions to the equation, making go a parameter towards 0.

Here we construct rational solutions of order N depending on $2N - 2$ parameters without the presence of a limit.

That provides an effective method to build an infinite hierarchy of rational solutions of order N depending on $2N - 2$ real parameters. We present here only the explicit rational solutions of order 4, depending on 6 real parameters, and the representations of their modulus in the plane of the coordinates (x, y) according to the real parameters $a_1, b_1, a_2, b_2, a_3, b_3$ and time t .

2 Rational solutions to the Johnson equation of order N depending on $2N - 2$ parameters

2.1 Families of rational solutions of order N depending on $2N - 2$ parameters

We need to define some notations. First of all, we define real numbers λ_j such that $-1 < \lambda_\nu < 1$, $\nu = 1, \dots, 2N$ which depend on a parameter ϵ which will be intended to tend towards 0; they can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N, \quad (2)$$

The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of λ_ν , $1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\begin{aligned} \kappa_j &= 2\sqrt{1 - \lambda_j^2}, & \delta_j &= \kappa_j \lambda_j, & \gamma_j &= \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \\ x_{r,j} &= (r - 1) \ln \frac{\gamma_j - i}{\gamma_j + i}, & r &= 1, 3, & \tau_j &= -12i\lambda_j^2 \sqrt{1 - \lambda_j^2} - 4i(1 - \lambda_j^2) \sqrt{1 - \lambda_j^2}, \\ \kappa_{N+j} &= \kappa_j, & \delta_{N+j} &= -\delta_j, & \gamma_{N+j} &= \gamma_j^{-1}, \\ x_{r,N+j} &= -x_{r,j}, & \tau_{N+j} &= \tau_j & j &= 1, \dots, N. \end{aligned} \quad (3)$$

e_ν $1 \leq \nu \leq 2N$ are defined in the following way :

$$\begin{aligned} e_j &= 2i \left(\sum_{k=1}^{1/2 M-1} a_k(je)^{2k+1} - i \sum_{k=1}^{1/2 M-1} b_k(je)^{2k+1} \right), \\ e_{N+j} &= 2i \left(\sum_{k=1}^{1/2 M-1} a_k(je)^{2k+1} + i \sum_{k=1}^{1/2 M-1} b_k(je)^{2k+1} \right), \quad 1 \leq j \leq N, \\ a_k, b_k &\in \mathbf{R}, \quad 1 \leq k \leq N. \end{aligned} \quad (4)$$

ϵ_ν , $1 \leq \nu \leq 2N$ are real numbers defined by :

$$\epsilon_j = 1, \quad \epsilon_{N+j} = 0 \quad 1 \leq j \leq N. \quad (5)$$

Let I be the unit matrix and $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$ the matrix defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(\kappa_\nu x + \left(\frac{\kappa_\nu y}{12} - 2\delta_\nu \right) yt + 4i\tau_\nu t + x_{r,\nu} + e_\nu). \quad (6)$$

Then we recall the following result¹ :

Theorem 2.1 *The function v defined by*

$$v(x, y, t) = -2 \frac{|n(x, y, t)|^2}{d(x, y, t)^2} \quad (7)$$

where

$$n(x, y, t) = \det(I + D_3(x, y, t)), \quad (8)$$

$$d(x, y, t) = \det(I + D_1(x, y, t)), \quad (9)$$

and $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$ the matrix

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(\kappa_\nu x + \left(\frac{\kappa_\nu y}{12} - 2\delta_\nu \right) yt + 4i\tau_\nu t + x_{r,\nu} + e_\nu). \quad (10)$$

is a solution to the Johnson equation (1), depending on $2N - 1$ parameters a_k, b_h , $1 \leq k \leq N - 1$ and ϵ .

¹The proof of this result is submitted to a review

We recall another result on the solutions to the Johnson equation obtained recently by the author in terms of wronskians. We need to define the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (11)$$

with the arguments

$$\Theta_{r,\nu} = \frac{-i\kappa_\nu x}{2} + i\left(\frac{-\kappa_\nu y}{24} + \delta_\nu\right)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \quad (12)$$

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (13)$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$ defined in (10).

Then we have the following statement :

Theorem 2.2 *The function v defined by*

$$v(x, y, t) = -2 \frac{|W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0))^2}$$

is a solution to the Johnson equation depending on $2N-1$ real parameters a_k, b_k and ϵ , with ϕ_ν^r defined in (11)

$$\begin{aligned} \phi_{r,\nu} &= \sin\left(\frac{-i\kappa_\nu x}{2} + i\left(\frac{-\kappa_\nu y}{24} + \delta_\nu\right)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad 1 \leq \nu \leq N, \\ \phi_{r,\nu} &= \cos\left(\frac{-i\kappa_\nu x}{2} + i\left(\frac{-\kappa_\nu y}{24} + \delta_\nu\right)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \end{aligned}$$

$\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$ being defined in (3), (2) and (4).

From those two preceding results, we can construct rational solutions to the Johnson equation as a quotient of two determinants.

We use the following notations :

$$\begin{aligned} X_\nu &= \frac{-i\kappa_\nu x}{2} + i\left(\frac{-\kappa_\nu y}{24} + \delta_\nu\right)yt - i\frac{x_{3,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}, \\ Y_\nu &= \frac{-i\kappa_\nu x}{2} + i\left(\frac{-\kappa_\nu y}{24} + \delta_\nu\right)yt - i\frac{x_{1,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}, \end{aligned}$$

for $1 \leq \nu \leq 2N$, with $\kappa_\nu, \delta_\nu, x_{r,\nu}$ defined in (3) and parameters e_ν defined by (4).

We define the following functions :

$$\begin{aligned} \varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k, \end{aligned} \quad (14)$$

for $1 \leq k \leq N$, and

$$\begin{aligned} \varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{aligned} \quad (15)$$

for $1 \leq k \leq N$.

We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N, 1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k .

$$\begin{aligned} \psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k, \end{aligned} \quad (16)$$

for $1 \leq k \leq N$, and

$$\begin{aligned} \psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{aligned} \quad (17)$$

for $1 \leq k \leq N$.

The following ratio

$$q(x, t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x, t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,k \in [1,2N]}}{\det(\psi_{j,k})_{j,k \in [1,2N]}}. \quad (18)$$

The terms λ_j depending on ϵ are defined by $\lambda_j = 1 - 2j\epsilon^2$. All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ . They can all be prolonged by continuity when $\epsilon = 0$.

We use the following expansions

$$\varphi_{j,k}(x, y, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l} \varphi_{j,1}}{\partial \epsilon^{2l}}(x, y, t, 0),$$

$$\varphi_{j,1}[0] = \varphi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\varphi_{j,N+k}(x, y, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l} \varphi_{j,N+1}}{\partial \epsilon^{2l}}(x, y, t, 0),$$

$$\varphi_{j,N+1}[0] = \varphi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1.$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\psi_{j,k}(x, y, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \psi_{j,1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,1}[l] = \frac{\partial^{2l} \psi_{j,1}}{\partial \epsilon^{2l}}(x, y, t, 0),$$

$$\psi_{j,1}[0] = \psi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\psi_{j,N+k}(x, y, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,N+1}[l] = \frac{\partial^{2l} \psi_{j,N+1}}{\partial \epsilon^{2l}}(x, y, t, 0),$$

$$\psi_{j,N+1}[0] = \psi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad N+1 \leq k \leq 2N..$$

Then we get the following result :

Theorem 2.3 *The function v defined by*

$$v(x, y, t) = -2 \frac{|\det((n_{jk})_{j,k \in [1,2N]})|^2}{\det((d_{jk})_{j,k \in [1,2N]})^2} \quad (19)$$

is a rational solution to the Johnson equation (1), where

$$\begin{aligned} n_{j1} &= \varphi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ d_{j1} &= \psi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ & & & 2 \leq k \leq N, \quad 1 \leq j \leq 2N \end{aligned} \quad (20)$$

The functions φ and ψ are defined in (14), (15), (16), (17).

Proof : In each column k (and $N+k$) of the determinants appearing in $q(x, t)$, we can successively eliminate the powers of ϵ strictly inferior to $2(k-1)$; then each common term in the numerator and denominator is factorized and simplified; in the end, we take the limit when ϵ goes to 0.

First of all, the components j of the columns 1 and $N+1$ are respectively equal by definition to

$\varphi_{j_1}[0] + 0(\epsilon)$ for C_1 , $\varphi_{j_{N+1}}[0] + 0(\epsilon)$ for C_{N+1} of Δ_3 , and $\psi_{j_1}[0] + 0(\epsilon)$ for C'_1 , $\psi_{j_{N+1}}[0] + 0(\epsilon)$ for C'_{N+1} of Δ_1 .

So we can replace the columns C_k by $C_k - C_1$ and C_{N+k} by $C_{N+k} - C_{N+1}$ for $2 \leq k \leq N$, for Δ_3 ; the same changes for Δ_1 are made. Each component j of the column C_k of Δ_3 can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l](k^{2l} - 1)\epsilon^{2l}$ and the column C_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l](k^{2l} - 1)\epsilon^{2l}$ for $2 \leq k \leq N$. For Δ_1 , we make the same reductions, each component j of the column C'_k can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,1}[l](k^{2l} - 1)\epsilon^{2l}$ and the column C'_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l](k^{2l} - 1)\epsilon^{2l}$ for $2 \leq k \leq N$.

The term $\frac{k^2-1}{2}\epsilon^2$ for $2 \leq k \leq N$ can be factorized in Δ_3 and Δ_1 in each column k and $N+k$, and so those common terms can be simplified in the numerator and denominator.

If we restrict the developments at order 1 in columns 2 and $N+2$, we respectively get $\varphi_{j_1}[1] + 0(\epsilon)$ for the component j of C_2 , $\varphi_{j_{N+1}}[1] + 0(\epsilon)$ for the component j of C_{N+2} of Δ_3 , and $\psi_{j_1}[1] + 0(\epsilon)$ for the component j of C'_2 , $\psi_{j_{N+1}}[1] + 0(\epsilon)$ for the component j of C'_{N+2} of Δ_1 . We can extend this algorithm up to the columns C_N , C_{2N} of Δ_3 and C'_N , C'_{2N} of Δ_1 .

Then we take the limit when ϵ tends to 0, and $q(x, y, t)$ can be replaced by $Q(x, y, t)$ defined by

$$Q(x, y, t) := \frac{\left\| \begin{array}{cccccc} \varphi_{1,1}[0] & \cdots & \varphi_{1,1}[N-1] & \varphi_{1,N+1}[0] & \cdots & \varphi_{1,N+1}[N-1] \\ \varphi_{2,1}[0] & \cdots & \varphi_{2,1}[N-1] & \varphi_{2,N+1}[0] & \cdots & \varphi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{2N,1}[0] & \cdots & \varphi_{2N,1}[N-1] & \varphi_{2N,N+1}[0] & \cdots & \varphi_{2N,N+1}[N-1] \end{array} \right\|^2}{\left(\begin{array}{cccccc} \psi_{1,1}[0] & \cdots & \psi_{1,1}[N-1] & \psi_{1,N+1}[0] & \cdots & \psi_{1,N+1}[N-1] \\ \psi_{2,1}[0] & \cdots & \psi_{2,1}[N-1] & \psi_{2,N+1}[0] & \cdots & \psi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2N,1}[0] & \cdots & \psi_{2N,1}[N-1] & \psi_{2N,N+1}[0] & \cdots & \psi_{2N,N+1}[N-1] \end{array} \right)^2} \quad (21)$$

So the solution to the Johnson equation takes the form :

$$v(x, y, t) = -2Q(x, y, t)$$

and we get the result. \square

3 Explicit expression of rational solutions of order 4 depending on 6 parameters

We explicitly construct rational solutions to the Johnson equation of order 4 depending on 6 parameters.

Because of the length of the expression, we only give the expression without parameters in the appendix.

We give patterns of the modulus of the solutions in the plane (x, y) of coordinates in function of the parameters $a_1, a_2, a_3, b_1, b_2, b_3$, and time t .

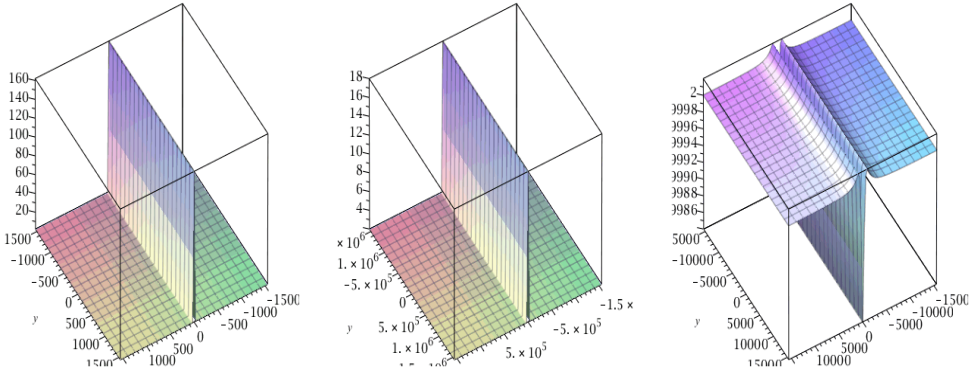


Figure 1. Solution of order 4 to (1), on the left for $t = 0$; in the center for $t = 0$, $a_2 = 10^4$; on the right for $t = 0$, $a_1 = 10^3$; all other parameters not mentioned equal to 0.

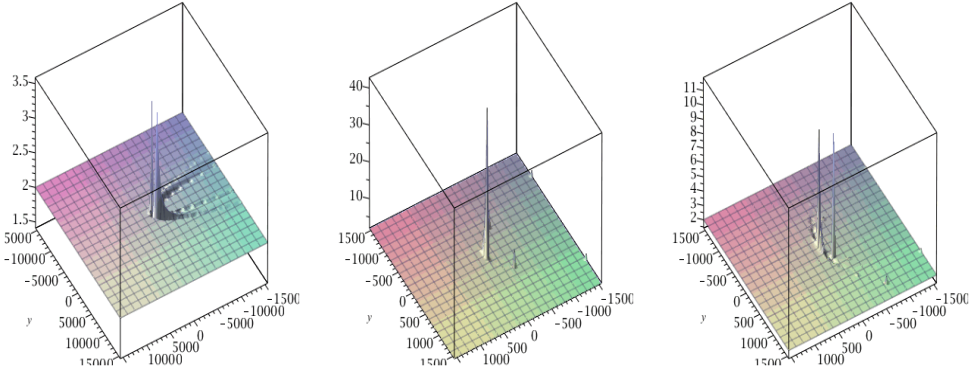


Figure 2. Solution of order 4 to (1), on the left for $t = 0,01$, $a_1 = 10^2$; in the center for $t = 0,01$, $a_2 = 10$; on the right for $t = 0,01$, $b_1 = 10$; all other parameters not mentioned equal to 0.

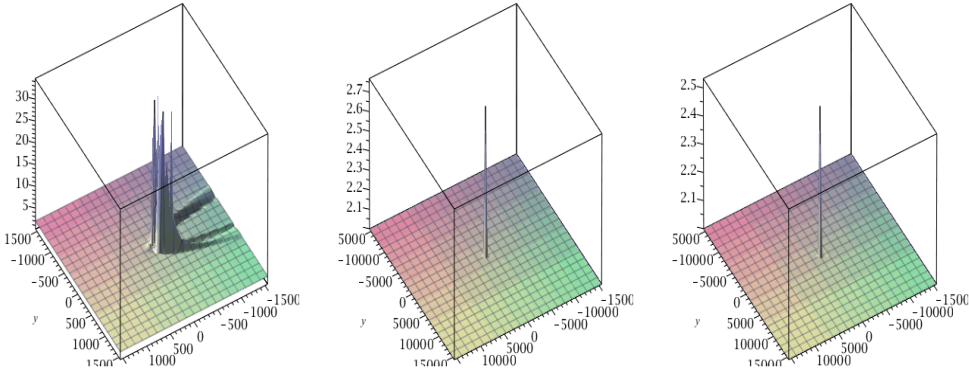


Figure 3. Solution of order 4 to (1), on the left for $t = 0,1$, $a_1 = 10^2$; in the center for $t = 0,1$, $b_1 = 10$; on the right for $t = 0,1$, $a_2 = 10^2$; all the other parameters to equal to 0.

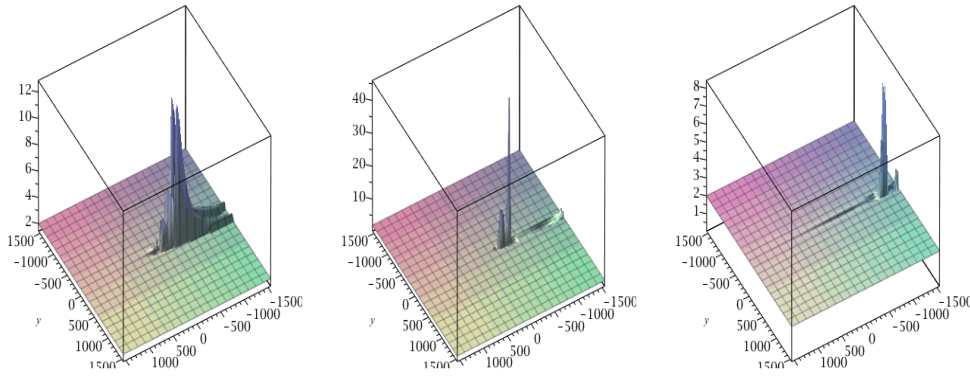


Figure 4. Solution of order 4 to (1), on the left for $t = 1$, $a_2 = 10^2$; in the center for $t = 10$, $a_1 = 10^2$; on the right for $t = 10$, $a_1 = 10^3$; all the other parameters to equal to 0.

In these constructions, we note that the initial rectilinear structure becomes deformed very quickly as time t increases. The heights of the peaks also decrease very quickly according to time t and of the various parameters. Because of the structure of the polynomials, one notices that the modulus of these solutions tend towards value 2 when time t and variables x and y tend towards the infinite.

4 Conclusion

From the previous results giving the solutions to the Johnson equation in terms of Fredholm determinants and wronskians, we succeed in obtaining rational solutions to the Johnson equation depending on $2N - 2$ real parameters. These solutions can be expressed in terms of a ratio of two polynomials of degree $2N(N + 1)$ in x , t and $4N(N + 1)$ in y . That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.

In the (x, y) plane of coordinates, different structures appear.

It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

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Appendix : Because of the length of the complete expression, we only give in this appendix the explicit expression of the rational solution of order 4 to the Johnson equation without parameters. They can be written as

$$v(x, y, t) = -2 \frac{\ln(x, y, t)}{(d(x, y, t))^2} \text{ with } n(x, y, t) = A(x, y, t) + iB(x, y, t), d(x, y, t) = C(x, y, t) + iD(x, y, t)$$

$$\text{with } A(x, y, t) = \sum_{k=0}^{20} a_k(y, t)x^k, B(x, y, t) = \sum_{k=0}^{20} b_k(y, t)x^k, C(x, y, t) = \sum_{k=0}^{20} c_k(y, t)x^k, D(x, y, t) = 0.$$

$$\begin{aligned} a_{20} &= 3833759992447475122176, & a_{19} &= (6389599987412458536960 y^2 + 1840204796374788058644480)t, & a_{18} &= (5058433323368196341760 y^4 + \\ & 2760307194562182087966720 y^2 + 419566693573451677370941440)t^2 + 28753199943356063416320, & a_{17} &= (2529216661684098170880 y^6 + \\ & 1955217596148212312309760 y^4 + 563102667690685145945210880 y^2 + 60417603874577041541415567360)t^3 + (43129799915034095124480 y^2 + \\ & 17941996764654183571783680)t, & a_{16} &= (895764234346451435520 y^8 + 868985598288094361026560 y^6 + 354699474501240398303723520 y^4 + \\ & 70277114464407698680986992640 y^2 + 6162595595206858237224387870720)t^4 + (30550274939815817379840 y^4 + 20242252760122668645089280 y^2 + \\ & 4786372675370823740534292480)t^2 - 161736749681377856716800, & a_{15} &= (238870462492387049472 y^{10} + 271557999465029487820800 y^8 + \\ & 139242162925692296437432320 y^6 + 40101742922599381373980508160 y^4 + 6486942731796692881288829337600 y^2 + 47328734171886712618832988471296)t^5 + \\ & (13577899973251474391040 y^6 + 10044451180212384820101120 y^4 + 4306079223517004057228083200 y^2 + 756809985376280836150363422720)t^3 + \\ & (-215648999575170475622400 y^2 - 45545068710276004451450880)t, & a_{14} &= (49764679685913968640 y^{12} + 63363533208506880491520 y^{10} + \\ & 38145911924852377465651200 y^8 + 13808896791996409592068177920 y^6 + 316397451869495596510973132800 y^4 + 435922551576737761622609331486720 y^2 + \\ & 2839724050271320275129979308277760)t^6 + (4243093741641085747200 y^8 + 2747527994587357170892800 y^6 + 1531970492982011058821529600 y^4 + \\ & 556477930423735908934090752000 y^2 + 81086784147458661016110366720000)t^4 + (-134780624734481547264000 y^4 - 72458063857257279809126400 y^2 - \\ & 3229559417637753042921062400)t^2 + 1186069497663437615923200, & a_{13} &= (8294113280985661440 y^{14} + 11440637940424853422080 y^{12} + \\ & 7737805584756487288258560 y^{10} + 3293353183912079028954071040 y^8 + 948485716966678760338772459520 y^6 + 184839709369548119511457584184320 y^4 + \\ & 22667972681990363604375685237309440 y^2 + 1363067544130233732342239006797332480)t^7 + (990055206382920007680 y^{10} + 382044832580703250022400 y^8 + \\ & 198588767608779211328716800 y^6 + 14375679869279844314300677600 y^4 + 49192649049458254349773622476800 y^2 + 632087699786269754352733306557440)t^5 + \\ & (-52414687396742823936000 y^6 - 51324461898890573198131200 y^4 - 8984799918299902696331673600 y^2 + 417358447817801931700568064000)t^3 + \\ & (138374774274010551910400 y^2 + 398519351214915038950195200)t, & a_{12} &= (1123161173466808320 y^{16} + 1634376848632121917440 y^{14} + \\ & 1203285919851743406981120 y^{12} + 576801970063697457048453120 y^{10} + 195462873061337238013099376640 y^8 + 4784226260496332187742689585280 y^6 + \\ & 8278254729452363179048963481272320 y^4 + 932625161773317816865742478335016960 y^2 + 53159634221079115561347321265095966720)t^8 + \\ & (178759967819138334720 y^{12} - 8075744428535190650880 y^{10} - 39248117922681026563276800 y^8 + 1648823497551810100545454080 y^6 + \\ & 9283906805902660747383747379200 y^4 + 316684912468983020943483878741760 y^2 + 374021549252840999472198806415605760)t^6 + (-14195644503284514816000 y^8 - \\ & 21804509957045014757376000 y^6 - 6831185242542533546449305600 y^4 - 417358447817801931700568064000 y^2 + 110683460361281072286990650572800)t^4 + \\ & (147590029773422382284800 y^4 + 470977415072172318759321600 y^2 + 7260297998497179436874652800)t^2 - 4953187958992196861952000, & a_{11} &= \\ & (291975685940756480 y^{18} + 187272347239097303040 y^{16} + 146517077489138458951680 y^{14} + 76751875048798451945963520 y^{12} + 29343905682992370383144878080 y^{10} + \\ & 8451044836701802670345724887040 y^8 + 1833438966925691173883327482429440 y^6 + 290302440061792555740071800854282240 y^4 + 30776630338519487956569501785055559680 y^2 + \\ & 1701108295074531697963114280483070935040)t^9 + (25537138259876904960 y^{14} - 16487978208259347578880 y^{12} - 23548870753608615937966080 y^{10} - \\ & 8817341281829796982995025920 y^8 - 761138147057354337664295239680 y^6 + 437525208016358121040339512852480 y^4 + 1540088732217580586061995058524072960 y^2 + \\ & 17253565492806379612016235849197813760)t^7 + (-2839128900656902963200 y^{10} - 6268796612650441742745600 y^8 - 2644719330789890713033113600 y^6 - \\ & 423155092926382514085298176000 y^4 + 16527394533584956495342495334400 y^2 + 118756842175868328538064320281600)t^5 + (249843343257807460761600 y^6 + \\ & 2551127664974266726612992000 y^4 + 82167444414129755303549337600 y^2 + 912160074286240440611304243200)t^3 + (-4953187958992196861952000 y^6 + \\ & 1462747164118381336146739200)t, & a_{10} &= (11439604544569344 y^{20} + 17340032151768268800 y^{18} + 1414457022674761006080 y^{16} + 7918844251066506946805760 y^{14} + \\ & 3318042598263440768740884480 y^{12} + 1079072537972782120301972422656 y^{10} + 275211725270362831122443922309120 y^8 + 54479329302934823452533159477903360 y^6 + \\ & 8071322172111728459198374242649374720 y^4 + 820710142360519678841853380934814924800 y^2 + 44909258989967636826226217004753072685056)t^{10} + \\ & (2926130425610895360 y^{16} - 4142023680111799173120 y^{14} - 5552074294617943572480000 y^{12} - 2774875482533482493430988800 y^{10} - 713283621327627974199245537280 y^8 - \\ & 66272811660597455181323981291520 y^6 + 15584231218868374978008283060050240 y^4 + 5783190341388465874191981544577433600 y^2 + 630920921939649503109674786593638973440)t^8 + \\ & (-43375804267026841600 y^{12} - 1299185384940598795960320 y^{10} - 624531483269670095762227200 y^8 - 139312704109553329979680358400 y^6 - \\ & 9508816636115587343911275724800 y^4 + 3613990271343910469881558979788800 y^2 + 808546584414229807682547419751383040)t^6 + (57255766163247543091200 y^8 + \\ & 8394736334623306815897600 y^6 + 42080020585102165249150156800 y^4 + 8142354162519524723084230656000 y^2 + 82973641815831142701617601579600)t^4 + \\ & (-227021147871423561728000 y^4 - 1611437149325461379088384000 y^2 - 236992212361124247935700172800)t^2 - 22289345815464885878784000, & a_9 &= \\ & (866636707921920 y^{22} + 1300502411382620160 y^{20} + 1089362019887559475200 y^{18} + 640868004293042629509120 y^{16} + 288170959432301880016896000 y^{14} + \\ & 102896891394092649087164743680 y^{12} + 2963430472149868293710346179840 y^{10} + 6883790993036019975202969485312000 y^8 + 1269787444522250115855195947831132160 y^6 + \\ & 17902745752472120445324728684310118400 y^4 + 17727339074987225062984033028192002375680 y^2 + 979838377962930258026753825558248858583040)t^{11} + \\ & (270938002371379200 y^{16} - 638011182995944243200 y^{14} - 840742678899288598118400 y^{12} - 482979213715195386711244800 y^{10} - 165537155961891038805413068800 y^8 - \\ & 3430686441062318785786694860800 y^6 - 3075319634693734127133359131852800 y^4 + 429548557965447646452665295883468800 y^2 + 169724064366835411525199458373468160000 y^2 + \\ & 184659820311169277394170107103333508000)t^9 + (-51637595746074624000 y^{14} - 201262693179900454502400 y^{12} - 96196093393826585405030400 y^{10} - \\ & 21999073276667124149826432000 y^8 - 2454299538973018581694729420800 y^6 + 374509647175174266712640376096000 y^4 + 257034039786313234315566487440588800 y^2 + \\ & 390365348072414650795875425897676800)t^7 + (9542627693874590515200 y^{10} + 1873825074433559557120000 y^8 + 12926719864534302219392064000 y^6 + \\ & 3607606490492998564718837760000 y^4 + 475997309736203103104497876992000 y^2 + 55341062407124027660404604377497600)t^5 + (-630614207742062100480000 y^6 - \\ & 78417613814266938896496000 y^4 - 252289921092987691135401984000 y^2 - 29497677796289438610295357440000)t^3 + (-18574454846220738232320000 y^2 - \\ & 10580009480470332497129472000)t, & a_8 &= (54164794245120 y^{24} + 78818327962583040 y^{22} + 67014124257127956480 y^{20} + 40891467820678187581440 y^{18} + \\ & 19425182661732300509675520 y^{16} + 7463870654096140365862010880 y^{14} + 2362145763180161945293931151360 y^{12} + 619083287533350266506058630430720 y^{10} + \\ & 13363967776023181938111111988510720 y^8 + 23333915812210853614131125536381992960 y^6 + 3171803314887420170582927478083372974080 y^4 + \\ & 309422645672504292008448576492078586920960 y^2 + 1763709803332744644481568860048479454494720)t^{12} + (20320350177853440 y^{20} - \\ & 70380130498353561600 y^{18} - 90716928206703899443200 y^{16} - 56579907888537320344780800 y^{14} - 22667297643345319033621708800 y^{12} - \end{aligned}$$

3313125919399052574720 y^{24} + 1431270397180390712279040 y^{22} + 412205874387952525136363520 y^{20} + 118715291823730327239272693760 y^{18} +
34190004045234334244910535802880 y^{16} + 6564480776684992175022822874152960 y^{14} + 1890570463685277746406572987756052480 y^{12} +
233350411517725710413611294488747048960 y^{10} + 672049185171050045899120052812759150100480 y^8 + 483875413323156033136643802518658807234560 y^6 +
1393561190370689375367353415125373736483553280 y^4 + 44593958091862060011755309284011959567473704960 y^2 + 12843059930456273283385529073795444355432427028480) t^{19} +
(- 77760 y^{34} - 948049920 y^{32} - 601974374400 y^{30} - 371504185344000 y^{28} - 214699698794004480 y^{26} - 44577649089136558080 y^{24} - 22363599955943604879360 y^{22} +
1192725330983658926899200 y^{20} + 1116390909800704755577651200 y^{18} + 81616763128814599976999769600 y^{16} + 438771718580507289476351876136960 y^{14} +
86979370291076146319052403082526720 y^{12} + 38993015813508853519635567872468582400 y^{10} + 4423934885023549926591380791349162803200 y^8 +
155131353576984052829687885761190381486080 y^6 + 11291345064325887616142807457929152566394880 y^4 + 23980865484295613001113206685282473048654479360 y^2 +
1086977728489137712786355663797791514457171558400) t^{17} + (100776960 y^{30} + 145118822400 y^{28} + 422586010828800 y^{26} + 234047636766720000 y^{24} +
514979696130549350400 y^{22} + 187450452408499521454080 y^{20} + 143350675717598507276697600 y^{18} + 48240968736965068957365043200 y^{16} +
16787084234449366586178404352000 y^{14} + 5539849092954375720620660254310400 y^{12} + 906821477291750247177762141099786240 y^{10} + 302638975007119851748989691555636838400 y^8 +
19014412438514680934484107824356497817600 y^6 + 6799247615597732887176599093165867139072000 y^4 + 19657438666253213845242615447732051404390400 y^2 +
23429247513107215123363629291795345944629739520) t^{15} + (- 126978969600 y^{26} - 36569943244800 y^{24} - 306602404164403200 y^{22} - 264155524760390860800 y^{20} +
2011386665622503424000 y^{18} + 976199029576883282490163200 y^{16} + 330891824948141577795010560000 y^{14} + 12530028066707786882832610099200 y^{12} -
19076286046097836452624204614860800 y^{10} - 11657402410510570332685793218265088000 y^8 - 1005288688942025129803104484212080640000 y^6 +
39336863316591613409763263334616714444800 y^4 + 17113992102713602891018756591334236304179200 y^2 + 28079055017927934734069847065830932401356800) t^{13} +
(102852965376000 y^{22} - 3730134210969600 y^{20} + 280260342646308864000 y^{18} - 335447932824409079808000 y^{16} + 335075448819066486128640000 y^{14} -
1920474146216657069015040000 y^{12} - 10352547314895034012920671769600 y^{10} - 905403504746789172334890057728000 y^8 - 12281496861304233544442647469359104000 y^6 -
478325635656213231096914470061998080000 y^4 + 176802863420658346919735085014244655104000 y^2 + 6018922927957413120437426621226377910681600) t^{11} +
(- 11939043983616000 y^{18} - 86376743146487808000 y^{16} + 20064997519065612288000 y^{14} + 10181553026817293549568000 y^{12} - 155899939946626398830985216000 y^{10} -
81240524632235791551591088128000 y^8 + 3532834921165738901362383519744000 y^6 - 2768098185909537159057208441307136000 y^4 + 287555750217841197328558039951736832000 y^2 +
35491883326517857046563034080311508992000) t^9 + (3643463445479424000 y^{14} - 2328972926320115712000 y^{12} + 4385635172024340971520000 y^{10} +
18995475822735074965585920000 y^8 + 25344947654615336154134740992000 y^6 + 2996874940684462536351843090432000 y^4 - 2505338697728524963401994050994176000 y^2 +
10482905655375375606485906180014080000) t^7 + (- 739001024697729024000 y^{10} + 870677570916597104640000 y^8 - 984446106849699126312960000 y^6 +
3075929722534154251272192000000 y^4 - 199684008143921582419979796480000 y^2 - 153112459270177668627739906670592000) t^5 + (15115930050635366400000 y^6 +
46145911258579646545920000 y^4 - 140506463684236774358384640000 y^2 - 25958172136690406094431846400000) t^3 + (- 544173481822873190400000 y^2 -
783609813824937394176000000) t , $c_0 = (440 + 829440 y^{36} + 309586821120 y^{32} + 68475651442606080 y^{28} + 9939377758197157724160 y^{24} +
98924098531086060327272448 y^{20} + 68380008090468668489821071605760 y^{16} + 3240977937746190422411267979010375680 y^{12} + 100807377775657506898680079219138725150720 y^8 +
1858081587160919167156471220167164981978071040 y^4 + 15411671916547527940062634888554533226518912434176) t^2 + (- 360 y^{36} - 4976640 y^{34} -
2418647040 y^{32} - 1651129712640 y^{30} - 1277974397583360 y^{28} - 293349690780124446720 y^{24} + 79515022065577261793280 y^{22} - 36497395128099963163115520 y^{20} +
16488234975518101005454540800 y^{18} - 2552378774210202035644362915840 y^{16} + 1641120194171248043755705718538240 y^{14} - 88620490485247394362808108801064960 y^{12} +
90747382256893331827515503412290519040 y^{10} - 350025617276588565620416941733120573440 y^8 + 2688196740684200183964802112510366004019200 y^6 +
70161934931857624801481335136520552704901120 y^4 + 33445468568896545008816481963008969675605278720 y^2 + 1538491554169241070405558170298412605077842821120) t^{18} +
(524880 y^{32} + 716636160 y^{30} + 2515392921600 y^{28} + 4175409839918284800 y^{24} - 103535184981220392960 y^{22} + 1492770297059235625697280 y^{20} -
28625407943607814245580800 y^{18} + 236696341933707114043146240000 y^{16} - 3561458754711909817178180812800 y^{14} + 19432743549210064726451025786961920 y^{12} -
236321307960659718300821623469506560 y^{10} + 821511946798243345743231168585872179200 y^8 - 8167264403120399864476395307106146713600 y^6 +
14932092833019268209366986734334923662950400 y^4 - 116130099197557447947279451260447811373629440 y^2 + 40195530584754571670752100667522498711697489920) t^{16} +
(- 755827200 y^{28} - 2779315686604800 y^{26} - 4069308244584038400 y^{22} - 9395009168413084876800 y^{20} + 10769279941746545275699200 y^{18} +
4318970241491608692326400000 y^{16} + 819259175346055643708522496000 y^{14} - 173537642602642043201096201011200 y^{12} - 30514578610371643313582653204070400 y^{10} -
46292667152176496966759579542722969600 y^8 - 126550060412933279150089973679207628800 y^6 + 253674381604210961415650955280872426700800 y^4 +
46749421443461168824262886737875583788646400 y^2 + 78377736220573711613723761592880358804684800) t^{14} + (714256704000 y^{24} - 292559545958400 y^{22} +
3515102944690176000 y^{20} - 8241472623939158016000 y^{18} + 3734971133050075742208000 y^{16} + 1388779557264097863008256000 y^{14} - 2460124362217543653021607526400 y^{12} -
849704701228349817215094305587200 y^{10} + 75831943176255521980334714388480000 y^8 - 77027514863609562478715068873703424000 y^6 +
28157761565235275682453576948383744000 y^4 + 812387581097882274019636445703714530918400 y^2 + 17077558446665307982248318921643942477824000) t^{12} +
(- 99424533196800 y^{20} - 921562569768960000 y^{18} + 516127699788890112000 y^{16} + 174278835941698895872000 y^{14} - 1770000314481841211965440000 y^{12} -
1692378844826005865408495616000 y^{10} - 762246513907490849854442176512000 y^8 + 8550839878891125976681238495232000 y^6 - 43296695260399539003892405263925248000 y^4 +
912652582481664962883231632980967424000 y^2 + 127960954009264970947967361215101717708800) t^{10} + (37952744223744000 y^{16} - 51186218160881664000 y^{14} +
15048748139299209216000 y^{12} - 700050564870897156489216000 y^{10} + 12650541111025250629362647040000 y^8 - 149575181220536202709979627520000 y^6 +
18822613842520643864361883926528000 y^4 - 26728222638946563171224213218394112000 y^2 + 98687999961135863191507666938542161920000) t^8 +
(- 10263903120801792000 y^{12} + 25797853953084358656000 y^{10} - 141243250393136863641600000 y^8 + 117093363350575533816545280000 y^6 -
1893346190948849266087428090000 y^4 - 22017588405210209231139080530296000 y^2 - 1713239420935328354202115104571392000) t^6 + (314915209388236800000 y^8 +
232180685577592279040000 y^6 + 3507089255652053137489920000 y^4 - 8516313249172823596565790720000 y^2 + 184989088234701892457815080960000) t^4 +
(- 2267389507595304960000 y^4 + 104481308509991682556800000 y^2 - 93155534667508557419642880000) t^2 + 81626022273430978560000$