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# REAL STRUCTURES ON SYMMETRIC SPACES

LUCY MOSER-JAUSLIN AND RONAN TERPEREAU

ABSTRACT. We obtain a necessary and sufficient condition for the existence of equivariant real structures on complex symmetric spaces for semisimple groups and discuss how to determine the number of equivalence classes for such structures.

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## INTRODUCTION

A (complex algebraic) *symmetric space* is a complex algebraic  $G$ -variety  $X = G/H$ , where  $G$  is a complex reductive algebraic group,  $\theta \in \text{Aut}_{\mathbb{C}}(G)$  is a non-trivial group involution, and  $H \subseteq G$  is a subgroup such that  $G^{\theta} \subseteq H \subseteq N_G(G^{\theta})$ . The historical motivation for the study of symmetric spaces comes from the *Riemannian symmetric spaces* (see [Hel78] for an exposition); those arise in a wide range of situations in both mathematics and physics, and local models are given by the real loci of certain (complex algebraic) symmetric spaces. Therefore, given a symmetric space, it is natural to ask whether it admits *equivariant real structures* (see § 1.2). The present note aims at providing an answer to this question. In this article, we restrict to the case where  $G$  is semisimple (see Rk. 0.2).

A homogeneous space  $G/H$  is *spherical* if a Borel subgroup of  $G$  acts with an open dense orbit; see [Tim11, Per14] for an exposition of the theory of spherical homogeneous spaces and their equivariant embeddings. Spherical homogeneous spaces are classified in terms of combinatorial data called *homogeneous spherical data* [Tim11, § 30.11]. By Vust [Vus74] symmetric spaces are spherical, thus symmetric spaces are also classified by the homogeneous spherical data. However, these data are quite complicated to handle and also, those corresponding to symmetric spaces have no particular features to distinguish them from those corresponding to non-symmetric spaces. Using these data, a criterion for the existence of equivariant real structures on general spherical homogeneous spaces was obtained by Borovoi and Gagliardi [BG, Th. 1.17] (generalizing results of Akhiezer and Cupit-Foutou

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[ACF14, Akh15, CF15]). However, this criterion can be difficult to apply in specific cases. In particular, in the case where the spherical homogeneous space is a symmetric space, the involution  $\theta$  is not used. The leading goal of this article is to obtain an independent practical criterion using the involution  $\theta$ .

Our main result in this note is the following (Th. 2.9 and Prop. 3.4):

**Theorem 0.1.** *Let  $(G, \sigma)$  be a complex semisimple group with a real group structure. Let  $\theta$  be a (non-trivial regular) group involution on  $G$ , and let  $G^\theta \subseteq H \subseteq N_G(G^\theta)$  be a symmetric subgroup. Then there exists a  $(G, \sigma)$ -equivariant real structure on the symmetric space  $G/H$  if and only if the following holds:*

- *the involutions  $\sigma \circ \theta \circ \sigma$  and  $\theta$  are conjugate by an inner automorphism;*
- *the  $\mathbb{Z}/2\mathbb{Z}$ -action on  $N_G(G^\theta)/G^\theta$  induced by  $\sigma$  (see Def. 2.8) stabilizes  $H/G^\theta$ ; and*
- *$\Delta_H(\sigma) = 0$ , where  $\Delta_H$  is the map defined by  $(*)$  at the end of § 1.1.*

*Moreover, if such a structure exists, then there are exactly  $2^n$  equivalence classes of  $(G, \sigma)$ -equivariant real structures on  $G/H$ , where  $n$  is a non-negative integer than can be calculated explicitly (see § 3 for details).*

*Remark 0.2.* The fact that  $G$  is assumed to be semisimple, and not just reductive, is crucial for Prop. 1.16, Prop. 2.3 (see also Rk. 2.4), and Cor. 2.7.

*Remark 0.3.* The fact that symmetric spaces are spherical is used in the proof of Prop. 1.16, to say that  $N_G(H)/H$  is an abelian group, and in the proof of Cor. 2.7, to apply Prop. 2.5 (which is the only result where some knowledge on the theory of equivariant embeddings for spherical homogeneous spaces is required).

*Remark 0.4.* Let  $X = G/H$  be a symmetric space with a  $(G, \sigma)$ -equivariant real structure  $\mu$  such that  $X(\mathbb{C})^\mu$  is non-empty. Then  $G(\mathbb{C})^\sigma$  acts on  $X(\mathbb{C})^\mu$  with finitely many orbits and a combinatorial description of these orbits using Galois cohomology is provided in [CFT18] (see also [BJ06, Chp. 6]).

In [MJT] we studied the equivariant real structures on *horospherical varieties* which are another class of spherical varieties. The main result [MJT, Th. 0.1] regarding the existence of equivariant real structures on horospherical homogeneous spaces is quite similar to Th. 0.1 but the case of horospherical homogeneous spaces differs greatly from the case of symmetric spaces for the following reasons:

- The homogeneous spherical data corresponding to horospherical homogeneous spaces are easy to discriminate and take a very simple form (see [MJT, § 3.1] for a recap) contrary to the case of symmetric spaces.
- The group  $\text{Aut}_{\mathbb{C}}^G(G/H) \simeq N_G(H)/H$ , which plays a key role when counting the number of equivalence classes of equivariant real structures on  $G/H$ , is a torus for horospherical homogeneous spaces while it is a finite abelian group for symmetric spaces (Prop. 1.16).
- In both cases, an equivariant real structure on  $G/H$  extends to a  $G$ -equivariant embedding  $G/H \hookrightarrow X$  if and only if the corresponding *colored fan* is stable for the induced action of the Galois group  $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$  (see [Hur11, Wed18]), but in the horospherical case the quotient  $X/\Gamma$  is always an algebraic variety while in the symmetric case it can be an algebraic space. Therefore the question of the existence of real forms for symmetric varieties is subtler than for horospherical varieties, and that is the reason why in this note we only consider the homogeneous case.

*Remark 0.5.* A homogeneous space  $G/H$  is *horosymmetric* if it is a homogeneous fibration over a flag variety  $G/P$ , whose fibers are symmetric spaces. This class of spherical homogeneous spaces, which contains both symmetric spaces and horospherical homogeneous spaces, was introduced by Delcroix in [Del]. It would be interesting to determine a nice criterion for the existence of equivariant real structures on horosymmetric spaces from Th. 0.1 and [MJT, Th. 0.1].

In § 1 we recall some definitions and results on real group structures, equivariant real structures, and symmetric spaces. Then in § 2 we prove the necessary and sufficient condition of Th. 0.1 for the existence of equivariant real structures on symmetric spaces (Th. 2.9). Finally, in § 3 we show how to determine the number of equivalence classes for such structures (Prop. 3.4).

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**Convention.** In this article we work over the field of real numbers  $\mathbb{R}$  and over the field of complex numbers  $\mathbb{C}$ . We denote by  $\Gamma$  the Galois group  $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z}$ . We will always denote by  $G$  a complex semisimple algebraic group. When we write *(semi)simple group* we always mean *connected (semi)simple algebraic group*. We refer the reader to [Hum75] for the standard background on algebraic groups.

## 1. GENERAL BACKGROUND ON REAL STRUCTURES AND SYMMETRIC SPACES

This first section is a short recollection of general results on real structures detailed in [MJT, §§ 1-2] and on symmetric spaces.

### 1.1. Real group structures.

**Definition 1.1.** A *real group structure* on the semisimple group  $G$  is an antiregular group involution  $\sigma : G \rightarrow G$ , i.e., a group involution over  $\text{Spec}(\mathbb{R})$  which makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbb{C}) & \xrightarrow{\text{Spec}(z \mapsto \bar{z})} & \text{Spec}(\mathbb{C}) \end{array}$$

Two real group structures  $\sigma$  and  $\sigma'$  on  $G$  are *equivalent* if there exists a (regular) group automorphism  $\varphi \in \text{Aut}_{gr}(G)$  such that  $\sigma' = \varphi \circ \sigma \circ \varphi^{-1}$ .

*Remark 1.2.* The *real locus*  $G(\mathbb{C})^\sigma$  of  $(G, \sigma)$  is a real Lie group.

We can always replace  $G$  by its universal covering space and assume that  $G$  is a simply-connected semisimple group, in which case the real group structures are easy to describe.

**Lemma 1.3.** [MJT, Lem. 1.7] *Assume that  $G = \prod_{i \in I} G_i$  is a simply-connected semisimple group, where the  $G_i$  are its simple factors, and let  $\sigma$  be a real group structure on  $G$ . Then, for a given  $i \in I$ , we have the following possibilities:*

- (i)  $\sigma(G_i) = G_i$  and  $\sigma|_{G_i}$  is a real group structure on  $G_i$ ; or
- (ii) there exists  $j \neq i$  such that  $\sigma(G_i) = G_j$ , then  $G_i \simeq G_j$  and  $\sigma|_{G_i \times G_j}$  is equivalent to  $(g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$ , where  $\sigma_0$  is any real group structure on  $G_i \simeq G_j$ .

The real group structures on complex simply-connected simple groups are well-known (a recap can be found in [MJT, Appendix]); they correspond to real Lie algebra structures on complex simple Lie algebras (see [Kna02, § VI.10] for the classification of those in terms of diagrams).

**Definition 1.4.** If there exists a Borel subgroup  $B \subseteq G$  such that  $\sigma(B) = B$ , then  $\sigma$  is called *quasi-split*. For  $c \in G$  we denote by  $\text{inn}_c$  the inner automorphism of  $G$  defined by

$$\text{inn}_c : G \rightarrow G, g \mapsto cgc^{-1}.$$

If  $\sigma_1$  and  $\sigma_2$  are two real group structures on  $G$  such that  $\sigma_2 = \text{inn}_c \circ \sigma_1$ , for some  $c \in G$ , then  $\sigma_2$  is called an *inner twist* of  $\sigma_1$ .

A quasi-split real group structure always preserves some maximal torus  $T \subseteq B$ . Moreover, any real group structure on  $G$  is equal to the inner twist of a quasi-split real group structure unique up to equivalence [Con14, Prop. 7.2.12].

**Example 1.5.** Let  $n \geq 2$ . There are exactly two inequivalent quasi-split real group structures on  $\text{SL}_{2n}$ . The first one is the split real group structure  $\sigma_{sp} : g \mapsto \bar{g}$ , whose real locus is  $\text{SL}_{2n}(\mathbb{R})$ , and the second is the real group structure defined by  $\sigma_{qs} : g \mapsto K_{n,n} \bar{g}^{-1} K_{n,n}$ , where  $K_{n,n} = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$  and whose real locus is  $\text{SU}(n, n, \mathbb{R})$ .

Recall that we denote  $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z}$ . Let  $\sigma_{qs}$  be a quasi-split real group structure on  $G$ . We consider the short exact sequence of  $\Gamma$ -groups

$$1 \rightarrow Z(G) \rightarrow G \rightarrow G/Z(G) \rightarrow 1,$$

where the  $\Gamma$ -action is induced by  $\sigma_{qs}$ . In other words, the element  $\gamma \in \Gamma$  acts on  $G$  and  $Z(G)$  by  $\sigma_{qs}$ , and on  $G/Z(G)$  by the induced real group structure. Let  $H^1(\Gamma, -)$  denote the first Galois cohomology pointed set (see [Ser02] for more details on Galois cohomology). Since  $Z(G)$  is an abelian group, there is a connecting map

$$\delta : H^1(\Gamma, G/Z(G)) \rightarrow H^2(\Gamma, Z(G)).$$

Tables where the map  $\delta$  is calculated when  $G$  is a simply-connected simple group can be found in [MJT, Appendix A].

**Let now  $H$  be a subgroup of  $G$  such that  $\sigma_{qs}(H) = H$  and  $N_G(H)/H$  is abelian.** Then  $\sigma_{qs}$  induces a real group structure on  $N_G(H)/H$ , namely  $\widehat{\sigma_{qs}}(nH) = \sigma_{qs}(n)H$ , and we can consider the second cohomology group  $H^2(\Gamma, N_G(H)/H)$ . Since it is abelian, we will use the additive notation for this cohomology group, with the neutral element equal to 0. The natural homomorphism  $\chi_H : Z(G) \rightarrow N_G(H)/H$ , induced by the inclusion  $Z(G) \rightarrow N_G(H)$ , yields an homomorphism between the second cohomology groups

$$\chi_H^* : H^2(\Gamma, Z(G)) \rightarrow H^2(\Gamma, N_G(H)/H).$$

In the rest of this article we will denote the composed map  $\chi_H^* \circ \delta$  by

$$(*) \quad \Delta_H : H^1(\Gamma, G/Z(G)) \rightarrow H^2(\Gamma, N_G(H)/H).$$

We recall that  $H^1(\Gamma, G/Z(G)) \simeq \{c \in G \mid c\sigma_{qs}(c) \in Z(G)\} / \equiv$ , where  $c \equiv c'$  if  $c^{-1}b^{-1}c'\sigma_{qs}(b) \in Z(G)$  for some  $b \in G$ .

**Notation 1.6.** If  $\sigma$  is a real group structure equivalent to  $\text{inn}_c \circ \sigma_{qs}$ , then we will write  $\Delta_H(\sigma)$  for the element  $\Delta_H([c])$  of  $H^2(\Gamma, N_G(H)/H)$ . (Note that if  $\text{inn}_c \circ \sigma_{qs} = \text{inn}_{c'} \circ \sigma_{qs}$ , then the classes  $[c]$  and  $[c']$  are equal.)

The element  $\Delta_H(\sigma) \in H^2(\Gamma, N_G(H)/H)$  is a *cohomological invariant* that plays a key-role in our criterion for the existence of equivariant real structures on symmetric spaces; see Th. 0.1.

**1.2. Equivariant real structures.** Let  $\sigma$  be a real group structure on  $G$ .

**Definition 1.7.** A  $(G, \sigma)$ -equivariant real structure on a  $G$ -variety  $X$  is an anti-regular involution  $\mu$  on  $X$  such that

$$\forall g \in G, \forall x \in X, \quad \mu(g \cdot x) = \sigma(g) \cdot \mu(x).$$

Two equivariant real structures  $\mu$  and  $\mu'$  on a  $(G, \sigma)$ -variety  $X$  are *equivalent* if there exists a  $G$ -automorphism  $\varphi \in \text{Aut}_G^G(X)$  such that  $\mu' = \varphi \circ \mu \circ \varphi^{-1}$ .

**Lemma 1.8.** [MJT, Lem. 2.4] *Let  $X = G/H$  be a homogeneous space. Then  $X$  has a  $(G, \sigma)$ -equivariant real structure if and only if there exists  $g \in G$  such that these two conditions hold:*

- (1)  $(G, \sigma)$ -compatibility condition:  $\sigma(H) = gHg^{-1}$
- (2) involution condition:  $\sigma(g)g \in H$

in which case such a structure  $\mu$  on  $X$  is given by  $\mu(kH) = \sigma(k)gH$  for all  $k \in G$ .

*Remark 1.9.* The first condition implies that  $\sigma(g)g \in N_G(H)$ , and so if  $N_G(H) = H$  the second condition is automatically fulfilled.

*Remark 1.10.* If  $H'$  is conjugate to  $H$ , then  $G/H$  has a  $(G, \sigma)$ -equivariant real structure if and only if  $G/H'$  has a  $(G, \sigma)$ -equivariant real structure.

**Proposition 1.11.** [MJT, Prop. 2.8] (see also [Bor, Th.1.6])

Let  $\sigma = \text{inn}_c \circ \sigma_{qs}$ , where  $c \in G$  and  $\sigma_{qs}$  is a quasi-split real groups structure on  $G$ . Assume that  $N_G(H)/H$  is abelian and that  $\sigma_{qs}(H) = H$ . Then

- (i)  $G/H$  has a  $(G, \sigma_{qs})$ -equivariant real structure; and
- (ii)  $G/H$  has a  $(G, \sigma)$ -equivariant real structure if and only if  $\Delta_H(\sigma) = 0$ ,

where  $\Delta_H$  is the map defined by (\*) at the end of § 1.1.

**1.3. Symmetric spaces.** In this section we recall some basic facts on symmetric spaces that we will need in the following. The interested reader is referred to [Tim11, § 26] for a detailed survey on symmetric spaces.

**Definition 1.12.** A subgroup  $H \subseteq G$  is *symmetric* if there exists a (non-trivial regular) group involution  $\theta$  on  $G$  such that  $G^\theta \subseteq H \subseteq N_G(G^\theta)$ . A homogeneous space  $G/H$  is *symmetric* if  $H$  is a symmetric subgroup of  $G$ .

**Example 1.13.** The group  $G$  itself can be viewed as a symmetric space for the action of  $G \times G$  by left and right multiplication. Indeed,  $G \simeq (G \times G)/H$ , where  $H = (G \times G)^\theta$  with  $\theta(g_1, g_2) = (g_2, g_1)$ .

**Example 1.14.** Let  $n \geq 2$  and let  $G = \text{SL}_{2n}$ . Then  $\theta : g \mapsto J({}^t g^{-1})^t J$ , with  $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ , is a group involution on  $G$ . We have  $G^\theta = \text{Sp}_{2n}$  and  $N_G(G^\theta) = \langle Z(G), G^\theta \rangle$ , and thus  $N_G(G^\theta)/G^\theta \simeq Z(G)/(Z(G) \cap G^\theta) \simeq \mathbb{Z}/n\mathbb{Z}$ .

As in the case of real group structures (see Lem. 1.3) we can always replace  $G$  by its universal covering space and assume that  $G$  is simply-connected, in which case the (regular) group involutions are easily described.

**Lemma 1.15.** *Let  $\theta$  be a group involution on a simply-connected semisimple group  $G = \prod_{i \in I} G_i$ , where the  $G_i$  are the simple factors of  $G$ . Then, for a given  $i \in I$ , we have the following possibilities:*

- (i)  $\theta(G_i) = G_i$  and  $\theta|_{G_i}$  is a group involution on  $G_i$ ; or
- (ii) there exists  $j \neq i$  such that  $\theta(G_i) = G_j$ , then  $G_i \simeq G_j$  and  $\theta|_{G_i \times G_j}$  is conjugate to  $(g_1, g_2) \mapsto (g_2, g_1)$ .

*Proof.* We use the fact that the factors  $G_i$  are the unique simple normal subgroups of  $G$  (see [Con14, Th. 5.1.19]). In particular, any group automorphism of  $G$  permutes the factors. Since  $\theta$  is a group involution, either  $\theta(G_i) = G_i$  and we get (i), or  $\theta(G_i) = G_j$  for some  $j \neq i$ . In the second case,  $G_i$  and  $G_j$  are then isomorphic. Therefore  $G_i \times G_j \simeq H \times H$ , for some simply-connected simple group  $H$ , and  $\theta|_{G_i \times G_j}$  identifies with  $\theta_{H \times H} : (h_1, h_2) \mapsto (\psi(h_2), \psi^{-1}(h_1))$  for some group automorphism  $\psi$  on  $H$ . But then it suffices to conjugate  $\theta_{H \times H}$  with the group automorphism defined by  $(h_1, h_2) \mapsto (\psi(h_2), h_1)$  to get the involution  $(g_1, g_2) \mapsto (g_2, g_1)$ .  $\square$

Conjugacy classes of (regular) group involutions on simple groups can be classified by using either Kac diagrams or Satake diagrams; see [Tim11, § 26.5] for more details on these classifications and [Tim11, Table 26.3] for the list of conjugacy classes of (regular) group involutions on simple groups.

**Proposition 1.16.** *For any symmetric subgroup  $G^\theta \subseteq H \subseteq N_G(G^\theta)$ , the quotient group  $N_G(H)/H$  is abelian and finite. Moreover, if  $G$  is simply-connected, then  $G^\theta$  is connected, and so  $G^\theta = H^0 = N_G(H)^0 = N_G(G^\theta)^0$ .*

*Proof.* Symmetric subgroups are spherical (see [Vus74] or [Tim11, Th. 26.14]), and if  $H$  is a spherical subgroup of  $G$ , then  $N_G(H)/H$  is abelian (see [BP87, § 5.2] or [Kno91, Th. 6.1]). The connectedness of  $G^\theta$  for  $G$  simply-connected was proved by Steinberg in [Ste68, § 8], and the finiteness of  $N_G(H)/H$  then follows from the work of De Concini and Procesi in [DCP83, § 1.7] (see also [Vus90, § 2.2]).  $\square$

## 2. EXISTENCE OF EQUIVARIANT REAL STRUCTURES ON SYMMETRIC SPACES

In this section we will always denote by  $\theta$  a (regular non-trivial) group involution on  $G$ , by  $\sigma_{qs}$  a quasi-split real group structure on  $G$ , and by  $\sigma = \text{inn}_c \circ \sigma_{qs}$  a real group structure on  $G$  obtained as an inner twist of  $\sigma_{qs}$ .

**Notation 2.1.** To simplify the notation we will denote  ${}^\psi\theta = \psi \circ \theta \circ \psi^{-1}$  for any (regular or antiregular) group automorphism  $\psi$ . Also, if  $\theta_1$  and  $\theta_2$  are two group involutions on  $G$ , we will write  $\theta_1 \sim \theta_2$  when they are conjugate by an inner automorphism of  $G$ .

The following example shows that the combinatorial invariants of the conjugacy class of  $\theta$  (such as Kac diagrams or Satake diagrams) are too coarse to determine the existence of a  $(G, \sigma)$ -equivariant real structure on the symmetric space  $G/G^\theta$ .

**Example 2.2.** Let  $G = \text{SL}_n^{\times 3}$  with  $n \geq 2$ , and let  $\sigma : (g_1, g_2, g_3) \mapsto (\overline{g_2}, \overline{g_1}, {}^t\overline{g_3}^{-1})$  be a real group structure on  $G$ . We give an example of two group involutions  $\theta$  and  $\theta'$  that are conjugate (by an outer automorphism of  $G$ ) such that  $G/G^\theta$  admits a  $(G, \sigma)$ -equivariant real structure but  $G/G^{\theta'}$  does not. Let  $\theta : (g_1, g_2, g_3) \mapsto (g_2, g_1, {}^t g_3^{-1})$ , let  $\psi : (g_1, g_2, g_3) \mapsto (g_3, g_2, g_1)$ , and let  $\theta' = {}^\psi\theta$ . Then  $\sigma(G^\theta) = G^\theta$  while  $\sigma(G^{\theta'})$  is not conjugate to  $G^{\theta'}$  in  $G$ , and we conclude with Lem. 1.8.

Therefore, a criterion for the existence of a  $(G, \sigma)$ -equivariant real structure on  $G/G^\theta$  should depend on  $\theta$  up to a conjugate by an **inner** automorphism of  $G$ , which is indeed the case in Th. 0.1.

The next result is well-known to specialists but we give a proof because of a lack of suitable reference. We thank Michael Bulois for indicating us the sketch of the proof.

**Proposition 2.3.** *The group involution  $\theta$  on the semisimple group  $G$  is uniquely determined by its fixed locus  $G^\theta$ .*

*Proof.* Any group involution on  $G$  lifts uniquely to a group involution on the universal covering space of  $G$ . Therefore, we can assume that  $G$  is simply-connected. As group involutions on simply-connected semisimple groups correspond to Lie algebra involutions on semisimple Lie algebras (see [Pro07, § 4.3.4]), it suffices to prove that if  $\Theta (= D_e \theta)$  is a Lie algebra involution on  $\mathfrak{g} = \text{Lie}(G)$ , then  $\mathfrak{g}^\Theta$  determines  $\Theta$ .

Since  $\mathfrak{g}$  is semisimple, it identifies with a direct sum of simple Lie algebras  $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ . Let  $\mathfrak{l}_i = \{(0, \dots, 0, *, 0, \dots, 0)\} \simeq \mathfrak{g}_i$  be the Lie subalgebra of  $\mathfrak{g}$  formed by elements whose all coordinates but the  $i$ -th vanish. Any Lie algebra automorphism of  $\mathfrak{g}$  permutes the  $\mathfrak{l}_i$ . Hence, either  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_i$  and  $\Theta|_{\mathfrak{l}_i}$  is a Lie algebra involution on  $\mathfrak{l}_i$  or  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_j$  for some  $i \neq j$ . We have  $\mathfrak{l}_i \cap \mathfrak{g}^\Theta \neq \{0\}$  if and only if  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_i$ , and so  $\mathfrak{g}^\Theta$  determines the set of indices  $I_0 = \{i \in I, \Theta(\mathfrak{l}_i) = \mathfrak{l}_i\}$ . Moreover, for all  $i, j \in I \setminus I_0$ , we have  $(\mathfrak{l}_i + \mathfrak{l}_j) \cap \mathfrak{g}^\Theta \neq 0$  if and only if  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_j$ , and so  $\mathfrak{g}^\Theta$  also determines the pairs of indices corresponding to the  $\mathfrak{l}_i$  that are switched by  $\Theta$ .

Let  $i < j$  such that  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_j$ . Then

$$(\mathfrak{l}_i + \mathfrak{l}_j) \cap \mathfrak{g}^\Theta = \{(0, \dots, 0, \varphi(z), 0, \dots, 0, z, 0, \dots, 0)\},$$

where  $\varphi$  is some Lie algebra isomorphism  $\mathfrak{l}_j \simeq \mathfrak{l}_i$ , and

$$\Theta|_{\mathfrak{l}_i + \mathfrak{l}_j}: \begin{array}{ccc} \mathfrak{l}_i + \mathfrak{l}_j & \rightarrow & \mathfrak{l}_i + \mathfrak{l}_j \\ (0, \dots, 0, x, 0, \dots, 0, y, 0, \dots, 0) & \mapsto & (0, \dots, 0, \varphi(y), 0, \dots, 0, \varphi^{-1}(x), 0, \dots, 0). \end{array}$$

So it remains only to prove that if  $\Theta(\mathfrak{l}_i) = \mathfrak{l}_i$ , then  $\mathfrak{l}_i^\Theta (= \mathfrak{g}^\Theta \cap \mathfrak{l}_i)$  determines  $\Theta|_{\mathfrak{l}_i}$ . Hence, we can assume that  $\mathfrak{g}$  is simple. Let  $\mathfrak{g}_0 = \mathfrak{g}^\Theta$ , and let  $\mathfrak{g}_1 \subseteq \mathfrak{g}$  be the subspace on which  $\Theta$  acts as the scalar  $-1$ . We want to show that the Lie subalgebra  $\mathfrak{g}_0$  determines the  $\mathfrak{g}_0$ -submodule  $\mathfrak{g}_1$ .

If the involution  $\Theta$  is an inner automorphism of  $\mathfrak{g}$ , then  $\mathfrak{g}_0$  contains a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  (see [Tim11, § 26.3]), and so  $\mathfrak{g}_1$  is an  $\mathfrak{h}$ -stable complement of  $\mathfrak{g}_0$  in  $\mathfrak{g}$ . Since the root subspaces of  $\mathfrak{g}$  are 1-dimensional, the  $\mathfrak{h}$ -submodule  $\mathfrak{g}_1$  is necessarily the sum of all the root subspaces not contained in  $\mathfrak{g}_0$ . Therefore  $\mathfrak{g}_1$  is uniquely determined by  $\mathfrak{g}_0$ , and so  $\Theta$  is uniquely determined by  $\mathfrak{g}^\Theta$ .

If the involution  $\Theta$  is an outer automorphism of  $\mathfrak{g}$  (only possible for ADE type), then  $\mathfrak{g}_1$  is an irreducible  $\mathfrak{g}_0$ -submodule of  $\mathfrak{g}$  by [Kac80, Prop. 3.1]. If the  $\mathfrak{g}_0$ -submodule  $\mathfrak{g}_0$  of  $\mathfrak{g}$  does not contain a summand isomorphic to  $\mathfrak{g}_1$ , then there is a unique  $\mathfrak{g}_0$ -stable complement of  $\mathfrak{g}_0$  in  $\mathfrak{g}$ , and so this complement must be  $\mathfrak{g}_1$ . Using the classification of symmetric spaces given in [Tim11, Table 26.3] we verify case by case that this is indeed the case.  $\square$



*Remark 2.4.*

- (1) Prop. 2.3 is true more generally for a connected reductive algebraic group  $L$  whose center  $Z(L)$  has dimension at most 1. But it is not true if  $\dim(Z(L)) \geq 2$ . Consider for instance  $L = Z(L) = \mathbb{G}_m^2$ , and the group involutions  $\theta_1(x, y) = (y, x)$  and  $\theta_2(x, y) = (x^2y^{-1}, x^3y^{-2})$ . Then  $L^{\theta_1} = L^{\theta_2} = \{(t, t), t \in \mathbb{G}_m\}$  but  $\theta_1 \neq \theta_2$ .
- (2) If  $\theta_1, \theta_2$  are two group involutions on a connected reductive algebraic group  $L$  such that  $L^{\theta_1} = L^{\theta_2}$ , then we can verify that  $\theta_1$  and  $\theta_2$  are conjugate in  $\text{Aut}_{gr}(L)$ . Hence, for connected reductive algebraic groups, the fixed locus determines the conjugacy class of a group involution.

The next result was proved by Akhiezer and Cupit-Foutou in [ACF14, Th. 4.4], but stated for a split real group structure on  $G$ . It was then generalized over arbitrary fields of characteristic zero by Snegirov in [Sne, Th. 1.1], but then the proof is more technical. For sake of completeness and for the reader's convenience, we reproduce their proof in our setting.

**Proposition 2.5.** *If  $H \subseteq G$  is a spherical subgroup satisfying  $N_G(N_G(H)) = N_G(H)$  and  $\sigma_{qs}(H) = gHg^{-1}$  for some  $g \in G$ , then there exists a subgroup  $H' \subseteq G$  conjugate to  $H$  such that  $\sigma_{qs}(H') = H'$ .*

*Proof.* (For this proof, and this proof only, the reader is assumed to be a bit familiar with the theory of equivariant embeddings for spherical homogeneous spaces; see [Tim11, Per14] for an exposition.)

Let  $N = N_G(H)$ . The condition  $\sigma_{qs}(H) = gHg^{-1}$  implies that  $\sigma_{qs}(N) = gNg^{-1}$ . Hence, by Lem. 1.8 and Rk. 1.9, the  $G$ -variety  $Y = G/N$  has a  $(G, \sigma_{qs})$ -equivariant real structure that we denote by  $\mu$ . Also, this real structure is unique since  $\text{Aut}_C^G(Y) \simeq N_G(N)/N = \{1\}$ .

By [Kno96, Cor. 7.2] the variety  $Y$  admits a wonderful compactification  $\bar{Y}$ , which is smooth, projective (see [Avd15, Prop. 3.18]), and has a unique closed orbit  $Y_0$ , which is therefore a flag variety  $G/P$ . The colored fan of the  $G$ -equivariant embedding  $Y \hookrightarrow \bar{Y}$  is determined by the cone  $(\mathcal{V}, \emptyset)$ , where  $\mathcal{V}$  is the valuation cone of  $Y$ , which is stable for the  $\Gamma$ -action on the set of colored cones induced by  $\sigma_{qs}$  (see [Hur11] or [Wed18] for details on this  $\Gamma$ -action). Hence, the equivariant real structure  $\mu$  on  $Y$  extends on  $\bar{Y}$  by [Wed18, Th. 9.1] (see also [Hur11, Th. 2.23]).

The restriction  $\mu_0 = \mu|_{Y_0}$  is a  $(G, \sigma_{qs})$ -equivariant real structure on  $Y_0 = G/P$ . Thus, by Lem. 1.8, the parabolic subgroups  $\sigma_{qs}(P)$  and  $P$  are conjugate. By [MJT, Prop. 3.9], there exists a parabolic subgroup  $P'$  conjugate to  $P$  such that  $\sigma_{qs}(P') = P'$  (since  $\sigma_{qs}$  is quasi-split). Hence,  $\mu'_0(kP') = \sigma_{qs}(k)P'$  is a  $(G, \sigma_{qs})$ -equivariant real structure on  $Y_0$ , equivalent to  $\mu_0$ , with a fixed point  $eP'$ . Thus  $\mu_0$  has a fixed point in  $Y_0$ ; in particular,  $\mu$  has a fixed point in  $\bar{Y}$ . By [Man17, Cor. 2.2.10], since  $\bar{Y}$  is smooth, the set of  $\mu$ -fixed points is Zariski dense in  $\bar{Y}$ , and so  $\mu$  has a fixed point in the open orbit  $Y = G/N$ .

Let  $g_0N$  be a  $\mu$ -fixed point in  $Y$ . Let  $\sigma'_{qs} = \text{inn}_{g_0}^{-1} \circ \sigma_{qs} \circ \text{inn}_{g_0}$ , and let  $\mu'$  be the  $(G, \sigma'_{qs})$ -equivariant real structure defined by  $\mu'(kN) = g_0^{-1}\mu(g_0kN)$ . Then  $\mu'(eN) = eN$ , and computing the stabilizers on both sides yields  $\sigma'_{qs}(N) = N$ . Thus

$$N = \sigma'_{qs}(N) = g_0^{-1}\sigma_{qs}(g_0)\sigma_{qs}(N)\sigma_{qs}(g_0^{-1})g_0 = g_0^{-1}\sigma_{qs}(g_0)gNg^{-1}\sigma_{qs}(g_0^{-1})g_0$$

and so  $g_0^{-1}\sigma_{qs}(g_0)g \in N_G(N) = N$ . It follows that  $\sigma_{qs}(H') = H'$ , where  $H' = g_0Hg_0^{-1}$ . This concludes the proof.  $\square$

*Remark 2.6.* We do not know whether the assumption  $N_G(N_G(H)) = N_G(H)$  can be dropped to get the conclusion of Prop. 2.5. Indeed, as remarked by Avdeev in [Avd13], it is not true that  $N_L(N_L(H)) = N_L(H)$  for any spherical subgroup  $H$  of a connected reductive algebraic group  $L$ . Here we give a simple counter-example pointed to us by Bart Van Steirteghem. Let

$$L = \mathrm{GL}_2 \text{ and } H = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \text{ with } a \in \mathbb{C}^* \right\}.$$

Then  $N_L(H) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ with } a, b \in \mathbb{C}^* \right\}$  and  $N_L(N_L(H))/N_L(H) \simeq \mathbb{Z}/2\mathbb{Z}$ .

We thank Jacopo Gandini for pointing to us that if  $H \subseteq G$  is a spherical subgroup such that  $N_G(H)/H$  is finite, then  $N_G(N_G(H)) = N_G(H)$ . This is crucial in the proof of the following result.

**Corollary 2.7.** *Let  $H$  be a symmetric subgroup of  $G$  such that  $\sigma_{qs}(H)$  is conjugate to  $H$ . Then there exists a subgroup  $H' \subseteq G$  conjugate to  $H$  such that  $\sigma_{qs}(H') = H'$ . Equivalently, there exists  $\sigma'_{qs}$  conjugate to  $\sigma_{qs}$  by an inner automorphism of  $G$  such that  $\sigma'_{qs}(H) = H$ .*

*Proof.* Since symmetric subgroups are spherical, it suffices to verify that  $N_G(N_G(H)) = N_G(H)$  and then to apply Prop. 2.5.

Denoting  $K = N_G(H^0)$ , we have the inclusions  $H^0 \subseteq H \subseteq N_G(H) \subseteq K$ . Then  $H$  is a normal subgroup of  $K$  if and only if  $H/H^0$  is a normal subgroup of  $K/H^0$ , which is true because  $H^0$  is a spherical subgroup of  $G$  (since  $H$  is spherical), and so  $K/H^0$  is an abelian group (see [BP87, § 5.2] or [Kno91, Th. 6.1]). Hence,  $K \subseteq N_G(H)$ , which yields  $K = N_G(H)$ . As  $H$  is a symmetric subgroup of a semisimple group, the group  $N_G(H)/H$  is finite (Prop. 1.16). Therefore  $H^0$  has finite index in  $K$ . It follows that  $K^0 = H^0$ , and thus  $N_G(K) \subseteq N_G(K^0) = N_G(H^0) = K$ .

Finally,  $H' = cHc^{-1}$  satisfies  $\sigma_{qs}(H') = H'$  if and only if  $\sigma'_{qs} = \mathrm{inn}_c^{-1} \circ \sigma_{qs} \circ \mathrm{inn}_c$  satisfies  $\sigma'_{qs}(H) = H$ , which proves the last statement of the corollary.  $\square$

Before stating Th. 2.9, which is the main result of this § 2, we need to define the action of the Galois group  $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$  on  $N_G(G^\theta)/G^\theta$ .

**Definition 2.8.** Let  $\sigma = \mathrm{inn}_c \circ \sigma_{qs}$  be a real group structure on  $G$ . If  ${}^\sigma\theta \sim \theta$ , then  $\sigma_{qs}(G^\theta) = gG^\theta g^{-1}$  for some  $g \in G$ . Hence, by Cor. 2.7, there exists a quasi-split real group structure  $\sigma'_{qs}$ , equivalent to  $\sigma_{qs}$ , such that  $\sigma'_{qs}(G^\theta) = G^\theta$ . Then  $\sigma'_{qs}(N_G(G^\theta)) = N_G(G^\theta)$ , and so  $\sigma'_{qs}$  induces a real group structure  $\tau$  on  $N_G(G^\theta)/G^\theta$  defined by  $\tau(nG^\theta) = \sigma'_{qs}(n)G^\theta$ . The  $\Gamma$ -action on  $N_G(G^\theta)/G^\theta$  that we will consider in the following is the one given by  $\tau$ . (Note that this  $\Gamma$ -action does not depend on the choice of  $\sigma'_{qs}$  in the conjugacy class of  $\sigma_{qs}$  by inner automorphisms.)

**Theorem 2.9.** *Let  $G$  be a complex semisimple group with a real group structure  $\sigma = \mathrm{inn}_c \circ \sigma_{qs}$ . Let  $\theta$  be a group involution on  $G$  and let  $G^\theta \subseteq H \subseteq N_G(G^\theta)$  be a symmetric subgroup. Then the following four conditions are equivalent:*

- (i)  $G/H$  has a  $(G, \sigma_{qs})$ -equivariant real structure;
- (ii)  ${}^\sigma\theta \sim \theta$  and the  $\Gamma$ -action on  $N_G(G^\theta)/G^\theta$  of Def. 2.8 stabilizes  $H/G^\theta$ ;
- (iii)  $H$  is conjugate to  $\sigma_{qs}(H)$ ;

(iv)  $H$  is conjugate to  $\sigma(H)$ .

Moreover  $G/H$  has a  $(G, \sigma)$ -equivariant real structure if and only if the (equivalent) conditions (i)-(iv) are satisfied and  $\Delta_H(\sigma) = 0$  with  $\Delta_H$  the map defined by (\*).

*Proof.* The equivalence of (i) and (iii) follows from Lem. 1.8 and Cor. 2.7. Indeed, if  $H$  is conjugate to  $\sigma_{qs}(H)$ , then we can find  $H'$  conjugate to  $H$  such that  $\sigma_{qs}(H') = H'$ . By Rk. 1.10, we can replace  $H$  by  $H'$  and then the two conditions of Lem. 1.8 are satisfied with  $g = 1$ .

The equivalence of (iii) and (iv) follows from the fact that  $\sigma = \text{inn}_c \circ \sigma_{qs}$ .

We now prove the equivalence of (ii) and (iii). We can always replace  $G$  by its universal covering space and assume that it is simply-connected. Then by Prop. 1.16 the group  $G^\theta$  is connected and  $N_G(G^\theta)^0 = H^0 = G^\theta$ . Also, by Rk. 1.10 and Cor. 2.7 we can replace (iii) by the condition (iii)' given by  $\sigma_{qs}(H) = H$ . (This boils down to conjugate  $\theta$  by some inner automorphism.) Then

$$\begin{aligned} \sigma_{qs}(H^0) = H^0 &\Leftrightarrow \sigma(H^0) = cH^0c^{-1} \Leftrightarrow \sigma(G^\theta) = cG^\theta c^{-1} \\ &\Leftrightarrow G^{\sigma\theta} = G^{\text{inn}_c\theta} \\ &\Leftrightarrow \sigma\theta = \text{inn}_c\theta \Leftrightarrow \sigma\theta \sim \theta \end{aligned}$$

where the penultimate equivalence comes from Prop. 2.3. Also,  $\sigma_{qs}(G^\theta) = G^\theta$  implies that  $\sigma_{qs}(N_G(G^\theta)) = N_G(G^\theta)$ , and so the  $\Gamma$ -action on  $N_G(G^\theta)/G^\theta$  stabilizes  $H/G^\theta$  if and only if  $\sigma_{qs}(H) = H$ . This finishes to prove the equivalence (ii)  $\Leftrightarrow$  (iii).

Finally, the last claim of the theorem follows from Prop. 1.11.  $\square$

*Remark 2.10.* If  $N_G(G^\theta)/G^\theta$  is a cyclic group, then the  $\Gamma$ -action stabilizes each subgroup of  $N_G(G^\theta)/G^\theta$ , which simplifies the condition (ii) in Th. 2.9.

**Corollary 2.11.** *Let  $G$ ,  $\sigma_{qs}$ , and  $\theta$  be as in Th. 2.9. Then  $G/G^\theta$  has a  $(G, \sigma_{qs})$ -equivariant real structure if and only if  $G/N_G(G^\theta)$  does, and this is the case if and only if  $\sigma\theta \sim \theta$ .*

*Proof.* If  $H = G^\theta$  or  $H = N_G(G^\theta)$ , then the  $\Gamma$ -action on  $N_G(G^\theta)/G^\theta$  trivially stabilizes  $H/G^\theta$ , and so the result follows from the equivalence of (i) and (ii) in Th. 2.9.  $\square$

**Example 2.12.** Let  $G = \text{SL}_n \times \text{SL}_n$  with  $n$  odd and  $n \geq 3$ , let  $\sigma: (g_1, g_2) \mapsto (\overline{g_2}, \overline{g_1})$ , and let  $\theta: (g_1, g_2) \mapsto ({}^t g_1^{-1}, {}^t g_2^{-1})$ . Then  $\sigma\theta = \theta$  and  $N_G(G^\theta)/G^\theta \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  on which  $\Gamma$  acts by  $\gamma \cdot (a, b) = (b^{-1}, a^{-1})$ . Thus, since  $\sigma$  is quasi-split (it preserves the usual Borel subgroup), it follows from Th. 2.9 that there exists a  $(G, \sigma)$ -equivariant real structure on the symmetric space  $G/H$  if and only if  $(a, b) \in H/G^\theta$  implies  $(b, a) \in H/G^\theta$ , that is,  $H/G^\theta$  is stable under the operation of exchanging the two factors of  $N_G(G^\theta)/G^\theta$ .

**Example 2.13.** Let  $n \geq 2$  and let  $G = \text{SL}_{2n}$ . Let  $\sigma$  be a real group structure on  $G$  obtained by an inner twist of  $\sigma_{qs}$ , where  $\sigma_{qs}$  is the quasi-split real group structure defined in Example 1.5, let  $\theta$  be the group involution defined in Example 1.14, and let  $G^\theta \subseteq H \subseteq N_G(G^\theta)$ . Then  $\sigma_{qs}\theta = \theta$  (and so  $\sigma\theta \sim \theta$ ), and the  $\Gamma$ -action on  $N_G(G^\theta)/G^\theta \simeq \mathbb{Z}/n\mathbb{Z}$  stabilizes  $H/G^\theta$  by Rk. 2.10. Hence, by Th. 2.9, the symmetric space  $G/H$  has a  $(G, \sigma_{qs})$ -equivariant real structure.

It remains to compute  $\Delta_H(\sigma)$  to determine whether  $G/H$  has a  $(G, \sigma)$ -equivariant real structure. Let  $S = \{0, \dots, n\}$ . The equivalence classes of the real group structures on  $G$  obtained as an inner twist of  $\sigma_{qs}$  are in bijection with  $S$ . For  $s \in S$ , we

denote by  $\sigma_s$  the real group structure whose real locus  $G(\mathbb{C})^{\sigma_s}$  is  $\mathrm{SU}(n+s, n-s, \mathbb{R})$ . Borovoi determined in [MJT, Table 2] that  $H^2(\Gamma, Z(G)) \simeq Z(G)/2Z(G) \simeq \mathbb{Z}/2\mathbb{Z}$  and that  $\delta(\sigma_s) = s \pmod{2}$ . Let  $\xi$  be a primitive  $2n$ -th root of unity. Then  $H = \langle \xi^r I_{2n}, G^\theta \rangle$ , for some positive integer  $r$  dividing  $2n$ , and  $A := N_G(H)/H \simeq Z(G)/(Z(G) \cap H) \simeq \mathbb{Z}/t\mathbb{Z}$  with  $t = \gcd(r, n)$ . We verify that the  $\Gamma$ -action on  $Z(G)$  (and so also on  $A$ ) is trivial, thus

$$H^2(\Gamma, A) \simeq A/2A \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } t \text{ is even;} \\ \{0\} & \text{if } t \text{ is odd.} \end{cases}$$

The map  $\chi_H^* : H^2(\Gamma, Z(G)) \simeq Z(G)/2Z(G) \rightarrow H^2(\Gamma, A) \simeq A/2A$  defined in § 1.1 is the map induced by the quotient map  $Z(G) \rightarrow A \simeq Z(G)/(Z(G) \cap H)$ , hence it is the identity map if  $t$  is even (resp. the trivial map if  $t$  is odd). It follows that  $\Delta_H(\sigma_s) = 0$  if and only if  $s$  is even or  $t$  is odd. Therefore,  $G/H$  has a  $(G, \sigma_s)$ -equivariant real structure if and only if  $s$  is even or  $t$  is odd.

### 3. NUMBER OF EQUIVALENCE CLASSES

As before let  $G$  be a semisimple group, let  $\sigma$  be a real group structure on  $G$ , and let  $X = G/H$  be a symmetric space. **We suppose that there exists a  $(G, \sigma)$ -equivariant real structure  $\mu$  on  $X$ .** Then  $\mu$  determines a  $\Gamma$ -action on  $A = \mathrm{Aut}_{\mathbb{C}}^G(X) \simeq N_G(H)/H$ ; indeed, the generator  $\gamma$  acts on  $A$  by  $\mu$ -conjugation.

**Notation 3.1.** In this section, and contrary to the previous examples, we will follow the usual conventions and use the multiplicative notation for the group law in  $A$ , even if  $A$  is a finite abelian group in our case by Prop. 1.16.

**Definition 3.2.** If  $A$  is a  $\Gamma$ -group, then the first Galois cohomology pointed set is  $H^1(\Gamma, A) = Z^1(\Gamma, A)/\sim$ , where  $Z^1(\Gamma, A) = \{a \in A \mid a^{-1} = \gamma a\}$  and two elements  $a_1, a_2 \in Z^1(\Gamma, A)$  satisfy  $a_1 \sim a_2$  if  $a_2 = b^{-1} a_1 \gamma b$  for some  $b \in A$ .

*Remark 3.3.* If  $A$  is an abelian group, then  $H^1(\Gamma, A)$  is an abelian group. Moreover,  $a^2 = a(a^{-1})^{-1} = a(\gamma a)^{-1} \sim 1$  for all  $a \in Z^1(\Gamma, A)$ . In the case where  $H^1(\Gamma, A)$  is finite, this implies that its cardinal is a power of 2.

By [MJT, Lem. 2.11] the set of equivalence classes of  $(G, \sigma)$ -equivariant real structures on  $X$  is in bijection with the set  $H^1(\Gamma, A)$ , hence our goal in this section is to determine the cardinal of  $H^1(\Gamma, A)$ .

Before stating the next result, we need some extra notation. Let  $\Gamma' = \{e, \gamma'\} \simeq \mathbb{Z}/2\mathbb{Z}$  acting on  $A$  by  $\gamma' a = \gamma a^{-1}$ . (This  $\Gamma'$ -action is well-defined since  $A$  is abelian.) For  $p$  a prime number, let  $A_p$  be the maximal  $p$ -subgroup of  $A$ .

**Proposition 3.4.** *Let  $G$  be a semisimple group with a real group structure  $\sigma$ , and let  $X = G/H$  be a symmetric space. We suppose that  $X$  has a  $(G, \sigma)$ -equivariant real structure  $\mu$ , and we consider the actions of  $\Gamma$  and  $\Gamma'$  on  $A$  defined above.*

- (i) *There exists  $n \geq 0$  such that  $H^1(\Gamma, A) \simeq H^1(\Gamma, A_2) \simeq (\mathbb{Z}/2\mathbb{Z})^n$ . In particular, there are  $2^n$  equivalence classes of  $(G, \sigma)$ -equivariant real structures on  $G/H$ .*
- (ii) *The integer  $n$  can be calculated explicitly as follows:  $|A_2^{\Gamma'}| \cdot |A_2^\Gamma|/|A_2| = 2^n$ .*

*Proof.* (i): In our situation, the group  $A$  is a finite abelian group (Prop. 1.16). Hence,  $A$  is isomorphic to a finite product of abelian  $p$ -groups  $A \simeq \prod_p A_p$ , and each  $A_p$  is  $\Gamma$ -stable since  $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$  acts on  $A$  by group involution. Thus  $H^1(\Gamma, A) = \prod_p H^1(\Gamma, A_p)$ . But each  $H^1(\Gamma, A_p)$  is itself an abelian  $p$ -group (by

definition of the Galois cohomology), and since every element of  $H^1(\Gamma, A)$  is 2-torsion (Rk. 3.3), we see that  $H^1(\Gamma, A_p) = \{1\}$  if  $p \neq 2$ . Therefore

$$H^1(\Gamma, A) \simeq H^1(\Gamma, A_2) \simeq (\mathbb{Z}/2\mathbb{Z})^n \text{ for some } n \geq 0.$$

(ii): In order to calculate  $n$  (or, more precisely,  $2^n$ ), we consider certain subgroups of  $A_2$ . Let us note that  $Z := Z^1(\Gamma, A_2) = A_2^{\Gamma'}$  is a subgroup of  $A_2$ , and  $H^1(\Gamma, A) = H^1(\Gamma, A_2) = Z/B$ , where  $B$  is the subgroup of  $Z$  defined by  $B = \{a \cdot \gamma a^{-1}; a \in A_2\}$ . The map  $\varphi : A_2 \rightarrow B$  given by  $\varphi(a) = a \cdot \gamma a^{-1}$  is a surjective group homomorphism (since  $A_2$  is abelian). The kernel is exactly  $A_2^\Gamma$ . Thus, the cardinality of  $H^1(\Gamma, A_2)$  is given by  $|A_2^{\Gamma'}|/|B|$ , and  $|B| = |A_2|/|A_2^\Gamma|$ . This proves the result.  $\square$

*Remark 3.5.* It is easy to give an upper-bound for  $n$ . Suppose that  $A_2$  is a product of  $r$  cyclic groups. Then  $Z = Z^1(\Gamma, A_2)$  is a subgroup of  $A_2$ , and therefore a product of  $r'$  cyclic groups, where  $r' \leq r$ . In particular,  $H^1(\Gamma, A_2)$  is a quotient of the group  $Z/Z^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{r'}$ . This shows that  $n \leq r' \leq r$ .

**Corollary 3.6.** *Let  $G$  be a semisimple group with a real group structure  $\sigma$ , and let  $X = G/H$  be a symmetric space with  $G^\theta \subseteq H \subseteq G$ . Suppose that  $X$  has a  $(G, \sigma)$ -equivariant real structure and that  $A = N_G(H)/H$  is cyclic of order  $m$ . If  $m$  is odd, then the real structure is unique up to equivalence, and if  $m$  is even, there are exactly 2 inequivalent real structures on  $X$ .*

*Proof.* If  $m$  is odd, then  $A_2$  is trivial, and the result holds. If  $m$  is even, then  $A_2$  is cyclic of order at least two. There are two possible  $\Gamma$ -actions on  $A_2$ . Either the action is trivial, or  $\gamma a = a^{-1}$  for all  $a \in A$ , in which case the  $\Gamma'$ -action is trivial. In both cases, since  $A_2$  has a unique element of order two, the result holds.  $\square$

**Example 3.7.** We pursue Example 2.13. We saw that  $G/H$  has a  $(G, \sigma_s)$ -equivariant real structure if and only if  $s$  is even or  $t$  is odd. Moreover, the group  $N_G(H)/H$  is cyclic, so Corollary 3.6 applies. We find that the the number of equivalence classes of  $(G, \sigma_s)$ -equivariant real structures on  $G/H$  is given by

$$|A_2^{\Gamma'}| \cdot |A_2^\Gamma|/|A_2| = |A_2^{\Gamma'}| = |\{a \in A | a^2 = 1\}| = \begin{cases} 1 & \text{if } t \text{ is odd;} \\ 2 & \text{if } s \text{ and } t \text{ are even.} \end{cases}$$

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