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On Classical Solutions to the Mean Field Game System of Controls

Z. Kobeissi*

April 24, 2019

Abstract

In this paper, we consider a class of mean field games in which the optimal strategy of a representative agent depends on the statistical distribution of the states and controls.

We prove some existence results for the forward-backward system of PDEs under rather natural assumptions. The main step of the proof consists of obtaining a priori estimates on the gradient of the cost function by Bernstein's method. Uniqueness is also proved under more restrictive assumptions.

The last section contains some examples to which the previously mentioned existence (and possibly uniqueness) results apply.

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1 - INTRODUCTION

The theory of Mean Field Games (MFG for short) has been introduced in the independent works of J.M. Lasry and P.L. Lions [19, 20, 21], and of M.Y. Huang, P.E. Caines and R.Malhamé [16, 17]. It aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. The agents are supposed to be rational (given a cost to be minimized, they always choose the optimal strategies), and indistinguishable. Furthermore, the agents interact via some empirical averages of quantities which depend on the state variable.

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At the limit when $N \rightarrow +\infty$, the game may be modeled by a system of two coupled partial differential equations (PDEs), which is named the MFG system. On the one hand, there is a Fokker-Planck-Kolmogorov equation describing the evolution of the statistical distribution m of the state variable; this equation is forward in time parabolic equation, and the initial distribution at time $t = 0$ is given. On the other hand, the optimal value of a generic agent at some time t and state x is noted $u(t, x)$ and is defined as the lowest cost that a representative agent can achieve from time t to T if it is at state x at time t . The value function satisfies a Hamilton-Jacobi-Bellman equation posed backward in time with a terminal condition involving a terminal cost. In the present work, we will restrict our attention to the case when the costs and the dynamics are periodic in the state variable, and we will work in the d -dimensional torus \mathbb{T}^d (as it is often done in the MFG literature for simplicity). We will take a finite horizon time $T > 0$, and will only consider second-order non-degenerate MFG systems. In this case, the MFG system is often written as:

$$\begin{cases} -\partial_t u(t, x) - \nu \Delta u(t, x) + H(x, \nabla_x u(t, x)) = f(x, m(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t m_t(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(x, \nabla_x u(t, x))m) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ u(T, x) = g(x, m(T)) & \text{in } \mathbb{T}^d, \\ m(0, x) = m_0(x) & \text{in } \mathbb{T}^d. \end{cases}$$

We refer the reader to [7] for some theoretical results on the convergence of the N -agents Nash equilibrium to the solutions of the MFG system. For a thorough study of the well-posedness of the MFG system, see the videos of P.L.Lions' lecture at the Collège de France, and some lecture notes [6].

There is also an important literature on the probabilistic aspects of MFGs, see [9, 18] for some examples and [10, 11] for a detailed presentation of the probabilistic viewpoint.

For applications of MFGs, numerical simulations are crucial because it is most often impossible to find explicit or semi-explicit solutions to the MFG system. We refer to [1] for a survey on finite difference methods and to [2] for applications to crowd motion.

Most of the literature on MFGs is focused on the case when the mean field interactions only involves the distributions of states. Here we will consider a more general situation in which the cost of an individual agent depends on the joint distribution μ of states and optimal strategies. To underline this, we choose to use the terminology *Mean Field Games of Controls (MFGCs)* for this class of MFGs; the latter terminology was introduced in [8].

For MFGCs, the forward-backward system takes the following form:

$$(1.1) \quad -\partial_t u(t, x) - \nu \Delta u(t, x) + H(x, \nabla_x u(t, x), \mu(t)) = f(x, m(t)) \quad \text{in } (0, T) \times \mathbb{T}^d,$$

$$(1.2) \quad \partial_t m_t(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(x, \nabla_x u(t, x), \mu(t))m) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d,$$

$$(1.3) \quad \mu(t) = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) \quad \text{in } [0, T],$$

$$(1.4) \quad u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{T}^d,$$

$$(1.5) \quad m(0, x) = m_0(x) \quad \text{in } \mathbb{T}^d.$$

In the first articles devoted to MFGCs, [15, 14], D. Gomes and his collaborators have given several existence results for MFGCs in various cases, using the name *extended MFGs* instead of *MFGCs*. In particular, [15] contains existence results for stationary games (infinite horizon) under the assumption that some of the parameters involved in the models are small. MFGCs are of great interest for studying a large and various range of models, see [8] for an application to optimal trading, or [13] in the case of competition between firms producing the same goods, or [3] for energy storage. In [8], existence results are proved with a degenerate diffusion. For MFGCs, uniqueness results usually require strong assumptions: for example, uniqueness has been obtained in [4] under the assumption that the Hamiltonian is invariant with respect to

the translations in the state variable. In [5], uniqueness and existence are proved for potential MFGCs, i.e. for which there exists a variational formulation. For a probabilistic point of view, a special section and several paragraphs in the books [10, 11] are devoted to MFGCs, see also [12].

In this paper, our main objective is to provide tools in order to prove existence of solutions for MFGCs under less restrictive assumptions on the Hamiltonian than those made in the available literature mentioned above.

In our opinion, one of the most interesting aspect in the examples discussed in Section 6 below, is the case when the agents favor a strategy close to the mainstream one. In this case, the monotonicity condition in [8] does not hold. Indeed, the latter translates the fact that the agents prefer directions opposite to the mainstream direction, which may be unrealistic in several situations, in particular in models of crowd motions. Without such an assumption, uniqueness will be unlikely. This explains why uniqueness results are not the main goal of the present paper, even if we will also give some uniqueness results for the sake of completeness. Besides, a deeper study of the non uniqueness including numerical simulations is the subject of a work in progress.

The paper is organised as follows. In Section 2, we introduce the notations and some functional spaces, then we present the assumptions that will be made in the whole paper, and we give some simple estimates on the joint distribution μ . In Section 3, we prove a priori estimates, supposing that there exists solutions of the MFGC system. The main theoretical results are contained in Section 4 in which we prove existence of solutions in several cases:

- with small parameter, see Theorem 4.6 and Corollary 4.7
- when the exponent in the estimates of H with respect to μ is sub-critical, see 4.8
- with short time horizon, see Theorem 4.9
- under some assumptions on the asymptotic behavior of H_x when the supports of μ and p grow to infinity (namely (4.17)) see 4.12.

We also present a uniqueness result in the quadratic case with short time horizon, see Theorem 4.10. In Section 5, we prove existence and uniqueness under the assumption that the Lagrangian L associated with the Hamiltonian H by Legendre's transform satisfies some monotonicity property with respect to μ **M1**. The final section 6 contains applications of the Theorems contained in Sections 4 and 5 to some examples that may be used in models of crowd motion.

2 - PRELIMINARIES CONSIDERATIONS

2.1 Notation

If K is a compact subset of an Euclidian space, we set

$$\|v\|_\infty = \sup_{x \in K} |v(x)|_E$$

for v a bounded map from K to a normed space E .

Let us introduce the functional space $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$ as the set of the functions $v \in C([0, T] \times \mathbb{T}^d; \mathbb{R})$ which have a gradient with respect to the state variable $\nabla_x v$ in $C([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$. It is a Banach space with the norm

$$\|v\|_{C^{0,1}} = \|v\|_\infty + \sum_{i=1}^d \|\partial_{x_i} v\|_\infty.$$

We shall need to use spaces of Hölder continuous functions in $[0, T] \times \mathbb{T}^d$: for $\beta \in (0, 1)$, $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ is classically defined by

$$C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}) = \left\{ v \in C([0, T] \times \mathbb{T}^d; \mathbb{R}), \exists C > 0 \text{ s.t. } \forall (t_1, x_1), (t_2, x_2) \in [0, T] \times \mathbb{T}^d, \right. \\ \left. |v(t_1, x_1) - v(t_2, x_2)| \leq C (|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}} \right\},$$

and we define the semi-norm

$$|v|_{C^{\frac{\beta}{2}, \beta}} = \sup_{(t_1, x_1) \neq (t_2, x_2)} \frac{|v(t_1, x_1) - v(t_2, x_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}}}.$$

The space $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ is a Banach space with the norm $\|v\|_{C^{\frac{\beta}{2}, \beta}} = |v|_{C^{\frac{\beta}{2}, \beta}} + \|v\|_{C^0}$. Then we introduce the space $C^{\frac{1+\beta}{2}, 1+\beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ of all the functions $v \in C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$ which admit a derivative in the x variable and such that $\partial_{x_i} v \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ for all $1 \leq i \leq d$, and for all $(t_1, x) \neq (t_2, x) \in [0, T] \times \mathbb{T}^d$, $|v(t_1, x) - v(t_2, x)| \leq C |t_1 - t_2|^{\frac{1+\beta}{2}}$ for some constant $C > 0$. The space $C^{\frac{1+\beta}{2}, 1+\beta}([0, T] \times \mathbb{T}^d)$ is a Banach space with the norm

$$\|v\|_{C^{\frac{1+\beta}{2}, 1+\beta}} = \|v\|_{C^0} + \sum_{i=1}^d \|\partial_{x_i} v\|_{C^{\frac{\beta}{2}, \beta}} + \sup_{(t_1, x) \neq (t_2, x) \in [0, T] \times \mathbb{T}^d} \frac{|v(t_1, x) - v(t_2, x)|}{|t_1 - t_2|^{\frac{1+\beta}{2}}}.$$

Definition 2.1. We define the Wassertein-1 distance on $\mathcal{P}(\mathbb{T}^d)$, by

$$d_1(m_1, m_2) = \inf \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y| d\Pi(x, y),$$

with $m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)$, where the infimum is taken over all $\Pi \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d)$ such that $\Pi(A, \mathbb{T}^d) = m_1(A)$, $\Pi(\mathbb{T}^d, A) = m_2(A)$, for any $A \in \mathcal{B}(\mathbb{T}^d)$ the set of Borelian sets in \mathbb{T}^d .

Then we note $B_{\mathbb{R}^d}(0, M)$ the ball in \mathbb{R}^d centered at 0 with radius $M > 0$, and we define the Wassertein-1 distance on $\mathcal{P}(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))$ by

$$d_1(\mu_1, \mu_2) = \inf \int_{(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))^2} |x - y| d\Pi(x, y),$$

with $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))$, where the infimum is taken over all $\Pi \in \mathcal{P}((\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))^2)$ such that $\Pi(A, \mathbb{T}^d \times B_{\mathbb{R}^d}(0, M)) = \mu_1(A)$, $\Pi(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M), A) = \mu_2(A)$, for any $A \in \mathcal{B}(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))$ the set of Borelian sets in $\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M)$.

The spaces of probability measures are equipped with the weak topology. Since \mathbb{T}^d is compact, the weak topology on $\mathcal{P}(\mathbb{T}^d)$ coincides with the metric topology induced by the Wassertein-1 distance d_1 defined in Definition 2.1.

Moreover, in the following, we will only consider measures on $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ which are compactly supported in $B_{\mathbb{R}^d}(0, M)$, for some $M > 0$. The weak convergence in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ of measures compactly supported in $\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M)$ coincides with the weak convergence in $\mathcal{P}(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))$. Thus they also coincide with the convergence in the Wassertein-1 distance on $\mathcal{P}(\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M))$.

Therefore, in the following, we use the Wassertein-1 distance to prove continuity with respect to variables in $\mathcal{P}(\mathbb{T}^d)$ or in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$.

Definition 2.2. For any $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, for $\gamma \in [1, \infty)$, we introduce

$$M_{\mu, \gamma} = \int_{\mathbb{T}^d \times \mathbb{R}^d} |\alpha|^\gamma d\mu(x, \alpha),$$

and $M_{\mu, \infty}$ defined as the radius of the smallest ball of $\mathbb{T}^d \times \mathbb{R}^d$ centered at the origin which contains the support of μ .

Definition 2.3. *The triple (u, m, μ) is a solution of (1.1)-(1.5) if $u \in C^{1,2}([0, T] \times \mathbb{T}^d) \cap C^0([0, T] \times \mathbb{T}^d)$ is a classical solution to the Hamilton-Jacobi-Bellman equation (1.1) with terminal condition (1.4), $m \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ is solution to the Fokker-Planck-Kolmogorov equation (1.2) in the sense of distribution with initial condition (1.5), and $\mu \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ verifies (1.3) at any $t \in [0, T]$.*

2.2 Assumptions

The constants entering in the assumptions below are $C_0 > 0, 0 \leq \lambda < 1, \beta_0 \in (0, 1), \gamma \in (1, 2]$. The conjugate exponent of γ is noted $\gamma' = \frac{\gamma}{\gamma-1}$. We assume that

- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{T}^d and we also name m_0 its density (abuse of notation). We assume that $m_0 \in C^{\beta_0}(\mathbb{T}^d)$.

- Hypothesis on H :

H1 H is convex with respect to p . H is differentiable with respect to (x, p) . H_p is differentiable with respect to x . H_p is locally $(\gamma - 1)$ -Hölder continuous with respect to p .

For all $x \in \mathbb{T}^d, p \in \mathbb{R}^d, \mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ compactly supported,

H2 $H_p(x, p, \mu) \cdot p - H(x, p, \mu) \geq -C_0 + C_0^{-1}|p|^\gamma$.

H3 $|H(x, p, \mu)| \leq C_0(1 + |p|^\gamma + M_{\mu, \gamma'})$.

H4 $|H_x(x, p, \mu)| \leq C_0(1 + |p|^\gamma + M_{\mu, \gamma'})$.

H5 $|H_p(x, p, \mu)| \leq C_0(1 + |p|^{\gamma-1}) + \lambda M_{\mu, 1}$.

H6 For $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ compactly supported,

$$|H_p(x, p, \mu^1) - H_p(x, p, \mu^2)| \leq \lambda d_1(\mu^1, \mu^2).$$

H7 For all $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ compactly supported, there exists $0 < C_1 = C_1(|p|, M_{\mu^1, \infty}, M_{\mu^2, \infty})$ such that

$$|H(x, p, \mu^1) - H(x, p, \mu^2)| \leq C_1 d_1(\mu^1, \mu^2).$$

- Hypothesis on f and g :

f1 $f : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \rightarrow \mathbb{R}$ is continuous, and we suppose that $x \mapsto f(x, m)$ is in $C^1(\mathbb{T}^d; \mathbb{R})$ and its C^1 -norm is uniformly bounded with respect to m , i.e.

$$\|f(\cdot, m)\|_{C^1} \leq C_0, \quad \forall m \in \mathcal{P}(\mathbb{T}^d).$$

f2 f is Lipschitz continuous with respect to m , i.e.

$$\|f(\cdot, m^1) - f(\cdot, m^2)\|_\infty \leq C_0 d_1(m^1, m^2).$$

g1 $g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \rightarrow \mathbb{R}$ is continuous, and we suppose that $x \mapsto g(x, m)$ is in $C^{2+\beta_0}(\mathbb{T}^d)$, with a norm bounded uniformly with respect to m , i.e.

$$\|g(\cdot, m)\|_{C^{2+\beta_0}} \leq C_0, \quad \forall m \in \mathcal{P}(\mathbb{T}^d).$$

2.3 Some preliminary estimates

Lemma 2.4. *We assume **H1**, **H5** and **H6**. Take p in $C(\mathbb{T}^d; \mathbb{R}^d)$, and $m \in \mathcal{P}(\mathbb{T}^d)$. Then the following two assertions are verified.*

(i) *There exists a unique $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ such that*

$$(2.1) \quad \mu = (I_d, -H_p(\cdot, p(\cdot), \mu)) \# m,$$

and

$$(2.2) \quad M_{\mu, \infty} \leq \frac{C_0}{1-\lambda} \left(1 + \|p\|_{\infty}^{\gamma-1}\right), \quad M_{\mu, \tilde{\gamma}} \leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(1 + \int_{\mathbb{T}^d} |p(x)|^{(\gamma-1)\tilde{\gamma}} m(dx)\right),$$

for any $\tilde{\gamma} \geq 1$.

(ii) *The map $(p, m) \mapsto \mu$ given by (2.1), is continuous from $C(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$ to $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. Recall that $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ is equipped with the topology of the weak convergence.*

Proof. (i) *Existence.*

We define the following sequences by induction,

$$\begin{cases} \mu_0 = m \otimes \delta_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d), \\ B_{k+1}(x) = -H_p(x, p(x), \mu_k), \text{ for } k \geq 0, \\ \mu_k = (I_d, B_k) \# m, \text{ for } k \geq 1. \end{cases}$$

We get from **H5**,

$$(2.3) \quad |B_1(x)| \leq C_0(1 + |p(x)|^{\gamma-1}).$$

Let us take X a random variable on \mathbb{T}^d whose law is m , then for $k \geq 1$, $\mathcal{L}(X, B_k(X)) = \mu_k$. Thus,

$$\begin{aligned} |B_2(x) - B_1(x)| &\leq \lambda d_1(\mu_1, \mu_0) \text{ from } \mathbf{H6} \\ &\leq \lambda \mathbb{E}[|B_1(X)|] \\ &= \lambda \int_{\mathbb{T}^d} |B_1(y)| dm(y) \\ &\leq \lambda C_0 \left(1 + \|p\|_{L^{\gamma-1}(m)}^{\gamma-1}\right) \text{ from (2.3),} \end{aligned}$$

and

$$\begin{aligned} |B_{k+2}(x) - B_{k+1}(x)| &= |H_p(x, p(x), \mu_{k+1}) - H_p(x, p(x), \mu_k)| \\ &\leq \lambda d_1(\mu_{k+1}, \mu_k) \text{ from } \mathbf{H6} \\ &\leq \lambda \mathbb{E}[|B_{k+1}(X) - B_k(X)|] \\ &= \lambda \int_{\mathbb{T}^d} |B_{k+1}(y) - B_k(y)| dm(y) \\ &\leq \lambda \|B_{k+1} - B_k\|_{\infty}. \end{aligned}$$

This implies that $(B_k)_{k \geq 1}$ converges uniformly to a continuous function B which satisfies

$$\begin{aligned} |B(x)| &\leq |B_1(x)| + \sum_{k=1}^{\infty} |B_{k+1}(x) - B_k(x)| \\ &\leq C_0(1 + |p(x)|^{\gamma-1}) + \frac{\lambda C_0}{1-\lambda} \left(1 + \|p\|_{L^{\gamma-1}(m)}^{\gamma-1}\right), \end{aligned}$$

and μ defined by $\mu = (I_d, B) \# m$ satisfies the fixed point relation (2.1).

This and **H5** yield

$$M_{\mu,\infty} \leq \|H_p(\cdot, p(\cdot), \mu)\|_\infty \leq C_0 \left(1 + \|p\|_\infty^{\gamma-1}\right) + \lambda M_{\mu,\infty},$$

since $M_{\mu,1} \leq M_{\mu,\infty}$. Then for $\tilde{\gamma} \geq 1$, we obtain

$$\begin{aligned} (2.4) \quad M_{\mu,\tilde{\gamma}} &= \int_{\mathbb{T}^d} |H_p(x, p(x), \mu)|^{\tilde{\gamma}} dm(x) \\ &\leq \int_{\mathbb{T}^d} \left(C_0 (1 + |p(x)|^{\gamma-1}) + \lambda M_{\mu,1}\right)^{\tilde{\gamma}} dm(x), \quad \text{by **H5**,} \\ &= \int_{\mathbb{T}^d} \left((1-\lambda) \frac{C_0}{1-\lambda} (1 + |p(x)|^{\gamma-1}) + \lambda M_{\mu,1}\right)^{\tilde{\gamma}} dm(x) \\ &\leq \frac{C_0^{\tilde{\gamma}}}{(1-\lambda)^{\tilde{\gamma}-1}} \int_{\mathbb{T}^d} (1 + |p(x)|^{\gamma-1})^{\tilde{\gamma}} dm(x) + \lambda M_{\mu,1}^{\tilde{\gamma}} \quad \text{by Jensen inequality} \\ &\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}-1}} \left(1 + \int_{\mathbb{T}^d} |p(x)|^{(\gamma-1)\tilde{\gamma}} dm(x)\right) + \lambda M_{\mu,\tilde{\gamma}}, \end{aligned}$$

which achieves the proof of (2.2).

Uniqueness

Suppose that $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ verify the fixed point equation (2.1). Then **H6** yields

$$\begin{aligned} d_1(\mu^1, \mu^2) &\leq \int_{\mathbb{T}^d} |H_p(x, p(x), \mu^2) - H_p(x, p(x), \mu^1)| dm(x) \\ &\leq \lambda d_1(\mu^1, \mu^2), \end{aligned}$$

hence $\mu^1 = \mu^2$ since $0 \leq \lambda < 1$. There is at most a solution of the fixed point equation (2.1).

(ii) Let $(p^n, m^n)_{n \geq 0} \in (C^0(\mathbb{T}^d \times \mathbb{R}^d); \mathcal{P}(\mathbb{T}^d))^{\mathbb{N}}$ be a convergent sequence to (p, m) in $C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$. We define $(\mu^n)_{n \geq 0}, \mu$ the fixed points obtained in (i) by

$$(2.5) \quad \begin{aligned} \mu^n &= (I_d, -H_p(\cdot, p^n(\cdot), \mu^n)) \# m^n, \quad \text{for } n \in \mathbb{N}, \\ \mu &= (I_d, -H_p(\cdot, p(\cdot), \mu)) \# m. \end{aligned}$$

The sequence $(p^n)_{n \geq 0}$ is uniformly bounded in $C(\mathbb{T}^d; \mathbb{R}^d)$, thus (2.2) yields that the supports of $(\mu^n)_{n \geq 1}$ are uniformly compactly supported. Thus the sequence (μ^n) is tight, so it is compact in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ endowed with the weak topology. Let $\tilde{\mu}$ be the limit of a subsequence $(\mu^{n'})$. By taking the limit when n' goes to infinity in (2.5), we prove that $\tilde{\mu}$ verifies the same fixed point relation as μ . By uniqueness of this fixed point, we deduce that $\tilde{\mu} = \mu$. It implies that the entire sequence (μ^n) tends to μ .

Therefore the map $(p, m) \mapsto \mu$ is continuous from $C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$ to $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. \square

Remark 2.5. In particular, if $(p, m, \mu) \in C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ verifies (2.1), the following inequality holds,

$$(2.6) \quad M_{\mu,\gamma'} \leq \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left(1 + \int_{\mathbb{T}^d} |p(x)|^\gamma dm(x)\right),$$

since $(\gamma-1)\gamma' = \gamma$.

Lemma 2.6. *We assume **H1**, **H5** and **H6**. Consider $(p^1, m^1), (p^2, m^2) \in C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$ such that p^1 is Lipschitz continuous, and $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ verifying the fixed point equation (2.1) of Lemma 2.4 associated respectively with (p^1, m^1) and (p^2, m^2) . Then we get*

$$(2.7) \quad d_1(\mu^1, \mu^2) \leq C \left(\|p^1 - p^2\|_\infty^{\gamma-1} + d_1(m^1, m^2)^{\gamma-1} \right),$$

with $C > 0$ depending on $C_0, \lambda, \|p^i\|_\infty, i = 1, 2$ and the Lipschitz constant of p^1 .

Proof. We take (X^1, X^2) two random variables with values in \mathbb{T}^d , respectively with law m^1 and m^2 , and such that

$$d_1(m^1, m^2) = \mathbb{E}[|X^1 - X^2|].$$

We introduce $\alpha^i = -H_p(X^i, p^i(X^i), \mu^i)$, for $i = 1, 2$. Thus we obtain that $\mathcal{L}(X^i, \alpha^i) = \mu^i$. By triangular inequality we get

$$(2.8) \quad \begin{aligned} \mathbb{E}[|\alpha^1 - \alpha^2|] &= \mathbb{E}[|H_p(X^1, p^1(X^1), \mu^1) - H_p(X^2, p^2(X^2), \mu^2)|] \\ &\leq \mathbb{E}[|H_p(X^1, p^1(X^1), \mu^1) - H_p(X^2, p^1(X^2), \mu^1)|] \\ &\quad + \mathbb{E}[|H_p(X^2, p^1(X^2), \mu^1) - H_p(X^2, p^2(X^2), \mu^1)|] \\ &\quad + \mathbb{E}[|H_p(X^2, p^2(X^2), \mu^1) - H_p(X^2, p^2(X^2), \mu^2)|]. \end{aligned}$$

We recall that p^1 is Lipschitz continuous and that $(x, p) \mapsto H_p(x, p, \mu^1)$ is $(\gamma - 1)$ -Hölder continuous on $\mathbb{T}^d \times B_{\mathbb{R}^d}\left(0, \max_{i=1,2} \|p^i\|_\infty\right)$ from **H1**. Let us consider a constant $C_1 > 0$ greater than the Lipschitz constant of p^1 , than the $(\gamma - 1)$ -Hölder constant of $x \mapsto H_p(x, p^1(x), \mu^1)$, and than the $(\gamma - 1)$ -Hölder constant of $p \in B_{\mathbb{R}^d}\left(0, \max_{i=1,2} \|p^i\|_\infty\right) \mapsto H(x, p, \mu)$. In particular C_1 depends on $\max_{i=1,2} \|p^i\|_\infty$. Then we obtain

$$(2.9) \quad \begin{aligned} \mathbb{E}[|H_p(X^1, p^1(X^1), \mu^1) - H_p(X^2, p^1(X^2), \mu^1)|] &\leq C_1 \mathbb{E}[|X^1 - X^2|^{\gamma-1}] \\ &\leq C_1 \mathbb{E}[|X^1 - X^2|^{\gamma-1}] \quad \text{because } 0 < \gamma - 1 \leq 1 \\ &\leq C_1 d_1(m^1, m^2)^{\gamma-1}, \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} \mathbb{E}[|H_p(X^2, p^1(X^2), \mu^1) - H_p(X^2, p^2(X^2), \mu^1)|] \\ = \int_{\mathbb{T}^d} |H_p(x, p^1(x), \mu^1) - H_p(x, p^2(x), \mu^1)| m^2(dx) \\ \leq C_1 \|p^1 - p^2\|_\infty^{\gamma-1}, \end{aligned}$$

since m^2 is a probability measure.

From **H6** we obtain,

$$(2.11) \quad \mathbb{E}[|H_p(X^2, p^2(X^2), \mu^1) - H_p(X^2, p^2(X^2), \mu^2)|] \leq \lambda d_1(\mu^1, \mu^2).$$

Moreover we get,

$$(2.12) \quad \begin{aligned} d_1(\mu^1, \mu^2) &\leq \mathbb{E}\left[\left(|X^1 - X^2|^2 + |\alpha^1 - \alpha^2|^2\right)^{\frac{1}{2}}\right] \\ &\leq \mathbb{E}[|X^1 - X^2| + |\alpha^1 - \alpha^2|] \end{aligned}$$

Thus (2.8), (2.9), (2.10), (2.11), and (2.12) yield

$$\mathbb{E}[|\alpha^1 - \alpha^2|] \leq (1 - \lambda)^{-1} \left(\lambda d_1(m^1, m^2) + C_1 d_1(m^1, m^2)^{\gamma-1} + C_1 \|p^1 - p^2\|_\infty^{\gamma-1} \right).$$

The d -dimensional torus \mathbb{T}^d has a finite diameter equal to $\sqrt{\frac{d}{2}}$, so $d_1(m^1, m^2) \leq \sqrt{\frac{d}{2}}$. This implies that

$$d_1(\mu^1, \mu^2) \leq C \left(d_1(m^1, m^2)^{\gamma-1} + \|p^1 - p^2\|_\infty^{\gamma-1} \right),$$

where $C = (1 - \lambda)^{-1} \left(C_1 + \lambda \left(\frac{d}{2} \right)^{\frac{2-\gamma}{2}} \right) + \left(\frac{d}{2} \right)^{\frac{2-\gamma}{2}}$. \square

Corollary 2.7. *With the same assumptions, consider $p \in C^\beta([0, T]; C^0(\mathbb{T}^d; \mathbb{R}^d))$ such that $p(t, \cdot)$ is Lipschitz continuous with respect to the state variable uniformly in $t \in [0, T]$, and $m \in C^\beta([0, T]; \mathcal{P}(\mathbb{T}^d))$, for $\beta \in (0, 1)$. For $t \in [0, T]$ let $\mu(t) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ be defined by the fixed point relation in Lemma 2.4. Then $\mu \in C^{\beta(\gamma-1)}([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$.*

3 - A PRIORI ESTIMATES

In this section we suppose that (u, m, μ) is a solution to (1.1)-(1.5) as introduced in Definition 2.3, and we will look for estimates. These a priori estimates will be used in the proof of existence.

Lemma 3.1. *Under Assumptions **H1**, **H3**, **H5**, **H6**, **f1** and **g1**, the function u verifies the following inequalities:*

$$(3.1) \quad \|u\|_\infty \leq C_0 \left(1 + 2T + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left(T + \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^\gamma dm(t, x) dt \right) \right).$$

Proof. We use the fundamental theorem of calculus on (1.1) to get

$$(3.2) \quad -\partial_t u(t, x) - \nu \Delta u(t, x) + \left[\int_0^t H_p(x, s \nabla_x u, \mu(s)) ds \right] \cdot \nabla_x u(t, x) = f(x, m(t)) - H(x, 0, \mu(t)).$$

The function u is the solution of a parabolic differential equation with coefficients in L^∞ and we can bound the right-hand side in absolute value using **H3** and **f1** the following way,

$$|f(x, m) - H(x, 0, \mu)| \leq C_0 + C_0 (1 + M_{\mu, \gamma'}).$$

And from **g1** $|u(T, \cdot)|$ is bounded by C_0 .

Thus the maximum principle for parabolic second order equation applied to u and $-u$, and (2.6) yield that

$$(3.3) \quad \begin{aligned} |u(t, x)| &\leq C_0 + \int_t^T C_0 (2 + M_{\mu, \gamma'}) ds \\ &\leq C_0 \left(1 + 2T + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left(T + \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^\gamma dm(s, x) ds \right) \right), \end{aligned}$$

for any $(t, x) \in [0, T] \times \mathbb{T}^d$. This achieves the proof. \square

Lemma 3.2. *We assume **H1**, **H2**, **H4**, **H5**, **H6**, **H7**, **f1**, **f2** and **g1**.*

*The function u is in $C^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{T}^d)$ for any $\beta \in (0, (\gamma-1)\beta_0)$ with β_0 introduced in **f1** and **g1**. Its $C^{1+\frac{\beta}{2}, 2+\beta}$ -norm can be bounded by a quantity depending only on $\|u\|_\infty, \beta, C_0, T, \lambda$.*

Proof. First step: estimate on $\nabla_x u$.

The proof comes from a Bernstein-like technique inspired by the lectures of P.L.Lions in [23] on November the 23rd 2018.

We take $\rho \in C^\infty([-\frac{1}{2}, \frac{1}{2}]^d)$ a non-negative function such that $\rho(x) = 0$ if $|x| \geq \frac{1}{4}$ and $\int_{\mathbb{T}^d} \rho(x) dx = 1$. By an abuse of notation, we also note $\rho \in C^\infty(\mathbb{T}^d)$ the composition of the

former ρ with the canonical injection from \mathbb{T}^d to $[-\frac{1}{2}, \frac{1}{2}]^d$. Let us introduce $\rho^\delta = \delta^{-d} \rho(\frac{\cdot}{\delta})$ and $u^\delta(t) = \rho^\delta \star u(t)$, for any $0 < \delta < 1$ and $t \in [0, T]$.

Thus u^δ depends smoothly on the state variable and its partial derivatives in space at any order have the same regularity in time as u , moreover it solves the following partial differential equation with final condition,

$$(3.4) \quad \begin{cases} -\partial_t u^\delta(t, x) - \nu \Delta u^\delta(t, x) + \rho^\delta \star (H(\cdot, \nabla_x u(t, \cdot), \mu(t))) (x) = \left(\rho^\delta \star f(\cdot, m(t)) \right) (x) & \text{in } (0, T) \times \mathbb{T}^d, \\ u^\delta(T, x) = \rho^\delta \star (g(\cdot, m(T, \cdot))) (x) & \text{in } \mathbb{T}^d, \end{cases}$$

We take the gradient with respect to the state variable of the latter equation and we take the scalar product with $\nabla_x u^\delta$,

$$(3.5) \quad -\frac{1}{2} \partial_t \left| \nabla_x u^\delta \right|^2 - \nu \nabla_x u^\delta \cdot \Delta \left(\nabla_x u^\delta \right) + \nabla_x u^\delta \cdot D_{x,x}^2 u^\delta H_p \left(x, \nabla_x u^\delta, \mu \right) \\ + \nabla_x u^\delta \cdot H_x^\delta \left(x, \nabla_x u, \mu \right) = \nabla_x u^\delta \cdot \nabla_x (f^\delta(x, m)) + \nabla_x u^\delta \cdot R^\delta(t, x),$$

where H^δ , f^δ and R^δ are defined by

$$\begin{aligned} H^\delta(x, p, \mu) &= \rho^\delta \star (H(\cdot, p(\cdot), \mu)) (x), \\ f^\delta(x, m) &= \left(\rho^\delta \star f(\cdot, m) \right) (x), \\ R^\delta(t, x) &= D_{x,x}^2 u^\delta H_p \left(x, \nabla_x u^\delta, \mu \right) - \rho^\delta \star (D_{x,x}^2 u H_p(\cdot, \nabla_x u, \mu)). \end{aligned}$$

By simple calculus, we notice that

$$\nabla_x \left| \nabla_x u^\delta \right|^2 = 2 D_{x,x}^2 u^\delta \nabla_x u^\delta,$$

and

$$\Delta \left| \nabla_x u^\delta \right|^2 = 2 \nabla_x u^\delta \cdot \Delta \left(\nabla_x u^\delta \right) + 2 \left| D_{x,x}^2 u^\delta \right|^2.$$

From (3.5) we obtain

$$(3.6) \quad -\frac{1}{2} \partial_t \left| \nabla_x u^\delta \right|^2 - \frac{\nu}{2} \Delta \left| \nabla_x u^\delta \right|^2 + \nu \left| D_{x,x}^2 u^\delta \right|^2 + \frac{1}{2} \nabla_x \left| \nabla_x u^\delta \right|^2 \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) \\ = \nabla_x u^\delta \cdot \nabla_x (f^\delta(x, m)) - \nabla_x u^\delta \cdot H_x^\delta \left(x, \nabla_x u, \mu \right) + \nabla_x u^\delta \cdot R^\delta(t, x).$$

Let us introduce the functions φ and w^δ defined by

$$(3.7) \quad \begin{aligned} \varphi(v) &= \exp(\exp(-\eta(v + \|u\|_\infty))), \text{ for } |v| \leq \|u\|_\infty \\ w^\delta(t, x) &= \varphi(u^\delta(T - t, x)) \left| \nabla_x u^\delta \right|^2(T - t, x), \end{aligned}$$

with $0 < \eta \leq 1$. The derivatives of φ are given by

$$(3.8) \quad \begin{aligned} \varphi'(v) &= -\eta e^{-\eta(v + \|u\|_\infty)} \exp\left(e^{-\eta(v + \|u\|_\infty)}\right), \\ \varphi''(v) &= \eta^2 e^{-\eta(v + \|u\|_\infty)} \exp\left(e^{-\eta(v + \|u\|_\infty)}\right) \left(1 + e^{-\eta(v + \|u\|_\infty)}\right), \end{aligned}$$

hence φ, φ' verify

$$(3.9) \quad \begin{aligned} 1 &\leq \varphi(v) \leq e, \\ \eta e^{-2\eta\|u\|_\infty} &\leq \left| \frac{\varphi'(v)}{\varphi(v)} \right| \leq \eta. \end{aligned}$$

Then w^δ verifies the partial differential equation on $(0, T) \times \mathbb{T}^d$ given below,

$$(3.10) \quad \begin{aligned} & \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) = 2\varphi(u^\delta) \nabla_x u^\delta \cdot R^\delta(t, x) \\ & + \varphi(u^\delta) \left[-2\nu \left| D_{x,x}^2 u^\delta \right|^2 + 2\nabla_x u^\delta \cdot \nabla_x (f^\delta(x, m)) - 2\nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) \right] - \nu \varphi''(u^\delta) \left| \nabla_x u^\delta \right|^4 \\ & + \varphi'(u^\delta) \left[-\partial_t u^\delta \left| \nabla_x u^\delta \right|^2 - \nu \Delta u^\delta \left| \nabla_x u^\delta \right|^2 - 2\nu \nabla_x u^\delta \cdot \nabla_x \left| \nabla_x u^\delta \right|^2 + \left| \nabla_x u^\delta \right|^2 \nabla_x u^\delta \cdot H_p \right]. \end{aligned}$$

Moreover we notice that

$$\nabla_x w^\delta = \nabla_x \left(\varphi(u^\delta) \left| \nabla_x u^\delta \right|^2 \right) = \varphi'(u^\delta) \left| \nabla_x u^\delta \right|^2 \nabla_x u^\delta + \varphi(u^\delta) \nabla_x \left| \nabla_x u^\delta \right|^2,$$

hence we obtain

$$(3.11) \quad \begin{aligned} \varphi'(u^\delta) \nabla_x u^\delta \cdot \nabla_x \left| \nabla_x u^\delta \right|^2 &= \frac{\varphi'(u^\delta)}{\varphi(u^\delta)} \left(\nabla_x w^\delta - \varphi'(u^\delta) \left| \nabla_x u^\delta \right|^2 \nabla_x u^\delta \right) \cdot \nabla_x u^\delta \\ &= \frac{\varphi'(u^\delta)}{\varphi(u^\delta)} \nabla_x w^\delta \cdot \nabla_x u^\delta - \frac{\varphi'(u^\delta)^2}{\varphi(u^\delta)} \left| \nabla_x u^\delta \right|^4. \end{aligned}$$

From the first line in (3.4), and (3.10), and (3.11), we get

$$(3.12) \quad \begin{aligned} & \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) + 2\nu \frac{\varphi'(u^\delta)}{\varphi(u^\delta)} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ & = 2\varphi(u^\delta) \nabla_x u^\delta \cdot R^\delta(t, x) - \varphi'(u^\delta) \left| \nabla_x u^\delta \right|^2 Q^\delta(t, x) \\ & + \varphi(u^\delta) \left[-2\nu \left| D_{x,x}^2 u^\delta \right|^2 + 2\nabla_x u^\delta \cdot \nabla_x (f^\delta(x, m)) - 2\nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) \right] \\ & + \varphi'(u^\delta) \left| \nabla_x u^\delta \right|^2 \left[-H \left(x, \nabla_x u^\delta, \mu \right) + f^\delta(x, m) + \nabla_x u^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) \right] \\ & + \nu \left(2 \frac{\varphi'(u^\delta)^2}{\varphi(u^\delta)} - \varphi''(u^\delta) \right) \left| \nabla_x u^\delta \right|^4 \end{aligned}$$

where Q^δ is defined by

$$Q^\delta(t, x) = \rho^\delta \star H(\cdot, \nabla_x u(t, \cdot), \mu(t))(x) - H \left(x, \nabla_x u^\delta(t, x), \mu(t) \right).$$

We notice that φ verifies

$$2 \frac{\varphi'(u^\delta)^2}{\varphi(u^\delta)} - \varphi''(u^\delta) \leq 0.$$

Let us suppose that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $\delta \leq \delta(\varepsilon)$,

$$(3.13) \quad \left\| R^\delta \right\|_\infty + \left\| Q^\delta \right\|_\infty \leq \frac{\varepsilon}{2e \left\| \nabla_x u \right\|_\infty + \nu \left\| \nabla_x u \right\|_\infty^2},$$

(this will be proved in the second step of the proof).

Let us take $\varepsilon > 0$ and $\delta \leq \delta(\varepsilon)$. This, **H2**, **H4**, **f1**, and (3.12) yield that w^δ verifies the following partial differential inequality,

$$(3.14) \quad \begin{aligned} & \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) + 2\nu \frac{\varphi'(u^\delta)}{\varphi(u^\delta)} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ & \leq \varepsilon + \varphi(u^\delta) \left[2C_0 \left| \nabla_x u^\delta \right| + 2C_0 \left| \nabla_x u^\delta \right| \left(1 + \left\| \nabla_x u(t) \right\|_\infty^\gamma + M_{\mu(t), \gamma'} \right) \right] \\ & \quad + 2C_0 \left| \varphi'(u^\delta) \right| \left| \nabla_x u^\delta \right|^2 + C_0^{-1} \varphi'(u^\delta) \left| \nabla_x u^\delta \right|^{2+\gamma}. \end{aligned}$$

Then from (2.6) and (3.9), we obtain,

$$(3.15) \quad \varphi(u^\delta) \left| \nabla_x u^\delta \right| M_{\mu, \gamma'} \leq e \|\nabla_x u\|_\infty \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} (1 + \|\nabla_x u(t)\|_\infty^\gamma).$$

Thus (3.7), (3.8), (3.9), (3.14) and (3.15) yield,

$$(3.16) \quad \begin{aligned} & \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p(x, \nabla_x u^\delta, \mu) - 2\nu\eta e^{-\eta(u^\delta + \|u\|_\infty)} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ & \leq \varepsilon + 2C_0 e \left(2 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \right) \|\nabla_x u\|_\infty + 2C_0 \eta e \|\nabla_x u\|_\infty^2 \\ & \quad + 2C_0 e \left(1 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \right) \|\nabla_x u(t)\|_\infty^{\gamma+1} - C_0^{-1} e^{-\frac{\gamma}{2}} \eta e^{-2\eta\|u\|_\infty} (w^\delta)^{1+\frac{\gamma}{2}}. \end{aligned}$$

And $\|w^\delta(0)\|_\infty \leq eC_0^2$. Thus a super-solution to (3.16) with initial condition eC_0^2 is given by

$$(3.17) \quad \max \left\{ eC_0^2, \left[C_0 \eta^{-1} e^{2\eta\|u\|_\infty} e^{\frac{\gamma}{2}} \left(\varepsilon + 2C_0 e \left(2 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \right) \|\nabla_x u\|_\infty + 2C_0 \eta e \|\nabla_x u\|_\infty^2 \right. \right. \right. \\ \left. \left. \left. + 2C_0 e \left(1 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \right) \|\nabla_x u\|_\infty^{\gamma+1} \right) \right]^{\frac{2}{2+\gamma}} \right\}.$$

By a comparison argument, either w^δ is bounded by eC_0^2 , or it verifies

$$(3.18) \quad \|w^\delta\|_\infty^{1+\frac{\gamma}{2}} \leq C \eta^{-1} e^{2\eta\|u\|_\infty} \left(\varepsilon + 1 + \|\nabla_x u\|_\infty^{\gamma+1} \right),$$

for some $C > 0$.

Let ε and δ tend to 0, $\|w^\delta\|_\infty \rightarrow \|w^0\|_\infty$, and $u^\delta \rightarrow u$ in $C^0([0, T] \times \mathbb{T}^d; \mathbb{R})$, and (3.18) yields

$$(3.19) \quad \|w^0\|_\infty^{1+\frac{\gamma}{2}} \leq \max \left[e^{1+\frac{\gamma}{2}} C_0^{2+\gamma}, C \eta^{-1} e^{2\eta\|u\|_\infty} \left(1 + \|w^0\|_\infty^{\frac{\gamma+1}{2}} \right) \right].$$

Let us introduce $r_\eta \geq 0$ the maximal root of

$$P_\eta(r) = r^{1+\frac{\gamma}{2}} - C \eta^{-1} e^{2\eta\|u\|_\infty} \left(1 + r^{\frac{\gamma+1}{2}} \right).$$

We may notice that $\lim_{\eta \rightarrow 0} r_\eta = +\infty$, and more precisely

$$r_\eta^{\frac{1}{2}} \sim_{\eta \rightarrow 0} C \eta^{-1} e^{2\eta\|u\|_\infty}.$$

Thus, there exists $\eta_0 > 0$ such that if $\eta \leq \eta_0$ then,

$$(3.20) \quad r_\eta^{\frac{1}{2}} \leq 2C \eta^{-1} e^{2\eta\|u\|_\infty}.$$

Hence from (3.19) and (3.20) we obtain,

$$\|w^0\|_\infty \leq \max(eC_0^2, r_\eta),$$

which implies that,

$$(3.21) \quad \|\nabla_x u\|_\infty \leq \max \left(C_0 e^{\frac{1}{2}}, 2C \eta^{-1} e^{2\eta\|u\|_\infty} \right),$$

for $\eta \leq \eta_0$. Therefore $\nabla_x u$ is bounded uniformly with respect to m and μ , the bound only depends on $\|u\|_\infty$ and the constants in the assumptions.

Second step: proof of the estimates on R^δ, Q^δ .

We take $\varepsilon > 0$.

We suppose that $\delta \leq 1$ so that $\delta \leq \delta^\beta$ for $\beta \in (0, 1)$.

We recall that if $h : \mathbb{T}^d \rightarrow \mathbb{R}$ is a continuous function with a modulus of continuity $\tilde{\varepsilon} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, then $\|\rho^\delta \star h - h\|_\infty \leq \tilde{\varepsilon}(\delta)$.

From **H1**, on the compact $\mathbb{T}^d \times B_{\mathbb{R}^d}(0, \|\nabla_x u\|_\infty) \times \mathcal{P}\left(\mathbb{T}^d \times B_{\mathbb{R}^d}\left(0, \sup_{t \in [0, T]} M_{\mu(t), \infty}\right)\right)$, H_p is $(\gamma - 1)$ -Hölder continuous with respect to p and Lipschitz continuous with respect to x , and H is Lipschitz continuous with respect to (x, p) . We take $C > 0$ a positive constant greater than any of the Lipschitz or Hölder constant described above. Moreover $\nabla_x u$ is Lipschitz continuous with Lipschitz constant $\|D_{x,x}^2 u\|_\infty$.

Hence we get

$$(3.22) \quad \begin{aligned} |Q^\delta| &\leq \left| \rho^\delta \star H(\cdot, \nabla_x u, \mu)(x) - H(x, \nabla_x u, \mu) \right| + \left| H(x, \nabla_x u, \mu) - H(x, \nabla_x u^\delta, \mu) \right| \\ &\leq \left(C + C \|D_{x,x}^2 u\|_\infty \right) \delta + C \left| \nabla_x (u^\delta - u) \right| \\ &\leq \left(2C \|D_{x,x}^2 u\|_\infty + 1 \right) \delta. \end{aligned}$$

Then the triangular inequality yields

$$(3.23) \quad \begin{aligned} |R^\delta| &\leq \left| \left(D_{x,x}^2 u - D_{x,x}^2 u^\delta \right) H_p(x, \nabla_x u^\delta, \mu) \right| + \left| D_{x,x}^2 u \left(H_p(x, \nabla_x u^\delta, \mu) - H_p(x, \nabla_x u, \mu) \right) \right| \\ &\quad + \left| D_{x,x}^2 u \left(H_p(x, \nabla_x u, \mu) - \rho^\delta \star H_p(\cdot, \nabla_x u, \mu) \right)(x) \right|. \\ &\leq \left| \left(D_{x,x}^2 u - D_{x,x}^2 u^\delta \right) H_p(x, \nabla_x u^\delta, \mu) \right| + C \|D_{x,x}^2 u\|_\infty^{\gamma-1} \delta + C \|D_{x,x}^2 u\|_\infty \left(1 + \|D_{x,x}^2 u\|_\infty^{\gamma-1} \right) \delta^{\gamma-1} \end{aligned}$$

Since $D_{x,x}^2 u$ is a continuous function defined on a compact set, by Heine theorem there exists $\tilde{\delta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the inverse of a uniform modulus of continuity on $[0, T] \times \mathbb{T}^d$, then

$$(3.24) \quad \left| \left(D_{x,x}^2 u - D_{x,x}^2 u^\delta \right) H_p(x, \nabla_x u^\delta, \mu) \right| \leq \varepsilon C.$$

if $\delta \leq \tilde{\delta}(\varepsilon)$.

From (3.22), (3.23) and (3.24) we deduce that we can chose an appropriate $\delta(\varepsilon)$ such that for any $\delta \leq \delta(\varepsilon)$, (3.13) is achieved.

Third step: End of the proof

We know that $\nabla_x u$ is bounded uniformly with respect to m and μ , then from (2.2), $M_{\mu(t), 1}$ and $M_{\mu(t), \infty}$ are also bounded by a constant which only depends on $\|u\|_\infty$ and the constants of the assumptions. Then looking to u as a solution to the heat equation with the right-hand side $f(x, m) - H(x, \nabla_x u, \mu)$ which is bounded in L^∞ , classical results (see for example Theorem 6.48 in [22]) give us that for any $\beta \in (0, 1)$, the $C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}$ -norm of u is bounded by a constant wich depends on the L^∞ -norm of the right-hand side, on $\|g(m(T))\|_{1+\beta}$ and β .

Then m verifies (1.2) and (1.5) with L^∞ coefficients, by Theorem 9.10 in [22] the L^∞ -norm of m is bounded by a constant which depends on $\|u\|_\infty$ and the constants of the assumptions. We fix $\beta \in (0, \beta_0)$, classical results (see for example Theorem 6.29 in [22]) on second order parabolic equation in divergence form yield that m is in $C^{\frac{\beta}{2}, \beta}(\mathbb{T}^d \times \mathbb{R}^d)$ and its associated norm is bounded by a constant depending on $\|u\|_\infty$ and $\|m^0\|_{C^{\beta_0}}$.

Let us differentiate (1.1) with respect to x_i for $i = 1, \dots, d$ and call $v_i = \partial_{x_i} u$, then

$$\begin{cases} -\partial_t v_i - \nu \Delta v_i + \nabla_x v_i \cdot H_p(x, \nabla_x u, \mu) = f_{x_i}(x, m) - H_{x_i}(x, \nabla_x u, \mu), \\ v_i(T, x) = g_{x_i}(x, m(T)). \end{cases}$$

The function v_i is the solution of a second-order parabolic equation with L^∞ coefficients and right-hand side, thus it is Lipschitz continuous with respect to the state variable, and its Lipschitz constant depends on $\|f_{x_i}\|_{L^\infty}$, $\|H_{x_i}(\cdot, \nabla_x u, \mu)\|_\infty$ and $\|H_p(\cdot, \nabla_x u, \mu)\|_\infty$.

Corollary 2.7 yields that $\mu \in C^{\frac{\beta(\gamma-1)}{2}}([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$.

Then assumptions **H1**, **H7** and the regularity properties of u and μ yield

$$\begin{aligned} & |H(x, \nabla_x u(t, x), \mu(t)) - H(y, \nabla_x u(s, y), \mu(s))| \\ & \leq |H(x, \nabla_x u(t, x), \mu(t)) - H(y, \nabla_x u(s, y), \mu(t))| \\ & \quad + |H(y, \nabla_x u(s, y), \mu(t)) - H(y, \nabla_x u(s, y), \mu(s))| \\ & \leq C \left(|t - s|^{\frac{\beta}{2}} + |x - y|^\beta \right) + C |t - s|^{\frac{\beta(\gamma-1)}{2}}. \end{aligned}$$

for $(t, x), (s, y) \in [0, T] \times \mathbb{T}^d$, where $C > 0$ is a constant which depends on the Lipschitz constant of H in the variables (x, p) when $|p| \leq \|\nabla_x u\|_\infty$, the $C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}$ -norm of u , the constant in assumption **H7**, and the $C^{\frac{\beta(\gamma-1)}{2}}([0, T]; \mathcal{P}([0, T] \times \mathbb{T}^d))$ -norm of μ .

Hence the map $(t, x) \mapsto H(x, \nabla_x u(t, x), \mu(t))$ is $C^{\frac{\beta(\gamma-1)}{2}, \beta(\gamma-1)}(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R})$.

Moreover we can prove that $(t, x) \mapsto f(x, m(t))$ is in $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d)$ using **f2**,

$$\begin{aligned} |f(x, m(t)) - f(y, m(s))| & \leq |f(x, m(t)) - f(y, m(t))| + |f(y, m(t)) - f(y, m(s))| \\ & \leq C_0 |x - y|^\beta + C_0 d_1(m(t), m(s)) \\ & \leq C_0 |x - y|^\beta + C_0 C_\beta |t - s|^{\frac{\beta}{2}}, \end{aligned}$$

for $(t, x), (s, y) \in [0, T] \times \mathbb{T}^d$, where $C_\beta > 0$ is the $C^{\frac{\beta}{2}, \beta}$ -norm of m . Thus u is the solution to the heat equation with right-hand side in $C^{\frac{\beta(\gamma-1)}{2}, \beta(\gamma-1)}$ and terminal condition in $C^{2+\beta_0}$. Classical results yield that u is in $C^{1 + \frac{\beta(\gamma-1)}{2}, 2 + \beta(\gamma-1)}$, and its $C^{1 + \frac{\beta(\gamma-1)}{2}, 2 + \beta(\gamma-1)}$ -norm depends on $\|g(\cdot, m(T))\|_{C^{2+\beta_0}}$ and the $C^{\frac{\beta(\gamma-1)}{2}, \beta(\gamma-1)}$ -norm of the right-hand side. We recall that β is any constant in $(0, \beta_0)$. The proof of the theorem is complete. \square

Lemma 3.3. *We assume **H1**, **H2**, **H4**, **H5**, **H6**, **f1** and **g1**.*

There exist constants $C, K > 0$ independent of (T, u, m, μ) such that for any $t \leq [0, T]$, at least one of the following assertions is true

(i)

$$(3.25) \quad \|u(t)\|_\infty \leq K.$$

(ii)

$$(3.26) \quad \|\nabla_x u(t)\|_\infty \leq C \max_{t \leq s \leq T} \|u(s)\|_\infty.$$

Proof. Let us prove (3.25) and (3.26) for $t = 0$. Since the proof does not use any information on the initial condition on m , it will be possible to repeat it for any $t > 0$.

Note that constants in (3.25) and (3.26) depend on C_0, λ, γ but not on t .

Let us take $\eta = (2\|u\|_\infty)^{-1}$. We recall (3.21),

$$\|\nabla_x u\|_\infty \leq \max \left(C_0 e^{\frac{1}{2}}, 4C e \|u\|_\infty \right),$$

if $\eta \leq \eta_0$. We make out three cases:

1. The first case is when $\eta > \eta_0$ i.e. $\|u\|_\infty < (2\eta_0)^{-1}$, and (3.25) holds.

2. The second case is when $|\nabla_x u| \leq C_0 e^{\frac{1}{2}}$ is bounded uniformly with respect to (u, m, μ) and so is u by a comparison argument. Therefore (3.25) is verified.
3. In the last case we have

$$\|\nabla_x u\|_\infty \leq 4Ce\|u\|_\infty,$$

thus (3.26) holds. □

Remark 3.4. *In order to prove Lemmas 2.4 and 3.3, we can relax **H3** by replacing $M_{\mu,1}$ with $M_{\mu,\infty}$ in **H3** and **H5**.*

Lemma 3.5. *We assume **H2**, **f1** and **g1**.*

The following inequality is verified

$$(3.27) \quad \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^\gamma m(t, dx) dt \leq C_0 (C_0(1+T) + \|u\|_\infty).$$

Proof. We multiply (1.1) by $-m$ and (1.2) by u , then we sum these two terms and integrate over $(0, T) \times \mathbb{T}^d$. After performing some integrations by part, we get

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} [H_p(x, \nabla_x u(t, x), \mu(t)) \cdot \nabla_x u - H(x, \nabla_x u(t, x), \mu(t))] m(t, dx) dt \\ &= \int_{\mathbb{T}^d} u(0, x) m^0(dx) - \int_{\mathbb{T}^d} g(x, m(T)) m(T, dx) - \int_0^T \int_{\mathbb{T}^d} f(x, m(t)) m(t, dx) dt. \end{aligned}$$

This, and **H2**, **f1** and **g1**, yield (3.27). □

4 - EXISTENCE AND UNIQUENESS RESULTS UNDER ADDITIONAL ASSUMPTIONS

Lemma 4.1. *We assume **H1**, **H2**, **H4**, **H5**, **H6**, **H7**, **f1**, **f2** and **g1**.*

Let $M > 0$ be a positive constant, there exists (u^M, m^M, μ^M) a solution in the sense of Definition 2.3, of the following system

$$(4.1) \quad -\partial_t u^M(t, x) - \nu \Delta u^M(t, x) + H(x, \nabla_x u^M(t, x), \mu^M(t)) = f(x, m^M(t)) \quad \text{in } (0, T) \times \mathbb{T}^d,$$

$$(4.2) \quad \partial_t m_t^M(t, x) - \nu \Delta m^M(t, x) - \operatorname{div}(H_p(x, \nabla_x u^M(t, x), \mu^M(t)) m^M) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d,$$

$$(4.3) \quad \mu^M(t) = [I_d, T_M(-H_p(\cdot, \nabla_x u^M(t, \cdot), \mu^M(t)))] \# m^M(t) \quad \text{in } [0, T],$$

$$(4.4) \quad u^M(T, x) = g(x, m^M(T)) \quad \text{in } \mathbb{T}^d,$$

$$(4.5) \quad m^M(0) = m_0,$$

where T_M is the truncation defined by

$$T_M(v) = \begin{cases} v & \text{if } |v| \leq M, \\ \frac{M}{|v|} v & \text{otherwise.} \end{cases}$$

Definition 4.2. *The triple (u^M, m^M, μ^M) is a solution of (4.1)-(4.5), for some $M > 0$, if $u^M \in C^{1,2}([0, T] \times \mathbb{T}^d) \cap C^0([0, T] \times \mathbb{T}^d)$ is a classical solution to the Hamilton-Jacobi-Bellman equation (4.1) with terminal condition (4.4), $m^M \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ is solution to the Fokker-Planck-Kolmogorov equation (4.2) in the sense of distribution with initial condition (4.5), and $\mu^M \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ verifies (4.3) at any $t \in [0, T]$.*

Remark 4.3. *The lemmas proved in the previous sections concerning estimates on solutions of (1.1)-(1.5) hold for solutions of (4.1)-(4.5). The same proof can be used and the involved constants are independent of M .*

Proof of Lemma (4.1). Let us take $(u, m) \in C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$. Then the results in Lemma 2.4 and Corollary 2.7 hold and we can define $\mu^M \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ as follows,

$$\mu^M(t) = [I_d, T_M(-H_p(\cdot, \nabla_x u(t, \cdot), \mu^M(t)))] \# m(t) \text{ in } [0, T].$$

Then we define u^M as the viscosity solution of the following backward Hamilton-Jacobi-Bellman equation with final condition

$$(4.6) \quad \begin{cases} -\partial_t u^M(t, x) - \nu \Delta u^M(t, x) + H(x, \nabla_x u^M(t, x), \mu^M(t)) = f(x, m(t)) \\ u(T, x) = g(x, m(T)). \end{cases}$$

We use the fundamental theorem of calculus,

$$(4.7) \quad -\partial_t u^M(t, x) - \nu \Delta u^M(t, x) + \nabla_x u^M(t, x) \cdot \int_0^1 H_p(x, s \nabla_x u^M(t, x), \mu^M(t)) ds \\ = f(x, m(t)) - H(x, 0, \mu^M(t)).$$

The right-hand side of (4.7) is bounded in C^0 -norm.

So by the maximum principle for second-order parabolic equation, u^M is bounded in C^0 -norm. Then with the same argument as for Lemma 3.2, u^M is bounded in $C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}$ -norm, for all $\beta \in (0, 1)$. Let us define m^M as the solution in the sense of distribution of the following Fokker-Planck-Kolmogorov equation with initial condition

$$\begin{cases} \partial_t m^M(t, x) - \nu \Delta m^M(t, x) + \operatorname{div}(b(t, x) m^M) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ m^M(0) = m_0, \end{cases}$$

with $b(t, x) = -H_p(x, \nabla_x u^M(t, x), \mu^M(t))$ which is a continuous function with respect to (t, x) . Moreover the initial condition is in $C^{\beta_0}(\mathbb{T}^d)$. Classical results on parabolic second-order equation in divergence form give us that $m \in L^\infty([0, T] \times \mathbb{T}^d; \mathbb{R})$ (see Theorem 9.11 in [22]) and then that $m \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ (see Theorem 6.29 in [22]), and that its $C^{\frac{\beta}{2}, \beta}$ -norm is bounded, for some $\beta \in (0, \beta_0)$.

The map $(u, m) \mapsto \mu^M$ is continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ to $C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ by Lemma 2.4 and Remark 4.3.

Then let us prove that the map $(m, \mu^M) \mapsto u^M$ is continuous from $C^0([0, T]; \mathcal{P}(\mathbb{T}^d)) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ to $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$. We take a convergent sequence $(\tilde{m}^n, \tilde{\mu}^n)_{n \in \mathbb{N}} \in (C^0([0, T]; \mathcal{P}(\mathbb{T}^d)) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)))^{\mathbb{N}}$ to $(\tilde{m}, \tilde{\mu})$. We define $(\tilde{u}^n)_{n \in \mathbb{N}}, \tilde{u} \in (C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)))$ as the solution of viscosity to (4.6) respectively with $(\tilde{m}^n, \tilde{\mu}^n)_{n \in \mathbb{N}}$ and $(\tilde{m}, \tilde{\mu})$. By stability of viscosity solution we have

$$(4.8) \quad \lim_{n \rightarrow \infty} \tilde{u}^n = \tilde{u} \text{ in } C^0([0, T] \times \mathbb{T}^d; \mathbb{R}),$$

since H is continuous with respect to μ and f is continuous with respect to m . Moreover we proved that \tilde{u}^n is bounded in $C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ uniformly with respect to n . Hence the sequence is relatively compact in $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$. Thus $\tilde{u}^n \rightarrow \tilde{u}$ in $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$.

Finally the map $(u^M, \mu^M) \mapsto m^M$ is also continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ to $C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ by linearity of the FPK equation and the $C^{\frac{\beta}{2}, \beta}$ -estimates, we have obtained above, with $\beta \in (0, \beta_0)$.

Thus the map $(u, m) \mapsto (u^M, m^M)$ is continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ to itself. The fixed points are exactly the solution to (4.1)-(4.5). The image of this map is a subset of a convex compact set. Thus there exists a fixed point by Schauder's theorem.

Using the same arguments as in the third step of the proof of Lemma 3.2, a fixed point verifies $u \in C^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ for any $\beta \in (0, (\gamma - 1)\beta_0)$. \square

Corollary 4.4. *Assume that any solution (u, m, μ) of (1.1)-(1.5) verifies the a priori estimate $\|\nabla_x u\|_\infty \leq C$, where $C > 0$ is a constant which does not depend on (u, m, μ) .*

Then there exists at least one solution of (1.1)-(1.5).

Proof. For $M > 0$ that will be defined later, there exists (u^M, m^M, μ^M) a solution to (4.1)-(4.5) by Lemma 4.1. In particular $\|\nabla_x u^M\|_\infty \leq C$ by Remark 4.3 and the new assumption in the corollary. From (2.6) and Remark 4.3, we obtain $M_{\mu^M(t), \gamma'} \leq \frac{C_0^{\gamma'} 2^{\gamma'-1} (1+C^\gamma)}{(1-\lambda)^{\gamma'}}$. This, **H5** and Remark 4.3 imply

$$\|H_p(x, \nabla_x u^M, \mu^M)\|_\infty \leq C_0 \left(1 + C^\gamma + \frac{C_0^{\gamma'} 2^{\gamma'-1} (1+C^\gamma)}{(1-\lambda)^{\gamma'}} \right).$$

Therefore, for the constant M defined by

$$M = 1 + C_0 \left(1 + C^\gamma + \frac{C_0^{\gamma'} 2^{\gamma'-1} (1+C^\gamma)}{(1-\lambda)^{\gamma'}} \right),$$

the truncation T_M leaves $-H_p(\cdot, \nabla_x u^M, \mu)$ unchanged. Hence (u^M, m^M, μ^M) is a solution to (1.1)-(1.5). \square

Remark 4.5. *We did not include the case $\gamma = 1$ to the hypothesis (i.e. when the Hamiltonian is Lipschitz continuous in p), however under assumptions **H1**, **H3-H7**, **f1 f2** and **g1**, the existence of solutions of (1.1)-(1.5) hold using the same arguments as in the proof of Lemma 4.1 since $-H_p$ the optimal control is a priori bounded, and so is the support of μ .*

Theorem 4.6 (*Existence of solution with small non-linearities*). *We assume **H1**, **H2**, **H3**, **H4**, **H5**, **H6**, **H7**, **f1**, **f2** and **g1**.*

If

$$(4.9) \quad 2^{\gamma'-1} C_0^{2+\gamma'} < (1-\lambda)^{\gamma'},$$

then there exists at least a solution to (1.1)-(1.5).

Proof. Let us suppose that (u, m, μ) is a solution to (1.1)-(1.5) and that (4.9) is verified.

From (3.1) and Lemma 3.5 we obtain

$$\|u\|_\infty \leq C_0 \left(1 + 2T + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} (T + C_0^2(1+T) + C_0 \|u\|_\infty) \right).$$

Thus if

$$\frac{2^{\gamma'-1} C_0^{2+\gamma'}}{(1-\lambda)^{\gamma'}} < 1,$$

the function u is uniformly bounded with respect to (m, μ) , so is $\nabla_x u$ by Lemma 3.2. Therefore corollary 4.4 yields the existence of solution of (1.1)-(1.5). \square

Corollary 4.7. *1. The result in Theorem (4.6) holds if we suppose that,*

$$(4.10) \quad C_0^{2+\gamma'} < (1-\lambda)^{\gamma'},$$

instead of (4.9).

2. If we suppose that H satisfies the following inequality,

$$|H(x, 0, \mu)| \leq C_0 + \delta M_{\mu, \gamma'},$$

for all $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, and for $\delta > 0$ such that

$$\delta < \frac{(1 - \lambda)^{\gamma'}}{C_0^{1 + \gamma'}}.$$

then there exists a solution to (1.1)-(1.5).

Proof. 1. We suppose (4.10). We need a finer estimate on $M_{\mu, \gamma'}$ with respect to $\nabla_x u$ than the one given in Lemma 2.4. We introduce $\delta \in (0, 1)$ defined by $C_0^{\gamma' + 2} = (1 - 2\delta)^{\gamma' - 1} (1 - \lambda)^{\gamma'}$. We recall the fourth line in (2.4) with $\tilde{\gamma} = \gamma'$,

$$M_{\mu, \gamma'} \leq \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma' - 1}} \left(1 + \int_{\mathbb{T}^d} |p(x)|^{\gamma' - 1} m(dx) \right)^{\gamma'} + \lambda M_{\mu, \gamma'}.$$

This implies

$$\begin{aligned} M_{\mu, \gamma'} &\leq \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'}} \left(1 + \int_{\mathbb{T}^d} |p(x)|^{\gamma' - 1} m(dx) \right)^{\gamma'} \\ &= \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'} (1 - \delta)^{\gamma'}} \left(\delta \frac{1 - \delta}{\delta} + (1 - \delta) \int_{\mathbb{T}^d} |p(x)|^{\gamma' - 1} m(dx) \right)^{\gamma'} \\ &\leq \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'} \delta^{\gamma'}} + \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'} (1 - \delta)^{\gamma' - 1}} \left(\int_{\mathbb{T}^d} |p(x)|^{\gamma' - 1} m(dx) \right)^{\gamma'} \\ &\leq \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'} \delta^{\gamma'}} + \frac{(1 - 2\delta)^{\gamma' - 1}}{C_0^2 (1 - \delta)^{\gamma' - 1}} \int_{\mathbb{T}^d} |p(x)|^{\gamma'} m(dx), \end{aligned}$$

where we used a convexity inequality in the third line, then Jensen's inequality and the definition of δ in the fourth line. Proceeding as in the proof of Theorem 4.6,

$$\|u\|_{\infty} \leq C_0 \left(1 + 2T + \frac{C_0^{\gamma'}}{(1 - \lambda)^{\gamma'} \delta^{\gamma'}} T + \frac{(1 - 2\delta)^{\gamma' - 1}}{(1 - \delta)^{\gamma' - 1}} (1 + T) \right) + \frac{(1 - 2\delta)^{\gamma' - 1}}{(1 - \delta)^{\gamma' - 1}} \|u\|_{\infty}.$$

Therefore $\|u\|_{\infty}$ is bounded by a constant depending on the constant in the assumptions. Then we can conclude by applying Lemma 3.2 and Corollary 4.4.

2. The proof is exactly the same as for Theorem 4.6 but the new assumption is used instead of **H3** in Lemma 3.1. □

Theorem 4.8. We assume **H1**, **H2**, **H3**, **H4**, **H5**, **H6**, **H7**, **f1**, **f2** and **g1**.

If we suppose that H verifies the inequality

$$|H(x, 0, \mu)| \leq C_0 (1 + M_{\mu, \tilde{\gamma}}),$$

for $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ with $\tilde{\gamma} < \gamma'$, then there exists a solution of (1.1)-(1.5).

Proof. We apply the same techniques as in the proof of Lemma 3.1. We get the following inequality:

$$(4.11) \quad \|u\|_{\infty} \leq C_0 \left(1 + 2T + \int_0^T M_{\mu(t), \tilde{\gamma}} dt \right).$$

From (2.2), we use two successive Hölder inequalities as follows:

$$\begin{aligned}
\int_0^T M_{\mu(t), \tilde{\gamma}} dt &\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(T + \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^{(\gamma-1)\tilde{\gamma}} m(t, dx) dt \right) \\
&\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(T + \int_0^T \left(\int_{\mathbb{T}^d} |\nabla_x u|^\gamma m(t, dx) \right)^{\frac{\tilde{\gamma}}{\gamma}} dt \right) \\
&\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(T + T^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} \left(\int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^\gamma m(t, dx) dt \right)^{\frac{\tilde{\gamma}}{\gamma}} \right).
\end{aligned}$$

This and Lemma 3.5 imply that

$$\begin{aligned}
\int_0^T M_{\mu(t), \tilde{\gamma}} dt &\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(T + T^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} (C_0^2(1+2T) + C_0 \|u\|_\infty)^{\frac{\tilde{\gamma}}{\gamma}} \right) \\
&\leq \frac{C_0^{\tilde{\gamma}} 2^{\tilde{\gamma}-1}}{(1-\lambda)^{\tilde{\gamma}}} \left(T + T^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} \left(C_0^{\frac{2\tilde{\gamma}}{\gamma}} (1+2T)^{\frac{\tilde{\gamma}}{\gamma}} + C_0^{\frac{\tilde{\gamma}}{\gamma}} \|u\|_\infty^{\frac{\tilde{\gamma}}{\gamma}} \right) \right),
\end{aligned}$$

since $\tilde{\gamma} < \gamma'$.

Thus for some $C > 0$ we have

$$\left(\|u\|_\infty^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} - C \right) \|u\|_\infty^{\frac{\tilde{\gamma}}{\gamma'}} = \|u\|_\infty - C \|u\|_\infty^{\frac{\tilde{\gamma}}{\gamma'}} \leq C.$$

This implies that, either $\|u\|_\infty \leq (C+1)^{\frac{\gamma'}{\gamma'-\tilde{\gamma}}}$, or $\|u\|_\infty^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} > C+1$ and

$$\|u\|_\infty^{\frac{\tilde{\gamma}}{\gamma'}} \leq \frac{C}{\|u\|_\infty^{\frac{\gamma'-\tilde{\gamma}}{\gamma'}} - C} \leq C.$$

In any case, u is uniformly bounded with respect to (m, μ) , which implies that there exists a solution to (1.1)-(1.5) by Corollary 4.4. \square

Theorem 4.9 (*Existence with short time horizon*). *We assume **H1, H2, H3, H4, H5, H6, H7, f1, f2** and **g1**.*

There exists $T_0 > 0$ such that, if $T < T_0$ then there exists a solution to (1.1)-(1.5).

Proof. We suppose that (u, m, μ) is a solution to (1.1)-(1.5).

By Lemma 3.3, there exists some constants $C, K > 0$ depending on the constants in the assumptions, such that for any $t \in [0, T]$,

$$\|\nabla_x u(t)\|_\infty \leq C \max \left(K, \max_{t \leq s \leq T} \|u(s)\|_\infty \right).$$

Remark 2.5 and (3.2) yield that,

$$\begin{aligned}
(4.12) \quad & -\partial_t u(t, x) - \nu \Delta u(t, x) + \nabla_x u(t, x) \cdot \int_0^1 H_p(x, s \nabla_x u(t, x), \mu(t)) ds \\
& = f(x, m(t)) - H(x, 0, \mu(t)) \\
& \leq C_0 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left(1 + \int_{\mathbb{T}^d} |\nabla_x u|^\gamma dm(t) \right) \\
& \leq C_0 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left[1 + C^\gamma \left(\max_{t \leq s \leq T} \|u(s)\|_\infty \right)^\gamma \right] \\
& \leq C_0 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left[1 + C^\gamma \left(\left(K + \max_{t \leq s \leq T} \|u(s)\|_\infty \right)^\gamma \right) \right].
\end{aligned}$$

This and an inequality of convexity imply

$$(4.13) \quad \begin{cases} -\partial_t u(t, x) - \nu \Delta u(t, x) + \nabla_x u(t, x) \cdot \int_0^1 H_p(x, s \nabla_x u(t, x), \mu(t)) ds \\ \leq C_0 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} \left[1 + C^\gamma 2^{\gamma-1} \left(K^\gamma + \max_{t \leq s \leq T} \|u(s)\|_\infty^\gamma \right) \right], \\ u(T, x) = g(x, m(T)). \end{cases}$$

We consider the following differential equation on $[0, T]$

$$\begin{cases} y'(t) = C_0 + \frac{C_0^{\gamma'} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} [1 + C^\gamma 2^{\gamma-1} (K^\gamma + y^\gamma)] \\ y(0) = C_0 \end{cases}$$

It has a bounded solution if $T < T_0$, for some $T_0 > 0$.

We suppose that $T < T_0$, then $(t, x) \mapsto y(T-t)$ is a super-solution to (4.13), hence by a comparison principle, $u \leq y$. We can also prove the same way that $u \geq -y$. Therefore u is bounded uniformly with respect to (m, μ) and there exists a solution to (1.1)-(1.5) by Corollary 4.4. \square

Theorem 4.10 (*Uniqueness with short time horizon*). *Under Assumptions **H1, H2, H3, H4, H5, H6, H7, f1, f2, g1**, and the additional assumptions that $\gamma = 2$, and that g satisfies the inequality below,*

$$(4.14) \quad \|g(\cdot, m^1) - g(\cdot, m^2)\|_\infty \leq C_0 d_1(m^1, m^2),$$

for any $m^1, m^2 \in \mathcal{P}(\mathbb{T}^d)$, there exists $T_1 > 0$ such that if $T < T_1$ then there is at most one solution to (1.1)-(1.5).

Proof. We take (u^1, m^1, μ^1) and (u^2, m^2, μ^2) two solutions of system (1.1)-(1.5).

We will suppose $T_1 \leq T_0$, where T_0 was defined in Theorem 4.9, so that we have uniform regularity estimates on u^i and $\nabla_x u^i$ and $D_{x,x} u^i$, for $i = 1, 2$. We introduce the two random processes X^1, X^2 which are solution to the following SDEs: for $i = 1, 2$,

$$\begin{cases} dX_t^i = \alpha^i dt + \sqrt{2\nu} dB_t \\ \alpha_t^i = -H_p(X_t^i, \nabla_x u^i(t, X_t^i), \mu^i(t)) \\ X_0^i = \xi, \text{ with } \mathcal{L}(\xi) = m^0, \end{cases}$$

Using Ito's integration, we see that

$$\mathbb{E} [|X_t^1 - X_t^2|] \leq \int_0^t \mathbb{E} [|\alpha_s^1 - \alpha_s^2|] ds.$$

Then with Lemma 4.11 below with $\gamma = 2$, we obtain

$$\sup_{t \in [0, T]} \mathbb{E} [|X_t^1 - X_t^2|] \leq CT \sup_{t \in [0, T]} (d_1(m^1(t), m^2(t)) + \|\nabla_x(u^1 - u^2)\|_\infty),$$

where C depends on $\max_{i=1,2} \sup_{t \in [0, T]} \|u^i(t, \cdot)\|_{C^2}$. Let us take $T_1 < C^{-1}$ then the inequality below holds,

$$(4.15) \quad \sup_{t \in [0, T]} d_1(m^1(t), m^2(t)) \leq CT(1-CT)^{-1} \sup_{t \in [0, T]} \|\nabla_x(u^1(t) - u^2(t))\|_\infty.$$

instead of (3.14). Then (2.2) and $\frac{\gamma'}{\gamma}(\gamma - 1) = 1$ yield

$$\begin{aligned} M_{\mu(t), \frac{\gamma'}{\gamma}} &\leq \frac{C_0^{\frac{\gamma'}{\gamma}} 2^{\frac{\gamma'}{\gamma}-1}}{(1-\lambda)^{\frac{\gamma'}{\gamma}}} \left(1 + \int_{\mathbb{T}^d} |\nabla_x u| dm(t, x) \right) \\ &\leq \frac{C_0^{\frac{\gamma'}{\gamma}} 2^{\frac{\gamma'}{\gamma}-1}}{(1-\lambda)^{\frac{\gamma'}{\gamma}}} (1 + \|\nabla_x u\|_\infty). \end{aligned}$$

This implies

$$\begin{aligned} \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) - 2\nu e^{-\eta(u^\delta + \|u\|_\infty)} \eta \nabla_x w^\delta \cdot \nabla_x u^\delta \\ \leq \varepsilon + 2C_0 e \left(2 + \frac{C_0^{\frac{\gamma'}{\gamma}} 2^{\frac{\gamma'}{\gamma}-1}}{(1-\lambda)^{\frac{\gamma'}{\gamma}}} \right) \|\nabla_x u\|_\infty \\ + 2C_0 e \eta \|\nabla_x u\|_\infty^2 + 2C_0 e \left(1 + \frac{C_0^{\frac{\gamma'}{\gamma}} 2^{\frac{\gamma'}{\gamma}-1}}{(1-\lambda)^{\frac{\gamma'}{\gamma}}} \right) \|\nabla_x u\|_\infty^2 - C_0^{-1} e^{-\frac{\gamma}{2}} \eta e^{-2\eta\|u\|_\infty} (w^\delta)^{1+\frac{\gamma}{2}}. \end{aligned}$$

Thus either w^δ is bounded by eC_0^2 , or it verifies

$$\|w^\delta\|_\infty^{1+\frac{\gamma}{2}} \leq C \eta^{-1} e^{2\eta\|u\|_\infty} \left(\varepsilon + 1 + \|\nabla_x u\|_\infty^2 \right),$$

for some $C > 0$. We let δ and ε tend to 0,

$$\|w^0\|_\infty^{1+\frac{\gamma}{2}} \leq \max \left[C_0^{2+\gamma} e^{1+\frac{\gamma}{2}}, C \eta^{-1} e^{2\eta\|u\|_\infty} (1 + \|w^0\|_\infty) \right].$$

We note s_η the maximal root of the following polynomial function,

$$R_\eta = r^{1+\frac{\gamma}{2}} - C \eta^{-1} e^{2\eta\|u\|_\infty} (1 + r).$$

Then there exists $\eta_0 > 0$ such that if $\eta \leq \eta_0$ then,

$$s_\eta^{\frac{\gamma}{2}} \leq 2C \eta^{-1} e^{2\eta\|u\|_\infty}.$$

We take $\eta = \|u\|_\infty^{-1}$. We make out two cases:

- $\eta > \eta_0$, i.e $\|u\|_\infty < \eta_0^{-1}$, then we can apply Lemma 3.2: there exists a solution to (1.1)-(1.5) by Corollary 4.4.
- $\eta \leq \eta_0$ and

$$\|w^0\|_\infty \leq \max \left(eC_0^2, (2Ce^2\|u\|_\infty)^{\frac{2}{\gamma}} \right).$$

This implies

$$\begin{aligned} \|\nabla_x u(0)\|_\infty^\gamma &\leq \max \left(e^{\frac{\gamma}{2}} C_0^\gamma, 2Ce^2 \max_{0 \leq s \leq T} \|u(s)\|_\infty \right) \\ &\leq e^{\frac{\gamma}{2}} C_0^\gamma + 2Ce^2 \max_{0 \leq s \leq T} \|u(s)\|_\infty. \end{aligned}$$

Since the constants in the latter equation do not depend on time and on the initial condition on m , we can repeat the previous arguments for any $t \in [0, T]$ and get

$$\|\nabla_x u(t)\|_\infty^\gamma \leq e^{\frac{\gamma}{2}} C_0^\gamma + 2Ce^2 \max_{t \leq s \leq T} \|u(s)\|_\infty.$$

From (1.1), u satisfies the heat equation with right-hand side $f(x, m(t)) - H(x, \nabla_x u(t), \mu(t))$ which can be bounded in absolute value using **H3**, **f1**, Remark 2.5, and (4):

$$\begin{aligned} |f(x, m(t)) - H(x, \nabla_x u, \mu(t))| &\leq C_0 + C_0 (1 + |\nabla_x u|^\gamma + M_{\mu(t), \gamma'}) \\ &\leq 2C_0 + C_0 \|\nabla_x u(t)\|_\infty^\gamma + \frac{C_0^{\gamma'+1} 2^{\gamma'-1}}{(1-\lambda)^{\gamma'}} (1 + \|\nabla_x u(t)\|_\infty^\gamma) \\ &\leq C \left(1 + \max_{t \leq s \leq T} \|u(s)\|_\infty \right), \end{aligned}$$

for some new constant $C > 0$ different than the one we used before. And from **g1**, $\|u(T)\|_\infty \leq C_0$.

We consider $y_+, y_- \in C^1([0, T]; \mathbb{R})$ defined as $y_+(t) = Ct + C_0 e^{Ct}$ and $y_-(t) = -Ct - C_0 e^{Ct}$ such that they are solution to the following differential equations

$$\begin{cases} y'_+(t) = C(1 + y_+(t)) \\ y_+(0) = C_0, \\ y'_-(t) = C(-1 + y_-(t)) \\ y_-(0) = -C_0. \end{cases}$$

By a comparison argument we obtain,

$$-CT - C_0 e^{CT} \leq y_-(T-t) \leq u(t, x) \leq y_+(t) \leq CT + C_0 e^{CT},$$

for $(t, x) \in [0, T] \times \mathbb{T}^d$.

Therefore by Lemma 3.2 $\|\nabla_x u\|_\infty$ is bounded by a constant depending on the constants of the assumptions. Thus there exists a solution to (1.1)-(1.5) by Corollary 4.4. □

5 - RESULTS WITH SOME MONOCITY CONDITIONS

The convex Lagrangian associated with the Hamiltonian H is defined as its Legendre transform

$$L(x, \alpha, \mu) = \sup_{p \in \mathbb{R}^d} -\alpha \cdot p - H(x, p, \mu).$$

Let us introduce some new assumptions of monotonicity.

M1 For $\mu^1 \neq \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$,

$$(5.1) \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} (L(x, \alpha, \mu^1) - L(x, \alpha, \mu^2)) d(\mu^1 - \mu^2)(x, \alpha) > 0.$$

M2 For $m^1, m^2 \in \mathcal{P}(\mathbb{T}^d)$, we suppose

$$\begin{cases} \int_{\mathbb{T}^d} (f(x, m^1) - f(x, m^2)) d(m^1 - m^2)(x) \geq 0, \\ \int_{\mathbb{T}^d} (g(x, m^1) - g(x, m^2)) d(m^1 - m^2)(x) \geq 0. \end{cases}$$

Theorem 5.1 (*Uniqueness with a monotonicity Assumptions*). *If **M1** and **M2** are verified, then there is at most one solution to (1.1)-(1.5).*

Proof. We suppose that (u^1, m^1, μ^1) and (u^2, m^2, μ^2) are two solutions to (1.1)-(1.5). We multiply by $m^1 - m^2$ the HJB equations (1.1) for u^1 and u^2 then subtract the two resulting identities, and we multiply by $u^2 - u^1$ the two FPK equations (1.2) for m^1 and m^2 the subtract the two resulting identities. We sum and integrate over $(0, T) \times \mathbb{T}^d$. After some integrations by part we get

$$(5.2) \quad \begin{aligned} 0 &= \int_0^T \int_{\mathbb{T}^d} [\nabla_x(u^1 - u^2) \cdot H_p(x, \nabla_x u^1, \mu^1) - H(x, \nabla_x u^1, \mu^1) + H(x, \nabla_x u^2, \mu^2)] dm^1(t, x) \\ &+ \int_0^T \int_{\mathbb{T}^d} [\nabla_x(u^2 - u^1) \cdot H_p(x, \nabla_x u^2, \mu^2) - H(x, \nabla_x u^2, \mu^2) + H(x, \nabla_x u^1, \mu^1)] dm^2(t, x) \\ &+ \int_0^T \int_{\mathbb{T}^d} (f(x, m^1(t)) - f(x, m^2(t))) d(m^1(t, x) - m^2(t, x)) dt \\ &+ \int_{\mathbb{T}^d} (g(x, m^1(T)) - g(x, m^2(T))) d(m^1(T, x) - m^2(T, x)). \end{aligned}$$

We introduce $\alpha^i(x) = -H_p(x, \nabla_x u^i, \mu^i)$ for $i = 1, 2$, then by the conjugacy relations, we get the two equalities below,

$$(5.3) \quad \begin{aligned} L(x, \alpha^i, \mu^i) &= \nabla_x u^i \cdot H_p(x, \nabla_x u^i, \mu^i) - H(x, \nabla_x u^i, \mu^i), \\ \nabla_x u^i &= -L_\alpha(x, \alpha^i, \mu^i). \end{aligned}$$

From **M2**, (5.2) and (5.3), we get

$$(5.4) \quad \begin{aligned} 0 &\geq \int_0^T \int_{\mathbb{T}^d} [L(x, \alpha^1, \mu^1) - L(x, \alpha^2, \mu^2) - (\alpha^1 - \alpha^2) \cdot L_\alpha(x, \alpha^2, \mu^2)] dm^1(t, x) dt \\ &+ \int_0^T \int_{\mathbb{T}^d} [L(x, \alpha^2, \mu^2) - L(x, \alpha^1, \mu^1) - (\alpha^2 - \alpha^1) \cdot L_\alpha(x, \alpha^1, \mu^1)] dm^2(t, x) dt. \end{aligned}$$

The function L is convex in α since it is a Legendre transform so

$$\begin{aligned} L(x, \alpha^1, \mu^2) - L(x, \alpha^2, \mu^2) - (\alpha^1 - \alpha^2) \cdot L_\alpha(x, \alpha^2, \mu^2) &\geq 0, \\ L(x, \alpha^2, \mu^1) - L(x, \alpha^1, \mu^1) - (\alpha^2 - \alpha^1) \cdot L_\alpha(x, \alpha^1, \mu^1) &\geq 0. \end{aligned}$$

The latter inequalities, (5.4) and the definition of μ^1 and μ^2 yield

$$\begin{aligned} 0 &\geq \int_0^T \int_{\mathbb{T}^d} [L(x, \alpha^1, \mu^1) - L(x, \alpha^1, \mu^2)] dm^1 dt + \int_0^T \int_{\mathbb{T}^d} [L(x, \alpha^2, \mu^2) - L(x, \alpha^2, \mu^1)] dm^2 dt \\ &\geq \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} [L(x, \alpha, \mu^1) - L(x, \alpha, \mu^2)] d(\mu^1 - \mu^2)(t, x, \alpha) dt. \end{aligned}$$

Then using **M1** and the continuity of μ^1 and μ^2 , we get that $\mu^1 = \mu^2$. Since m^1, m^2 are the first marginals of respectively μ^1, μ^2 , we have $m^1 = m^2$. Finally $u^1 = u^2$ by uniqueness of the solution of (1.1), (1.4). \square

Corollary 5.2. *The uniqueness holds if the $>$ symbol in (5.1) is replaced by \geq , and we suppose that L verifies the inequality below,*

$$(5.5) \quad L(x, \alpha^2, \mu) - L(x, \alpha^1, \mu) - (\alpha^2 - \alpha^1) \cdot L_\alpha(x, \alpha^1, \mu) \geq C_0^{-1} |\alpha^2 - \alpha^1|^{\gamma'},$$

for $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, $x \in \mathbb{T}^d$ and $\alpha^1, \alpha^2 \in \mathbb{R}^d$.

Theorem 5.3 (*Existence with a monotonicity assumption*). We assume **H1**, **H2**, **H4**, **H5**, **H6**, **H7**, **f1**, **f2**, **g1**, **M1** and that the inequality below is verified for any $(x, \alpha, m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$,

$$(5.6) \quad L(x, \alpha, m \otimes \delta_0) \geq C_0^{-1} |\alpha|^{\gamma'} - C_0,$$

where $\delta_0 \in \mathcal{P}(\mathbb{R}^d)$ is the Dirac measure at 0 on \mathbb{R}^d . Then there exists at least one solution to (1.1)-(1.5).

Proof. We suppose that (u, m, μ) is a solution to (1.1)-(1.5).

We take (X, α) defined as

$$\begin{cases} \alpha_t = -H_p(X_t, \nabla_x u(t, X_t), \mu(t)), \\ dX_t = \alpha_t dt + \sqrt{2\nu} dB_t, \\ X_0 = \xi \sim m_0, \end{cases}$$

where $(B_t)_{t \in [0, T]}$ is a Brownian motion independent of ξ .

The function u is the value function of a controlled problem, i.e. the lowest cost that a representative agent can achieve from time t to T if at t it is at state x , when the probability measures m, μ are imposed (this optimality condition does not assume a priori that m is the law of X , or that μ is the joint law of (X, α)):

$$(5.7) \quad \mathbb{E} \left[\int_0^T L(X_t, \alpha_t, \mu(t)) + f(X_t, m(t)) dt + g(X_T, m(T)) \right] \\ = \min_{\alpha'} \mathbb{E} \left[\int_0^T L(X_t^{\alpha'}, \alpha'_t, \mu(t)) + f(X_t^{\alpha'}, m(t)) dt + g(X_T^{\alpha'}, m(T)) \right],$$

where for some control α' , we define

$$\begin{cases} dX^{\alpha'} = \alpha' dt + \sqrt{2\nu} dB'_t, \\ X_0^{\alpha'} = \xi' \sim m_0, \end{cases}$$

and $(B'_t)_{t \in [0, T]}$ is a Brownian motion independent of ξ' .

From (1.3), we know that $\mathcal{L}(X_t, \alpha_t) = \mu(t)$, and $m(t)$ is the law of X_t by (1.2), for any $t \in [0, T]$.

We introduce as \tilde{X} the stochastic process defined as

$$\begin{cases} d\tilde{X}_t = \sqrt{2\nu} dB_t, \\ \tilde{X}_0 = \xi \sim m_0, \end{cases}$$

and we note $\tilde{\mu} = \mathcal{L}(\tilde{X}, 0)$.

We apply the optimality criterion of (5.7) to $\alpha' = 0$,

$$\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\mu(t, x, \alpha) dt + \int_0^T \int_{\mathbb{T}^d} f(x, m(t)) dm(t, x) dt + \int_{\mathbb{T}^d} g(x, m(T)) m(T, dx) \\ \leq \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) dt + \int_0^T \int_{\mathbb{T}^d} f(x, m(t)) d\tilde{m}(t, x) dt + \int_{\mathbb{T}^d} g(x, m(T)) d\tilde{m}(T, x).$$

This, **f1** and **g1** imply

$$(5.8) \quad \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\mu(t, x, \alpha) dt \\ \leq \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) dt + 2C_0(1 + T).$$

We use assumption **M1**, with $(\mu, \tilde{\mu})$,

$$(5.9) \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) + \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \tilde{\mu}(t)) d\mu(t, x, \alpha) \\ \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \mu(t)) d\mu(t, x, \alpha) + \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \tilde{\mu}(t)) d\tilde{\mu}(t, x, \alpha).$$

Then L is continuous on the compact set $\mathbb{T}^d \times \{0\} \times \{m' \otimes \delta_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d), m' \in \mathcal{P}(\mathbb{T}^d)\}$, thus it is bounded from above on that set by a constant $C > 0$. Hence

$$(5.10) \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \tilde{\mu}(t)) \tilde{\mu}(t, d(x, \alpha)) = \int_{\mathbb{T}^d} L(x, 0, \tilde{\mu}(t)) \tilde{m}(t, dx) \leq C.$$

Therefore from (5.6), (5.8), (5.9) and (5.10), we obtain,

$$\int_0^T \int_{\mathbb{T}^d} (C_0^{-1} |\alpha|^{\gamma'} - C_0) dm(t, x) \leq \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, \alpha, \tilde{\mu}(t)) d\mu(t, x, \alpha) dt \\ \leq 2C_0(1 + T) + CT.$$

From the latter inequality we deduce that

$$(5.11) \quad \int_0^T M_{\mu(t), \gamma'} dt \leq C_0^2(3 + 2T) + C_0CT.$$

Then, by (3.3) and (5.11) we obtain

$$\|u\|_\infty \leq C_0(1 + 2T) + C_0^3(3 + 2T) + C_0^2CT.$$

Thus, from Lemma 3.2, $\nabla_x u$ is bounded by a constant which depends on the constants of the assumptions.

Therefore there exists a solution to (1.1)-(1.5) by Corollary 4.4. \square

6 - EXAMPLES

In this section we make assumptions **f1**, **f2**, **g1**. We recall that the initial distribution m_0 is absolutely continuous with respect to the Lebesgue measure on \mathbb{T}^d . Its density, still named m_0 , is assumed to be β_0 -Hölder continuous, with the same exponent β_0 as in **(f1)** and **(g1)**.

Hereafter (x, α, μ) is a generic element of $\mathbb{T}^d \times \mathbb{R}^d \times \mathcal{L}(\mathbb{T}^d \times \mathbb{R}^d)$. We consider the system obtained with the following Lagrangian,

$$(6.1) \quad \begin{cases} \tilde{L}(x, \alpha, \mu) = L(\alpha, V(x)) = \frac{\theta}{a'} |\alpha - \tilde{\lambda} V(x)|^{a'} + \frac{1 - \theta}{b'} |\alpha|^{b'}, \\ V(x) = \frac{1}{Z(x)} \int_{\Omega \times \mathbb{R}^d} \tilde{\alpha} k(x, y) d\mu(y, \tilde{\alpha}), \\ Z(x) = \int_{\Omega} k(x, y) dm(y), \end{cases}$$

with $2 \leq a', b'$, $-1 < \tilde{\lambda} < 1$, and a Markovian kernel k . We suppose that k is differentiable with respect to x with a differential bounded uniformly with respect to $y \in \mathbb{T}^d$. For example $k(x, y) = \tilde{Z}^{-1} \exp(-\frac{\rho}{2} |x - y|_{\mathbb{T}^d}^2)$, where $|x - y|_{\mathbb{T}^d}$ is the distance on \mathbb{T}^d induced by the euclidean norm on \mathbb{R}^d , and $\tilde{Z} = \int_{\mathbb{T}^d} \exp(-\frac{\rho}{2} |x|_{\mathbb{T}^d}^2) dx$.

We define a and b the respective conjugate exponents of a' and b' by $a = \frac{a'}{a'-1}$ and $b = \frac{b'}{b'-1}$.

It is convenient to introduce the Hamiltonian $H(p, V)$ as the Legendre transform of L , i.e.

$$(6.2) \quad H(p, V) = \max_{\alpha \in \mathbb{R}^d} -p \cdot \alpha - L(\alpha, V),$$

for $(p, V) \in \mathbb{R}^d \times \mathbb{R}^d$.

A set of hypotheses that ensure that **H1-H7** are satisfied when H is defined as in (6.3) is proposed in the following paragraph:

H'1 H is convex with respect to p , H is differentiable with respect to (p, V) , H_p is differentiable with respect to V , H_p is locally $(\gamma - 1)$ -Hölder continuous with respect to p , for some γ , $1 < \gamma \leq 2$.

There exists $C_0 > 0$ and λ : $0 \leq \lambda < 1$ such that for any $(p, V, \tilde{V}) \in \mathbb{R}^{3d}$,

H'2 $H_p(p, V) \cdot p - H(p, V) \geq -C_0 + C_0^{-1}|p|^\gamma$.

H'3 $|H(p, V)| \leq C_0 (1 + |p|^\gamma + |V|^{\gamma'})$.

H'4 $|H_V(p, V)| \leq C_0 (1 + |p| + |V|^{\gamma'-1})$.

H'5 $|H_p(p, V)| \leq C_0 (1 + |p|^{\gamma-1}) + \lambda|V|$.

H'6

$$\left| H_p(p, V) - H_p(p, \tilde{V}) \right| \leq \lambda |V - \tilde{V}|$$

H'7 H is locally Lipschitz w.r.t. V , i.e. there exists $0 < C_1 = C_1(|p|, \max(|V|, |\tilde{V}|))$ such that

$$\left| H(p, V) - H(p, \tilde{V}) \right| \leq C_1 |V - \tilde{V}|.$$

Definition 6.1. *The Lagrangian L is strictly convex with respect to α . Thus for any $(p, V) \in \mathbb{R}^{2d}$, we can define $\alpha = \alpha(p, V) \in \mathbb{R}^d$ as the unique optimal control in (6.2),*

$$(6.3) \quad H(p, V) = -p \cdot \alpha - L(\alpha, V).$$

We have the identity $\alpha(p, V) = -H_p(p, V)$.

Lemma 6.2. *If $a = b = 2$, the Hamiltonian H can be written explicitly in the following way:*

$$(6.4) \quad H(p, V) = \frac{|p|^2}{2} - \tilde{\lambda}\theta p \cdot V - \frac{\tilde{\lambda}^2\theta(1-\theta)}{2}|V|^2,$$

and it verifies assumptions **H'1-H'7**.

Proof. Take $(p, V) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\alpha = \alpha(p, V)$ the optimal control in definition 6.1. It verifies

$$0 = -p - \theta(\alpha - \tilde{\lambda}V) - (1 - \theta)\alpha,$$

hence

$$\alpha = p - \tilde{\lambda}\theta V.$$

This implies (6.4). Then assumptions **H'1-H'7** can be checked by straightforward calculus. \square

Remark 6.3. *We will not consider the case $\theta = 0$ since it is the linear quadratic model which has already been treated several times in the MFG literature, see for example [23].*

Proposition 6.4. *If $\theta = 1$, the Hamiltonian H can be written explicitly the following way,*

$$(6.5) \quad H(p, V) = \frac{1}{a}|p|^a - \tilde{\lambda}p \cdot V,$$

and there exists a solution to (1.1)-(1.5).

Proof. Take $(p, V) \in \mathbb{R}^d \times \mathbb{R}^d$, and let $\alpha = \alpha(p, V)$ be the optimal control in Definition 6.1. It satisfies (A.1), hence

$$|p| = \left| \alpha - \tilde{\lambda}V \right|^{a'-1},$$

and

$$\begin{aligned} \alpha &= \tilde{\lambda}V - |\alpha - \tilde{\lambda}V|^{2-a'}p \\ &= \tilde{\lambda}V - |p|^{\frac{2-a'}{a'-1}}p. \end{aligned}$$

Thus H is given by (6.5), **H'1-H'7** hold and $H(0, \cdot) = 0$. Therefore there exists a solution to (1.1)-(1.5) by Theorem 4.8. \square

Proposition 6.5. *If k is the constant function $k(x, y) = 1$, then there exists at least one solution to (1.1)-(1.5).*

Proof. Lemmas A.1 and A.4, which are stated and proved in the appendix, yield that **H'1-H'7** hold.

Here V does not depend on x , thus there exists a solution to (1.1)-(1.5) by Theorem 4.12. \square

Proposition 6.6. *We suppose that k is the constant function $k(x, y) = 1$, $a = b = 2$, $\tilde{\lambda} < 0$ and **M2**. There exists a unique solution to (1.1)-(1.5).*

Proof. For $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, we define

$$V^i = \int_{\mathbb{T}^d \times \mathbb{R}^d} \alpha d\mu^i(x, \alpha),$$

for $i = 1, 2$, and m^1, m^2 as the first marginals of μ^1, μ^2 respectively.

Then we notice that the equation below holds,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} [L(\alpha, \mu^1) - L(\alpha, \mu^2)] d(\mu^1 - \mu^2)(x, \alpha) = -\lambda\theta |V^1 - V^2|^2.$$

Moreover L verifies (5.5) for some C_0 . Hence there is a unique solution to (1.1)-(1.5) by Corollary 5.2 and Proposition 6.6. \square

Proposition 6.7. *If $a \neq b$, there exists a solution to (1.1)-(1.5).*

Proof. Lemmas A.1 and A.4, which are stated and proved in the appendix, yield that **H'1-H'7** hold.

Take $V \in \mathbb{R}^d$, and $\alpha = \alpha(0, V)$. We are going to study the behaviour of α as $|V|$ tends to $+\infty$.

From (A.2) and (A.3), we know that $\lim_{V \rightarrow +\infty} |\alpha(0, V)| = +\infty$.

- if $a > b$ then $\frac{(a'-2)(b'-1)}{a'-1} < b' - 2$, and $|\alpha| = o_{+\infty}(|V|)$. Then (A.3) yields

$$|\alpha|^{b'-1 - \frac{(a'-2)(b'-1)}{a'-1}} = O_{+\infty}(|V|),$$

and $b' - 1 - \frac{(a'-2)(b'-1)}{a'-1} = \frac{a-1}{b-1} > 1$, so we obtain

$$|\alpha| = O_{+\infty}\left(|V|^{\frac{b-1}{a-1}}\right),$$

which yields

$$H(0, V) = O_{+\infty}\left(|V|^{a'}\right) + O_{+\infty}\left(|V|^{\frac{b-1}{a-1}b'}\right),$$

with $a' < b'$, and $\frac{b-1}{a-1}b' < b'$, and $b = \gamma$.

- if $a < b$ then $\frac{(a'-2)(b'-1)}{a'-1} > b' - 2$, and $\alpha = \tilde{\lambda}V + \underset{+\infty}{O}(|V|)$. Then (A.3) yields

$$\left(1 + \underset{+\infty}{O}\left(|V|^{b'-2-\frac{(a'-2)(b'-1)}{a'-1}}\right)\right) \alpha = \tilde{\lambda}V.$$

We notice that $b' - 2 - \frac{(a'-2)(b'-1)}{a'-1} = \frac{b'-a'}{a'-1} < 0$, and we obtain

$$\alpha = \tilde{\lambda}V + \underset{+\infty}{O}\left(|V|^{1+\frac{b'-a'}{a'-1}}\right) = \tilde{\lambda}V + \underset{+\infty}{O}\left(|V|^{\frac{a-1}{b-1}}\right).$$

This implies

$$H(0, V) = \underset{+\infty}{O}\left(|V|^{\frac{b'-1}{a'-1}a'}\right) + \underset{+\infty}{O}\left(|V|^{b'}\right),$$

with $b' < a'$, and $\frac{a-1}{b-1}a' < a'$, and $a = \gamma$.

Thus for any $a \neq b$, the assumptions of Theorem 4.8 are satisfied and there exists a solution to (1.1)-(1.5). \square

Proposition 6.8. *We suppose that $1 < a = b \leq 2$.*

If $\tilde{\lambda} \in (-1, 1)$, there exists $\theta_0 \in (0, 1]$ such that if $\theta \in [0, \theta_0] \cup (1 - \theta_0, 1]$ then (1.1)-(1.5) admits at least one solution.

Moreover if $\theta \in [0, 1]$, there exists $\lambda_0 \in (0, 1)$ such that if $|\tilde{\lambda}| \leq \lambda_0$ then (1.1)-(1.5) has a solution.

Proof. Take $V \in \mathbb{R}^d$, and $\alpha = \alpha(0, V)$.

In this case we recall (A.4) in which α is given by an explicit formula. Thus $H(0, V)$ is given by

$$\begin{aligned} H(0, V) &= -\frac{\theta}{a'} \left| \alpha - \tilde{\lambda}V \right|^{a'} - \frac{1-\theta}{a'} |\alpha|^{a'} \\ &= -\frac{\lambda^{a'} \theta (1-\theta)^a + (1-\theta)\theta^a}{a' \left((1-\theta)^{a-1} + \theta^{a-1} \right)^{a'}} |V|^{a'}. \end{aligned}$$

We notice that $\left((1-\theta)^{a-1} + \theta^{a-1} \right)^{a'} \geq (1-\theta + \theta)^{a'} \geq 1$ since $\theta \in [0, 1]$. Then we can apply Corollary 4.7 with $\delta = \frac{\lambda^{a'} \theta (1-\theta)^a + (1-\theta)\theta^a}{a' \left((1-\theta)^{a-1} + \theta^{a-1} \right)^{a'}}$, there exists a solution to (1.1)-(1.5) if

$$(6.6) \quad \frac{\lambda^{a'} \theta (1-\theta)^a + (1-\theta)\theta^a}{a' \left((1-\theta)^{a-1} + \theta^{a-1} \right)^{a'}} C_0^{a'} < \frac{(1-\lambda)^{a'}}{C_0^{1+a'}},$$

which is equivalent to

$$h(\theta) = \frac{\theta(1-\theta)^a + (1-\theta)\theta^a}{\left((1-\theta)^{a-1} + \theta^{a-1} \right)^{a'}} < \frac{a'(1-\lambda)^{a'}}{\lambda^{a'} C_0^{1+2a'}}.$$

The function $h \in C^0([0, 1]; \mathbb{R})$ defined in the latter equation takes its values in $[0, 1]$, it verifies $h(0) = h(1) = 0$ and $h\left(\frac{1}{2}\right) = 1$.

Therefore if $\frac{a'(1-\lambda)^{a'}}{\lambda^{a'} C_0^{1+2a'}} > 1$, then (6.6) is satisfied for any θ so we choose $\theta_0 = 1$.

Otherwise there exists $\theta_0 \in \left(0, \frac{1}{2}\right)$ such that

$$h(\theta_0) = \frac{a'(1-\lambda)^{a'}}{\lambda^{a'} C_0^{1+2a'}}.$$

Hence for any $\theta \in [0, \theta_0) \cup (\theta_0, 1]$, there exists a solution to (1.1)-(1.5). We notice that θ_0 depends on $\tilde{\lambda}$.

The inequality below is also equivalent to (6.6),

$$\frac{\lambda}{1-\lambda} < \left(\frac{((1-\theta)^{a-1} + \theta^{a-1})^{a'}}{\theta(1-\theta)^a + (1-\theta)\theta^a} \frac{a'}{C_0^{1+2a'}} \right)^{\frac{1}{a'}} = \frac{\lambda_0}{1-\lambda_0}.$$

The quantity $\lambda_0 \in (0, 1)$ is well-posed if $\theta \in (0, 1)$. Hence there exists a solution to (1.1)-(1.5) if $|\tilde{\lambda}| \leq \lambda_0$. We recall that λ_0 depends on θ . \square

Proposition 6.9. *We suppose $1 < a = b \leq 2$.*

There exists $T_0 > 0$ such that if $T < T_0$ then (1.1)-(1.5) admits at least one solution.

Moreover if g verifies (4.14), there exists $T_1 < T_0$ such that if $T < T_1$ the solution of (1.1)-(1.5) is unique.

This is a direct corollary of Theorems 4.9 and 4.10.

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A - VERIFICATION OF THE ASSUMPTIONS ON THE EXAMPLES

Here we consider L, H defined by (6.1) and (6.2).

Lemma A.1. *If $\theta \in (0, 1)$, H satisfies **H'1**, with $\gamma = \min(a, b)$ and $\lambda = |\tilde{\lambda}|$.*

Proof. Take $(p, V) \in \mathbb{R}^{2d}$ and $\alpha = \alpha(p, V)$ in Definition 6.1. The optimal control α verifies

$$(A.1) \quad p = -D_\alpha L(\alpha, V) = -\theta|\alpha - \tilde{\lambda}V|^{a'-2}(\alpha - \tilde{\lambda}V) - (1-\theta)|\alpha|^{b'-2}\alpha,$$

and $D_\alpha L$ is in $C^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and its differential with respect to α at (α, V) is invertible if $\alpha \neq 0$ or $\alpha \neq \tilde{\lambda}V$, which holds whenever $(p, V) \neq (0, 0)$. Therefore α is C^1 on $(\mathbb{R}^d \times \mathbb{R}^d) \setminus (0, 0)$ by the inverse mapping theorem.

Proof that α is $(\gamma - 1)$ -Hölder with respect to p at $(0, 0)$.

We take $p \in \mathbb{R}^d$ that is bound to tend to 0 and $\alpha = \alpha(p, 0)$. From (A.1) we obtain,

$$p = -\theta|\alpha|^{a'-2}\alpha - (1-\theta)|\alpha|^{b'-2}\alpha,$$

then

$$|p| \geq |\alpha|^{\min(a'-1, b'-1)} + o\left(|\alpha|^{\min(a'-1, b'-1)}\right),$$

and

$$|\alpha| \leq |p|^{\max(a-1, b-1)} + o\left(|p|^{\max(a-1, b-1)}\right),$$

since $(a' - 1)^{-1} = a - 1$ and $(b' - 1)^{-1} = b - 1$. Thus α is $\max(a - 1, b - 1)$ -Hölder. With $\gamma = \min(a, b)$, and $\gamma - 1 \leq \max(a - 1, b - 1)$, and α is $(\gamma - 1)$ -Hölder with respect to p .

Proof that α is differentiable with respect to V at $(0, 0)$.

We take $V \in \mathbb{R}^d$ that will eventually tend to 0 and $\alpha = \alpha(0, V)$. From (A.1) we obtain

$$0 = \theta|\alpha - \tilde{\lambda}V|^{a'-2}(\alpha - \tilde{\lambda}V) + (1-\theta)|\alpha|^{b'-2}\alpha,$$

which implies

$$(A.2) \quad \theta|\alpha - \tilde{\lambda}V|^{a'-1} = (1-\theta)|\alpha|^{b'-1},$$

and then

$$(A.3) \quad \left(\theta^{a-1}(1-\theta)^{2-a}|\alpha|^{\frac{(a'-2)(b'-1)}{a'-1}} + (1-\theta)|\alpha|^{b'-2} \right) \alpha = \tilde{\lambda}\theta^{a-1}(1-\theta)^{2-a}|\alpha|^{\frac{(a'-2)(b'-1)}{a'-1}}V.$$

- if $a > b$ then $\frac{(a'-2)(b'-1)}{a'-1} < b' - 2$, and we obtain the following expansion as $|V|$ tends to 0,

$$\alpha = \tilde{\lambda}V + o(|V|).$$

- if $a = b$ we obtain,

$$(A.4) \quad \alpha = \tilde{\lambda} \frac{\theta^{a-1}}{\theta^{a-1} + (1-\theta)^{a-1}}V.$$

- if $a < b$ then $\frac{(a'-2)(b'-1)}{a'-1} > b' - 2$, and we obtain the following estimate as $|V|$ tends to 0,

$$\alpha = o(|V|).$$

We deduce the differential of α with respect to V in any of the above three cases:

$$(A.5) \quad D_V \alpha(0,0) = \begin{cases} \tilde{\lambda}I_d & \text{if } b < a \\ \tilde{\lambda} \frac{\theta^{a-1}}{\theta^{a-1} + (1-\theta)^{a-1}}I_d & \text{if } b = a \\ 0 & \text{if } b > a. \end{cases}$$

End of the proof.

We have already seen that

$$\alpha(p, V) = -H_p(p, V).$$

We proved that α is differentiable with respect to p , and that it is locally $(\gamma-1)$ -Hölder continuous with respect to V . We deduce the desired property for H_p . Then by (6.3), and since L is C^1 , H is differentiable with respect to V on $\mathbb{R}^d \times \mathbb{R}^d$.

This concludes the proof that **H'1** is satisfied. \square

Remark A.2. We proved that H and H_p are C^1 with respect to p on $(\mathbb{R}^d \times \mathbb{R}^d) \setminus (0,0)$ and are differentiable with respect to p at $(0,0)$. However in general, these functions are not C^1 with respect to p on $\mathbb{R}^d \times \mathbb{R}^d$. We can even prove that the two following assertions are equivalent,

- (i) $D_V H$ and $D_V H_p$ are continuous at $(0,0)$.
- (ii) $a = b = 2$.

Lemma A.3. If $\theta \in (0,1)$, H satisfies **H'2-H'7**, with $\gamma = \min(a,b)$ and $\lambda = |\tilde{\lambda}|$.

Proof. Proof that **H'2** holds

If $(p, V) \neq (0,0)$, (A.1) yields

$$(A.6) \quad \alpha = \frac{-p + \tilde{\lambda}\theta|\alpha - \tilde{\lambda}V|^{a'-2}V}{\theta|\alpha - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\alpha|^{b'-2}},$$

and

$$(A.7) \quad \alpha - \tilde{\lambda}V = \frac{-p + \tilde{\lambda}(1-\theta)|\alpha|^{b'-2}V}{\theta|\alpha - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\alpha|^{b'-2}}.$$

From (A.1), we deduce that either

$$\theta|\alpha - \tilde{\lambda}V|^{a'-1} \geq \frac{1}{2}|p|,$$

or

$$(1 - \theta)|\alpha|^{b'-1} \geq \frac{1}{2}|p|.$$

We recall that $\alpha = -H_p(p, V)$, hence

$$\begin{aligned} H_p(p, V) \cdot p - H(p, V) &= L(\alpha, V) \\ &= \frac{\theta}{a'}|\alpha - \tilde{\lambda}V|^{a'} + \frac{1 - \theta}{b'}|\alpha|^{b'}, \\ &\geq \min\left(\frac{|p|^a}{2^a a' \theta^{a-1}}, \frac{|p|^b}{2^b b' (1 - \theta)^{b-1}}\right). \end{aligned}$$

Thus **H'2** holds.

*Proof that **H'3** holds*

First we use that $\alpha' = 0$ is a sub-optimal control in (6.3), hence

$$(A.8) \quad H(p, V) \geq -\frac{\theta|\tilde{\lambda}|^{a'}}{a'}|V|^{a'}.$$

We recall the following inequality of interpolation,

$$(A.9) \quad |z^1 \cdot z^2| \leq \frac{|z^1|^q}{q} + \frac{|z^2|^{q'}}{q'},$$

for any $z^1, z^2 \in \mathbb{R}^d$, and $q, q' > 1$ such that $\frac{1}{q} + \frac{1}{q'} = 1$.

Then if $b \leq a$, we can apply (A.9) with $q = b$ to the first term in (6.3),

$$H(p, V) \leq \frac{|p|^b}{(1 - \theta)^{\frac{b}{b'}}} + \frac{(1 - \theta)|\alpha|^{b'}}{b'} - \frac{\theta}{a'}|\alpha - \tilde{\lambda}V|^{a'} - \frac{1 - \theta}{b'}|\alpha|^{b'},$$

thus

$$(A.10) \quad H(p, V) \leq \frac{|p|^b}{(1 - \theta)^{b-1}b}.$$

And if $a < b$, we apply (A.9) with $q = a$ to (6.3),

$$\begin{aligned} H(p, V) &= -p \cdot (\alpha - \tilde{\lambda}V) - \tilde{\lambda}p \cdot V - \frac{\theta}{a'}|\alpha - \tilde{\lambda}V(x)|^{a'} - \frac{1 - \theta}{b'}|\alpha|^{b'}, \\ &\leq \frac{|p|^a}{\theta^{\frac{a}{a'}}a} + \frac{\theta|\alpha - \tilde{\lambda}V|^{a'}}{a'} + \frac{|\tilde{\lambda}|}{a}|p|^a + \frac{|\tilde{\lambda}|}{a'}|V|^{a'} - \frac{\theta}{a'}|\alpha - \tilde{\lambda}V(x)|^{a'} - \frac{1 - \theta}{b'}|\alpha|^{b'}, \\ (A.11) \quad H(p, V) &\leq \left(\frac{1}{\theta^{a-1}a} + \frac{\lambda}{a}\right)|p|^a + \frac{\lambda}{a'}|V|^{a'}, \end{aligned}$$

since $\lambda = |\tilde{\lambda}|$. We deduce from (A.8), (A.10), (A.11), that **H'3** is verified.

Proof that the norm of $D_V \alpha = (\partial_{V_j} \alpha^i)_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d}$ is lower than or equal to λ .

Here we consider the following norm on the space of square matrices of order d

$$\|A\| = \sup_{X \neq 0} \frac{|AX|}{|X|}.$$

Let us introduce

$$\begin{aligned} v_1 &= \mathbf{1}_{\alpha - \tilde{\lambda}V \neq 0} \frac{\alpha - \tilde{\lambda}V}{|\alpha - \tilde{\lambda}V|}, \\ v_2 &= \mathbf{1}_{\alpha \neq 0} \frac{\alpha}{|\alpha|}, \\ B &= I_d + (a' - 2)v_1v_1^T, \\ C &= I_d + (b' - 2)v_2v_2^T. \end{aligned}$$

We recall that if $v_i \neq 0$, then $v_iv_i^T$ is the orthogonal projection onto $\mathbb{R}v_i$ for $i = 1, 2$.

Let us suppose that $a' \neq 2, b' \neq 2, \alpha - \tilde{\lambda}V \neq 0, \alpha \neq 0$. Then we differentiate the i -th component of (A.1) with respect to V^j ,

$$\begin{aligned} 0 &= \theta |\alpha - \tilde{\lambda}V|^{a'-2} \left(\partial_{V^j} \alpha^i - \tilde{\lambda} \delta_{i,j} \right) \\ &\quad + \theta (a' - 2) |\alpha - \tilde{\lambda}V|^{a'-4} \sum_{k=1}^d \left(\partial_{V^j} \alpha^k - \tilde{\lambda} \delta_{k,j} \right) \left(\alpha^i - \tilde{\lambda}V^i \right) \left(\alpha^k - \tilde{\lambda}V^k \right) \\ &\quad + (1 - \theta) |\alpha|^{b'-2} \partial_{V^j} \alpha^i + (1 - \theta) (b' - 2) |\alpha|^{b'-4} \sum_{k=1}^d \partial_{V^j} \alpha^k \alpha^i \alpha^k. \end{aligned}$$

This implies

$$0 = \theta |\alpha - \tilde{\lambda}V|^{a'-2} B \left(D_V \alpha - \tilde{\lambda} I_d \right) + (1 - \theta) |\alpha|^{b'-2} C D_V \alpha,$$

and thus

$$(A.12) \quad D_V \alpha = \tilde{\lambda} \left[I_d + \frac{(1 - \theta) |\alpha|^{b'-2}}{\theta |\alpha - \tilde{\lambda}V|^{a'-2}} B^{-1} C \right]^{-1}.$$

We can check that this last equation holds in the general case for any $(\alpha, V) \neq (0, 0), a', b'$.

- If $\alpha = \tilde{\lambda}V = 0$ then $(p, V) = (0, 0)$, we see on (A.5) that $D_V \alpha$ is a non-negative semi-definite matrix with eigenvalues in $[-\lambda, \lambda]$.

From now on, we suppose that $(\alpha, V) \neq (0, 0)$, thus (A.12) is satisfied.

- If $(a' - 2)v_1 = 0$ (i.e. $B = I_d$) or $(b' - 2)v_2 = 0$ (i.e. $C = I_d$), then (A.12) yields that $D_V \alpha$ is a symmetric positive definite matrix with eigenvalues in $(-\lambda, \lambda)$.
- We assume that $(a' - 2)v_1 \neq 0$ and $(b' - 2)v_2 \neq 0$, and v_1, v_2 are aligned. Then B and C commute and $B^{-1}C$ is a symmetric positive definite matrix. Then (A.12) yields that $D_V \alpha$ is a symmetric positive definite matrix with eigenvalues in $(-\lambda, \lambda)$.
- We assume that $(a' - 2)v_1 \neq 0$ and $(b' - 2)v_2 \neq 0$, and v_1, v_2 are linearly independent. Let us note $k = \frac{(1 - \theta) |\alpha|^{b'-2}}{\theta |\alpha - \tilde{\lambda}V|^{a'-2}} > 0$. The two orthogonal subspaces $\text{span}(v_1, v_2)$ and $\{v_1, v_2\}^\perp$ are stable by $D_V \alpha, B, C$. The restriction of $D_V \alpha$ to $\{v_1, v_2\}^\perp$ is symmetric positive definite with eigenvalues in $(-\lambda, \lambda)$.

Let us name $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ the restriction of $D_V \alpha, B, C$ to $\text{span}(v_1, v_2)$ respectively. We notice that

$$\tilde{B}^{-1} = I_d + ((a' - 1)^{-1} - 1) v_1 v_1^\perp,$$

thus the eigenvalues of \tilde{B}^{-1} are 1 and $(a' - 1)^{-1} \leq 1$ since $a' \geq 2$. The eigenvalues of \tilde{C} are 1 and $(b' - 1) \geq 1$. Lemma A.4 below yields that $M = (I_d + k\tilde{B}^{-1}\tilde{C})(I_d + k\tilde{C}\tilde{B}^{-1})$ is a symmetric positive definite matrix with eigenvalues greater than or equal to 1. This implies

$$\begin{aligned}\|\tilde{A}X\|^2 &= \lambda^2 \langle M^{-1}X, X \rangle \\ &\leq \lambda^2 \|X\|^2.\end{aligned}$$

This concludes the proof that the norm of $D_V \boldsymbol{\alpha}$ is lower than or equal to λ .

Proof of H'6.

We take $(p, V_1, V_2) \in \mathbb{R}^{3d}$ and $\boldsymbol{\alpha}_i = -H_p(p, V_i)$, $i = 1, 2$, then

$$\begin{aligned}|H_p(p, V_1) - H_p(p, V_2)| &\leq \sup_{s \in [0,1]} \{ \|D_V \boldsymbol{\alpha}(p, sV_1 + (1-s)V_2)\| \} |V_1 - V_2| \\ &\leq \lambda |V_1 - V_2|,\end{aligned}$$

thus **H'6** is satisfied.

Proof of H'5.

Let $(p, V) \in \mathbb{R}^{2d}$, we take $\boldsymbol{\alpha} = -H_p(p, V)$.

- We suppose $b' \geq a'$.

Either $|\boldsymbol{\alpha}| \leq |p|^{b-1}$ or $|\boldsymbol{\alpha}| > |p|^{b-1} = |p|^{\frac{1}{b'-1}}$. In the latter case, we use (A.6),

$$\begin{aligned}|\boldsymbol{\alpha}| &\leq \left| \frac{-p + \tilde{\lambda}\theta|\boldsymbol{\alpha} - \tilde{\lambda}V|^{a'-2}V}{\theta|\boldsymbol{\alpha} - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\boldsymbol{\alpha}|^{b'-2}} \right| \\ &\leq \frac{|p|}{(1-\theta)|\boldsymbol{\alpha}|^{b'-2}} + \lambda|V| \\ &\leq (1-\theta)^{-1}|p|^{1-\frac{b'-2}{b'-1}} + \lambda|V|.\end{aligned}$$

We recall that $1 - \frac{b'-2}{b'-1} = b - 1$, so

$$(A.13) \quad |H_p(p, V)| = |\boldsymbol{\alpha}| \leq (1-\theta)^{-1}|p|^{b-1} + \lambda|V|.$$

- Now we suppose that $b' < a'$.

Either $|\boldsymbol{\alpha} - \tilde{\lambda}V| \leq |p|^{a-1}$ or $|\boldsymbol{\alpha} - \tilde{\lambda}V| > |p|^{\frac{1}{a'-1}}$. In the latter case, we use (A.6),

$$\begin{aligned}|\boldsymbol{\alpha}| &\leq \left| \frac{-p + \tilde{\lambda}\theta|\boldsymbol{\alpha} - \tilde{\lambda}V|^{a'-2}V}{\theta|\boldsymbol{\alpha} - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\boldsymbol{\alpha}|^{b'-2}} \right| \\ &\leq \frac{|p|}{\theta|\boldsymbol{\alpha} - \tilde{\lambda}V|^{a'-2}} + \lambda|V| \\ &\leq \theta^{-1}|p|^{1-\frac{a'-2}{a'-1}} + \lambda|V|.\end{aligned}$$

From the identity $1 - \frac{a'-2}{a'-1} = a - 1$, we deduce

$$(A.14) \quad |H_p(p, V)| = |\boldsymbol{\alpha}| \leq \theta^{-1}|p|^{a-1} + \lambda|V|.$$

We deduce from (A.13) and (A.14) that **H'5** is verified.

Proof of H'4.

We differentiate (6.3) with respect to V ,

$$H_V(p, V) = -(D_V \alpha)^T (p - L_\alpha(\alpha, V)) - L_V(\alpha, V) = -L_V(\alpha, V) = \tilde{\lambda} \theta (\alpha - \tilde{\lambda} V) \left| \alpha - \tilde{\lambda} V \right|^{a'-2}.$$

From (A.1) we obtain

$$\theta |\alpha - \lambda V|^{a'-1} \leq |p| + (1 - \theta) |\alpha|^{b'-1}.$$

The two latter inequalities and **H'5** yield that **H'4** is verified.

Proof of H'7.

We recall (6.3) and that α is locally Lipschitz continuous with respect to V by **H'6** and L is C^1 . Therefore H is also locally Lipschitz with respect to V and **H'7** is verified. \square

Lemma A.4. *Let $B, C \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ be two symmetric positive definite matrices with eigenvalues $(1, r)$ and $(1, s)$ respectively, and $0 < r \leq 1, s \geq 1$. Then for any $k > 0$ the matrix M defined as*

$$M = I_d + k(BC + CB) + k^2 BC^2 B,$$

is symmetric definite positive with eigenvalues greater than or equal to 1.

Proof. We can suppose that B, C have the following form:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad B = U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T, \quad \text{with } U \in \mathcal{O}_2(\mathbb{R}),$$

since the eigenvalues of M are invariant if we replace B and C by taking there respective conjugate with the same orthogonal matrix. By the same argument and by noticing that C commute with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we can assume that U admits a positive determinant, and thus we can write it as

$$U = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix},$$

with $\chi \in [0, 2\pi)$.

Then

$$\begin{aligned} M &= I_d + k(BC + CB) + k^2 BC^2 B \\ &\sim I_d + kU^T \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} + k \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} U + k^2 \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}. \end{aligned}$$

We name \widetilde{M} the matrix in the right-hand side of the latter equation, M and \widetilde{M} have the same eigenvalues. Let us compute \widetilde{M}

$$\widetilde{M} = \begin{pmatrix} \cos^2 \chi (1+k)^2 + \sin^2 \chi (1+ks)^2 & -k(s-1)[1+r+kr(1+s)] \cos \chi \sin \chi \\ -k(s-1)[1+r+kr(1+s)] \cos \chi \sin \chi & \cos^2 \chi (1+krs)^2 + \sin^2 \chi (1+kr)^2 \end{pmatrix},$$

and then its trace is

$$\text{tr}(\widetilde{M}) = \cos^2 \chi (1+k)^2 + \sin^2 \chi (1+kr)^2 + \cos^2 \chi (1+krs)^2 + \sin^2 \chi (1+ks)^2,$$

and its determinant is

$$\begin{aligned} \det(\widetilde{M}) &= (1+k)^2 (1+krs)^2 \cos^4 \chi + (1+kr)^2 (1+ks)^2 \sin^4 \chi \\ &\quad + 2(1+k)(1+kr)(1+ks)(1+krs) \cos^2 \chi \sin^2 \chi \\ &= [(1+k)(1+krs) \cos^2 \chi + (1+kr)(1+ks) \sin^2 \chi]^2. \end{aligned}$$

The eigenvalues of \widetilde{M} are the roots of the following second-order polynomial function,

$$X^2 - \text{tr}(\widetilde{M})X + \det(\widetilde{M}),$$

its smallest root is

$$\frac{1}{2} \left(\text{tr}(\widetilde{M}) - \sqrt{\text{tr}^2(\widetilde{M}) - 4 \det(\widetilde{M})} \right),$$

which is greater than or equal to 1 if and only if

$$\text{tr}^2(\widetilde{M}) - 4 \det(\widetilde{M}) \leq (\text{tr}(\widetilde{M}) - 2)^2.$$

Thus we only have to check that $\text{tr}(\widetilde{M}) \leq \det(\widetilde{M}) + 1$, let us consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} f(x) &= (1+k)^2(1+krs)^2x^2 + (1+kr)^2(1+ks)^2(1-x)^2 \\ &\quad + 2(1+k)(1+kr)(1+ks)(1+krs)x(1-x) \cos^2 \chi(1+k)^2 \\ &\quad + \sin^2 \chi(1+kr)^2 + \cos^2 \chi(1+krs)^2 + \sin^2 \chi(1+ks)^2 + 1. \end{aligned}$$

The function f is second-order polynomial in x with

$$\begin{aligned} f(0) &= ((1+kr)^2 - 1) ((1+ks)^2 - 1) \geq 0 \\ f(1) &= ((1+k)^2 - 1) ((1+krs)^2 - 1) \geq 0, \\ f''(x) &= 2[(1+k)(1+krs) - (1+kr)(1+ks)]^2. \end{aligned}$$

If $(1+k)(1+krs) - (1+kr)(1+ks) = 0$, then f is linear and thus $f(x) \geq 0$ for all $x \in [0, 1]$.

If $(1+k)(1+krs) - (1+kr)(1+ks) \neq 0$, then the minimum of this polynomial function on \mathbb{R} is obtained at x_{\min} defined as

$$\begin{aligned} x_{\min} &= \frac{(1+k)^2 + (1+krs)^2 - (1+ks)^2 - (1+kr)^2}{2[(1+k)(1+krs) - (1+kr)(1+ks)]^2} \\ &= \frac{(1-r^2)(1-s^2)k^2 + 2(1-r)(1-s)k}{2[(1+k)(1+krs) - (1+kr)(1+ks)]^2} \leq 0, \end{aligned}$$

since $0 < r \leq 1, s \geq 1$ and $k > 0$. Thus f has no local minimum on $[0, 1]$, then $f(x) \geq 0$ for all $x \in [0, 1]$ since $f(0) \geq 0$ and $f(1) \geq 0$.

Since $\det(\widetilde{M}) - \text{tr}(\widetilde{M}) + 1 = f(\cos^2 \chi) \geq 0$, we proved that the smallest eigenvalue of \widetilde{M} is greater than or equal to 1. The same property is true for M since they are similar.

This achieves the proof of the lemma. \square

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