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From set relations to belief function relations

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Abstract. In uncertainty theories, a common problem is to define how we can extend relations between sets (e.g., inclusion, ranking, consistency, ...) to corresponding notions between uncertainty representations. Such definitions can then be used to perform the same operations as those that are done for sets: comparing information content, ordering alternatives or checking consistency, to name a few. In this paper, we propose a general way to extend set relations to belief functions, using constrained stochastic matrices to identify those belief functions in relation. We then study some properties of our proposal, as well as its connections with existing works focusing on specific relations.

Keywords: set relations, belief functions, specificity, ranking, consistency.

1 Introduction

One can define many relations between two (or more) subsets A, B of some finite set X , i.e. between elements of some boolean algebra $(2^X, \cap, \cup, \cdot^c)$. Such relations can check whether the sets are consistent ($A \cap B \neq \emptyset$); whether one set is more informative than another, or implies it ($A \subseteq B$); when the space on which they are defined is ordered, whether one set is “higher” than another ($A \prec B$); etc. These relations can then be related to practical problems such as restoring consistency or ranking alternatives.

To address the same questions in those uncertainty theories that formally generalise set theory (based, e.g., on possibility distributions, belief functions or sets of probabilities [12]), it is desirable to carry over relations between sets to uncertainty representations. Given the higher expressiveness of such theories, the problem is ill-posed in the sense that there is not a unique way to do so. We can cite as a typical example the notion of inclusion between belief functions, that has many definitions [15]. Yet, the works that deal with such issues usually focus

39 on extending one particular relation (e.g., inclusion, non-empty intersection) in
40 meaningful ways.

41 In this paper, we propose a simple way to extend any set relation to an equiv-
42 alent relation between belief functions, in the sense that the relation is exactly
43 recovered when considering categorical belief functions (i.e., belief functions hav-
44 ing a single focal element), that are equivalent to sets. Basically, for a pair of
45 belief functions to be in relation, we require that there must exist at least one
46 (left) stochastic matrix such that one of these belief functions is obtained as
47 the dot product of the matrix with the other belief function. Additionally, the
48 matrix is constrained to have null entries on pairs of focal sets not satisfying the
49 relation to extend.

50 To our knowledge, no systematic ways of extending set relations has been
51 proposed in the literature before, and while there may be other ways to perform
52 such an extension, the presented solution has the advantage to be a formal ex-
53 tension (as the relation is exactly recovered for the case of sets), and to connect
54 with other more specific proposals of the literature. The proposal is presented in
55 Section 2, along with the necessary reminders. To which extent it can preserve
56 properties of the initial relation, including its compatibility with (multivariate)
57 functions, is studied in Sections 3 (properties on initial spaces) and 5 (compat-
58 ibility property). To make the approach more concrete, Section 4 relates it to
59 existing works on specific relations, while Section 6 illustrates the results by
60 applying them to simple examples, sometimes inspired from applications (sys-
61 tem reliability and multi-criteria decision making). Finally, Section 7 discusses
62 a mean to make the relation no longer binary but gradual, building first connec-
63 tions to fuzzy relations.

64 2 Main proposal

65 This section recalls the basic tools that are necessary to understand this paper,
66 and present our main proposal. The next sections will then focus on studying
67 its properties and connection with other works.

68 2.1 Relations and their properties

69 Given some (here finite) space X , a relation \mathbf{R} between subsets of X (i.e., on
70 the power set 2^X) is just a subset $\mathbf{R} \subseteq 2^X \times 2^X$ that specifies which pair of
71 subsets are related to each others. For convenience, we will write $\mathbf{A}\mathbf{R}\mathbf{B}$ whenever
72 $(A, B) \in \mathbf{R}$, and $\neg\mathbf{A}\mathbf{R}\mathbf{B}$ whenever $(A, B) \notin \mathbf{R}$.

Example 1. As an illustration, let us consider the binary space $X = \{a, b\}$, and
the strict inclusion relation $\mathbf{R} = \subset$. Then we have

$$\mathbf{R} = \{(\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a, b\}), (\{a\}, \{a, b\}), (\{b\}, \{a, b\})\}$$

73 and the fact that $(\{a\}, \{a, b\}) \in \mathbf{R}$ can be denoted $\{a\}\mathbf{R}\{a, b\}$. The fact that
74 $(\{a\}, \{b\}) \notin \mathbf{R}$ is denoted $\neg\{a\}\mathbf{R}\{b\}$.

Such relations can have many different properties, the main ones that can be found in the literature being the following:

1. Symmetry: \mathbf{R} is symmetric iff $\mathbf{A}\mathbf{R}\mathbf{B} \implies \mathbf{B}\mathbf{R}\mathbf{A}$ for all $A, B \subseteq X$
2. Antisymmetry: \mathbf{R} is antisymmetric iff $\mathbf{A}\mathbf{R}\mathbf{B} \wedge \mathbf{B}\mathbf{R}\mathbf{A} \implies A = B$ for all $A, B \subseteq X$
3. Asymmetry: \mathbf{R} is asymmetric iff $\mathbf{A}\mathbf{R}\mathbf{B} \implies \neg(\mathbf{B}\mathbf{R}\mathbf{A})$ for all $A, B \subseteq X$
4. Reflexivity: \mathbf{R} is reflexive iff $\mathbf{A}\mathbf{R}\mathbf{A}$ for all $A \subseteq X$
5. Irreflexivity: \mathbf{R} is irreflexive iff $\neg(\mathbf{A}\mathbf{R}\mathbf{A})$ for all $A \subseteq X$
6. Transitivity: \mathbf{R} is transitive iff $\mathbf{A}\mathbf{R}\mathbf{B} \wedge \mathbf{B}\mathbf{R}\mathbf{C} \implies \mathbf{A}\mathbf{R}\mathbf{C}$ for all $A, B, C \subseteq X$
7. Completeness: \mathbf{R} is complete, or total, iff $\mathbf{A}\mathbf{R}\mathbf{B} \vee \mathbf{B}\mathbf{R}\mathbf{A}$ for all $A, B \subseteq X$

In addition to those properties, more complex relations have been defined as combination of those properties, that play an important role in many problems. These are, for instance, equivalence relations as well as order relations of different types. They are summarised in Table 1, together with the properties they satisfy.

Name	1	2	3	4	5	6	7
Tolerance	✓		✓				
Partial equivalence	✓					✓	
Equivalence	✓		✓		✓	✓	
Preorder				✓	✓		
Total Preorder				✓	✓	✓	✓
Partial order		✓	✓	✓		✓	
Total order		✓	✓	✓	✓	✓	✓

Table 1. Complex relations

2.2 Belief functions

Belief functions or their equivalent representations as mathematical tools can be traced back at least to Choquet [4], but their use as uncertainty representation was popularised first by Dempster [6] and Shafer [27], before being used by Smets [29] in his Transferable Belief Model.

Their mathematical properties makes them interesting uncertainty models, as they generalise a number of uncertainty representations [9] (possibility measures, sets of cumulative distributions, probabilities), while remaining of limited complexity when compared to more complex models such as lower previsions or desirable gambles [7].

Formally, a belief function on a finite space $X = \{x_1, \dots, x_K\}$ is in one-to-one correspondence with a mass function $m : 2^X \rightarrow [0, 1]$ that satisfies $\sum_{A \subseteq X} m(A) = 1$. From such a mass function, the belief and plausibility of an event $A \subseteq X$ respectively read

$$Bel(A) = \sum_{\emptyset \neq E \subseteq A} m(E) \text{ and } Pl(A) = \sum_{E \cap A \neq \emptyset} m(E). \quad (1)$$

103 If $m(\emptyset) = 0$, they can be interpreted as bounds of the probability $P(A)$ of A ,
 104 inducing the probability set

$$\mathcal{P} = \{P : Bel(A) \leq P(A) \leq Pl(A), \forall A \subseteq X\}. \quad (2)$$

105 Within this latter interpretation and in contrast with the works set within the so-
 106 called Dempster-Shafer theory, the mass function is not a central tool, but merely
 107 a possible transformation of the lower envelope of \mathcal{P} given by the belief function.
 108 As the mass function m plays a fundamental role in our proposal, the current
 109 work is more in-line with the Dempster-Shafer interpretation of belief functions,
 110 however it does not prevent it to have links with an imprecise probabilistic
 111 interpretation.

112 We denote by \mathcal{B}^X the set of all belief functions on X . A particularly interest-
 113 ing subclass of belief functions for this study are categorical ones. A categorical
 114 mass function, denoted m_B , is such that $m_B(B) = 1$.

115 2.3 Extending set relations to belief functions

116 Let \mathbf{R} be a relation on 2^X (equivalently a subset of $2^X \times 2^X$). We then propose
 117 the following simple definition to extend this relation to belief functions, i.e. into
 118 a relation on \mathcal{B}^X :

Definition 1. *Given two mass functions m_1, m_2 and a subset relation \mathbf{R} , we say that $m_1 \tilde{\mathbf{R}} m_2$ iff there is a (left)¹ stochastic matrix S such that $\forall A, B \subseteq X$*

$$m_1(A) = \sum_{B \subseteq X} S(A, B) m_2(B) \quad (3)$$

$$\text{with } S(A, B) > 0 \wedge m_2(B) > 0 \implies \mathbf{R}A, B. \quad (4)$$

Definition 1 states that $m_1 \tilde{\mathbf{R}} m_2$ iff m_1 can be obtained from m_2 by transferring each mass $m_2(B)$ to a subset A such that $\mathbf{R}A, B$. It is easily checked that $\tilde{\mathbf{R}}$ is a generalisation of \mathbf{R} in the sense that

$$m_A \tilde{\mathbf{R}} m_B \Leftrightarrow \mathbf{R}A, B, \forall A, B \subseteq X. \quad (5)$$

119 Indeed, if $\mathbf{R}A, B$, we can choose $S(E, F) = m_A(E)$ for all $F \subseteq X$, and this
 120 matrix matches the conditions of Definition 1, hence $m_A \tilde{\mathbf{R}} m_B$. Conversely, if
 121 $m_A \tilde{\mathbf{R}} m_B$, then (3) implies $S(A, B) = 1$ and (4) then gives $\mathbf{R}A, B$.

122 Also, there is only one relation $\tilde{\mathbf{R}}$ on belief functions spanned by Definition
 123 1 from a given set relation \mathbf{R} . To see this, suppose two such belief function
 124 relations exist. If a matrix matching the conditions of Definition 1 was found for
 125 the first one then the same matrix also works for the other and the relations are
 126 equivalent. Likewise, two relations \mathbf{R} and \mathbf{R}' defined on sets cannot lead, through
 127 Definition 1, to the same relation $\tilde{\mathbf{R}}$ on belief functions. This is an immediate
 128 consequence of (5). Consequently and by a small abuse of notation, we will use

¹ We use left-stochasticity only throughout the paper.

129 the same notation for a relation \mathbf{R} on the subset or belief function side in the
 130 remainder of the paper, as it introduces no ambiguity. However, in general, the
 131 stochastic matrix S involved in Definition 1 is not unique when $m_1 \mathbf{R} m_2$ holds.

132 Definition 1 is inspired from previous works on specificity of belief functions
 133 [15, 16, 30], as well as on recent proposals dealing with set ordering [24].
 134 In particular, Definition 1 can be endowed with an interpretation similar to the
 135 one given in [15], as $S(A, B)$ can be seen as the ratio of $m(B)$ that flows from
 136 B to A , with the flow being possibly non-null only when $\mathbf{A} \mathbf{R} B$.

137 *Remark 1.* Readers that are familiar with the belief function literature may wonder
 138 why the condition $m_2(B) > 0$ is necessary in (4), as this condition does not
 139 appear in related works. This condition is necessary to generalise any relation
 140 on sets that is not inverse serial, i.e. a relation such that there is a B_* with
 141 $\neg(\mathbf{A} \mathbf{R} B_*)$, $\forall A \subseteq X$. For such sets B_* , left stochasticity is incompatible with
 142 the implication $S(A, B_*) > 0 \implies \mathbf{A} \mathbf{R} B_*$, and without checking $m_2(B) > 0$
 143 in Definition 1 the relation on belief functions of a not inverse serial \mathbf{R} would
 144 always be empty. By checking $m_2(B) > 0$ in Definition 1, we can induce a non
 145 empty relation on belief functions. When B_* is a focal element of m_2 , we have
 146 $\neg(m_1 \mathbf{R} m_2)$, which makes perfect sense. When $m(B_*) = 0$, then a null mass can
 147 be distributed to any set A without harm.

148 As the above mentioned related works dealt with directional, or rather asym-
 149 metric relations, Definition 1 is naturally asymmetric. However, Proposition 1
 150 shows that it has a somehow symmetric counterpart.

151 **Proposition 1.** *Consider two mass functions m_1, m_2 and a belief function re-*
 152 *lation \mathbf{R} . Then the two following conditions are equivalent:*

1. *there is a stochastic matrix $S(A, B)$ such that*

$$m_1(A) = \sum_{B \subseteq X} S(A, B) m_2(B),$$

with $S(A, B) > 0 \wedge m_2(B) > 0 \implies \mathbf{A} \mathbf{R} B$.

2. *there is a joint mass function $m_{12}(A, B)$ on $2^X \times 2^X$ such that*

$$m_{12}(A, B) > 0 \implies \mathbf{A} \mathbf{R} B, \tag{6}$$

$$m_1(A) = \sum_B m_{12}(A, B), \tag{7}$$

$$m_2(B) = \sum_A m_{12}(A, B). \tag{8}$$

Proof. 1. \implies 2. First, consider the matrix $S(A, B)$, that we know exists if
 $m_1 \mathbf{R} m_2$. Let us now simply define the joint m_{12} as

$$m_{12}(A, B) = m_2(B) S(A, B) \text{ for any } A, B.$$

We clearly have $m_{12}(A, B) > 0$ only if $\mathbf{AR}B$, since $S(A, B) > 0$ means that either $\mathbf{AR}B$ or $m_2(B) = 0$ (in the other cases it is null), and moreover

$$\sum_B m_{12}(A, B) = \sum_B m_2(B)S(A, B) = m_1(A),$$

$$\sum_A m_{12}(A, B) = \sum_A m_2(B)S(A, B) = m_2(B) \sum_A S(A, B) = m_2(B)$$

153 with the last equality following from S being stochastic.

2. \implies 1. Again, consider the joint $m_{12}(A, B)$ satisfying constraints (6)-(8), that we know exists by assumption. If we assume that this implies the existence of matrix S , we get

$$m_1(A) = \sum_B m_{12}(A, B) = \sum_B m_2(B)S(A, B).$$

154 For any B s.t. $m_2(B) > 0$, we thus define

$$S(A, B) = \frac{m_{12}(A, B)}{m_2(B)}. \quad (9)$$

The other entries of S are set to arbitrary values provided that these latter are compliant with left stochasticity. For those entries which are set according to (9), i.e. when $m_2(B) > 0$, we can now check that $S(A, B)$ satisfies the required properties, as

$$\sum_A S(A, B) = \frac{\sum_A m_{12}(A, B)}{m_2(B)} = \frac{m_2(B)}{m_2(B)} = 1,$$

$$S(A, B) > 0 \Leftrightarrow \frac{m_{12}(A, B)}{m_2(B)} > 0 \Rightarrow \mathbf{AR}B$$

□

This proposition shows, in particular, that any stochastic matrix S can be associated to a unique joint mass function m_{12} , and vice-versa. Also note that, using a transformation similar to the one of the second part of the proof, we can alternatively build a stochastic matrix S' such that

$$S'(B, A) = \begin{cases} \frac{m_{12}(A, B)}{m_1(A)} & \text{if } m_1(A) > 0 \\ \lambda_B^{(A)} & \text{if } m_1(A) = 0 \end{cases},$$

with $\sum_B \lambda_B^{(A)} = 1$. S' is such that

$$m_2(B) = \sum_{A \subseteq X} S'(B, A)m_1(A).$$

155 Moreover, $S'(A, B) > 0$ and $m_1(A) > 0$ imply $\mathbf{BR}A$ but gives no guarantee on
156 $\mathbf{AR}B$.

157 *Remark 2.* Proposition 1 shows that we can view our definition of relations in
 158 two different ways: as a "transfer" matrix S allowing to go from m_2 to m_1
 159 without violating the relation on sets, or as the existence of a joint structure
 160 consistent with m_1, m_2 and the relation \mathbf{R} . Although we consider that the joint
 161 structure is more intuitive and easier to explain, both views have been adopted
 162 in the past and are in our opinion useful, as:

- 163 – there are settings where one mathematical tool is more natural than the
 164 other. For instance, Smets' matrix computations [28] make a heavy use of the
 165 first view, while recent works about consistency adopt the second view [11];
- 166 – mathematically, it may also be more convenient to use one or the other, for
 167 instance in proofs. For example, most of our negative proofs and examples
 168 use joint matrices and the second view, but Propositions 7 and 10 are simpler
 169 to prove using the first view.

170 Finally, let us note that the relation \mathbf{R} on belief functions can be interpreted
 171 in exactly the same way as the relation on sets it extends, this interpretation
 172 varying according to the application and pursued goal. For instance, the relation
 173 $\mathbf{R}AB$ iff $A \cap B \neq \emptyset$ will often be used when A, B concern the same object of inter-
 174 ests but are issued from different sources, and when one wants to check whether
 175 they are consistent. In contrast, ranking relations between A, B will often be
 176 used when A, B concern different objects or alternatives evaluated on the same
 177 scale (e.g., movies given a finite number of stars). Generally speaking, mass func-
 178 tions are random set distributions [26] and relation \mathbf{R} is one way to propagate
 179 a relation (and its interpretation) on sets to their random counterparts.

180 3 Property preservation

181 3.1 Preservation of simple properties

182 We may now wonder how many of the initial relation \mathbf{R} properties between sets
 183 are preserved when extended to belief functions according to Definition 1. We will
 184 now provide a series of results for common properties, either by providing proofs
 185 or counter-examples. We will keep the proposition/proof format, to provide a
 186 uniform presentation.

Proposition 2 (Preserved symmetry). *If \mathbf{R} is symmetric on sets, it is so on belief functions:*

$$m_1 \mathbf{R} m_2 \implies m_2 \mathbf{R} m_1, \forall m_1, m_2.$$

Proof. Let us assume that $S(A, B)$ is a stochastic matrix satisfying Definition 1 for $m_1 \mathbf{R} m_2$, and m_{12} is its associated joint mass. Then we can see that

$$S'(A, B) > 0 \text{ and } m_1(A) > 0 \implies B \mathbf{R} A \Leftrightarrow A \mathbf{R} B,$$

since \mathbf{R} is symmetric. Matrix S' satisfies the conditions of Definition 1, hence $m_2 \mathbf{R} m_1$. \square

Proposition 3 (Unpreserved antisymmetry). *If \mathbf{R} is antisymmetric on sets, it is not necessarily so on belief functions, as*

$$m_1 \mathbf{R} m_2 \wedge m_2 \mathbf{R} m_1 \not\Rightarrow m_2 = m_1$$

Proof. Consider two mass functions that are positive only on subsets A, B, C and such that

$$\begin{aligned} m_1(A) &= 0.3, m_1(B) = 0.5, m_1(C) = 0.2, \\ m_2(A) &= 0.4, m_2(B) = 0.3, m_2(C) = 0.3, \end{aligned}$$

as well as the antisymmetric relation \mathbf{R} on those subsets summarised by the matrix

$$\begin{array}{c} A \quad B \quad C \\ \begin{array}{l} A \\ B \\ C \end{array} \left[\begin{array}{ccc} \mathbf{R}A & \mathbf{R}B & \\ & \mathbf{R}B & \mathbf{R}C \\ \mathbf{R}A & & \mathbf{R}C \end{array} \right]. \end{array}$$

We can then consider the joint mass function

$$\begin{aligned} m_{12}(A, A) &= 0.3, m_{12}(B, B) = 0.3, \\ m_{12}(B, C) &= 0.2, m_{12}(C, A) = 0.1, m_{12}(C, C) = 0.1, \end{aligned}$$

that shows that we have $m_1 \mathbf{R} m_2$, while the joint mass function

$$\begin{aligned} m_{12}(A, A) &= 0.2, m_{12}(B, B) = 0.3, \\ m_{12}(A, C) &= 0.1, m_{12}(B, A) = 0.2, m_{12}(C, C) = 0.2, \end{aligned}$$

shows that $m_2 \mathbf{R} m_1$, hence we can have both without $m_1 = m_2$. \square

Proposition 4 (Unpreserved asymmetry). *If \mathbf{R} is asymmetric on sets, it is not necessarily so on belief functions, as*

$$m_1 \mathbf{R} m_2 \not\Rightarrow \neg(m_2 \mathbf{R} m_1)$$

Proof. Simply consider two mass functions m_1, m_2 that are positive only on subsets A, B, C, D, E and such that

$$\begin{aligned} m_1(A) &= 0.2, m_1(B) = 0.3, m_1(C) = 0.2, m_1(D) = 0.1, m_1(E) = 0.2, \\ m_2(A) &= 0.2, m_2(B) = 0.1, m_2(C) = 0.3, m_2(D) = 0.3, m_2(E) = 0.1 \end{aligned}$$

as well as the asymmetric relation \mathbf{R} on those subsets summarised by the matrix

$$\begin{array}{c} A \quad B \quad C \quad D \quad E \\ \begin{array}{l} A \\ B \\ C \\ D \\ E \end{array} \left[\begin{array}{ccccc} & & \mathbf{R}C & \mathbf{R}D & \\ \mathbf{R}A & & & & \mathbf{R}E \\ & \mathbf{R}B & & \mathbf{R}D & \\ \mathbf{R}A & \mathbf{R}B & & & \mathbf{R}E \\ \mathbf{R}A & & \mathbf{R}C & & \end{array} \right] \end{array}$$

We can then consider the joint mass function

$$m_{12}(A, C) = 0.1, m_{12}(A, D) = 0.1, m_{12}(B, A) = 0.2$$

$$m_{12}(B, E) = 0.1, m_{12}(C, D) = 0.2, m_{12}(D, B) = 0.1, m_{12}(E, C) = 0.2$$

that shows that we have $m_1 \mathbf{R} m_2$, while the joint mass function

$$m_{12}(A, B) = 0.1, m_{12}(A, E) = 0.1, m_{12}(B, C) = 0.2$$

$$m_{12}(B, D) = 0.1, m_{12}(C, A) = 0.2, m_{12}(D, C) = 0.1, m_{12}(E, D) = 0.2$$

shows that $m_2 \mathbf{R} m_1$, hence we can have both. □

Proposition 5 (Preserved reflexivity). *If \mathbf{R} is reflexive on sets, it is so on belief functions:*

$$\forall m, \text{ we have } m \mathbf{R} m$$

Proof. Simply observe that, if \mathbf{R} is reflexive ($A \mathbf{R} A$ for any subset) and if $m_1 = m_2 = m$, we can always define the joint mass function such that for any A we have $m_{12}(A, A) = m(A)$, that satisfies Equations (6)-(8). □

187 **Proposition 6 (Unpreserved irreflexivity).** *If \mathbf{R} is irreflexive on sets, it is*
188 *not necessarily so on belief functions, as we may have $m \mathbf{R} m$ for some $m \in \mathcal{B}^X$.*

Proof. Consider the following mass function

$$m(A) = 0.5, m(B) = 0.5$$

and the relation \mathbf{R} summarised in the following matrix

$$\begin{array}{c} A \\ B \end{array} \begin{array}{cc} A & B \\ \left[\begin{array}{cc} & A \mathbf{R} B \\ B \mathbf{R} A & \end{array} \right] \end{array}$$

which is irreflexive. However, the joint $m(A, B) = m(B, A) = 0.5$ shows that we have $m \mathbf{R} m$, hence \mathbf{R} may not be irreflexive for belief functions. □

Proposition 7 (Preserved transitivity). *If \mathbf{R} is transitive on sets, it is so on belief functions:*

$$m_1 \mathbf{R} m_2 \wedge m_2 \mathbf{R} m_3 \implies m_1 \mathbf{R} m_3$$

Proof. If we have $m_1 \mathbf{R} m_2 \wedge m_2 \mathbf{R} m_3$, this means that there are two matrices S_{12} and S_{23} satisfying Definition 1 and such that

$$m_1(A) = \sum_B S_{12}(A, B) m_2(B),$$

$$m_2(B) = \sum_C S_{23}(B, C) m_3(C).$$

We therefore have

$$\begin{aligned} m_1(A) &= \sum_B S_{12}(A, B) \sum_C S_{23}(B, C) m_3(C) \\ &= \sum_B \sum_C S_{12}(A, B) S_{23}(B, C) m_3(C) \\ &= \sum_C m_3(C) \sum_B S_{12}(A, B) S_{23}(B, C) \end{aligned}$$

Now, let us define the matrix S_{13} elements as

$$S_{13}(A, C) = \sum_B S_{12}(A, B) S_{23}(B, C),$$

189 meaning that $S_{13} = S_{12} \cdot S_{23}$ is the result of a matrix product. One can then
190 show that S_{13} satisfies Definition 1 and that $m_1 \mathbf{R} m_3$ as

- 191 – S_{13} is stochastic, being the product of stochastic matrices ;
– we have that

$$\begin{aligned} S_{13}(A, C) > 0 &\Leftrightarrow \exists(A, B) \text{ and } (B, C) \text{ s.t. } S_{12}(A, B) S_{23}(B, C) > 0 \\ &\Rightarrow \mathbf{A} \mathbf{R} \mathbf{B} \wedge \mathbf{B} \mathbf{R} \mathbf{C} \\ &\Rightarrow \mathbf{A} \mathbf{R} \mathbf{C}. \end{aligned}$$

□

192 **Proposition 8 (Unpreserved completeness).** *If \mathbf{R} is complete (or total) on*
193 *sets, it is not necessarily so on belief functions: for any two m_1, m_2 we may have*
194 *neither $m_1 \mathbf{R} m_2$ nor $m_2 \mathbf{R} m_1$.*

Proof. Consider the relation \mathbf{R} = “having a lower cardinality than” on the space
 $X = \{a, b, c\}$, meaning that $\mathbf{A} \mathbf{R} \mathbf{B} \Leftrightarrow |A| \leq |B|$, which is a complete relation on
sets. Consider now the two mass functions

$$\begin{aligned} m_1(\{a\}) &= 0.6, & m_1(\{a, b\}) &= 0.4, \\ m_2(\{a\}) &= 0.8, & m_2(X) &= 0.2. \end{aligned}$$

195 Then, we have neither $m_1 \mathbf{R} m_2$, nor $m_2 \mathbf{R} m_1$, as indeed all stochastic matrices
196 such that $m_1 = S \cdot m_2$ or $m_2 = S \cdot m_1$ must contain non-null value on pairs of
197 subsets A, B with $\neg(\mathbf{A} \mathbf{R} \mathbf{B})$. Consider for instance the case

$$m_1 : \begin{matrix} \{a\} \\ \{a, b\} \\ X \end{matrix} \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 & 0 \\ 1/4 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0 \\ 0.2 \end{pmatrix} : m_2.$$

198 We only display the submatrix of S corresponding to focal elements of the mass
199 functions. Entries in green can be set to either 0 or some positive number. Entries
200 in red cannot be assigned a positive number. It is clear that at least some non-
201 null value must be given to $S(\{a, b\}, \{a\})$, hence $\neg m_1 \mathbf{R} m_2$. A similar observation
202 can be made for the reverse case.

□

The expected cardinality [17] of a belief function, defined as

$$C(m) = \sum_{A \subseteq X} m(A)|A|,$$

203 yields a complete relation between belief functions that is a generalisation of the
 204 relation \mathbf{R} defined in the above proof. This is therefore also an illustration that
 205 not all binary relations on belief functions can be retrieved via the mechanism
 206 under study.

207 Table 2 summarises our obtained results so far, and in particular which prop-
 208 erties existing on subsets is guaranteed to be preserved when considering them
 209 on the richer language of belief functions. It should be noted that even if a prop-
 210 erty is not guaranteed to be preserved in general, it may be preserved in specific
 211 cases: for instance, the inclusion relation is antisymmetric, and its generalisation
 212 to belief functions, called specialisation [15], is too. The same remark is true for
 213 strict inclusion, that is asymmetric. We will see later on that all partial orders
 214 (among which inclusion) are in fact preserved by Definition 1.

\mathbf{R} on 2^X is	\implies \mathbf{R} on \mathcal{B}^X is
Symmetric	Yes
Antisymmetric	No
Asymmetric	No
Reflexive	Yes
Irreflexive	No
Transitive	Yes
Complete	No

Table 2. Summary of properties preservation

215 Finally, we can also consider two different binary relations \mathbf{R} and \mathbf{R}' and
 216 check whether a property for this pair of relations is preserved. There is mainly
 217 one such property which is implication.

218 **Proposition 9 (Preserved implication).** *If \mathbf{R} and \mathbf{R}' are such that $ARB \implies$
 219 $AR'B$ for any subsets A and B , it is so on belief functions.*

220 *Proof.* Let m_1 and m_2 denote two mass functions and S is a stochastic matrix
 221 compliant with definition 1 for relation \mathbf{R} . Obviously, S is also compliant with
 222 definition 1 for relation \mathbf{R}' because when $m_2(B) > 0$, then $S(A, B) > 0 \implies$
 223 $ARB \implies AR'B$.

224 3.2 Preservation of classical relations

225 In this section, we study whether some classical relations composed of multi-
 226 ple properties are preserved by our definition. We will limit ourselves to the

relations recalled in Table 1, as those are the most common, but clearly others could be studied, such as specific order relations (semi-orders, interval orders, tournaments, ...) [19].

A very first remark is that, if the relation is defined by a set of properties that are all preserved by our definition, then it is immediate that it is preserved when considering it on belief functions. Among other things, this means that

- Tolerance relations
- (Partial) Equivalence relations
- Preorder relations

are preserved when extended to belief functions. However, total preorders are usually not preserved when extended to belief functions. A simple example is given in the proof of Proposition 8. In this proof we examine the relation “having a lower cardinality than” which is a total preorder on sets while the induced relation on belief functions fails to be complete.

Moreover, it is easy to see that total orders are not preserved either, as any refinement of the relation “having a lower cardinality than” into a total order would constrain even more the element on which the stochastic matrix has to be positive. Another very common class of binary relations on sets are partial orders, for which we can show that they are preserved:

Proposition 10 (Preserved partial order). *If \mathbf{R} is a partial order on sets, it is so on belief functions.*

See Appendix A for a proof of the above proposition.

4 Related works

4.1 Inclusion and consistency

In the case where the relations are either inclusion or consistency, then we retrieve well-known results of the literature:

- in the case of inclusion we have $\mathbf{A}\mathbf{R}\mathbf{B}$ iff $A \subseteq B$, and Definition 1 is then essentially equivalent to that of specialisation [15]. The only difference amounts to checking if $m_2(B) > 0$ in condition (4), that we need to handle generic relations, but that is not needed in the specific case of specialisation (as in this case, for any B there is always a subset A such that $\mathbf{A}\mathbf{R}\mathbf{B}$). Beyond this difference, the notion of specialisation and the extension of inclusion to belief functions proposed in this paper are actually formally equivalent in the sense that for any mass functions m_1 and m_2 , m_1 is a specialisation of m_2 if and only if $m_1 \mathbf{R} m_2$.
- in the case of consistency, we have $\mathbf{A}\mathbf{R}\mathbf{B}$ iff $A \cap B \neq \emptyset$, and one can see that $m_1 \mathbf{R} m_2$ iff there is a joint mass assigning positive mass to pairs of sets having a non-empty intersection. This is equivalent to require $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \emptyset$, with \mathcal{P}_i the probability set induced by m_i [3].

266 **4.2 Rankings**

267 When the space $X = \{x_1, \dots, x_n\}$ is ordered (with $x_i \leq x_{i+1}$) and possibly
 268 infinite, it makes sense to consider relations of the kind “higher than” in order
 269 to compare sets. There are many ways to rank two sets A, B , such as:

- 270 – Single-bound dominance, that can be declined itself into four notions:
 - 271 • loose dominance: $\mathbf{AR}_{\leq LD} B$ if $\min A \leq \max B$
 - 272 • lower bound: $\mathbf{AR}_{\leq LB} B$ if $\min A \leq \min B$
 - 273 • upper bound: $\mathbf{AR}_{\leq UB} B$ if $\max A \leq \max B$
 - 274 • strict dominance: $\mathbf{AR}_{\leq SD} B$ if $\max A \leq \min B$
- 275 – Pairwise-bound or lattice dominance: $\mathbf{AR}_{\leq PD} B$ if $\min A \leq \min B$ and $\max A \leq$
 276 $\max B$, whose extension to belief functions studied in [24] correspond to our
 277 proposal.

278 Extensions of this kind of relations to belief functions have already been in-
 279 vestigated in [24], and are connected to the extensions of stochastic dominance
 280 explored in [8] for belief functions, and in [25] for the general case of sets of cumu-
 281 lative distributions. In fact, let us first define the following stochastic dominance
 282 notions:

- stochastic loose dominance:

$$m_1 \prec_{LD}^{St} m_2 \text{ iff } Pl_1([x_1, \dots, x_i]) \geq Bel_2([x_1, \dots, x_i]), \forall x_i \in X$$

- stochastic lower bound:

$$m_1 \prec_{LB}^{St} m_2 \text{ iff } Pl_1([x_1, \dots, x_i]) \geq Pl_2([x_1, \dots, x_i]), \forall x_i \in X$$

- stochastic upper bound:

$$m_1 \prec_{UB}^{St} m_2 \text{ iff } Bel_1([x_1, \dots, x_i]) \geq Bel_2([x_1, \dots, x_i]), \forall x_i \in X$$

- stochastic strict dominance:

$$m_1 \prec_{SD}^{St} m_2 \text{ iff } Bel_1([x_1, \dots, x_i]) \geq Pl_2([x_1, \dots, x_i]), \forall x_i \in X$$

- stochastic lattice dominance:

$$m_1 \prec_{PD}^{St} m_2 \text{ iff } (m_1 \prec_{LB}^{St} m_2) \wedge (m_1 \prec_{UB}^{St} m_2)$$

283 We then have the following strong relationships between the extensions of
 284 ranking to belief functions and the stochastic dominance relations:

Proposition 11. *For $y \in \{LD, LB, UB, SD\}$, we have that*

$$m_1 \mathbf{R}_{\leq y} m_2 \Leftrightarrow m_1 \prec_y^{St} m_2$$

285 *Proof.* We will only demonstrate the relation for one of the y , that is SD (the
286 strongest relation), as proofs for the other cases are analogous.

287 \Leftarrow First, let us remind that if $m_1 \prec_{SD}^{St} m_2$, it means that the cumulative
288 distribution induced by the minimal values of the focal elements of m_2 stochas-
289 tically dominates the one induced by the maximal values of m_1 . Let us denote
290 A_1, \dots, A_n and B_1, \dots, B_m the focal sets of m_1, m_2 , and assume without loss of
291 generality that they are ordered according to their maximal values for m_1 , and
292 their minimal values for m_2 , that is $\max A_i \leq \max A_{i+1}$ for any $i = 1, \dots, n-1$
293 and $\min B_i \leq \min B_{i+1}$ for any $i = 1, \dots, m-1$.

Let us denote by $\alpha_i = \sum_{j=1}^i m_1(A_j)$ and $\beta_i = \sum_{j=1}^i m_2(B_j)$ the cumulated weights of the first i elements of m_1 and m_2 , assuming all α_i, β_i are all distinct for easiness. We denote by

$$\gamma_1, \dots, \gamma_{n+m-1} = \{\alpha_1, \dots, \alpha_n\} \cup \{\beta_1, \dots, \beta_m\}$$

the union of all distinct possible cumulative values of masses, assuming that they are also ordered, i.e., $\gamma_i \leq \gamma_{i+1}$ (we have $m+n-1$ distinct values because $\alpha_n = \beta_m = 1$). Let us now define the following joint mass function m_{12} such that, for any $i = 1, \dots, m+n-1$,

$$m_{12}(A_{\gamma_i}, B_{\gamma_i}) = \gamma_i - \gamma_{i-1}$$

with $\gamma_0 = 0$, and the following definitions for the focal sets:

$$A_{\gamma_i} = \{A_i : (\sum_{j=1}^i m_1(A_j) \geq \gamma_i) \wedge (\sum_{j=1}^{i-1} m_1(A_j) < \gamma_i)\},$$

$$B_{\gamma_i} = \{B_i : (\sum_{j=1}^i m_2(B_j) \geq \gamma_i) \wedge (\sum_{j=1}^{i-1} m_2(B_j) < \gamma_i)\},$$

294 that by construction satisfy Equations (7)-(8). The construction is illustrated in
295 Figure 1 for the case of $n = 3$ and $m = 2$. This comes down to construct the
296 joint mass in a level-wise manner, and since we also have that $m_1 \prec_{SD}^{St} m_2$, we
297 have that for any i , $\max A_{\gamma_i} \leq \min B_{\gamma_i}$, hence $A_{\gamma_i} \mathbf{R}_{\leq SD} B_{\gamma_i}$

\Rightarrow if $m_1 \mathbf{R}_{\leq SD} m_2$, this means that there is a joint $m_{12}(A, B)$ that is positive only if $\max A \leq \min B$. Let us now show that this implies, for any $x_i \in X$,

$$Bel_1([x_1, \dots, x_i]) \geq Pl_2([x_1, \dots, x_i]),$$

which is equivalent to

$$\sum_{A: \max A \leq x_i} m_1(A) \geq \sum_{B: \min B \leq x_i} m_2(B).$$

Using the relation between m_{12}, m_1 and m_2 , we get

$$\sum_{A: \max A \leq x_i} \sum_B m_{12}(A, B) \geq \sum_{B: \min B \leq x_i} \sum_A m_{12}(A, B)$$

but since $m_{12}(A, B) > 0$ only if $\max A \leq \min B$, we can write

$$\sum_{A: \max A \leq x_i} \sum_B m_{12}(A, B) \geq \sum_{B: \min B \leq x_i} \sum_{A: \max A \leq x_i} m_{12}(A, B)$$

as all the elements on the right-hand side summation are also in the left-hand side, this latter can only be bigger. \square

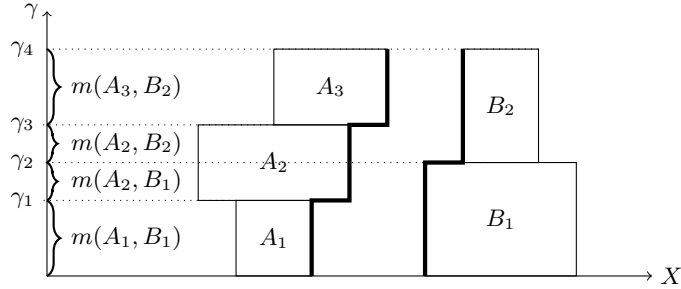


Fig. 1. Illustration of proof of Proposition 11 (construction of joint mass).

298 The above proposition shows a clear relation between ranking relations (when
 299 extended according to Definition 1) and the corresponding stochastic dominance
 300 relation. While this confirms the interest of our proposal and its links with
 301 existing, more specific works, this also provides an efficient computational way to
 302 check whether m_1, m_2 are in a ranking relation, as checking stochastic dominance
 303 is easier than checking whether a relation holds (which can be done by solving
 304 a linear programming problem, as suggested in Section 7).

305 The next example however shows that the property is not true for pairwise
 306 bounds, essentially because focal elements are usually not totally ordered with
 307 respect to pairwise bounds.

Example 2. Let us consider the space $X = \{x_1, \dots, x_{12}\}$ and the two following mass functions

$$\begin{aligned} m_1(\{x_1, \dots, x_7\} = A_1) &= 1/3, & m_2(\{x_2, \dots, x_{12}\} = B_1) &= 1/3, \\ m_1(\{x_3, \dots, x_9\} = A_2) &= 1/3, & m_2(\{x_4, \dots, x_8\} = B_2) &= 1/3, \\ m_1(\{x_5, \dots, x_{11}\} = A_3) &= 1/3, & m_2(\{x_6, \dots, x_{10}\} = B_3) &= 1/3. \end{aligned}$$

The pairs of sets satisfying the relation $\mathbf{R}_{\leq PD}$ is summarised in the matrix

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{ccc} B_1 & B_2 & B_3 \\ \left[\begin{array}{ccc} \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} \\ & \mathbf{R}_{\leq PD} & \\ & & \mathbf{R}_{\leq PD} \end{array} \right] \end{array}$$

308 which shows that there are no m_{12} for which $m_1 \mathbf{R}_{\leq PD} m_2$, since at least a positive
 309 number must be put on the third row. However, we do have $m_1 \prec_{PD}^{St} m_2$, as the
 310 bounds of each focal elements, once increasingly re-ordered separately, satisfy
 311 the pairwise dominance notion.

312 However, that the converse holds (if $m_1 \mathbf{R}_{\leq PD} m_2$, then $m_1 \prec_{PD}^{St} m_2$) has
 313 been shown in [24]. Finally, from Proposition 9 and the existing implications
 314 between the different rankings, we can easily conclude that:

$$m_1 \mathbf{R}_{\leq SD} m_2 \Rightarrow m_1 \mathbf{R}_{\leq PD} m_2 \Rightarrow \left\{ \begin{array}{l} m_1 \mathbf{R}_{\leq UB} m_2 \\ m_1 \mathbf{R}_{\leq LB} m_2 \end{array} \right\} \Rightarrow m_1 \mathbf{R}_{\leq LD} m_2 \quad (10)$$

315 5 Preservation through functional mapping

316 5.1 Univariate functions

317 This section investigates whether a function that is compatible (in the sense
 318 of Definition 2 below) with set relations given respectively on its domain and
 319 codomain, is also compatible with the extensions of these relations to belief func-
 320 tions. We consider first the case of univariate functions; multivariate functions
 321 are handled in Section 5.2.

322 Let f be some function with domain X and codomain Y , i.e., $f : X \rightarrow Y$.
 323 We recall that the image $f(A)$ of some subset $A \subseteq X$ under f is the subset
 324 $f(A) = \{f(x) : x \in A\} \subseteq Y$. More generally, the image $f(m)$ of some mass
 325 function $m \in \mathcal{B}^X$ under f is the mass function $f(m) \in \mathcal{B}^Y$ defined, for all
 326 $B \subseteq Y$, as

$$f(m)(B) = \sum_{f(A)=B} m(A). \quad (11)$$

Definition 2. Let $f : X \rightarrow Y$. Let \mathbf{R}^X and \mathbf{R}^Y be relations on 2^X and 2^Y , respectively. The function f is said to be $(\mathbf{R}^X; \mathbf{R}^Y)$ -compatible if

$$A \mathbf{R}^X B \Rightarrow f(A) \mathbf{R}^Y f(B), \forall A, B \subseteq X.$$

327 *Example 3.* Let \mathbf{R}_{\subseteq}^X be the relation corresponding to inclusion on X , i.e., $A \mathbf{R}_{\subseteq}^X B$
 328 iff $A \subseteq B$, $A, B \subseteq X$. Similarly, let \mathbf{R}_{\subset}^X and \mathbf{R}_{\cap}^X denote the relations correspond-
 329 ing, respectively, to strict inclusion and consistency on X , and let \mathbf{R}_{\subseteq}^Y denote
 330 inclusion on Y .

331 Since for any function f and any $A, B \subseteq X$ such that $A \subseteq B$ it holds that
 332 $f(A) \subseteq f(B)$, any function f is $(\mathbf{R}_{\subseteq}^X; \mathbf{R}_{\subseteq}^Y)$ -compatible. Similarly, any function f
 333 is $(\mathbf{R}_{\subseteq}^X; \mathbf{R}_{\subseteq}^Y)$ -compatible.

334 However, not all functions f are $(\mathbf{R}_{\cap}^X; \mathbf{R}_{\cap}^Y)$ -compatible. For instance, if f is
 335 a constant function, i.e. $f(x) = y$ for some $y \in Y$ and all $x \in X$, then f is
 336 $(\mathbf{R}_{\cap}^X; \mathbf{R}_{\subseteq}^Y)$ -compatible (in this case we have $f(A) \subseteq f(B)$ for all $A, B \subseteq X$ such
 337 that $A \cap B \neq \emptyset$ since $f(A) = f(B) = \{y\}$). However, if f is the identity function,
 338 i.e. $X = Y$ and $f(x) = x$ for all $x \in X$, then f is not $(\mathbf{R}_{\cap}^X; \mathbf{R}_{\subseteq}^Y)$ -compatible
 339 (in this case $f(A) = A$ for all $A \subseteq X$ and in general $A \cap B \neq \emptyset \not\Rightarrow A \subseteq B$,
 340 $A, B \subseteq X$).

341 Similarly, let X and Y be two ordered spaces and let $\mathbf{R}_{\leq PD}^X$ and $\mathbf{R}_{\leq PD}^Y$ be the
 342 relations corresponding to pairwise-bound dominance on X and on Y , respec-
 343 tively. Then, not all functions f are $(\mathbf{R}_{\leq PD}^X; \mathbf{R}_{\leq PD}^Y)$ -compatible. For instance, if
 344 f is decreasing, i.e. $f(x) \leq f(x')$ for all $x \in X$ and $x' \in X$ such that $x \geq x'$,
 345 then we have $f(A) \geq_{PD} f(B)$ for all $A, B \subseteq X$ such that $A \leq_{PD} B$, and
 346 thus f is not $(\mathbf{R}_{\leq PD}^X; \mathbf{R}_{\leq PD}^Y)$ -compatible since in general we have in this case
 347 $A \leq_{PD} B \not\Rightarrow f(A) \leq_{PD} f(B)$. However, if f is monotonically non-decreasing,
 348 then it is $(\mathbf{R}_{\leq PD}^X; \mathbf{R}_{\leq PD}^Y)$ -compatible since if $f(x) \leq f(x')$ for all $x \in X$ and
 349 $x' \in X$ such that $x \leq x'$ then we have $A \leq_{PD} B \Rightarrow f(A) \leq_{PD} f(B)$.

350 **Proposition 12 (Preserved compatibility).** *If f is $(\mathbf{R}^X; \mathbf{R}^Y)$ -compatible, it*
 351 *is so on belief functions:*

$$m_1 \mathbf{R}^X m_2 \Rightarrow f(m_1) \mathbf{R}^Y f(m_2). \quad (12)$$

352 *Proof.* Since $m_1 \mathbf{R}^X m_2$, there exists a joint mass function m_{12} on X^2 satisfy-
 353 ing (6)-(8) for \mathbf{R}^X . Consider the joint mass function m on Y^2 defined as

$$m(A', B') = \sum_{f(A)=A', f(B)=B'} m_{12}(A, B), \quad \forall A', B' \subseteq Y. \quad (13)$$

354 Since $m_{12}(A, B) > 0 \Rightarrow \mathbf{A} \mathbf{R}^X B$ and $\mathbf{A} \mathbf{R}^X B \Rightarrow f(A) \mathbf{R}^Y f(B)$, then $m(A', B') >$
 355 $0 \Rightarrow A' \mathbf{R}^Y B'$. Besides,

$$\begin{aligned} \sum_{B'} m(A', B') &= \sum_{B'} \sum \{m_{12}(A, B) | f(A) = A', f(B) = B'\} \\ &= \sum_{f(A)=A'} \sum_{B'} \sum_{f(B)=B'} m_{12}(A, B) \\ &= \sum_{f(A)=A'} \sum_B m_{12}(A, B) \\ &= \sum_{f(A)=A'} m_1(A) \\ &= (f(m_1))(A') \end{aligned}$$

356 Similarly, $\sum_{A'} m(A', B') = (f(m_2))(B')$.

In sum, when $m_1 \mathbf{R}^X m_2$ and f is $(\mathbf{R}^X; \mathbf{R}^Y)$ -compatible, there exists a joint mass function m on Y^2 such that $m(A', B') > 0 \Rightarrow A' \mathbf{R}^Y B'$, for all $A', B' \subseteq Y$, and whose marginals are $f(m_1)$ and $f(m_2)$, hence $f(m_1) \mathbf{R}^Y f(m_2)$. \square

357 **Corollary 1.** For any function f , $m_1 \mathbf{R}_{\subseteq}^X m_2 \Rightarrow f(m_1) \mathbf{R}_{\subseteq}^Y f(m_2)$, which follows
358 from the $(\mathbf{R}_{\subseteq}^X; \mathbf{R}_{\subseteq}^Y)$ -compatibility of any f .

359 Corollary 1 was already known [16, Proposition 2]. Proposition 12 is a generali-
360 sation of this latter result.

361 5.2 Multivariate functions

362 These results can be extended to functions having more than one argument:

363 **Definition 3.** Let $f : X_1 \times X_2 \rightarrow Y$. Let \mathbf{R}^{X_1} , \mathbf{R}^{X_2} and \mathbf{R}^Y be relations on 2^{X_1} ,
364 2^{X_2} and 2^Y , respectively. The function f is said to be $(\mathbf{R}^{X_1}, \mathbf{R}^{X_2}; \mathbf{R}^Y)$ -compatible
365 if, for all $A_1, B_1 \subseteq X_1$ and all $A_2, B_2 \subseteq X_2$

$$A_1 \mathbf{R}^{X_1} B_1 \wedge A_2 \mathbf{R}^{X_2} B_2 \Rightarrow f(A_1, A_2) \mathbf{R}^Y f(B_1, B_2). \quad (14)$$

366 *Example 4.* Since for any function f and any $A_1, B_1 \subseteq X_1$ and $A_2, B_2 \subseteq X_2$,
367 such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ it holds that $f(A_1, A_2) \subseteq f(B_1, B_2)$, any
368 function is $(\mathbf{R}_{\subseteq}^{X_1}, \mathbf{R}_{\subseteq}^{X_2}; \mathbf{R}_{\subseteq}^Y)$ -compatible.

Let X_1 , X_2 and Y be ordered spaces. If f is non-decreasing in both its arguments (for short, *non-decreasing*), i.e., for all $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$,

$$x_1 \leq x'_1 \wedge x_2 \leq x'_2 \Rightarrow f(x_1, x_2) \leq f(x'_1, x'_2),$$

369 then for $y \in \{LD, LB, UB, SD, PD\}$ we have $f(A_1, A_2) \leq_y f(B_1, B_2)$ for all
370 $A_1, B_1 \subseteq X_1$ and $A_2, B_2 \subseteq X_2$ such that $A_1 \leq_y B_1$ and $A_2 \leq_y B_2$, i.e. f is
371 $(\mathbf{R}_{\leq_y}^{X_1}, \mathbf{R}_{\leq_y}^{X_2}; \mathbf{R}_{\leq_y}^Y)$ -compatible. This can easily be shown as follows (we provide
372 only the proof for the case $y = LD$, the other cases being similar). Since f is non-
373 decreasing, we have $\min f(A_1, A_2) = f(\min A_1, \min A_2)$ and $\max f(B_1, B_2) =$
374 $f(\max B_1, \max B_2)$. Besides, since $\min A_1 \leq \max B_1$ and $\min A_2 \leq \max B_2$ and
375 f is non-decreasing, we obtain $\min f(A_1, A_2) \leq \max f(B_1, B_2)$.

We remind that the image $f(m_{12})$ of some joint mass function $m_{12} \in \mathcal{B}^{X_1 \times X_2}$ under f is the mass function $f(m_{12}) \in \mathcal{B}^Y$ defined, for all $B \subseteq Y$, as [16]:

$$(f(m_{12}))(B) = \sum_{f(A_1, A_2) = B} m_{12}(A_1, A_2).$$

376 Let us also recall that if m_{12} satisfies $m_{12}(A_1, A_2) = m_1(A_1)m_2(A_2)$ with m_i
377 the marginal of m_{12} on X_i , $i = 1, 2$, then m_1 and m_2 are said to be independent.
378 This independence notion is the main one used in evidence theory, but can also be
379 interpreted and used as an outer approximation within imprecise probability [18,
380 5].

381 **Proposition 13 (Preserved compatibility, several arguments).** *Let m_{12}*
 382 *(resp. m'_{12}) denote the joint mass function on $X_1 \times X_2$ obtained from independent*
 383 *mass functions m_1 and m_2 (resp. m'_1 and m'_2) defined on X_1 and X_2 , respectively.*

If f is $(\mathbf{R}^{X_1}, \mathbf{R}^{X_2}; \mathbf{R}^Y)$ -compatible, it is so on belief functions:

$$m_1 \mathbf{R}^{X_1} m'_1 \wedge m_2 \mathbf{R}^{X_2} m'_2 \Rightarrow f(m_{12}) \mathbf{R}^Y f(m'_{12}).$$

384 *Proof.* Since $m_i \mathbf{R}_i^X m'_i$, $i = 1, 2$, there exist joint mass functions $m_{11'}$ and $m_{22'}$
 385 satisfying

$$\begin{aligned} m_{11'}(A_1, B_1) > 0 &\implies A_1 \mathbf{R}_1^X B_1, \\ m_{22'}(A_2, B_2) > 0 &\implies A_2 \mathbf{R}_2^X B_2. \end{aligned}$$

386 Furthermore, let $m_{11'22'}$ denote the joint mass function on $X_1 \times X_1 \times X_2 \times$
 387 X_2 obtained from independent marginals $m_{11'}$ and $m_{22'}$. Mass function $m_{11'22'}$
 388 satisfies

$$m_{11'22'}(A_1, B_1, A_2, B_2) > 0 \implies A_1 \mathbf{R}_1^X B_1 \wedge A_2 \mathbf{R}_2^X B_2. \quad (15)$$

389 Moreover, we have

$$\begin{aligned} \sum_{B_1, B_2} m_{11'22'}(A_1, B_1, A_2, B_2) &= \sum_{B_1, B_2} m_{11'}(A_1, B_1) m_{22'}(A_2, B_2) \\ &= \sum_{B_1} m_{11'}(A_1, B_1) \cdot \sum_{B_2} m_{22'}(A_2, B_2) \\ &= \sum_{B_1} m_{11'}(A_1, B_1) \cdot m_2(A_2) \\ &= m_1(A_1) \cdot m_2(A_2) \\ &= m_{12}(A_1, A_2) \end{aligned}$$

390 and similarly

$$\sum_{A_1, A_2} m_{11'22'}(A_1, B_1, A_2, B_2) = m'_{12}(B_1, B_2).$$

391 In other words, $m_{11'22'}$ has m_{12} and m'_{12} as marginals.

Consider the joint mass function m on Y^2 defined as, for any $A', B' \subseteq Y$,

$$m(A', B') = \sum_{f(A_1, A_2)=A', f(B_1, B_2)=B'} m_{11'22'}(A_1, B_1, A_2, B_2).$$

392 Since Eqs. (15) and (14) hold, then $m(A', B') > 0 \Rightarrow A' \mathbf{R}^Y B'$. Besides,

$$\begin{aligned}
\sum_{B'} m(A', B') &= \sum_{B'} \sum_{B'} \{m_{11'22'}(A_1, B_1, A_2, B_2) | f(A_1, A_2) = A', f(B_1, B_2) = B'\} \\
&= \sum_{f(A_1, A_2)=A'} \sum_{B'} \sum_{f(B_1, B_2)=B'} m_{11'22'}(A_1, B_1, A_2, B_2) \\
&= \sum_{f(A_1, A_2)=A'} \sum_{B_1, B_2} m_{11'22'}(A_1, B_1, A_2, B_2) \\
&= \sum_{f(A_1, A_2)=A'} m_{12}(A_1, A_2) \\
&= (f(m_{12}))(A')
\end{aligned}$$

Similarly, $\sum_{A'} m(A', B') = (f(m'_{12}))(B')$. □

393 **Corollary 2.** For any function f , $m_1 \mathbf{R}_{\subseteq}^{X_1} m'_1 \wedge m_2 \mathbf{R}_{\subseteq}^{X_2} m'_2 \Rightarrow f(m_{12}) \mathbf{R}_{\subseteq}^Y f(m'_{12})$,
394 which follows from any f being $(\mathbf{R}_{\subseteq}^{X_1}, \mathbf{R}_{\subseteq}^{X_2}; \mathbf{R}_{\subseteq}^Y)$ -compatible.

395 **Corollary 3.** For X_1, X_2 and Y ordered spaces and f any monotonically non
396 decreasing function, $m_1 \mathbf{R}_{\leq_y}^{X_1} m'_1 \wedge m_2 \mathbf{R}_{\leq_y}^{X_2} m'_2 \Rightarrow f(m_{12}) \mathbf{R}_{\leq_y}^Y f(m'_{12})$, for $y \in$
397 $\{LD, LB, UB, SD, PD\}$, which follows from the $(\mathbf{R}_{\leq_y}^{X_1}, \mathbf{R}_{\leq_y}^{X_2}; \mathbf{R}_{\leq_y}^Y)$ -compatibility
398 of any such functions for $y \in \{LD, LB, UB, SD, PD\}$.

399 Corollary 2 was already known [16, Proposition 3], and Corollary 3 generalises
400 a result in [24, proof of Proposition 5], where f is the addition of integers and
401 $y = PD$. Proposition 13, which can be readily extended to the case of functions
402 having more than two arguments, is thus a generalisation of these results of [16,
403 24].

404 We note that the setting of Corollary 3, *i.e.*, monotonic non-decreasing func-
405 tions on ordered spaces, is commonly encountered in multi-criteria decision mak-
406 ing [23, 10], reliability analysis [13], and optimisation problems [21] hence this
407 corollary may be useful for such problems when function arguments are tainted
408 with uncertainty. Next section provides such examples.

409 6 Illustrative applications

410 6.1 System reliability

411 In multi-state system reliability assessment, one main issue is to assess the avail-
412 ability, or the performance of a whole system, given the performances of its
413 components. Usually, the system is assumed to have n components x_i that take
414 values on a finite, ordered scale X_i , that we will denote here by natural num-
415 bers. The performance of the system then depends on the state of each of its
416 component, and is usually modelled by structure function $f(x_1, \dots, x_n)$ that is
417 non-decreasing, as the system performance can only increase or stay the same if
418 a component performance increases.

Let us consider a simple communication system made of one source and one receiver, with a channel in between made of n repeaters x_i , where each of them can be in different states $X_i = \{1, 2, \dots, K\}$ which is the maximal number of messages this repeater can store and send. Given this, we may be interested in the global capacity of a channel, which is

$$f(x_1, \dots, x_n) = \min(x_1, \dots, x_n),$$

419 as this is a series system. Now, assume we want to compare the capacity of two
 420 different channels in order to choose the best one, with repeaters whose state is
 421 uncertain due to the fact that they have degraded over time. If this uncertainty is
 422 modelled by belief functions and that m_i^j models the state of the i^{th} repeater of
 423 the j^{th} channel, then if we have $m_i^1 \mathbf{R}_{\leq y} m_i^2$ for any $y \in \{LD, LB, UB, SD, PD\}$,
 424 then we know from Corollary 3 that channel 1 achieves at most the same level
 425 of performance than channel 2, without even computing the propagated mass
 426 function through f . Note that this would be true, whatever the function f is (as
 427 long as it is non-decreasing).

Example 5. Assume we have four identical repeaters x_1, x_2, x_3 and x_4 working independently with $X_i = \{1, 2, 3\}$, and are considering two different, partially known technologies at our disposal to build the system. Then, if our knowledge of these two technologies is such that, $\forall i$

$$\begin{aligned} m_i^1(\{2\}) &= 0.5 & m_i^2(\{3\}) &= 0.5 \\ m_i^1(\{1, 2\}) &= 0.3 & m_i^2(\{2, 3\}) &= 0.3 \\ m_i^1(\{1, 2, 3\}) &= 0.2 & m_i^2(\{1, 2, 3\}) &= 0.2 \end{aligned}$$

428 we can easily conclude that $f^1 \mathbf{R}_{\leq PD} f^2$ and this without performing any com-
 429 putation, as $m_i^1 \mathbf{R}_{\leq PD} m_i^2$.

430 6.2 Multi-criteria decision making

431 In multi-criteria decision making, it is quite common to consider as variables
 432 x_i the utilities of the criteria. For instance, these could be scores obtained by
 433 students over different courses, or the evaluation of students regarding the qual-
 434 ity of courses (e.g., with respect to interest, teaching quality and study time
 435 required). It could be that, for some reasons, those assessments are uncertain
 436 (e.g., because students are allowed to provide imprecise assessments in case of
 437 hesitation).

438 One common function to aggregate utilities is the weighted average, that is
 439 to have

$$f(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_i \quad (16)$$

440 or one of its extension such as the Choquet integral. Again, if we want to compare
 441 two courses, and we have $m_i^1 \mathbf{R}_{\leq PD} m_i^2$ for all x_i , then we know from Corollary 3,

442 without any computation, that the second course is considered better by the
 443 students. Besides, if there is a third course such that $m_i^2 \mathbf{R}_{\leq PD} m_i^3$ for all x_i ,
 444 then, without any computation, we know from Corollary 3 that this course is
 445 preferred over the second one but also over the first one as the set relation $\mathbf{R}_{\leq PD}$
 446 is transitive, a property that we know is preserved thanks to Proposition 7.

Example 6. Assume three courses evaluated by students against two criteria x_1 and x_2 with $X_i = \{0, \dots, 10\}$. Let m_i^j be the mass function representing the uncertain evaluation of course j according to criterion i . Suppose we have for criterion x_1

$$m_1^1(\{2, 3, 4\}) = 1, \quad m_1^2(\{2, 3, 4\}) = 0.6, \quad m_1^3(\{6, 7, 8\}) = 1, \\ m_1^2(\{3, 4\}) = 0.4,$$

and for criterion x_2

$$m_2^1(\{4\}) = 0.13, \quad m_2^2(\{5, 6\}) = 0.65, \quad m_2^3(\{5, 6, 7\}) = 0.3, \\ m_2^1(\{7\}) = 0.07, \quad m_2^2(\{7\}) = 0.35, \quad m_2^3(\{7, 8\}) = 0.2, \\ m_2^1(\{5, 6\}) = 0.8, \quad m_2^3(\{8\}) = 0.5.$$

If the weighted average (16) is used to aggregate these evaluations, then whatever the weights w_i , we can easily conclude that the overall uncertain score f^1 of the first course will be such that $f^1 \mathbf{R}_{\leq PD} f^2$, with f^2 the uncertain score of the second course, since $m_i^1 \mathbf{R}_{\leq PD} m_i^2$ for all x_i . Indeed, for the first criterion the only joint mass function obtainable from m_1^1 and m_1^2 is

$$m_1^{12}(\{2, 3, 4\}, \{2, 3, 4\}) = 0.6, \quad m_1^{12}(\{2, 3, 4\}, \{3, 4\}) = 0.4$$

and we can easily see that $\{2, 3, 4\} \mathbf{R}_{\leq PD} \{2, 3, 4\}$ and $\{2, 3, 4\} \mathbf{R}_{\leq PD} \{3, 4\}$. For the second criterion, we can consider the joint mass function

$$m_2^{12}(\{5, 6\}, \{5, 6\}) = 0.65, \quad m_2^{12}(\{5, 6\}, \{7\}) = 0.15, \\ m_2^{12}(\{7\}, \{7\}) = 0.07, \quad m_2^{12}(\{4\}, \{7\}) = 0.13.$$

447 where every pair of sets satisfy the relation $\mathbf{R}_{\leq PD}$. From those two facts and
 448 Corollary 3, we can conclude that $f^1 \mathbf{R}_{\leq PD} f^2$ for any increasing function of the
 449 two criteria. Similarly, we obtain $f^2 \mathbf{R}_{\leq PD} f^3$ and $f^1 \mathbf{R}_{\leq PD} f^3$. Since $\mathbf{R}_{\leq PD}$ is
 450 transitive, the latter comparison could have been deduced from $f^1 \mathbf{R}_{\leq PD} f^2$ and
 451 $f^2 \mathbf{R}_{\leq PD} f^3$.

As an illustration, let us confirm that for the first two courses and the simple weighted average with $w_1 = 0.5$ and $w_2 = 0.5$, we do have $f^1 \mathbf{R}_{\leq PD} f^2$: denoting by m_f^j the propagated evaluation of course j , we get

$$m_f^1(\{3, 3.5, 4\} = A_1) = 0.13, \quad m_f^2(\{3.5, 4, 4.5, 5\} = B_1) = 0.39 \\ m_f^1(\{4.5, 5, 5.5\} = A_2) = 0.07, \quad m_f^2(\{4.5, 5, 5.5\} = B_2) = 0.21 \\ m_f^1(\{3.5, 4, 4.5, 5\} = A_3) = 0.8, \quad m_f^2(\{4, 4.5, 5\} = B_3) = 0.26 \\ m_f^2(\{5, 5.5\} = B_4) = 0.14$$

With the following matrix summarising the pairs of sets where the relation $\mathbf{R}_{\leq PD}$ holds

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} \\ & \mathbf{R}_{\leq PD} & & \mathbf{R}_{\leq PD} \\ \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} & \mathbf{R}_{\leq PD} \end{bmatrix}.$$

452 This means that any joint mass function where $m(A_2, B_1)$ and $m(A_2, B_3)$ are
 453 null satisfy our definition, and it is easy to see that such a mass function exists,
 454 for example by taking $m(A_2, B_2) = 0.07$.

455 6.3 Equivalence relations in taxonomies

456 Let us now assume that the elements of space X are concepts linked together by
 457 a taxonomy, modelled as a rooted tree. Then, one possible question about two
 458 uncertain observations of such a taxonomy is whether they belong to the same
 459 general sub-concept of interest, or in other words whether they belong to the
 460 same branch of the tree. In practice, this comes down to define a corresponding
 461 partition C_1, \dots, C_K of X , and to say that $\mathbf{A}\mathbf{R}\mathbf{B}$ iff $A \cup B \subseteq C_i$ for some i .

Example 7. Assume we have the space $X = \{(M)otorcycle, (T)ruck, (C)at, (D)og\}$ together with the taxonomy provided by Figure 2. The partition defined by the concepts of the first level (Vehicle and Animal) is $C_1 = \{M, T\}$ and $C_2 = \{C, D\}$. We could then wonder whether two uncertain objects belong to the same category, given this granularity. For instance, consider the three mass functions

$$\begin{aligned} m_1(\{C\}) &= 0.6, & m_2(\{T, C\}) &= 0.2, & m_3(\{D\}) &= 0.4, \\ m_1(\{D, C\}) &= 0.4, & m_2(X) &= 0.8, & m_3(\{D, C\}) &= 0.6. \end{aligned}$$

We do have $m_1\mathbf{R}m_3$, as $\{C\}\mathbf{R}\{D\}\mathbf{R}\{D, C\}$, but not $m_1\mathbf{R}m_2$ nor $m_3\mathbf{R}m_2$, concluding that while the first and third objects belong to the same category, m_2 does not. To see that $m_1\mathbf{R}m_3$, one can consider the joint mass function

$$m_{13}(\{D, C\}, \{D\}) = 0.4, \quad m_{13}(\{C\}, \{D, C\}) = 0.6,$$

462 and to see that $\neg m_1\mathbf{R}m_2$ and $\neg m_3\mathbf{R}m_2$, it is sufficient to observe that $\neg X\mathbf{R}A$
 463 for any A , and that the mass $m_2(X)$ is strictly positive, hence that some joint
 464 mass must be given to it. Since we know that \mathbf{R} is an equivalence relation,
 465 that is preserved when considering belief functions, we could have deduced from
 466 $\neg m_1\mathbf{R}m_2$ that $\neg m_3\mathbf{R}m_2$.

467 Such notions could be used for example in formal concept analysis [22], where
 468 we may want to know the most specific common concept to which two uncertain
 469 objects belong. Another possibility includes for instance hierarchical classifica-
 470 tion [1], in which a usually very large number of classes are structured according
 471 to a taxonomy that is used to find the right class (leaf) of an object. Being able to
 472 tell whether two classifiers agree that a particular instance belong to a common
 473 sub-tree may then be a helpful item of information.

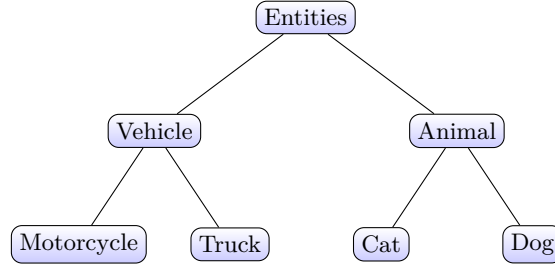


Fig. 2. Taxonomy of Example 7

474 7 From binary to gradual relations

475 So far, we have been concerned with the problem of assessing whether or not
 476 two mass functions m_1, m_2 were in relation, viewing this as a binary value that
 477 could only takes values 0 ($\neg m_1 \mathbf{R} m_2$) or 1 ($m_1 \mathbf{R} m_2$). It can be interesting to
 478 relax this assumption by allowing the relation to be gradual, that is to take any
 479 value between 0 and 1.

An easy way to do that is to follow an optimistic principle and to say that
 for any two mass functions m_1, m_2 , the degree $\alpha_{\mathbf{R}}$ to which m_1 is in relation \mathbf{R}
 with m_2 is the solution of the optimisation problem

$$1 - \alpha_{\mathbf{R}} = \min \sum_{A, B: \neg(ARB)} m_{12}(A, B)$$

$$m_1(A) = \sum_B m_{12}(A, B),$$

$$m_2(B) = \sum_A m_{12}(A, B).$$

480 This generalises the approach taken so far, as we will have $m_1 \mathbf{R} m_2$ iff $\alpha_{\mathbf{R}} = 1$,
 481 that is if the degree to which they are in relation is maximal. Conversely, $\alpha_{\mathbf{R}} = 0$
 482 iff there is no pair of subsets A, B having positive mass such that ARB .

483 For instance, if we consider again the “having a lower cardinality than”
 484 example from the proof of Proposition 8, we would have that $m_1 \alpha_{\mathbf{R}} m_2$ with
 485 $\alpha_{\mathbf{R}} = 1 - 0.2 = 0.8$, a quite high value. Such gradual relations have been pro-
 486 posed in the past, for example the conflict measure κ_m^2 proposed in [11] is nothing
 487 else but the solution of the optimisation problem applied to the relation ARB
 488 iff $A \cap B \neq \emptyset$.

489 Studying the properties and implications of using such gradual relations in
 490 detail goes out of the scope of the current paper, yet a clear first step would be
 491 to relate such a gradual view to the large literature concerning fuzzy relations.
 492 Indeed, fuzzy relations are also $[0, 1]$ -valued, and researchers of this field have
 493 come with various proposals of how classical properties can be extended to this
 494 case, e.g., to deal with fuzzy preferences [20, 14] or fuzzy equivalence relations [2].

495 8 Conclusions

496 In this paper, a universal way to generalise a binary relation from sets to belief
 497 functions is introduced. Several results are provided showing which properties of
 498 the relation are preserved through this mechanism, including its compatibility
 499 with functions. Our proposal is also connected to more specific generalisation of
 500 binary relations such as the notion of specialisation. Consequently, our results
 501 are also a generalisation of pre-existing ones for specific relations.

502 There are however several questions that remain to address. A first one is to
 503 consider not relations on the same space, but more general relations on different
 504 spaces, including compositions of such relations.

505 Finally, we have also proposed a way to transform the initial binary relation
 506 on sets into a gradual relation on belief functions. This also opens up a whole
 507 avenue of research, as this directly connect our proposal to the various notions of
 508 fuzzy relations, that consist in providing a number in the unit interval reflecting
 509 how much a relation holds. Performing such a study goes beyond the actual
 510 scope of the present paper.

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517 A Proof of Proposition 10

518 In this appendix, we give a proof that if \mathbf{R} is a partial order on sets then the
 519 mechanism described in Definition 1 yields a partial order on belief functions. We
 520 already know that the induced relation on belief functions inherits the reflexivity
 521 and transitivity properties (c.f. Propositions 5 and 7). So we only need to prove
 522 that antisymmetry holds.

Suppose there exist two mass functions m_1 and m_2 such that $m_1 \mathbf{R} m_2$ and
 $m_2 \mathbf{R} m_1$. This means that there are two matrices S_1 and S_2 compliant with
 Definition 1 and such that $m_1 = S_1 \cdot m_2$ and $m_2 = S_2 \cdot m_1$ (mass functions are
 seen as vectors here). By plugging these two equations together, we obtain

$$m_2 = S_2 \cdot S_1 \cdot m_2.$$

523 To complete the proof, we will be needing the following intermediate result:

524 **Lemma 1.** *If \mathbf{R} is a partial order on sets, then there is an indexation of sub-*
 525 *sets of X such that for any pair of mass functions $(m_1; m_2)$ with $m_1 \mathbf{R} m_2$, the*
 526 *stochastic matrix S satisfying Definition 1 is upper triangular.*

Proof. Define a refinement \mathbf{R}_* of \mathbf{R} such that \mathbf{R}_* is a total order (subsets that cannot be ordered using \mathbf{R} can be ordered in an arbitrary way). Let N denote the cardinality of 2^X . Let $(A_i)_{i=1}^N$ denote all subsets of X indexed using \mathbf{R}_* so that

$$A_i \mathbf{R}_* A_j \Leftrightarrow i \leq j.$$

527 Now suppose S_* is a stochastic matrix compliant with Definition 1 for relation
528 \mathbf{R}_* for some pair of mass functions m_1 and m_2 . If $m_2(A_j) > 0$, we need to have
529 $S_*(A_i, A_j) = 0$ whenever $\neg(A_i \mathbf{R}_* A_j) \Leftrightarrow i > j$.

Suppose S is a stochastic matrix compliant with Definition 1 for relation \mathbf{R} for the same pair of mass functions m_1 and m_2 . If $m_2(A_j) = 0$, we can reassign entries of the j^{th} column of S as we want while remaining compliant with Definition 1. For instance, we can set

$$S(A_i, A_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Provided that the above reassignment is completed, the matrix S is upper triangular because when $m_2(A_j) > 0$,

$$i > j \Leftrightarrow \neg(A_i \mathbf{R}_* A_j) \Rightarrow \neg(A_i \mathbf{R} A_j) \Rightarrow S(A_i, A_j) = 0.$$

□

530 Based on the above lemma, we can require that S_1 and S_2 are upper triangu-
531 lar and consequently $S_{21} = S_2 \cdot S_1$ as well. Since left stochasticity is preserved by
532 matrix product, we know that S_{21} is also left stochastic. Consequently, we have
533 that S_{21} coincides with the identity matrix I on every column corresponding to
534 a focal element of m_2 . In other words, if $m_2(A_j) > 0$, then

$$S_{21}(A_i, A_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad (17)$$

where $(A_i)_{i=1}^N$ are all subsets of X ordered in the way arising from the definitions of matrices S_1 and S_2 and their upper triangularities. To prove this, observe that

$$m_2(A_i) = \sum_{j=1}^N S_{21}(A_i, A_j) m_2(A_j), \quad (18)$$

$$= \sum_{j=i}^N S_{21}(A_i, A_j) m_2(A_j) \quad (19)$$

Let A_{k_1} denote the focal element of m_2 with maximal index, i.e. A_i is not a focal element of m_2 if $i > k_1$. We necessarily have that

$$m_2(A_{k_1}) = S_{21}(A_{k_1}, A_{k_1}) m_2(A_{k_1}).$$

535 This obviously implies that $S_{21}(A_{k_1}, A_{k_1}) = 1$ and that all other entries of the
 536 column corresponding to A_{k_1} are null because S_{21} is left stochastic. Now, let A_{k_2}
 537 be the focal element of m_2 with the second maximal index, i.e. if $i > k_2$, then
 538 A_i is either A_{k_1} or not a focal element of m_2 . We have now

$$m_2(A_{k_2}) = S_{21}(A_{k_2}, A_{k_2})m_2(A_{k_2}) + S_{21}(A_{k_2}, A_{k_1})m_2(A_{k_1}).$$

539 $S_{21}(A_{k_2}, A_{k_1})$ is in the k_1^{th} column of S_{21} therefore we deduce that $S_{21}(A_{k_2}, A_{k_2}) =$
 540 1 and that all other entries of the corresponding column are null. We can continue
 541 to iterate on the focal elements of m_2 to obtain (17).

Furthermore, the upper triangularities of S_1 and S_2 give that $S_{21}(A_i, A_i) =$
 $S_2(A_i, A_i) \times S_1(A_i, A_i), \forall i$. When $m_2(A_i) > 0$, we know that $S_{21}(A_i, A_i) =$
 1 and we deduce that $S_1(A_i, A_i) > 0$ and $S_2(A_i, A_i) > 0$. From the upper
 triangularity of the matrices, we also have

$$S_{21}(A_{i-1}, A_i) = S_2(A_{i-1}, A_{i-1})S_1(A_{i-1}, A_i) + S_2(A_{i-1}, A_i)S_1(A_i, A_i).$$

542 When $m_2(A_i) > 0$, we know that $S_{21}(A_{i-1}, A_i) = 0$ and we deduce that
 543 $S_1(A_{i-1}, A_i) = 0$. We can iterate and show that $S_1(A_{i-k}, A_i) = 0$ for any
 544 $k \in \{1; \dots; i-1\}$ which in turn implies that $S_1(A_i, A_i) = 1$ because S_1 is
 545 stochastic. Since $m_1 = S_1 \cdot m_2$, we see that, for any focal element A of m_2 , we
 546 have $m_1(A) \geq m_2(A)$. Finally, we necessarily have that $m_1 = m_2$ because the
 547 masses of m_1 need to sum to one.

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