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► **To cite this version:**

Frédéric Paletou. On Milne–Barbier–Unsöld relationships. *Open Astronomy*, 2018, 27 (1), pp.76-79. <10.1515/astro-2018-0018>. <hal-02105946>

HAL Id: hal-02105946

<https://hal.science/hal-02105946v1>

Submitted on 23 Jul 2025

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Commentary

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On Milne–Barbier–Unsöld relationships

<https://doi.org/10.1515/astro-2018-0018>

Received Oct 27, 2017; accepted Dec 23, 2017

Abstract: This short review aims to clarify upon the origins of so-called Eddington–Barbier relationships, which relate the emergent specific intensity and the flux to the photospheric source function at specific optical depths. Here we discuss the assumptions behind the original derivation of Barbier (1943). We also point to the fact that Milne had already formulated these two relations in 1921.

Keywords: Radiative transfer, analytical methods, education, history

1 Introduction

The theory of radiation transfer is fundamental in astrophysics. Besides the in situ exploration of various bodies in the solar system, cosmic rays astrophysics and the recent spectacular advent of gravitational wave detections, remote sensing of radiation remains the primary means through which we advance our knowledge of celestial bodies.

Analytical solutions were successively derived, mostly during the first half of the XXth century, and beyond the advent of numerical computing, into the 1960's. After pioneering contributions of Schuster (1905) and Schwarzschild (1906) (see also Menzel & Milne 1966), important results were further established, in particular for the case of stellar photospheres in radiative equilibrium. They relate, for instance, to the temperature distribution in a so-called “gray atmosphere”, and to the associated limb-darkening law of radiation. One may also mention the $\sqrt{\varepsilon}$ surface value for simplified radiation transfer out of local thermodynamical equilibrium (see e.g., Hubeny 1987; Lambert *et al.* 2016).

Fundamental elements of radiative transfer can be found in the texts of Rutten (2003) and Hubeny and Michals (2014).

2 Eddington–Barbier relationships

The so-called Eddington–Barbier relationships constitute fundamental analytic results, systematically presented in most textbooks and lectures about radiative transfer in astrophysics. In most cases, they are introduced and derived assuming that the source function is just a linear function of the optical depth:

$$S(\tau) = a + b\tau, \quad (1)$$

where a and b are arbitrary coefficients.

It is then easy to derive the emergent specific intensity, from a plane-parallel semi-infinite atmosphere, according to:

$$I(\mu) = \int_0^{\infty} S(\tau) e^{-(\tau/\mu)} (d\tau/\mu). \quad (2)$$

Here μ is the usual cosine of the angle of the ray to the vertical direction. Given equation (1), the specific intensity is merely:

$$I(\mu) = a + b\mu, \quad (3)$$

that is, the source function at optical depth $\tau = \mu$ i.e.,

$$I(\mu) = S(\tau = \mu). \quad (4)$$

In other words, this means also that, for a given line of sight the emergent intensity equals the source function at a depth found after crossing an optical depth unity along the line of sight.

A second relationship can also be derived for the emergent flux defined as:

$$\mathcal{F} = 2\pi \int_0^1 I(\mu) \mu d\mu. \quad (5)$$

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The quantity \mathcal{F} is relevant to spatially unresolved objects, like most stars (besides the Sun), and it is easy to show that the emergent flux is then characterized by the source function at optical depth $\tau = 2/3$.

Many commonly-read textbooks such as Athay (1972); Mihalas (1978), the very popular *e*-book of Rutten (2003), and even the recent Hubeny and Mihalas (2014) omit however to cite any original publication establishing first these two classical relationships.

3 Original derivation

The origin of the derivation of these relationships can be found in an article of French astronomer Daniel Barbier published in 1943, although he did not address explicitly the case of the specific intensity there. The original derivation of Barbier starts with the following Taylor series expansion for the source function:

$$S(\tau) = S(\tau_*) + (\tau - \tau_*)S'(\tau_*) + \frac{1}{2}(\tau - \tau_*)^2 S''(\tau_*) \quad (6)$$

that we truncate here at 2nd order. In this expression S' and S'' are respectively the first and the second derivatives vs. τ of the source function. This expansion is introduced into Eq. (2), and straightforward integrations give the following expression for the emergent specific intensity:

$$I(\mu) = S(\tau_*) + (\mu - \tau_*)S'(\tau_*) + (\mu^2 - \mu\tau_* + \frac{1}{2}\tau_*^2)S''(\tau_*). \quad (7)$$

In his original article of 1943, D. Barbier does not give an expression for the emergent specific intensity, but does give the emergent “total flux”, where intensity is integrated over μ (equation 5). However, we can adopt his argument, and choose τ_* which makes the term in S' vanish, and which minimizes that in S'' . It is therefore obvious that:

$$\tau_* = \mu. \quad (8)$$

In such a case, the emergent specific intensity is:

$$I(\mu) = S(\tau = \mu) + \frac{1}{2}\mu^2 S''(\tau = \mu), \quad (9)$$

which is indeed identical to $S(\tau = \mu)$, if one assumes that the source function is no more than linear in the optical depth, so that $S''(\tau) = 0$.

4 Discussion

Barbier (1943) cites Eddington quite precisely, pointing at p. 330 of his famous textbook *The internal constitution of*

stars (1926). In this chapter, *The Outside of a Star*, Eddington states that the effective temperature of the angle-dependent atmospheric radiation should be the temperature of the layer where $\tau \approx \mu$. This may have inspired Barbier to adopt a Taylor series expansion method.

Soon after Barbier’s contribution, Unsöld (1948; 1949, in English) makes explicit these classical relationships, both for the specific intensity and for the emergent flux. Unsöld (1955) gives direct credit to Barbier in his famous textbook, *Physik der Sternatmosphären*. However, he wrongly cites Barbier (1944), instead of Barbier (1943)! About Barbier’s method based on a Taylor series expansion of the source function S around a certain optical depth τ_* which has to be determined, he writes, originally in German, that: “*a method of approximation, proposed by A.S. Eddington and better argued by D. Barbier is still very useful and interesting*”. Unsöld coins it also the “*x = cos ϑ -Methode*” of Eddington and Barbier, where x is used for optical thickness τ , and $\mu = \cos \vartheta$.

Kourganoff’s (1952) textbook gives a proper citation and description of Barbier’s original contribution in his §18.2: “*After giving Barbier’s demonstration, which is known as the τ_* -method*”, which is however immediately followed by: “*we shall explain why it seems to us to be unsatisfactory*”... Modern texts reflect Kourganoff’s clear statement that: “*Now Barbier’s demonstration (or a direct calculation) shows that all of the Eddington–Barbier relations are rigorously true if the source function is a linear function of τ* ”. And despite a critical discussion in the remaining of §18, Kourganoff concludes with: “*The Eddington–Barbier relations, apart from the applications which have already been made by Barbier himself and by Unsöld, are extremely useful when one wants to represent the connexion between the source function and certain observable quantities like $I(0, \mu)$ and $\mathcal{F}(0)$* ”.

With time, citations to Barbier (1943) vanish, although they appear in textbooks such as those of Zirin (§6.10., 1988) and Castor (§5.7., 2004). This is, surprisingly, not the case for the famous and comprehensive Mihalas (1978) textbook. However, in its §2-2 an exercise (2-5) is directly inspired by the method used by Barbier (1943). It may be unfortunate though, that the revised and expanded textbook by Hubeny and Mihalas (2014) does not elaborate on the original derivation of one of the most famous result of analytical radiation transfer.

5 A lost contribution of Milne?

During the course of our investigations on the original contributions leading to the so-called Eddington–Barbier relations, we also went back to an article of V.V. Ivanov (1991) in the proceedings of the Trieste conference *Stellar Atmospheres: beyond classical models*.

At the end of section *History of ART*, where ART stands for “Analytical Radiative Transfer”, Ivanov writes this somewhat intriguing statement: “*The standard Eddington approximate form of the temperature distribution in a grey atmosphere,*

$$T^4 = (3/4)T_{\text{eff}}^4(\tau + 2/3), \quad (10)$$

belongs not to Eddington, but to Milne. In 1917, he introduced the approximation known today as the Eddington approximation.”

However, it seems that Milne did not publish any astrophysical result before 1921, according to Tayler (1996) for instance. This led us to look back with some care to the early contributions of Milne in the domain of radiative transfer and stellar atmospheres. And his first article of 1921, *Radiative equilibrium in the outer layers of a star: the Temperature Distribution and the Law of Darkening* contains in Eqs. (36) and (37) both Eddington–Barbier relations, for the specific intensity and for the flux, as shown in Figure 1.

In this article, the derivation is formally distinct from the one used by Barbier (1943). Milne is inspired by the method already used by Schwarzschild and Jeans. He uses, in particular, simplifications for lower and upper boundaries incident radiation which are assumed to be distinct

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For completeness we will now obtain briefly the corresponding first approximations. In this case the temperature distribution is taken to be

$$B(\tau) = a + 2b\tau \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and the corresponding escaping radiation is given by

$$I(\circ, \theta) = a + 2b \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The corresponding flux at the boundary is

$$2\pi \int_0^{\frac{1}{2}\pi} I(\circ, \theta) \sin \theta \cos \theta d\theta = \pi(a + \frac{2}{3}b) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which should be equal to πF or $\frac{8}{3}\pi b$; and by the condition of radiative equilibrium at the boundary the expression

Figure 1. In this excerpt of Milne’s (1921) article, $B(\tau)$ should also read as the source function and Eqs. (36) and (37) are just the Eddington–Barbier relationships respectively on the specific intensity and on the flux, published more than twenty years before Barbier (1943).

but independent of direction. Then he gives an integral form of the “temperature distribution” for a semi-infinite atmosphere, with no incoming radiation on its surface. First, he shows that a linear source function is a possible solution at large optical depths, and even at this stage the first part of his Eq. (24) already contains the Eddington–Barbier relationship for the specific intensity, when (inappropriately though) setting... $\tau = 0$. He goes however further and implements a method of successive approximations, which leads to a second approximation for the source function of the form:

$$S(\tau) = a + 2b\tau + \frac{1}{2}e^{-\tau}(b - a - b\tau) + \frac{1}{2}(a\tau + b\tau^2) \int_{\tau}^{\infty} \frac{e^{-y}}{y} dy. \quad (11)$$

This expression has indeed the same form as the first approximation, linear in τ , for large optical depths but now, the first exponential integral $E_1(\tau)$ appears, as in the Schwarzschild–Milne equations (see e.g., Hubeny and Mihalas 2014). It is also more accurate in the sense that it exhibits the often forgotten singularity in $dS/d\tau$ at $\tau = 0$ (see e.g., Chevallier *et al.* 2003 and references therein).

Both the so-called Eddington–Barbier relationships appear once Milne, “for completeness”, goes back to his first approximation, when the source function is just linear in the optical depth. At this stage, Milne does not comment any further on his Eqs. (36) and (37). This may also explain why these intermediate results of Milne were neither used, nor cited further. Perhaps they may also have been eclipsed by other aspects of Milne’s important contributions in the early 1920’s?

For instance, following the remark of Ivanov (1991), in his first 1921 article Milne establishes also a limb-darkening law such that:

$$\frac{I(0, \mu)}{I(0, 1)} = \frac{3}{5}(\mu + 2/3), \quad (12)$$

improving on Schwarzschild and Jeans. It is also fully consistent with the assumption known as the “Eddington approximation”, which leads to the (constant) $2/3$ appearing in Eq. (10).

6 Conclusion

One may argue that Unsöld (1948; 1949, in English) makes the classical relations explicit, for the first time, both for the specific intensity and for the emergent flux, following Barbier (1943). However they were both expressed in Milne

(1921), but cited neither by Barbier, nor by Unsöld (or anyone else, to the best of our knowledge) more than twenty years after.

After our investigations, we would therefore propose to the astrophysical community to shift, at last, from the “Eddington–Barbier” usual designation to the fairer “Milne–Barbier–Unsöld” relationships.

Finally, we also report that despite its legacy, Barbier (1943) has been cited only four times so far, according to the ADS service! This is also the case for Unsöld (1948; and it is even worse for his 1949 article, although written in English)...

Acknowledgment: This review was initiated after a question of one of *Université de Toulouse* astrophysics master student, Mrs Alexandra Le Reste, during a lecture of Sept. 2017. We also wish to thank Prof. R.J. “Rob” Rutten for fruitful and enjoyable discussions, as well as Drs Céline Reylé (*Observatoire de Besançon*, France) and Torsten Böhm (CNRS, IRAP, Toulouse, France) for their kind assistance. This research has made use of NASA’s Astrophysics Data System, a fantastic tool to search and read again historical astrophysical literature.

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