



THE DISCRETE LAPLACIAN OF A 3-SIMPLICIAL COMPLEX

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THE DISCRETE LAPLACIAN OF A 3-SIMPLICIAL COMPLEX

BAALAL AZEDDINE AND HATIM KHALID

ABSTRACT. In this paper, We introduce the notion of oriented tetrahedrons especially tetrahedrons in a connected oriented locally finite graph. This framework then permits to define the Laplace operator on this structure of the 3-simplicial complex. We develop the notion of χ -completeness for the graphs, based on the cut-off functions. Moreover, We study essential self-adjointness of the discrete Laplacian from the χ -completeness geometric hypothesis.

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1. INTRODUCTION

The impact of the geometry on the essential self-adjointness of the Laplacians is studied in many areas of mathematics on Riemannian manifolds; see ([C], [EL],[M1]) and also on one-dimensional simplicial complexes; see ([AT],[CTT],[FLW],[M2],[T]). Indeed, Laplacians on Riemannian manifolds and simplicial complexes share a lot of common elements. Despite of this, various geometric notions such as distance and completeness in the Riemannian framework have no immediate analogue in the discrete setting. Combinatorial Laplacians were originally studied on graphs, beginning with Kirchhoff and his study of electrical networks [K]. Simplicial complexes can be viewed as generalizations of graphs, we can form complex, a 3-simplicial complex whose faces and tetrahedrons correspond to the cliques of the graph. In this article, we take a connected oriented locally finite graph and we introduce the oriented tetrahedrons, so we can regard it as a three-dimensional simplicial complex. This work presents a more general framework for the Laplacians defined in terms of the combinatorial structure of a simplicial complex. The main result of this work gives a geometric hypothesis to ensure essential self-adjointness for the discrete Laplacian.

Key words and phrases. infinite graph, 3-simplicial complex, difference operator, Laplacian on forms, essential self-adjointness.

We develop the χ -completeness hypothesis for 3-simplicial complex. This hypothesis on locally finite graphs covers many situations that have been already studied in [AT]. The authors prove that the χ -completeness is satisfied by graphs which are complete for some intrinsic metric, as defined in [FLW] and [HKMW]. The paper is structured as follows: In the second section, we will first present the basic concepts about graphs or rather one-dimensional simplicial complexes and about structure of 2-simplicial complex that called triangulation. Next, we introduce the notion of oriented tetrahedrons. This special structure of 3-simplicial complex. We permit to define the Gauss-Bonnet operator $T = d + \delta$ acting on triplets of functions, 1-forms, 2-forms and 3-forms. After that, we define the discrete Laplacian by $L = T^2$ which admits a decomposition according to the degree $L = L_0 \oplus L_1 \oplus L_2 \oplus L_3$. In the third, the fourth and fifth sections, we study the closability of the operators which are used in the following sections. Next, we get started with refer to [AT] for the notion χ -completeness of the graphs and we develop this geometric hypothesis for the 3-simplicial complex in Definition. In the case of complete manifolds, there is a result of Chernoff ; see [C], and we also have for the discrete setting; see [AT], which conclude that the Dirac operator is essentially self-adjoint. As a result, they prove essential self-adjointness of the Laplace-Beltrami operator. So, we take this idea to make the relationship between T and L about the essential self-adjointness, when the 3-simplicial complex is χ -complete. We can extend the results in this paper to more general 3-simplicial complex, where the oriented 3-simplex are not necessarily tetrahedrons. Particularly we can give a more general expression of the operator d^2 .

2. PRELIMINARIES

2.1. The basic concepts. A graph K is a pair (V, E) , where V is the countable set of vertices and E the set of oriented edges, considered as a subset of $V \times V$. When two vertices x and y are connected by an edge e , we say they are neighbors. We denote $x \sim y$ and $e = (x, y) \in E$. We assume that E is symmetric, ie. $(x, y) \in E \Rightarrow (y, x) \in E$. An oriented graph K is given by a partition of E : $E = E^- \cup E^+$ such that $(x, y) \in E^- \Leftrightarrow (y, x) \in E^+$. In this case for $e = (x, y) \in E^-$, we define the origin $e^- = x$, the termination $e^+ = y$ and the opposite edge $-e = (y, x) \in E^+$. Let $c : V \rightarrow (0, \infty)$ the weight on the vertices. We also have $r : E \rightarrow (0, \infty)$ the weight on the oriented edges with $\forall e \in E$, $r(-e) = r(e)$. A path between two vertices $x, y \in V$ is a finite set of oriented edges e^1, \dots, e^n , $n \geq 1$ such that $e_1^- = x$, $e_n^+ = y$ and, if $n \geq 2$, $\forall j, 1 \leq j \leq n-1 \Rightarrow e_j^+ = e_{j+1}^-$. The path is called a cycle or closed when the origin and the end are identical, ie. $e_1^- = e_n^+$, with $n \geq 3$. If no cycles appear more than once in a path, the path is called a simple path. The graph K is connected if any two vertices x and y can be connected by a path with $e_1^- = x$ and $e_n^+ = y$. We say that the graph K is locally finite if each vertex belongs to a finite number of edges. The graph K is without loops if there is not the type of edges (x, x) , ie. $\forall x \in V \Rightarrow (x, x) \notin E$. The set of neighbors of $x \in V$ is denoted by $V(x) = \{y \in V : y \sim x\}$. The degree of $x \in V$ is by definition $\deg(x)$, the number of neighbors of x . The combinatorial distance d_{comb} on K is

$$d_{comb}(x, y) = \min \{n, \{e_i\}_{1 \leq i \leq n} \subseteq E \text{ a path between the two vertices } x \text{ and } y\}$$

Let B be a finite subset of V . We define the edge boundary $\partial_E B$ of B by $\partial_E B := \{e \in E \text{ such that } \{e^-, e^+\} \cap B \neq \emptyset \text{ and } \{e^-, e^+\} \cap B^c \neq \emptyset\}$. In the sequel, we

assume that K is **without loops, connected, locally finite and oriented**. An *oriented triangular face* of K is a surface limited by a simple closed path of length equals 3, considered as an element of E^3 , i. e. ϖ is an oriented triangular face $\Rightarrow \varpi = (e_1, e_2, e_3) \in E^3$ such that $\{e_i\}_{1 \leq i \leq 3} \subseteq E$ is a simple closed path. Let F be the set of all oriented faces of K . In the sequel we will represent the oriented faces by their vertices. For a face $\varpi = (x, y, z) \in F$. Let us set $\varpi = (x, y, z) = (y, z, x) = (z, x, y) \in F \Rightarrow -\varpi = (y, x, z) = (x, z, y) = (z, y, x) \in F$. A *triangulation* T is a 2-simplicial complex such that all the faces are triangular. To define weighted triangulations we need weights, let us give $s : F \rightarrow (0, \infty)$ the weight on oriented faces such that for all $\varpi \in F$, $s(-\varpi) = s(\varpi)$. The weighted triangulation (T, c, r, s) is given by the triangulation $T = (V, E, F)$. We say that T is simple if the weights of the vertices, the edges and faces equals 1. For an edge $e \in E$, we also denote the oriented face (e^-, e^+, x) by (e, x) , with $x \in V(e^-) \cap V(e^+)$. The set of vertices belonging to the edge $e \in E$ is given by

$$F_e = \{x \in V, (e, x) \in F\} = V(e^-) \cap V(e^+)$$

An *oriented tetrahedron* of K is a volume limited by four oriented triangular faces of F , considered as an element of V^4 , i. e. $\bar{\Omega}$ is an oriented tetrahedron $\Rightarrow \bar{\Omega} = (x, y, z, v) \in V^4$. Let Δ_3 be the set of all oriented tetrahedron of K . We consider the pair (T, Δ_3) as a 3-simplicial complex, we denote it by Δ . We can denote also $\Delta = (V, E, F, \Delta_3)$. *Odd permutation* means we exchange the position of any two vertices an odd number of times. *Even permutation* means we exchange the position of any two vertices an even number of times. For a $(\bar{\Omega}, \bar{\Pi}) \in \Delta_3^2$ we have : $\bar{\Omega} = \bar{\Pi} \iff \bar{\Pi}$ is obtained from $\bar{\Omega}$ by an even permutation. $\bar{\Omega} = -\bar{\Pi} \iff \bar{\Pi}$ is obtained from $\bar{\Omega}$ by an odd permutation. To define weighted 3-simplicial complex we need weights, let us give $t : \Delta_3 \rightarrow (0, \infty)$ the weight on oriented tetrahedrons such that for all $\Omega \in \Delta_3$, $t(-\bar{\Omega}) = t(\bar{\Omega})$. The weighted 3-simplicial complex (V, c, r, s, t) is giving by the 3-simplicial complex $\Delta = (V, E, F, \Delta_3)$. We say that Δ is simple if the weights of the vertices, the edges, the triangular faces and tetrahedrons equals 1. The set of vertices belonging to the tetrahedron where $(x, y, z) \in F$ is an oriented triangular face is giving by:

$$\Delta_3(x, y, z) = \{v \in V, (x, y, z, v) \in \Delta_3\} = V(x) \cap V(y) \cap V(z).$$

2.2. Functions spaces. We denote the set of 0-cochains or functions on V by:

$$C(V) = \{f : V \rightarrow \mathbb{C}\}$$

and the set of functions of finite support by $C_c(V)$.

Similarly, we denote the set of 1-cochains or 1-forms on E by:

$$C(E) = \{\varphi : E \rightarrow \mathbb{C}, \varphi(-e) = -\varphi(e)\}$$

and the set of 1-forms of finite support by $C_c(E)$.

Moreover, we denote the set of 2-cochains or 2-forms on F by:

$$C(F) = \{\phi : F \rightarrow \mathbb{C}, \phi(-\varpi) = -\phi(\varpi)\}$$

and the set of 2-forms of finite support by $C_c(F)$.

Further, we denote the set of 3-cochains or 3-forms on Δ_3 by:

$$C(\Delta_3) = \{\rho : \Delta_3 \rightarrow \mathbb{C}, \rho(-\bar{\Omega}) = -\rho(\bar{\Omega})\}$$

and the set of 3-forms of finite support by $C_c(\Delta_3)$.

Let us define the Hilbert spaces $l^2(V)$, $l^2(E)$, $l^2(F)$ and $C(\Delta_3)$ as the sets of cochains with finite norm, we have

$$l^2(V) = \left\{ f \in C(V); \sum_{x \in V} c(x) |f(x)|^2 < \infty \right\},$$

with the inner product

$$\langle f, g \rangle_{l^2(V)} = \sum_{x \in V} c(x) f(x) \bar{g}(x).$$

$$l^2(E) = \{ \varphi \in C(E); \sum_{e \in E} r(e) |\varphi(e)|^2 < \infty \},$$

with the inner product

$$\langle \varphi, \psi \rangle_{l^2(E)} = \frac{1}{2} \sum_{e \in E} r(e) \varphi(e) \bar{\psi}(e).$$

$$l^2(F) = \left\{ \phi \in C(F); \sum_{\varpi \in F} s(\varpi) |\phi(\varpi)|^2 < \infty \right\},$$

with the inner product

$$\langle \phi_1, \phi_2 \rangle_{l^2(F)} = \frac{1}{6} \sum_{(x,y,z) \in F} s(x,y,z) \phi_1(x,y,z) \phi_2(x,y,z).$$

$$l^2(\Delta_3) = \left\{ \rho \in C(\Delta_3); \sum_{\bar{\Omega} \in \Delta_3} c(\bar{\Omega}) |\rho(\bar{\Omega})|^2 < \infty \right\},$$

with the inner product

$$\langle \rho_1, \rho_2 \rangle_{l^2(\Delta_3)} = \frac{1}{24} \sum_{(x,y,z,v) \in \Delta_3} t(x,y,z,v) \rho_1(x,y,z,v) \bar{\rho}_2(x,y,z,v)$$

The direct sum of the spaces $l^2(V)$, $l^2(E)$, $l^2(F)$ and $l^2(\Delta_3)$ can be considered as a new Hilbert space denoted by H , that is $H = l^2(V) \oplus l^2(E) \oplus l^2(F) \oplus l^2(\Delta_3)$, with the norm $\forall F = (f, \phi, \varphi, \psi) \in H$,

$$\|F\|_H^2 = \|f\|_{l^2(V)}^2 + \|\phi\|_{l^2(E)}^2 + \|\varphi\|_{l^2(F)}^2 + \|\psi\|_{l^2(\Delta_3)}^2.$$

2.3. Operators. We give in this part the expressions of the operators introduced on graphs and on triangulations which are already well known and we also give other operators acting on 3-simplicial complex.

The difference operator.

$$d^0 : C_c(V) \longrightarrow C_c(E)$$

by

$$\forall f \in C_c(V), d^0(f)(e) = f(e^+) - f(e^-).$$

The co-boundary operator. It is the formal adjoint of d^0 , denoted δ^0

$$\delta^0 : C_c(E) \longrightarrow C_c(V)$$

acts as

$$\forall \varphi \in C_c(E), \delta^0(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+=x} r(e)\varphi(e).$$

The exterior derivative. It is the operator d^1 .

$$\begin{aligned} d^1 : C_c(E) &\longrightarrow C_c(F), \\ \forall \psi \in C_c(E), d^1(\psi)(x, y, z) &= \psi(x, y) + \psi(y, z) + \psi(z, x). \end{aligned}$$

The co-exterior derivative. It is the formal adjoint of d^1 , denoted δ^1 .

$$\delta^1 : C_c(F) \longrightarrow C_c(E),$$

which satisfies

$$\langle d^1\psi, \phi \rangle_{l^2(F)} = \langle \psi, \delta^1\phi \rangle_{l^2(E)}, \forall (\psi, \phi) \in C_c(E) \times C_c(F).$$

Operators acting on a 3-simplicial complex.

The operator d^2 .

$$d^2 : C_c(F) \longrightarrow C_c(\Delta_3),$$

given by $\forall \varphi \in C_c(F)$

$$d^2(\varphi)(x, y, z, v) = \varphi(y, z, v) + \varphi(v, z, x) + \varphi(x, y, v) + \varphi(z, y, x).$$

The formal adjoint of d^2 , denoted δ^2 .

$$\delta^2 : C_c(\Delta_3) \longrightarrow C_c(F)$$

wich satisties

$$\langle d^2\varphi, \psi \rangle_{l^2(\Delta_3)} = \langle \varphi, \delta^2\psi \rangle_{l^2(F)}, \forall (\varphi, \psi) \in C_c(F) \times C_c(\Delta_3).$$

Lemma 1. *The formal adjoint*

$$\delta^2 : C_c(\Delta_3) \longrightarrow C_c(F)$$

is given by

$$\forall \psi \in C_c(\Delta_3), \delta^2(\psi)(y, z, v) = \frac{1}{s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \psi(x, y, z, v).$$

Proof. Let $(\varphi, \psi) \in C_c(F) \times C_c(\Delta_3)$

We have

$$\langle d^2\varphi, \psi \rangle_{l^2(\Delta_3)} = \langle \varphi, \delta^2\psi \rangle_{l^2(F)}$$

And

$$\begin{aligned} \langle d^2\varphi, \psi \rangle_{l^2(\Delta_3)} &= \frac{1}{24} \sum_{(x, y, z, v) \in \Delta_3} t(x, y, z, v) d^2\varphi(x, y, z, v) \bar{\psi}(x, y, z, v) \\ &= \frac{1}{24} \sum_{(x, y, z, v) \in \Delta_3} t(x, y, z, v) [\varphi(y, z, v) + \varphi(v, z, x) + \varphi(x, y, v) + \varphi(z, y, x)] \bar{\psi}(x, y, z, v) \end{aligned}$$

The expression of d^2 contributing to the first sum is divided into four similar parts. So we obtain

$$\langle d^2\varphi, \psi \rangle_{l^2(\Delta_3)} = \frac{1}{6} \sum_{(x, y, z, v) \in \Delta_3} t(x, y, z, v) \varphi(y, z, v) \bar{\psi}(x, y, z, v)$$

$$= \frac{1}{6} \sum_{(y,z,v) \in F} \left[\varphi(y, z, v) \sum_{x \in \Delta_3(y,z,v)} t(x, y, z, v) \bar{\psi}(x, y, z, v) \right]$$

We have

$$\langle \varphi, \delta^2 \psi \rangle_{l^2(F)} = \frac{1}{6} \sum_{(y,z,v) \in F} s(y, z, v) \varphi(y, z, v) \overline{\delta^2 \psi}(y, z, v)$$

Then

$$\delta^2(\psi)(y, z, v) = \frac{1}{s(y, z, v)} \sum_{x \in \Delta_3(y,z,v)} t(x, y, z, v) \psi(x, y, z, v).$$

□

2.3.1. *Gauss-Bonnet operator on Δ .* By analogy to Riemannian geometry, we use

the decomposition of the operators in [EL] to define the Gauss-Bonnet operator. Let us begin by defining the operator

$$d : C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3) \hookrightarrow$$

by

$$\forall (f, \phi, \varphi, \psi) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3),$$

$$d(f, \phi, \varphi, \psi) = (0, d^0 f, d^1 \phi, d^2 \varphi),$$

and δ the formal adjoint of d . Thus it satisfies

$$\langle d(f_1, \phi_1, \varphi_1, \psi_1), (f_2, \phi_2, \varphi_2, \psi_2) \rangle_H = \langle (f_1, \phi_1, \varphi_1, \psi_1), \delta(f_2, \phi_2, \varphi_2, \psi_2) \rangle_H,$$

for all $(f_1, \phi_1, \varphi_1, \psi_1), (f_2, \phi_2, \varphi_2, \psi_2) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.

Lemma 2. *Let $\Delta = (K, F, \Delta_3)$ be a 3-simplicial complex. Then*

$$\delta : C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3) \hookrightarrow$$

is given by

$$\delta(f, \phi, \varphi, \psi) = (\delta^0 \phi, \delta^1 \varphi, \delta^2 \psi, 0),$$

$$\forall (f, \phi, \varphi, \psi) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3).$$

Proof. Let $(f, \phi, \varphi, \psi), (f_1, \phi_1, \varphi_1, \psi_1) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.

We have

$$\begin{aligned} \langle d(f_1, \phi_1, \varphi_1, \psi_1), (f, \phi, \varphi, \psi) \rangle_H &= \langle (f_1, \phi_1, \varphi_1, \psi_1), \delta(f, \phi, \varphi, \psi) \rangle_H \\ \text{And } \langle d(f_1, \phi_1, \varphi_1, \psi_1), (f, \phi, \varphi, \psi) \rangle_H &= \langle (0, d^0 f_1, d^1 \phi_1, d^2 \varphi_1), (f, \phi, \varphi, \psi) \rangle_H \\ &= \langle 0, f \rangle_{l^2(V)} + \langle d^0 f_1, \phi \rangle_{l^2(E)} + \langle d^1 \phi_1, \varphi \rangle_{l^2(F)} + \langle d^2 \varphi_1, \psi \rangle_{l^2(\Delta_3)} \\ &= \langle f_1, \delta^0 \phi \rangle_{l^2(V)} + \langle \phi_1, \delta^1 \varphi \rangle_{l^2(E)} + \langle \varphi_1, \delta^2 \psi \rangle_{l^2(F)} + \langle \psi_1, 0 \rangle_{l^2(\Delta_3)} \\ &= \langle (f_1, \phi_1, \varphi_1, \psi_1), (\delta^0 \phi, \delta^1 \varphi, \delta^2 \psi, 0) \rangle_H \end{aligned}$$

Hence

$$\langle (f_1, \phi_1, \varphi_1, \psi_1), (\delta^0 \phi, \delta^1 \varphi, \delta^2 \psi, 0) \rangle_H = \langle (f_1, \phi_1, \varphi_1, \psi_1), \delta(f, \phi, \varphi, \psi) \rangle_H$$

Then

$$\delta(f, \phi, \varphi, \psi) = (\delta^0 \phi, \delta^1 \varphi, \delta^2 \psi, 0).$$

□

Definition 1. Let $\Delta = (K, F, \Delta_3)$ be a 3-simplicial complex, the Gauss-Bonnet operator defined as

$$T = d + \delta : C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3) \rightarrow$$

given by

$$T(f, \phi, \varphi, \psi) = (\delta^0 \phi, d^0 f + \delta^1 \varphi, d^1 \phi + \delta^2 \psi, d^2 \varphi).$$

for all $(f, \phi, \varphi, \psi) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$. Moreover, the matrix representation of T is given by

$$T = \begin{pmatrix} 0 & \delta^0 & 0 & 0 \\ d^0 & 0 & \delta^1 & 0 \\ 0 & d^1 & 0 & \delta^2 \\ 0 & 0 & d^2 & 0 \end{pmatrix}$$

Lemma 3. If $\Delta = (K, F, \Delta_3)$ is a 3-simplicial complex then $d^2 d^1 = \delta^1 \delta^2 = 0$.

Proof. Let $f \in C_c(V)$, we have that

$$\begin{aligned} d^2(d^1 f)(x, y, z, v) &= d^1 f(y, z, v) + d^1 f(v, z, x) + d^1 f(x, y, v) + d^1 f(z, y, x) \\ &= f(y, z) + f(z, v) + f(v, y) + f(v, z) + f(z, x) + f(x, v) + f(x, y) + f(y, v) \\ &\quad + f(v, x) + f(z, y) + f(y, x) + f(x, z). \end{aligned}$$

Then

$$d^2(d^1 f)(x, y, z, v) = 0.$$

Since $d^2 d^1 = 0$ and the operator $\delta^1 \delta^2$ is the formal adjoint of $d^2 d^1$.

Then

$$\delta^1 \delta^2 = 0.$$

□

Before giving an important result for $f \in C(V)$, we define the three operators

$$\sim : C(V) \longrightarrow C(E)$$

by $f \longrightarrow \tilde{f}$

$$\approx : C(V) \longrightarrow C(F)$$

by $f \longrightarrow \tilde{\tilde{f}}$ and

$$\cong : C(V) \longrightarrow C(\Delta_3)$$

by $f \longrightarrow \tilde{\tilde{\tilde{f}}}$ where

$$\begin{aligned} \tilde{f}(e) &= \frac{1}{2} (f(e^+) + f(e^-)). \\ \tilde{\tilde{f}}(x, y, z) &= \frac{1}{3} (\tilde{f}(x, y) + \tilde{f}(y, z) + \tilde{f}(z, x)) = \frac{1}{3} d^1 \tilde{f}(x, y, z). \\ \tilde{\tilde{\tilde{f}}}(x, y, z, v) &= \frac{1}{4} (\tilde{\tilde{f}}(y, z, v) + \tilde{\tilde{f}}(v, z, x) + \tilde{\tilde{f}}(x, y, v) + \tilde{\tilde{f}}(z, y, x)) \\ &= \frac{1}{4} d^2 \tilde{\tilde{f}}(x, y, z, v). \end{aligned}$$

The exterior product of two 2-forms defined as

$$\wedge_d : C(F) \times C(F) \longrightarrow C(\Delta_3),$$

is giving by

$$\begin{aligned}
(\Psi \wedge_d \varphi)(x, y, z, v) &= \varphi(y, z, v)(\Psi(x, y, z) + \Psi(x, z, v) - \Psi(x, y, v)) \\
&\quad + \varphi(v, z, x)(\Psi(y, v, z) + \Psi(y, z, x) - \Psi(y, v, x)) \\
&\quad + \varphi(x, y, v)(\Psi(z, x, y) + \Psi(z, y, v) - \Psi(z, x, v)) \\
&\quad + \varphi(z, y, x)(\Psi(v, z, y) + \Psi(v, y, x) - \Psi(v, z, x)).
\end{aligned}$$

Lemma 4. (*Derivation property*) Let $(f, \varphi) \in C(V) \times C(F)$

$$d^2 \left(\tilde{f} \varphi \right) (x, y, z, v) = 4 \tilde{f} (x, y, z, v) d^2 \varphi (x, y, z, v) + \frac{1}{3} \left(d^1 \tilde{f} \wedge_d \varphi \right) (x, y, z, v).$$

Proof. Let $(f, \varphi) \in C(V) \times C(F)$

We have

$$\begin{aligned}
d^2(\tilde{f}\varphi)(x, y, z, v) &= \tilde{f}\varphi(y, z, v) + \tilde{f}\varphi(v, z, x) + \tilde{f}\varphi(x, y, v) + \tilde{f}\varphi(z, y, x) \\
&= \tilde{f}(y, z, v)\varphi(y, z, v) + \tilde{f}(v, z, x)\varphi(v, z, x) + \tilde{f}(x, y, v)\varphi(x, y, v) + \tilde{f}(z, y, x)\varphi(z, y, x)
\end{aligned}$$

$$\text{Since } \tilde{f}(x, y, z, v) = \frac{1}{4} \left(\tilde{f}(y, z, v) + \tilde{f}(v, z, x) + \tilde{f}(x, y, v) + \tilde{f}(z, y, x) \right)$$

$$\text{Then } \tilde{f}(y, z, v) = 4\tilde{f}(x, y, z, v) - \tilde{f}(v, z, x) - \tilde{f}(x, y, v) - \tilde{f}(z, y, x)$$

$$\tilde{f}(v, z, x) = 4\tilde{f}(x, y, z, v) - \tilde{f}(y, z, v) - \tilde{f}(x, y, v) - \tilde{f}(z, y, x)$$

$$\tilde{f}(x, y, v) = 4\tilde{f}(x, y, z, v) - \tilde{f}(y, z, v) - \tilde{f}(v, z, x) - \tilde{f}(z, y, x)$$

$$\tilde{f}(z, y, x) = 4\tilde{f}(x, y, z, v) - \tilde{f}(y, z, v) - \tilde{f}(v, z, x) - \tilde{f}(x, y, v)$$

$$\begin{aligned}
\text{Thus } d^2(\tilde{f}\varphi)(x, y, z, v) &= 4\tilde{f}(x, y, z, v) [\varphi(y, z, v) + \varphi(v, z, x) + \varphi(x, y, v) + \varphi(z, y, x)] \\
&\quad + \varphi(y, z, v) \left[\tilde{f}(x, y, z) + \tilde{f}(x, z, v) - \tilde{f}(x, y, v) \right] + \varphi(v, z, x) \left[\tilde{f}(y, v, z) + \tilde{f}(y, z, x) - \tilde{f}(y, v, x) \right] \\
&\quad + \varphi(x, y, v) \left[\tilde{f}(z, x, y) + \tilde{f}(z, y, v) - \tilde{f}(z, x, v) \right] + \varphi(z, y, x) \left[\tilde{f}(v, z, y) + \tilde{f}(v, y, x) - \tilde{f}(v, z, x) \right] \\
&= 4\tilde{f}(x, y, z, v) d^2 \varphi (x, y, z, v) + \frac{1}{3} \left[\varphi(y, z, v) \left[d^1 \tilde{f}(x, y, z) + d^1 \tilde{f}(x, z, v) - d^1 \tilde{f}(x, y, v) \right] \right. \\
&\quad + \varphi(v, z, x) \left[d^1 \tilde{f}(y, v, z) + d^1 \tilde{f}(y, z, x) - d^1 \tilde{f}(y, v, x) \right] + \varphi(x, y, v) \left[d^1 \tilde{f}(z, x, y) + d^1 \tilde{f}(z, y, v) \right. \\
&\quad \left. - d^1 \tilde{f}(z, x, v) \right] + \varphi(z, y, x) \left[d^1 \tilde{f}(v, z, y) + d^1 \tilde{f}(v, y, x) - d^1 \tilde{f}(v, z, x) \right] \left. \right]
\end{aligned}$$

Then

$$d^2 \left(\tilde{f} \varphi \right) (x, y, z, v) = 4 \tilde{f} (x, y, z, v) d^2 \varphi (x, y, z, v) + \frac{1}{3} \left(d^1 \tilde{f} \wedge_d \varphi \right) (x, y, z, v).$$

□

Remark 1. Let $(f, \varphi) \in C(V) \times C(F)$

We have

$$d^2 \left(\tilde{f} \varphi \right) (x, y, z, v) = 4 \tilde{f} (x, y, z, v) d^2 \varphi (x, y, z, v) + \frac{1}{3} \left(d^1 \tilde{f} \wedge_d \varphi \right) (x, y, z, v).$$

$$\text{Since } \tilde{f}(x, y, z) = \frac{1}{3} d^1 \tilde{f}(x, y, z) \text{ then } d^1 \tilde{f}(x, y, z) = 3 \tilde{f}(x, y, z).$$

Thus

$$d^2 \left(\tilde{\tilde{f}} \varphi \right) (x, y, z, v) = 4 \tilde{\tilde{f}} (x, y, z, v) d^2 \varphi (x, y, z, v) + \left(\tilde{\tilde{f}} \wedge_d \varphi \right) (x, y, z, v).$$

Lemma 5. (*Derivation property*) Let $(f, \phi) \in C(V) \times C(\Delta_3)$

$$\begin{aligned} \delta^2 \left(\tilde{\tilde{f}} \phi \right) (y, z, v) &= \frac{1}{4} \tilde{\tilde{f}} (y, z, v) \delta^2 \phi (y, z, v) + \frac{1}{12s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \phi(x, y, z, v) \\ &\quad \left[d^1 \tilde{\tilde{f}} (v, z, x) + d^1 \tilde{\tilde{f}} (x, y, v) + d^1 \tilde{\tilde{f}} (z, y, x) \right]. \end{aligned}$$

Proof. Let $(f, \phi) \in c(V) \times c(\Delta_3)$

We have

$$\begin{aligned} \delta^2 \left(\tilde{\tilde{f}} \phi \right) (y, z, v) &= \frac{1}{s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \left(\tilde{\tilde{f}} \phi \right) (x, y, z, v) \\ &= \frac{1}{4s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \phi(x, y, z, v) \left[\tilde{\tilde{f}} (y, z, v) + \tilde{\tilde{f}} (v, z, x) + \tilde{\tilde{f}} (x, y, v) + \tilde{\tilde{f}} (z, y, x) \right] \\ &= \frac{1}{4s(y, z, v)} \tilde{\tilde{f}} (y, z, v) \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \phi(x, y, z, v) + \frac{1}{4s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \phi(x, y, z, v) \\ &\quad \left[\tilde{\tilde{f}} (v, z, x) + \tilde{\tilde{f}} (x, y, v) + \tilde{\tilde{f}} (z, y, x) \right] \end{aligned}$$

Since $\tilde{\tilde{f}} (x, y, z) = \frac{1}{3} d^1 \tilde{\tilde{f}} (x, y, z)$

Then

$$\begin{aligned} \delta^2 \left(\tilde{\tilde{f}} \phi \right) (y, z, v) &= \frac{1}{4} \tilde{\tilde{f}} (y, z, v) \delta^2 \phi (y, z, v) + \frac{1}{12s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \phi(x, y, z, v) \\ &\quad \left[d^1 \tilde{\tilde{f}} (v, z, x) + d^1 \tilde{\tilde{f}} (x, y, v) + d^1 \tilde{\tilde{f}} (z, y, x) \right]. \end{aligned}$$

□

2.3.2. Laplacian. Through the Gauss-Bonnet operator T , we can define the discrete Laplacian on T . So, Lemma 2.8 induces the following definition.

Definition 2. Let $\Delta = (K, F, \Delta_3)$ be a 3-simplicial complex, the Laplacian on Δ defined as

$$L = T^2 : C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3) \hookrightarrow$$

is given by

$$L(f, \varphi, \phi, \psi) = (\delta^0 d^0 f, (d^0 \delta^0 + \delta^1 d^1) \varphi, (d^1 \delta^1 + \delta^2 d^2) \phi, d^2 \delta^2 \psi)$$

for all $(f, \varphi, \phi, \psi) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.

Remark 2. We can write

$$L = L_0 \oplus L_1 \oplus L_2 \oplus L_3.$$

where L_0 is the discrete Laplacian acting on functions given by

$$L_0(f)(x) = \delta^0 d^0 f(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e) d^0 f(e),$$

with $f \in C_c(V)$. And where L_1 is the discrete Laplacian acting on 1-forms given by

$$\begin{aligned} L_1(\varphi)(x, y) &= (d^0\delta^0 + \delta^1d^1)\varphi(x, y) \\ &= \frac{1}{c(y)} \sum_{e, e^+=y} r(e)\varphi(e) - \frac{1}{c(x)} \sum_{e, e^+=x} r(e)\varphi(e) \\ &\quad + \frac{1}{r(x, y)} \sum_{z \in F_{(x, y)}} s(x, y, z) d^1\varphi(e)(x, y, z). \end{aligned}$$

with $\varphi \in C_c(E)$. And where also L_2 is the discrete Laplacian acting on 2-forms given by

$$\begin{aligned} L_2\phi(y, z, v) &= (d^1\delta^1 + \delta^2d^2)\phi(y, z, v) \\ &= \delta^1\phi(y, z) + \delta^1\phi(z, v) + \delta^1\phi(v, y) + \delta^2d^2\phi(y, z, v) \end{aligned}$$

with $\phi \in C_c(F)$. And where also L_3 is the discrete Laplacian acting on 3-forms given by

$$\begin{aligned} L_3\psi(x, y, z, v) &= d^2\delta^2\psi(x, y, z, v) \\ &= \delta^2\psi(y, z, v) + \delta^2\psi(v, z, x) + \delta^2\psi(x, y, v) + \delta^2\psi(z, y, x) \\ &= \frac{1}{s(y, z, v)} \sum_{u \in \Delta_3(y, z, v)} t(u, y, z, v)\psi(u, y, z, v) + \frac{1}{s(v, z, x)} \sum_{w \in \Delta_3(v, z, x)} t(w, v, z, x) \\ &\quad \psi(w, v, z, x) + \frac{1}{s(x, y, v)} \sum_{b \in \Delta_3(x, y, v)} t(b, x, y, v)\psi(b, x, y, v) + \frac{1}{s(z, y, x)} \\ &\quad \sum_{a \in \Delta_3(z, y, x)} t(a, z, y, x)\psi(a, z, y, x). \end{aligned}$$

with $\psi \in C_c(\Delta_3)$.

Remark 3. The operator L_1 is called the full Laplacian and defined as $L_1 = L_1^- + L_1^+$, where $L_1^- = d^0\delta^0$ (resp. $L_1^+ = \delta^1d^1$) and $L_2 = L_2^- + L_2^+$, where $L_2^- = d^1\delta^1$ (resp. $L_2^+ = \delta^2d^2$).

3. CLOSABILITY

On a connected locally finite graph, the operators d^0, δ^0, d^1 and δ^1 are closable (see [AT] and [YC]). The next lemma proves the closability of the operators d^2 and δ^2 on a 3-simplicial complex.

Lemma 6. Let $\Delta = (k, F, \Delta_3)$ be a weighted 3-simplicial complex. Then the operators d^2 and δ^2 are closable.

Proof. • Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence from $C_c(\delta F)$ and $\phi \in l^2(\Delta_3)$ such that $\lim_{n \rightarrow \infty} (\|\varphi_n\|_{l^2(F)} + \|d^2\varphi_n - \phi\|_{l^2(\Delta_3)}) = 0$. Then for each face ϖ , $\varphi_n(\varpi)$ converges to 0 and for each tetrahedron $\bar{\Omega}$, $d^2\varphi_n(\bar{\Omega})$ converges to $\phi(\bar{\Omega})$. But by the expression of d^2 and local finiteness of Δ , for each tetrahedron $\bar{\Omega}$, $d^2\varphi_n(\bar{\Omega})$ converges to 0. Thus we have that $\phi = 0$. Then the operator d^2 is closable.

- Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence from $C_c(F)$ and $\varphi \in l^2(\Delta_3)$ such that $\lim_{n \rightarrow \infty} (\|\phi_n\|_{l^2(\Delta_3)} + \|\delta^2 \phi_n - \varphi\|_{l^2(F)}) = 0$. Then for each face tetrahedron $\bar{\Omega}$, $\phi_n(\bar{\Omega})$ converges to 0 and for each face ϖ , $\delta^2 \phi_n(\varpi)$ converges to $\varphi(\varpi)$. But by the expression of δ^2 and local finiteness of Δ , for each face ϖ , $\delta^2 \phi_n(\varpi)$ converges to 0. Thus we have that $\varphi = 0$. Then the operator δ^2 is closable. \square

The smallest extension is the closure (see [S],[RSv1]), denoted $\bar{d}^0 = d_{min}^0$ (resp. $\bar{\delta}^0 = \delta_{min}^0$, $\bar{d}^1 = d_{min}^1$, $\bar{\delta}^1 = \delta_{min}^1$, $\bar{d}^2 = d_{min}^2$, $\bar{\delta}^2 = \delta_{min}^2$, $\bar{T} = T_{min}$, $\bar{L} = L_{min}$) has the domain

$$\begin{aligned} Dom(d_{min}^0) &= \{f \in l^2(V), \exists (f_n)_{n \in \mathbb{N}}, f_n \in C(V), \lim_{n \rightarrow \infty} \|f_n - f\|_{l^2(V)} = 0, \\ &\quad \lim_{n \rightarrow \infty} d^0 f_n \text{ exists in } l^2(E)\} \end{aligned}$$

for such an f , one puts $d_{min}^0 f = \lim_{n \rightarrow \infty} d^0(f_n)$. We notice that $d_{min}^0 f$ is independent of the sequence $(f_n)_{n \in \mathbb{N}}$ because d^0 is closable. The largest is $d_{max}^0 = (d^0)^*$, the adjoint operator of δ_{min}^0 (resp. $\delta_{max}^0 = (d^0)^*$, the adjoint operator of d_{min}^0). We denote $d_{max}^1 = (\delta^1)^*$, the adjoint operator of δ_{min}^1 (resp. $\delta_{max}^1 = (d^1)^*$, the adjoint operator of d_{min}^1). We also denote $d_{max}^2 = (\delta^2)^*$, the adjoint operator of δ_{min}^2 (resp. $\delta_{max}^2 = (d^2)^*$, the adjoint operator of d_{min}^2).

Proposition 1. *Let $\Delta = (K, F, \Delta_3)$ be a weighted 3-simplicial complex. then*

$$\begin{aligned} Dom(T_{min}) &\subseteq Dom(d_{min}^0) \oplus (Dom(\delta_{min}^0) \cap Dom(d_{min}^1)) \oplus \\ &\quad (Dom(\delta_{min}^1) \cap Dom(d_{min}^2)) \oplus Dom(\delta_{min}^2) \end{aligned}$$

Proof. Let $F = (f, \varphi, \phi, \psi) \in Dom(\Delta_{min})$. so there exists a sequence

$$(F_n)_n = ((f_n, \varphi_n, \phi_n, \psi_n))_n \subseteq C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$$

such that $\lim_{n \rightarrow \infty} F_n = F$ in H and $(TF_n)_{n \in \mathbb{N}}$ converges in H . Let us denote by $l_0 = (f_0, \varphi_0, \phi_0, \psi_0)$ this limit.

Therefore

$$\begin{aligned} \|TF_n - l_0\|_H^2 &= \|\delta^0 \varphi_n - f_0\|_{l^2(V)}^2 + \|(d^0 + \delta^1)(f_n, \phi_n) - \varphi_0\|_{l^2(E)}^2 \\ &\quad + \|(d^1 + \delta^2)(\varphi_n, \psi_n) - \phi_0\|_{l^2(F)}^2 + \|d^2 \phi_n - \psi_0\|_{l^2(\Delta_3)}^2 \end{aligned}$$

Hence $\delta^0 \varphi_n \rightarrow f_0$ and $d^2 \phi_n \rightarrow \psi_0$ respectively in $l^2(V)$ and $l^2(\Delta_3)$. So, by definition,

$$f_0 = \delta_{min}^0 \varphi, \psi_0 = d_{min}^2 \phi, \varphi \in Dom(\delta_{min}^0) \cap Dom(d_{min}^1)$$

and $\phi \in Dom(\delta_{min}^1) \cap Dom(d_{min}^2)$. Moreover, we have

$$\|(d^0 + \delta^1)(f, \phi)\|_{l^2(E)}^2 = \|d^0 f\|_{l^2(E)}^2 + \|\delta^1 \phi\|_{l^2(E)}^2$$

$\forall n \in \mathbb{N}, \forall (f, \phi) \in C_c(V) \times C_c(F)$. Since $((d^0 + \delta^1)(f_n, \phi_n))_n$ converges in $l^2(E)$, then by completeness of $l^2(E)$. $(d^0(f_n))_n$ and $(\delta^1(\phi_n))_n$ are convergent in $l^2(E)$. Thus, we conclude that $\phi \in Dom(\delta_{min}^1)$ and $f \in Dom(d_{min}^0)$.

Further, we have

$$\| (d^1 + \delta^2)(\varphi, \psi) \|_{l^2(F)}^2 = \| d^1 \varphi \|_{l^2(F)}^2 + \| \delta^2 \psi \|_{l^2(F)}^2, \forall n \in \mathbb{N}, \forall (\varphi, \psi) \in C_c(E) \times C_c(\Delta_3).$$

Since $((d^1 + \delta^2)(\varphi_n, \psi_n))_n$ converges in $l^2(F)$, then by completeness of $l^2(F)$. $(d^1(\varphi_n))_n$ and $(\delta^2(\psi_n))_n$ are convergent in $l^2(F)$. Thus, we conclude that $\psi \in \text{Dom}(\delta_{min}^2)$ and $\varphi \in \text{Dom}(d_{min}^1)$. Then

$$\begin{aligned} F = & (f, \varphi, \phi, \psi) \in \text{Dom}(d_{min}^0) \oplus (\text{Dom}(\delta_{min}^0) \cap \text{Dom}(d_{min}^1)) \oplus (\text{Dom}(\delta_{min}^1) \cap \text{Dom}(d_{min}^2)) \\ & \oplus \text{Dom}(\delta_{min}^2) \end{aligned}$$

Thus

$$\begin{aligned} \text{Dom}(T_{min}) \subseteq & \text{Dom}(d_{min}^0) \oplus (\text{Dom}(\delta_{min}^0) \cap \text{Dom}(d_{min}^1)) \oplus (\text{Dom}(\delta_{min}^1) \cap \text{Dom}(d_{min}^2)) \\ & \oplus \text{Dom}(\delta_{min}^2). \end{aligned}$$

□

Proposition 2. Let $\Delta = (k, F, \Delta_3)$ be a weighted 3-simplicial complex. then

$$\begin{aligned} \text{Dom}(L_{min}) \subseteq & \text{Dom}(\delta_{min}^0 d_{min}^0) \oplus (\text{Dom}(d_{min}^0 \delta_{min}^0) \cap \text{Dom}(\delta_{min}^1 d_{min}^1)) \oplus \\ & (\text{Dom}(d_{min}^1 \delta_{min}^1) \cap \text{Dom}(\delta_{min}^2 d_{min}^2)) \oplus \text{Dom}(d_{min}^2 \delta_{min}^2). \end{aligned}$$

Proof. (1) We have $(L_0)_{min} \subseteq \delta_{min}^0 d_{min}^0$ and $(L_1)_{min} \subseteq (d_{min}^0 \delta_{min}^0 + \delta_{min}^1 d_{min}^1)$ see [YC proposition 3.3]

(2) Since for any $\phi, \Theta \in C_c(F)$. We have

$$\begin{aligned} \langle L_2 \phi, \Theta \rangle_{l_2(F)} &= \langle (d^1 \delta^1 + \delta^2 d^2) \phi, \Theta \rangle_{l_2(F)} \\ &= \langle d^1 \delta^1 \phi, \Theta \rangle_{l_2(F)} + \langle \delta^2 d^2 \phi, \Theta \rangle_{l_2(F)} \\ &= \langle \delta^1 \phi, \delta^1 \Theta \rangle_{l_2(E)} + \langle d^2 \phi, d^2 \Theta \rangle_{l_2(\Delta_3)}. \end{aligned}$$

Using the same meyhod as in [YC proposition 3.3.i] we obtain that

$$(L_2^-)_{min} \subseteq d_{min}^1 \delta_{min}^1 \text{ and } (L_2^+)_{min} \subseteq \delta_{min}^2 d_{min}^2.$$

Now we need to show that $(L_2)_{min} \subseteq (L_2^-)_{min} + (L_2^+)_{min}$.

let $\varphi \in \text{Dom}((L_2)_{min})$, So there exists a sequence $(\varphi_n)_n \subseteq C_c(F)$ such that $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ in $l^2(F)$ and $(L_2 \varphi_n)_{n \in \mathbb{N}}$ converges in $l^2(F)$. Thus, we obtain

$$\| (L_2^- + L_2^+) (\varphi_n) \|_{l^2(F)}^2 = \| L_2^- (\varphi_n) \|_{l^2(F)}^2 + \| L_2^+ (\varphi_n) \|_{l^2(F)}^2, \forall n \in \mathbb{N}.$$

Then $(L_2^- \varphi_n)_n$ and $(L_2^+ \varphi_n)_n$ are convergent in $l^2(F)$. Moreover, by the closability of L_2^- and L_2^+ . We conclude that

$$\varphi \in \text{Dom}((L_2^-)_{min}) \cap \text{Dom}((L_2^+)_{min}). \text{ Then}$$

$$(L_2)_{min} \subseteq (L_2^-)_{min} + (L_2^+)_{min}.$$

(3) We will show that $(L_3)_{min} \subseteq d_{min}^2 \delta_{min}^2$. Since, for any $\psi, \Theta \in C_c(\Delta_3)$, we have

$$\langle L_2 \psi, \Theta \rangle_{l_2(\Delta_3)} = \langle d^2 \delta^2 \psi, \Theta \rangle_{l_2(\Delta_3)} = \langle \delta^2 \psi, \delta^2 \Theta \rangle_{l_2(F)}$$

Using the same method as in [YC proposition 3.3.i] we obtain that $(L_3)_{min} \subseteq d_{min}^2 \delta_{min}^2$.
then

$$\begin{aligned} Dom(L_{min}) \subseteq & Dom(d_{min}^0 d_{min}^0) \oplus (Dom(d_{min}^0 \delta_{min}^0) \cap Dom(\delta_{min}^1 d_{min}^1)) \oplus \\ & (Dom(d_{min}^1 \delta_{min}^1) \cap Dom(\delta_{min}^2 d_{min}^2)) \oplus Dom(d_{min}^2 \delta_{min}^2). \end{aligned}$$

□

4. GEOMETRIC HYPOTHESIS

In this section, we give the geometric hypothesis for the 3-simplicial complex Δ . First we recall the definition of χ -completeness given in [AT] for the case of graphs and also for the case of triangulations. A graph $K = (V, E)$ is χ -complete if there exists an increasing sequence of finite sets $(B_n)_{n \in \mathbb{N}}$ such that $V = \bigcup_{n \in \mathbb{N}} B_n$ and there exist related functions χ_n satisfying the following three conditions:

- i) $\chi_n \in C_c(V)$, $0 \leq \chi_n \leq 1$.
- ii) $x \in B_n \implies \chi_n(x) = 1$.
- iii) $\exists C > 0$ such that $\forall n \in \mathbb{N}$, $x \in V$

$$\frac{1}{c(x)} \sum_{e \in E, e^\pm = x} r(e) |d^0 \chi_n(e)|^2 \leq C.$$

Moreover, a triangulation $T = (K, F)$ is χ -complete, if

- 1- K is χ -complete.
- 2- $\exists M > 0$, $\forall n \in \mathbb{N}$, $e \in E$ such that

$$\frac{1}{r(e)} \sum_{x \in F(e)} s(e, x) |d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 \leq M.$$

Definition 3. A 3-simplicial complex $\Delta = (T, \Delta_3)$ is χ -complete, if

- C_1) T is χ -complete.
- C_2) $\exists N > 0$, $\forall n \in \mathbb{N}$, $(y, z, v) \in F$ such that

$$\frac{1}{s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(y, z, v, x) |d^1 \tilde{\chi}_n(y, z, x) + d^1 \tilde{\chi}_n(z, v, x) + d^1 \tilde{\chi}_n(v, y, x)|^2 \leq N.$$

Proposition 3. Let Δ be a 3-simplicial complex of bounded degree, i.e $\exists \lambda > 0$, $\forall x \in V$, $\deg(x) \leq \lambda$. Then Δ is χ -complete 3-simplicial complex.

Proof. Let us consider Δ an infinite 3-simplicial complex. Given $\mu \in V$, let B_n be a ball of radius $n \in \mathbb{N}$ centered by the vertex μ . $B_n = \{x \in V, d_{comb}(\mu, x) \leq n\}$.

We set the cut-off function $\chi_n \in C_c(V)$ as follow:

$$\chi_n(x) = \left(\frac{2n - d_{comb}(\mu, x)}{n} \vee 0 \right) \wedge 1, \forall n \in \mathbb{N}^*.$$

- if $x \in B_n \implies \chi_n(x) = 1$ and $x \in B_{2n}^c \implies \chi_n(x) = 0$.

- For $(x, y) \in E$, we have that

$$\begin{aligned} |d^0 \chi_n(x, y)| &= |\chi_n(y) - \chi_n(x)| \\ &\leq \frac{1}{n} |d_{comb}(\mu, y) - d_{comb}(\mu, x)| = \frac{1}{n}. \end{aligned}$$

$$\text{Then } \forall u \in V, \sum_{(x, y) \in F, (x, y)^\pm = u} |d^0 \chi_n(x, y)|^2 \leq \frac{\lambda}{n^2}$$

$$\text{Thus } \forall (x, y) \in E, \sum_{u \in F(x, y)} |d^0 \chi_n(x, u) + d^0 \chi_n(y, u)|^2 \leq \frac{2\lambda}{n^2}$$

So, the triangulation $T = (K, F)$ is χ -complete.

- For $(y, z, v) \in F$, we have that

$$\begin{aligned} |d^1 \tilde{\chi}_n(y, z, v)| &= |\tilde{\chi}_n(y, z) + \tilde{\chi}_n(z, v) + \tilde{\chi}_n(v, y)| = |\chi_n(y) + \chi_n(z) + \chi_n(v)| \\ &\leq \frac{1}{n} |d_{comb}(\mu, y) + d_{comb}(\mu, z) + d_{comb}(\mu, v)| \end{aligned}$$

Then

$$\forall (y, z, v) \in F, \quad \sum_{x \in \Delta_3(y, z, v)} |d^1 \tilde{\chi}_n(y, z, x) + d^1 \tilde{\chi}_n(z, v, x) + d^1 \tilde{\chi}_n(v, y, x)|^2 \leq \frac{3\lambda}{n^2}.$$

Hence Δ is χ -complete 3-simplicial complex. \square

5. ESSENTIAL SELF-ADJOINTNESS

In [AT], the authors use the χ -completeness hypothesis on a graph to ensure essential self-adjointness for the Gauss-Bonnet operator and the Laplacian. In this section, with the same idea we will prove the main result, when the 3-simplicial complex is χ -complete. Let us begin from

Proposition 4. *Let $\Delta = (K, F, \Delta_3)$ be a χ -complete 3-simplicial complex then the operator $d^2 + \delta^2$ is essentially self-adjoint on $C_c(F) \oplus C_c(\Delta_3)$.*

Proof. It suffices to show that $d_{min}^2 = \delta_{min}^2$ and $d_{max}^2 = \delta_{max}^2$. Indeed, $d^2 + \delta^2$ is a direct sum and if $F = (\varphi, \phi) \in \text{Dom}((d^2 + \delta^2)_{max})$ then $\varphi \in \text{Dom}(d_{max}^2)$ and $\phi \in \text{Dom}(\delta_{max}^2)$.

By hypothesis, we have $\varphi \in \text{Dom}(d_{min}^2)$ and $\phi \in \text{Dom}(\delta_{min}^2)$, thus $F \in \text{Dom}((d^2 + \delta^2)_{min})$.

1) Let $\varphi \in \text{Dom}(d_{max}^2)$, we will show that

$$\|\varphi - \tilde{\chi}_n \varphi\|_{l^2(F)} + \|d^2(\varphi - \tilde{\chi}_n \varphi)\|_{l^2(\Delta_3)} \rightarrow 0 \text{ when } n \rightarrow \infty$$

We know that

$$\forall p \in \mathbb{N}, \exists n_p, \forall n \geq n_p, \|\varphi - \tilde{\chi}_n \varphi\|_{l^2(F)} \leq \sum_{(x, y, z) \in F_p^c} s(x, y, z) |\varphi(x, y, z)|^2$$

$$\text{So } \lim_{n \rightarrow \infty} \|\varphi - \tilde{\chi}_n \varphi\|_{l^2(F)} = 0.$$

From the derivation formula in lemma 4, we have

$$\begin{aligned} d^2(\varphi - \tilde{\chi}_n \varphi)(x, y, z, v) &= d^2\left(\varphi(1 - \tilde{\chi}_n)\right)(x, y, z, v) \\ &= d^2\left(\widetilde{(1 - \chi_n)\varphi}\right)(x, y, z, v) \\ &= 4\left(1 - \tilde{\chi}_n\right) d^2\varphi(x, y, z, v) + \frac{1}{3} (d^1(1 - \tilde{\chi}_n) \wedge_d \varphi)(x, y, z, v) \\ &= 4\left(1 - \tilde{\chi}_n\right) d^2\varphi(x, y, z, v) + \\ &\quad \frac{1}{3} [d^1(1 - \tilde{\chi}_n)(x, y, z) + d^1(1 - \tilde{\chi}_n)(x, z, v) - d^1(1 - \tilde{\chi}_n)(x, v, y)] \varphi(y, z, v) \\ &\quad + \frac{1}{3} [d^1(1 - \tilde{\chi}_n)(y, v, z) + d^1(1 - \tilde{\chi}_n)(y, z, x) - d^1(1 - \tilde{\chi}_n)(y, v, x)] \varphi(v, z, x) \\ &\quad + \frac{1}{3} [d^1(1 - \tilde{\chi}_n)(z, x, y) + d^1(1 - \tilde{\chi}_n)(z, y, v) - d^1(1 - \tilde{\chi}_n)(z, x, v)] \varphi(x, y, v) \\ &\quad + \frac{1}{3} [d^1(1 - \tilde{\chi}_n)(v, z, y) + d^1(1 - \tilde{\chi}_n)(v, y, x) - d^1(1 - \tilde{\chi}_n)(v, z, x)] \varphi(z, y, x) \\ &= 4\left(1 - \tilde{\chi}_n\right) d^2\varphi(x, y, z, v) + \frac{1}{3} [d^1 \tilde{\chi}_n(x, z, y) + d^1 \tilde{\chi}_n(x, v, z) - d^1 \tilde{\chi}_n(x, v, y)] \\ &\quad \varphi(y, z, v) + \frac{1}{3} [d^1 \tilde{\chi}_n(y, z, v) + d^1 \tilde{\chi}_n(y, x, z) - d^1 \tilde{\chi}_n(y, x, v)] \varphi(v, z, x) \\ &\quad + \frac{1}{3} [d^1 \tilde{\chi}_n(z, y, x) + d^1 \tilde{\chi}_n(z, v, y) - d^1 \tilde{\chi}_n(z, v, x)] \varphi(x, y, v) \\ &\quad + \frac{1}{3} [d^1 \tilde{\chi}_n(v, y, z) + d^1 \tilde{\chi}_n(v, x, y) - d^1 \tilde{\chi}_n(v, x, z)] \varphi(z, y, x). \end{aligned}$$

Since $d^2\varphi \in l^2(\Delta_3)$, one has $\lim_{n \rightarrow \infty} \|(1 - \tilde{\chi}_n) d^2\varphi\|_{l^2(\Delta_3)} = 0$.

On the other hand,

$$\begin{aligned}
& \sum_{(x,y,z,v) \in \Delta_3} t(x,y,z,v) |\varphi(y,z,v)|^2 |d^1 \tilde{\chi}_n(x,z,y) + d^1 \tilde{\chi}_n(x,v,z) - d^1 \tilde{\chi}_n(x,v,y)|^2 = \\
& \sum_{(y,z,v) \in F} |\varphi(y,z,v)|^2 \\
& \sum_{x \in \Delta_3(y,z,v)} t(y,z,v,x) |d^1 \tilde{\chi}_n(y,z,x) + d^1 \tilde{\chi}_n(z,v,x) + d^1 \tilde{\chi}_n(v,y,x)|^2 \leq \\
& N \sum_{(y,z,v) \in F^*(n)} s(y,z,v) |\varphi(y,z,v)|^2
\end{aligned}$$

We conclude that this term tends to 0 as $n \rightarrow \infty$. Applying the same process to other terms.

2) Let $\phi \in \text{Dom}(\delta_{max}^2)$, we will show that

$$\|\phi - \tilde{\chi}_n \phi\|_{l^2(\Delta_3)} + \|d^2(\phi - \tilde{\chi}_n \phi)\|_{l^2(F)} \rightarrow 0 \text{ when } n \rightarrow \infty.$$

We have that

$$\begin{aligned}
\|\phi - \tilde{\chi}_n \phi\|_{l^2(\Delta_3)} &= \frac{1}{24} \sum_{(x,y,z,v) \in \Delta_3} t(x,y,z,v) |1 - \tilde{\chi}_n(x,y,z,v)|^2 |\phi(x,y,z,v)|^2 \\
&\leq \sum_{(x,y,z,v) \in \Delta_3} t(x,y,z,v) |\phi(x,y,z,v)|^2 \rightarrow 0 \text{ when } n \rightarrow \infty.
\end{aligned}$$

By the derivation formula in lemme (5), we have

$$\begin{aligned}
\delta^2(\phi - \tilde{\chi}_n \phi)(y,z,v) &= \delta^2\left(\left(1 - \tilde{\chi}_n\right)\phi\right)(y,z,v) \\
&= \frac{1}{4}\left(1 - \tilde{\chi}_n\right)(y,z,v) \delta^2 \phi(y,z,v) + \frac{1}{12s(y,z,v)} \sum_{x \in \Delta_3(y,z,v)} t(x,y,z,v) \phi(x,y,z,v) \\
&\quad [d^1(1 - \tilde{\chi}_n)(v,z,x) + d^1(1 - \tilde{\chi}_n)(x,y,v) + d^1(1 - \tilde{\chi}_n)(z,y,x)] \\
&= \frac{1}{4}\left(1 - \tilde{\chi}_n\right)(y,z,v) \delta^2 \phi(y,z,v) + \frac{1}{12s(y,z,v)} \sum_{x \in \Delta_3(y,z,v)} t(x,y,z,v) \phi(x,y,z,v) \\
&\quad [d^1 \tilde{\chi}_n(x,z,v) + d^1 \tilde{\chi}_n(v,y,x) + d^1 \tilde{\chi}_n(x,y,z)].
\end{aligned}$$

We know that $\lim_{n \rightarrow \infty} \|(1 - \tilde{\chi}_n) \delta^2 \phi\| = 0$. Because $\delta^2 \phi \in l^2(F)$. For the seconde

term, we use the definition 3 and the Cauchy-Schwarz inequality. Fix $(y,z,v) \in F$, then

$$\begin{aligned}
& \left| \sum_{x \in \Delta_3(y,z,v)} t(x,y,z,v) \phi(x,y,z,v) [d^1 \tilde{\chi}_n(x,z,v) + d^1 \tilde{\chi}_n(v,y,x) + d^1 \tilde{\chi}_n(x,y,z)] \right|^2 \leq \\
& N s(y,z,v) \sum_{x \in \Delta_3^*(y,z,v)(n)} t(y,z,v,x) |\phi(y,z,v,x)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{(y,z,v) \in F} s(y,z,v) \left| \frac{1}{s(y,z,v)} (d^1 \tilde{\chi}_n(x,z,v) + d^1 \tilde{\chi}_n(v,y,x) + d^1 \tilde{\chi}_n(x,y,z)) \right|^2 \leq \\
& N \sum_{(y,z,v) \in F} \sum_{x \in \Delta_3^*(y,z,v)(n)} t(y,z,v,x) |\phi(y,z,v,x)|^2 \text{ this term tends to 0.}
\end{aligned}$$

□

Corollary 1. *Let $\Delta = (K, F, \Delta_3)$ be a χ -complete 3-simplicial complex then the operator $L_2^+ \oplus L_3$ is essentially self-adjoint on $C_c(F) \oplus C_c(\Delta_3)$.*

Proof. First we have that $L_2^+ \oplus L_3 = (d^2 + \delta^2)^2$ and $L_2^+ \oplus L_3(C_c(F) \oplus C_c(\Delta_3)) \subseteq C_c(F) \oplus C_c(\Delta_3)$. As proposition 13 in [AT] we prove that $d^2 + \delta^2$ is essentially self-adjoint if and only if $L_2^+ \oplus L_3$ is essentially self-adjoint. □

Theorem 1. *Let $\Delta = (K, F, \Delta_3)$ be a χ -complete 3-simplicial complex then the operator T is essentially self-adjoint on $C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.*

Proof. • We will show that $Dom(T_{min}) = Dom(d_{min}^0) \oplus (Dom(\delta_{min}^0) \cap Dom(d_{min}^1)) \oplus (Dom(\delta_{min}^1) \cap Dom(d_{min}^2)) \oplus Dom(\delta_{min}^2)$.

Let $F = (f, \varphi, \phi, \psi) \in Dom(d_{min}^0) \oplus (Dom(\delta_{min}^0) \cap Dom(d_{min}^1)) \oplus (Dom(\delta_{min}^1) \cap Dom(d_{min}^2)) \oplus Dom(\delta_{min}^2)$

Then there exist $(f_n)_n \subseteq C_c(V)$ and $(\psi_n)_n \subseteq C_c(\Delta_3)$ such that:

$f_n \rightarrow f$ in $l^2(V)$ and $d_{min}^0 f_n \rightarrow d_{min}^0 f$ in $L^2(E)$.

$\psi_n \rightarrow \psi$ in $l^2(\Delta_3)$ and $\delta_{min}^2 \psi_n \rightarrow \delta_{min}^2 \psi$ in $L^2(F)$.

On the other hand, let $\varphi \in Dom(\delta_{min}^0) \cap Dom(d_{min}^0)$ and $\phi \in Dom(\delta_{min}^1) \cap Dom(d_{min}^1)$. From [YC theorem 5-3 first step] we have

$\|\varphi - \tilde{\chi}_n \varphi\|_{l^2(E)} + \|\delta^0(\varphi - \tilde{\chi}_n \varphi)\|_{l^2(V)} + \|\delta^1(\varphi - \tilde{\chi}_n \varphi)\|_{l^2(F)} \rightarrow 0$, when $n \rightarrow \infty$. By the χ -completeness of Δ , we consider the sequence $(\tilde{\chi}_n \phi)_n \subseteq C_c(F)$. It remains to show that

$\|\phi - \tilde{\chi}_n \phi\|_{l^2(F)} + \|\delta^1(\phi - \tilde{\chi}_n \phi)\|_{l^2(E)} + \|\delta^2(\phi - \tilde{\chi}_n \phi)\|_{l^2(\Delta_3)} \rightarrow 0$, when $n \rightarrow \infty$.

The first and the third terms has already been shown in Proposition 4. For the following we need a derivation formula of δ^1 taken in [YC]. Let $e \in E$, for each $(f, \phi) \in C_c(V) \times C_c(F)$ we have $\delta^1(\tilde{f}\phi)(e) = \tilde{f}(e)\delta^1\phi(e) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)(d^0\chi_n(e^-, x) + d^0\chi_n(e^+, x))\phi(e, x)$.

Thereore, by derivation formula, we get

$$\delta^1(\phi - \tilde{\chi}_n \phi)(e) = (1 - \tilde{\chi}_n)\phi(e)\delta^1\phi(e) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)(d^0\chi_n(x, e^-) + d^0\chi_n(x, e^+))\phi(e, x)$$

$\phi(e, x)$

as a consequence, because $\delta^1\phi \in l^2(E)$, we have $\lim_{n \rightarrow \infty} \|(1 - \tilde{\chi}_n)\delta^1\phi\|_{l^2(E)} = 0$.

0. For the second term, we combine the property 2) of χ -completeness for a triangulation with the Cauchy-Schwarz inequality to obtain for all

$$\begin{aligned} & x \in V, \\ & \left| \sum_{x \in F_e} s(e, x)(d^0\chi_n(x, e^-) + d^0\chi_n(x, e^+))\phi(e, x) \right|^2 \leq \sum_{x \in F_e} s(e, x)|d^0\chi_n(x, e^-) + d^0\chi_n(x, e^+)|^2 \\ & \sum_{e \in \text{supp}(d^0\chi_n), x \in F_e} s(e, x)|\phi(e, x)|^2 \\ & \leq Mr(e) \sum_{e \in \text{supp}(d^0\chi_n), x \in F_e} s(e, x)|\phi(e, x)|^2. \end{aligned}$$

So,

$$\begin{aligned} & \sum_{e \in E} \frac{1}{r(e)} \left| \sum_{x \in F_e} s(e, x)(d^0\chi_n(x, e^-) + d^0\chi_n(x, e^+))\phi(e, x) \right|^2 \leq \sum_{e \in E} M \\ & \sum_{e \in \text{supp}(d^0\chi_n), x \in F_e} s(e, x)|\phi(e, x)|^2 \\ & \leq M \sum_{e \in \text{supp}(d^0\chi_n)} s(e, x)|\phi(e, x)|^2 \rightarrow 0 \text{ when } n \rightarrow \infty. \end{aligned}$$

Hence $F_n \rightarrow F$ in H , $TF_n \rightarrow T_{min}F$ in H , where $F_n = (f_n, \tilde{\chi}_n\varphi, \tilde{\chi}_n\phi, \psi_n)$ and

$$T_{min}F(f, \varphi, \phi, \psi) = (\delta_{min}^0\varphi, d_{min}^0f + \delta_{min}^1\phi, d_{min}^1\varphi + \delta_{min}^2\psi, d_{min}^2\phi).$$

So $F \in Dom(T_{min})$.

Then

$$\begin{aligned} Dom(T_{min}) &= Dom(d_{min}^0) \oplus (Dom(\delta_{min}^0) \cap Dom(d_{min}^1)) \oplus \\ &\quad (Dom(\delta_{min}^1) \cap Dom(d_{min}^2)) \oplus Dom(\delta_{min}^2). \end{aligned}$$

- To show that T is essentially self-adjoint, we will prove that $T_{max} = T_{min}$. By the first step, Theorem 1 in [AT] and Proposition 4 it remains to show that:

$$\begin{aligned} Dom(T_{max}) &\subseteq Dom(d_{max}^0) \oplus (Dom(\delta_{max}^0) \cap Dom(d_{max}^1)) \oplus \\ &\quad (Dom(\delta_{max}^1) \cap Dom(d_{max}^2)) \oplus Dom(\delta_{max}^2). \end{aligned}$$

Let $F = (f, \varphi, \phi, \psi) \in Dom(T_{max})$ then $TF \in H$. This implies that $\delta^0\varphi \in l^2(V)$, $d^0f + \delta^1\phi \in l^2(E)$, $d^1\varphi + \delta^2\psi \in l^2(F)$ and $d^2\phi \in l^2(\Delta_3)$. As consequence, by the definition of δ_{max}^0 and d_{max}^1 we have $\varphi \in Dom(\delta_{max}^0) \cap Dom(d_{max}^1)$. Moreover, by χ -completeness of Δ , there exists a sequence of cut-off functions $(\chi_n)_n \subseteq C_c(V)$. Then, the parallelogram identity with lemma 2 – 8 in [YC] we get $\|d^0(\chi_n f) + \delta^1(\tilde{\chi}_n \phi)\|_{l^2(E)}^2 = \|d^0\chi_n f\|_{l^2(E)}^2 + \|\delta^1\tilde{\chi}_n \phi\|_{l^2(E)}^2$. We have $d^0(\chi_n f) + \delta^1(\tilde{\chi}_n \phi)$ converges in $l^2(E)$ from theorem 5 – 3 seconde step in [YC]. Now, By the definition of δ_{max}^1 and d_{max}^2 . We have $\phi \in Dom(\delta_{max}^1) \cap Dom(d_{max}^2)$. Moreover, by χ -completeness of Δ , there exists a sequence of cut-off functions $(\chi_n)_n \subseteq C_c(V)$. Then, the parallelogram identity with Lemma 3 in we get $\|d^1(\tilde{\chi}_n \tilde{f}) + \delta^2(\tilde{\chi}_n \psi)\|_{l^2(F)}^2 = \|d^1\tilde{\chi}_n \tilde{f}\|_{l^2(F)}^2 + \|\delta^2\tilde{\chi}_n \psi\|_{l^2(F)}^2$. It remains to prove that $d^1(\tilde{\chi}_n \tilde{f}) + \delta^2(\tilde{\chi}_n \psi)$ converges in $l^2(F)$. From lemma 4 and 5, we get

$$d^1(\tilde{\chi}_n \tilde{f}) = \tilde{\chi}_n d^1 \tilde{f} + \frac{1}{6} (d^0 \chi_n \wedge_{disc} \tilde{f}).$$

$$\delta^2(\tilde{\chi}_n \psi)(y, z, v) = \frac{1}{4} \tilde{\chi}_n(y, z, v) \delta^2 \psi(y, z, v) + \frac{1}{12s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v)$$

$$\psi(x, y, z, v) [d^1 \tilde{\chi}_n(v, z, x) + d^1 \tilde{\chi}_n(x, y, v) + d^1 \tilde{\chi}_n(z, y, x)]$$

We put

$$\begin{aligned} K_n(y, z, v) &= \frac{1}{12s(y, z, v)} \sum_{x \in \Delta_3(y, z, v)} t(x, y, z, v) \psi(x, y, z, v) [d^1 \tilde{\chi}_n(v, z, x) + d^1 \tilde{\chi}_n(x, y, v) \\ &\quad + d^1 \tilde{\chi}_n(z, y, x)]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|d^1(\tilde{f} - \tilde{\chi}_n \tilde{f}) + \delta^2(\psi - \tilde{\chi}_n \psi)\|_{l^2(F)}^2 &= \|(1 - \tilde{\chi}_n)(d^1 \tilde{f} + \delta^2 \psi) + \tilde{f} d^1 \tilde{\chi}_n + \\ K_n\|_{l^2(F)}^2 &\leq 4 \left[\|(1 - \tilde{\chi}_n)(d^1 \tilde{f} + \delta^2 \psi)\|_{l^2(F)}^2 + \|\tilde{f} d^1 \tilde{\chi}_n\|_{l^2(F)}^2 + \|K_n\|_{l^2(F)}^2 \right] \end{aligned}$$

Because $d^1 \tilde{f} + \delta^2 \psi \in l^2(F)$, we have $\lim_{n \rightarrow \infty} \|(1 - \tilde{\chi}_n)(d^1 \tilde{f} + \delta^2 \psi)\|_{l^2(F)}^2 =$

0

By proposition 4 we have $\lim_{n \rightarrow \infty} \|K_n\|_{l^2(F)}^2 = 0$. Moreover, by the hypothesis 2 of χ -completeness we have

$$\begin{aligned}
\|\tilde{f}d^1\tilde{\chi}_n\|_{l^2(F)}^2 &= \frac{1}{6} \sum_{(y,z,v) \in F} s(y,z,v) |\tilde{f}(y,z,v) d^1\tilde{\chi}_n(y,z,v)|^2 \\
&= \frac{1}{6} \sum_{(y,z,v) \in F} s(y,z,v) |\tilde{f}(y,z,v)|^2 |\tilde{\chi}_n(y,z) + \tilde{\chi}_n(z,v) + \tilde{\chi}_n(v,y)|^2 \\
&= \frac{1}{6} \sum_{(y,z,v) \in F} s(y,z,v) |\tilde{f}(y,z,v)|^2 |\chi(y) + \chi(z) + \chi(v)|^2 \\
&\leq \sum_{(y,z,v) \in F} s(y,z,v) |\tilde{f}(y,z)|^2 |d^0\chi_n(y,v) + d^0\chi_n(z,v)|^2 \\
&= \sum_{(y,z) \in E} |\tilde{f}(y,z)|^2 \sum_{v \in F(y,z)} |s(y,z,v) (d^0\chi_n(y,v) + d^0\chi_n(z,v))|^2 \\
&\leq M \sum_{(y,z) \in E_n} r(y,z) |\tilde{f}(y,z)|^2 \\
\text{Where } E_n &= \{(y,z) \in E, \exists \varpi \in \text{supp}(d^1\tilde{\chi}_n) \text{ such that } \varpi \in F_{(y,z)}\} \\
\text{This term tends to 0.} \\
\text{So}
\end{aligned}$$

$$\begin{aligned}
\text{Dom}(T_{max}) \subseteq & \text{Dom}(d_{max}^0) \oplus (\text{Dom}(\delta_{max}^0) \cap \text{Dom}(d_{max}^1)) \oplus \\
& (\text{Dom}(\delta_{max}^1) \cap \text{Dom}(d_{max}^2)) \oplus \text{Dom}(\delta_{max}^2).
\end{aligned}$$

then the operator T is essentially self-adjoint on $C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.

□

Theorem 2. *Let $\Delta = (K, F, \Delta_3)$ be a χ -complete 3-simplicial complex. Then the operator T is essentially self-adjoint on $C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$ if and only if L is essentially self-adjoint on $C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.*

Proof. Since $T(C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)) \subseteq C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$, using the same technique in the proof of Proposition 13 in [AT], the result holds. □

Corollary 2. *Let $\Delta = (K, F, \Delta_3)$ be a χ -complete 3-simplicial complex then the operator L is essentially self-adjoint on $C_c(V) \oplus C_c(E) \oplus C_c(F) \oplus C_c(\Delta_3)$.*

REFERENCES

- [AT] C. Anné and N. Torki-Hamza: The Gauss-Bonnet operator of an infinite graph, Anal. Math. Phys. 5 no. 2, 137-159, 2015.
- [YC] Chebbi, Y.: The discrete Laplacian of a 2-Simplicial complex, Potential Anal., 49, No. 2, 331-358(2018).
- [TYC] Chebbi, Y.: Laplacien discret d'un 2-complexe simplicial, Doc. Thesis (P.H.D), Universit de Nantes, Facult des sciences et des techniques; Universit de Carthage (Tunisie) (2018).
- [BG] M. Bonnefont and S. Golénia: Essential spectrum and Weyl asymptotics for discrete Laplacians, arXiv:1406.5391v1 [math.SP], 2014.
- [BGJ] H. Baloudi, S. Golénia and A. Jeribi: The adjacency matrix and the discrete Laplacian acting on forms, arXiv:1505.06109v1 [math.SP], 2015.
- [C] P.R. Chernoff: Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. 12, 401-414, 1973.

- [CTT] Y. Colin De Verdière, N. Torki-Hamza and F. Truc: Essential self-adjointness for combinatorial schrödinger operators II-metrically non complete graphs, *Mathematical Physics Analysis and Geometry* 14, 1 (2011) 21-38.
- [D] J. Dodziuk: Elliptic operators on infinite graphs, in *analysis, geometry and topology of elliptic operators* (World Scientific, 2006), pp. 353-368.
- [EL] K. D. Elworthy and Xue-Mei Li: Special Itô maps and an L2-Hodge theory for one forms on path spaces. In *Stochastic processes, physics and geometry: new interplays, I* (Leipzig, 1999), pp. 145-162. Amer. Math. Soc., 2000.
- [FLW] L. Frank, D. Lenz and D. Wingert: Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory, *J. Funct. Anal* 266 (2014), 4765-4808.
- [G] M. Gaffney, Hilbert space methods in the theory of harmonic integrals, *Ann. Math.* 78 (1955), 426-444.
- [K] G. Kirchhoff: "Über die Auflösung der Gleichungen auf welche man bei der Untersuchen der linearen Vertheilung galvanischer Ströme geführt wird, *Ann. der Phys. und Chem.* 72 (1847), 495-508.
- [GS] S. Golénia and C. Schumacher: The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs, *J. Math. Phys.* 52, no. 6, pp. 1-17, 2011.
- [HKMW] X. Huang, M. Keller, J. Masamune, R.K. Wojciechowski, A note on self-adjoint extensions of the Laplacian on weighted graphs, *J. Funct. Anal.* 265 no. 8 (2013), 1556-1578.
- [M1] J. Masamune: Analysis of the Laplacian of an incomplete manifold with almost polar boundary, *Rend. Mat. Appl.* (7) 25 (2005), no. 1, 109-126.
- [M2] J. Masamune: A Liouville property and its application to the Laplacian of an infinite graph. *Contemp. Math.* 484, pp. 103-115, 2009.
- [RSv1] M. Reed , B. Simon: *Methods of Modern Mathematical Physics, Vol. 1*, New York Academic Press, 1980.
- [RSv2] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Vol. 2*, New York Academic Press, 1975.
- [LP] R. Lyons and Y. Peres: *Probability on trees and networks*. Cambridge University Press, 2014.
- [S] K. Schmüdgen: *Unbounded self-adjoint operators on Hilbert space*, Graduate texts in mathematics 265, c Springer 2012.
- [T] N. Torki-Hamza: Laplaciens de graphes infinis I-Graphes métriquement complets, *Confluentes Math.* 2 (2010), no. 3, 333-350.
- [W] R. K. Wojciechowski: *Stochastic compactness of graph*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)-City University of New York.

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