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# On the representation as exterior differentials of closed forms with $L^1$ -coefficients

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## Abstract

Let  $N \geq 2$ . If  $g \in L_c^1(\mathbf{R}^N)$  has zero integral, then the equation  $\operatorname{div} X = g$  need not have a solution  $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$  (Wojciechowski 1999) or even  $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$  (Bourgain and Brezis 2003). Using these results, we prove that, whenever  $N \geq 3$  and  $2 \leq \ell \leq N - 1$ , there exists some  $\ell$ -form  $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$  such that  $df = 0$  and the equation  $d\lambda = f$  has no solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ . This provides a negative answer to a question raised by Baldi, Franchi and Pansu (2019).

## Résumé

### Sur la représentation comme différentielles extérieures des formes fermées à coefficients $L^1$

Soit  $N \geq 2$ . Si  $g \in L_c^1(\mathbf{R}^N)$  est d'intégrale nulle, alors en général il n'est pas possible de résoudre l'équation  $\operatorname{div} X = g$  avec  $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$  (Wojciechowski 1999), ou même  $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$  (Bourgain et Brezis 2003). En utilisant ces résultats, nous prouvons que, pour  $N \geq 3$  et  $2 \leq \ell \leq N - 1$ , il existe une  $\ell$ -forme  $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$  avec  $df = 0$  et telle que l'équation  $d\lambda = f$  n'ait pas de solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ . Ceci donne une réponse négative à une question posée par Baldi, Franchi et Pansu (2019).

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## Version française abrégée

Répondant à une question posée par Baldi, Franchi et Pansu ([1]), nous montrons le résultat suivant :

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**Théorème 1** Soit  $N, \ell$  entiers tels que  $N \geq 3$  et  $2 \leq \ell \leq N - 1$ . Il existe une  $\ell$ -forme différentielle fermée à coefficients  $L^1$ ,  $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$ , telle que l'équation  $d\lambda = f$  n'ait pas de solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ .

La preuve est faite par l'absurde et repose sur un résultat de non-existence bien connu pour l'équation  $\operatorname{div} X = g$ . Plus précisément, il existe des fonctions  $g \in L^1_c((0, 1)^\ell; \mathbf{R})$  avec  $\int g = 0$  et telles que l'équation  $\operatorname{div} X = g$  n'ait pas de solution  $X \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$  (voir Bourgain et Brezis [2]). À partir d'une telle fonction  $g$ , nous construisons une forme différentielle explicite  $f$ , fermée, à support compact et à coefficients  $L^1$ . En supposant le Théorème 1 faux, nous obtenons l'existence d'une fonction  $G \in C^2_c((0, 1)^\ell; \mathbf{R})$  et d'un champ  $Y \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$  tels que  $\operatorname{div} Y = g + G$ , ce qui contredit les propriétés de la fonction  $g$ .

En combinant ce résultat avec les résultats de [6], [2], nous obtenons la conséquence suivante.

**Corollaire 2** Soient  $N \geq 2$  et  $1 \leq \ell \leq N$ . Soit  $\mathcal{A}$  la classe des  $\ell$ -formes  $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$  satisfaisant la condition de compatibilité  $df = 0$  (si  $1 \leq \ell \leq N - 1$ ), respectivement  $\int f = 0$  (si  $\ell = N$ ). Alors nous avons l'équivalence  $1 \iff 2$ , où

1. l'équation  $d\lambda = f$  a une solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$  pour tout  $f \in \mathcal{A}$ .
2.  $\ell = 1$ .

## 1 Introduction

We consider the Hodge system

$$d\lambda = f \text{ in } \mathbf{R}^N, \tag{1}$$

where  $f$  and  $\lambda$  are  $\ell$  and  $(\ell - 1)$ -forms respectively,  $f$  being given and satisfying the compatibility condition  $df = 0$ . We focus on the case where  $f$  has  $L^1$  coefficients.

To start with, let us recall some known facts about the cases  $\ell = N$  and  $\ell = 1$ .

In the case  $\ell = N$ , (1) reduces to the divergence equation. It was first shown by Wojciechowski [6] that there exists  $g \in L^1_c(\mathbf{R}^N)$ , with zero integral, such that the equation  $\operatorname{div} X = g$  has no solution  $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$ . On the other hand, Bourgain and Brezis [2] proved, using a different method, the following: there exists  $g \in L^1_c(\mathbf{R}^N)$  with zero integral, such that the equation  $\operatorname{div} X = g$  has no solution  $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$ . In view of the embedding  $W_{loc}^{1,1} \hookrightarrow L_{loc}^{N/(N-1)}$ , this improves [6].

In the case  $\ell = 1$ , (1) reduces to the following “gradient” equation

$$\nabla \lambda = f, \tag{2}$$

where  $f$  is a vector field satisfying the compatibility condition  $\nabla \times f = 0$  and  $\lambda$  is a function. Unlike the case  $\ell = N$ , this time (2) has a solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N)$ . Actually, any solution of (2) belongs to  $W_{loc}^{1,1}$  and, moreover, if  $f$  is compactly supported then we may choose  $\lambda \in W^{1,1}$ .

The question of the solvability in  $W_{loc}^{1,1}$  of the system (1) with datum in  $L^1$  in the remaining cases, i.e.,  $2 \leq \ell \leq N - 1$ , has been recently raised by Baldi, Franchi and Pansu [1]. Our main result settles this problem.

**Theorem 3** *Let  $N \geq 3$ . Let  $2 \leq \ell \leq N - 1$ . Then there exists some  $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$  such that  $df = 0$  and the equation  $d\lambda = f$  has no solution  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ .*

The proof of Theorem 3 we present is a simplification, communicated to the author by P. Mironescu, of the original one. This simplified version has the advantage of being relatively self-contained and elementary.

## 2 Proof of Theorem 3

We start with some auxiliary results.

**Lemma 4** *Let  $1 \leq \kappa \leq N - 1$  and  $f \in L_c^1(\mathbf{R}^N; \Lambda^\kappa)$  be such that  $df = 0$ . Then there exists some  $\omega \in L_{loc}^q(\mathbf{R}^N; \Lambda^{\kappa-1})$ , for all  $1 \leq q < N/(N - 1)$ , such that  $d\omega = f$ .*

*Proof.* Let  $E$  be “the” fundamental solution of  $\Delta$  and set  $\eta := E * f$ . Let  $\omega := d^*\eta$ . First,  $\eta \in W_{loc}^{1,q}(\mathbf{R}^N)$  (by elliptic regularity) and thus  $\omega \in L_{loc}^q(\mathbf{R}^N)$ ,  $1 \leq q < N/(N - 1)$ . Next,  $d\eta = E * df = 0$ . Finally,

$$d\omega = dd^*\eta = (dd^* + d^*d)\eta = \Delta\eta = f.$$

Hence,  $\omega$  has the required properties. □

A similar argument leads to the following.

**Lemma 5** *Let  $1 < r < \infty$ ,  $k \in \mathbf{N}$ . Let  $1 \leq \kappa \leq N - 1$ . Let  $f \in W_c^{k,r}(\mathbf{R}^N; \Lambda^\kappa)$  be such that  $df = 0$ . Then there exists some  $\omega \in W_{loc}^{k+1,r}(\mathbf{R}^N; \Lambda^{\kappa-1})$  such that  $d\omega = f$ .*

We next recall the following “inversion of  $d$  with loss of regularity”. It is folklore, and one possible proof consists of using Bogovskii’s formula (see for example [4, Corollary 3.3 and Corollary 3.4] for related arguments).

**Lemma 6** *Let  $1 \leq \kappa \leq N-1$ . Let  $Q$  be an open cube in  $\mathbf{R}^N$ . Then there exists some integer  $m = m(N, \kappa)$  such that if  $f \in C_c^k(Q; \Lambda^{\kappa-1})$ , with  $k \in \{m, m+1, \dots\} \cup \{\infty\}$ , satisfies  $df = 0$ , then there exists some  $\omega \in C_c^{k-m}(Q; \Lambda^{\kappa-1})$  such that  $d\omega = f$ .*

Combining Lemmas 4–6, we obtain the following

**Proposition 7** *Let  $1 \leq \kappa \leq N-1$ . Let  $Q$  be an open cube in  $\mathbf{R}^N$ . Let  $f \in L_c^1(Q; \Lambda^\kappa)$  be such that  $df = 0$ . Then there exists some  $\omega \in L_c^q(Q; \Lambda^{\kappa-1})$ , for all  $1 \leq q < N/(N-1)$ , such that  $d\omega = f$ .*

*Proof.* Set  $f_0 := f$ . We consider a sequence  $(\zeta_j)_{j \geq 0}$  in  $C_c^\infty(Q; \mathbf{R})$  such that  $\zeta_0 = 1$  on  $\text{supp } f_0$  and, for  $j \geq 1$ ,  $\zeta_j = 1$  on  $\text{supp } \zeta_{j-1}$ . We let  $\eta_0$  be a solution of  $d\eta_0 = f_0$ , constructed as in Lemma 4. We set  $\omega_0 := \zeta_0 \eta_0$ , so that  $\omega_0 \in L_c^q(Q; \Lambda^{\kappa-1})$ ,  $1 \leq q < N/(N-1)$  and

$$d\omega_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 d\eta_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 f_0 = \underbrace{d\zeta_0 \wedge \eta_0}_{-f_1} + f_0.$$

Let us note that  $df_1 = -d^2\omega_0 + df_0 = 0$  and that  $f_1 \in L_c^q(Q; \Lambda^\kappa)$ ,  $1 \leq q < N/(N-1)$ .

Fix some  $1 < r < N/(N-1)$ . By Lemma 5, there exists some  $\eta_1 \in W_{loc}^{1,r}(\mathbf{R}^N; \Lambda^{\kappa-1})$  such that  $d\eta_1 = f_1$ . Set  $\omega_1 := \zeta_1 \eta_1$ . Then  $\omega_1 \in W_c^{1,r}(Q; \Lambda^{\kappa-1})$  and, as above,  $f_2 := f_1 - d\omega_1$  satisfies  $df_2 = 0$  and  $f_2 \in W_c^{1,r}(Q; \Lambda^\kappa)$ . Applying again Lemma 5, we may find  $\eta_2 \in W_{loc}^{2,r}(\mathbf{R}^N)$  such that  $d\eta_2 = f_2$ .

Iterating the above, we have

$$\begin{aligned} \omega_0 + \dots + \omega_j &\in L_c^q(Q; \Lambda^{\kappa-1}), \quad 1 \leq q < N/(N-1), \\ d(\omega_0 + \dots + \omega_j) &= f_0 - f_j, \quad \text{with } df_j = 0 \text{ and } f_j \in W_c^{j,r}(Q; \Lambda^\kappa). \end{aligned}$$

Let now  $j$  be such that  $W^{j,q}(Q) \hookrightarrow C^m(Q)$ , with  $m$  as in Lemma 6. Let  $\xi \in C_c^0(Q; \Lambda^{\kappa-1})$  be such that  $d\xi = -f_j$ . Set  $\omega := \omega_0 + \dots + \omega_j + \xi$ . Then  $\omega$  has all the required properties.  $\square$

Let us note the following consequence of hypoellipticity of  $\Delta$  and of the proofs of Proposition 7 and Lemmas 4 and 5 (but *not* of their statements).

**Corollary 8** *Assume, in addition to the hypotheses of Proposition 7, that  $f \in C^\infty(U)$  for some open set  $U \subset Q$ . Let  $s \in \mathbf{N}$ . Then we may choose  $\omega$  such that, in addition,  $\omega \in C^s(U)$ .*

*Proof of Theorem 3.* We write the variables in  $\mathbf{R}^N$  as follows:  $x = (x', x'')$ , with  $x' \in \mathbf{R}^\ell$  and  $x'' \in \mathbf{R}^{N-\ell}$ .

Pick some  $g \in L_c^1((0, 1)^\ell; \mathbf{R})$  with zero integral, such that the equation  $\operatorname{div} X = g$  has no solution  $X \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$  (see [6], [2]). Clearly, for any  $G \in C_c^2((0, 1)^\ell; \mathbf{R})$ ,

$$\text{the equation } \operatorname{div} Y = g + G \text{ has no solution } Y \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell). \quad (3)$$

Let  $\psi \in C_c^\infty((0, 1)^{N-\ell})$  be such that  $\psi \equiv 1$  in some nonempty open set  $V \subset (0, 1)^{N-\ell}$ . Set  $Q := (0, 1)^N$  and  $\eta := g(x')\psi(x'')dx' \in L_c^1(Q; \Lambda^\ell)$ . We note that  $d\eta = g(x')d\psi(x'') \wedge dx' \in L_c^1(Q; \Lambda^{\ell+1})$ . Let us also note that  $d\eta = 0$  in  $\mathbf{R}^\ell \times V$ . By Corollary 8 with  $U = (0, 1)^\ell \times V$ , there exists some  $\omega \in L_c^q(Q; \Lambda^\ell)$ ,  $1 \leq q < N/(N-1)$ , such that  $d\omega = d\eta$  and  $\omega \in C^2((0, 1)^\ell \times V)$ .

Consider now the closed form  $f := \eta - \omega \in L_c^1(Q; \Lambda^\ell)$ . We claim that there exists no  $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$  such that  $d\lambda = f$ . Argue by contradiction and let  $\lambda_i$  denote the coefficient, in  $\lambda$ , of  $dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_\ell$ ,  $1 \leq i \leq \ell$ . Let  $\omega_0$  denote the coefficient of  $dx'$  in  $\omega$ . Then, in  $\mathbf{R}^\ell \times V$ , we have

$$\sum_{i=1}^{\ell} (-1)^{i+1} \partial_i \lambda_i(x', x'') = g(x')\psi(x'') - \omega_0(x', x'') = g(x') - \omega_0(x', x''). \quad (4)$$

Hence, for a.e.  $x'' \in V$ , the following equation is satisfied in  $\mathcal{D}'(\mathbf{R}^\ell)$ :

$$\sum_{i=1}^{\ell} (-1)^{i+1} \partial_i \lambda'_i = g - \omega'_0, \quad (5)$$

with

$$\lambda'_i := \lambda_i(\cdot, x'') \in W_{loc}^{1,1}(\mathbf{R}^\ell) \text{ and } \omega'_0 = \omega_0(\cdot, x'') \in C_c^2((0, 1)^\ell). \quad (6)$$

The above properties (5) and (6), combined with the embedding  $W_{loc}^{1,1}(\mathbf{R}^\ell) \hookrightarrow L^{\ell/(\ell-1)}(\mathbf{R}^\ell)$ , contradict (3).  $\square$

*Remark 1* *We have actually proved the following improvement of Theorem 3. Let  $N \geq 3$  and  $2 \leq \ell \leq N-1$ . Then there exists some  $f \in L_c^1(\mathbf{R}^d; \Lambda^\ell)$*

satisfying  $df = 0$  and such that the system  $d\lambda = f$  has no solution

$$\lambda \in L_{loc}^1(\mathbf{R}^{N-\ell}; L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \Lambda^{\ell-1})).$$

*Remark 2* A similar question can be raised in  $L^\infty$ . We have the following analogue of Theorem 3.

**Theorem 9** *Let  $N \geq 3$ . Let  $2 \leq \ell \leq N - 1$ . Then there exists some  $f \in L_c^\infty(\mathbf{R}^N; \Lambda^\ell)$  such that  $df = 0$  and the equation  $d\lambda = f$  has no solution  $\lambda \in W_{loc}^{1,\infty}(\mathbf{R}^N; \Lambda^{\ell-1})$ .*

*The proof of Theorem 9 is very similar to the one of Theorem 3. The main difference is the starting point, in dimension  $\ell$ . Here, we use the fact that there exists some  $g \in L_c^\infty(\mathbf{R}^\ell)$ , with zero integral, such that the equation  $\operatorname{div} X = g$  has no solution  $X \in W_{loc}^{1,\infty}(\mathbf{R}^\ell; \mathbf{R}^\ell)$  (see [5]).*

### 3 Solution in $L^{N/(N-1)}$ when $1 \leq \ell \leq N - 1$

As mentioned in the introduction, when  $\ell = N$ , the system (1) with right-hand side  $f \in L^1$  need not have a solution  $\lambda \in L_{loc}^{N/(N-1)}$ . In view of Theorem 3 and of Proposition 7, it is natural to ask whether, in the remaining cases  $1 \leq \ell \leq N - 1$ , given a closed  $\ell$ -form  $f \in L_c^1$ , it is possible to solve (1) with  $\lambda \in L_{loc}^{N/(N-1)}$ . This is clearly the case when  $\ell = 1$  (by the Sobolev embedding  $W_{loc}^{1,1} \hookrightarrow L_{loc}^{N/(N-1)}$ ). Moreover, we may pick  $\lambda \in W^{1,1}$ . The remaining cases are settled by our next result. In what follows, we do not make any support assumption on  $f$ , and therefore the case where  $\ell = 1$  is also of interest.

**Proposition 10** *Let  $N \geq 2$  and  $1 \leq \ell \leq N - 1$ . Then, for every  $f \in L^1(\mathbf{R}^N; \Lambda^\ell)$  with  $df = 0$ , there exists some  $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$  such that  $f = d\lambda$ .*

*Proof.* Suppose  $f \in L^1(\mathbf{R}^N; \Lambda^{\ell-1})$  with  $df = 0$  as above. According to Bourgain and Brezis [3] (see Corollary 20 in [3] for a very similar statement; see also Theorem 3 in [7]), we have

$$\left| \int_{\mathbf{R}^d} \langle \psi, f \rangle \right| \lesssim \|f\|_{L^1} \|d^* \psi\|_{L^N}, \quad \forall \psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell). \quad (7)$$

Consider the functional

$$L_f : S = \{d^* \psi; \quad \psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell)\} \rightarrow \mathbf{R}, \quad L_f(d^* \psi) := \int_{\mathbf{R}^d} \langle \psi, f \rangle.$$

Here,  $S$  is endowed with the  $L^N$ -norm. The inequality (7) shows that  $L_f$  is well-defined and bounded. By the Hahn-Banach theorem, there exists an extension  $\tilde{L}_f : L^N(\mathbf{R}^N; \Lambda^{\ell+1}) \rightarrow \mathbf{R}$  of  $L_f$  with  $\|\tilde{L}_f\| = \|L_f\|$ . Hence, there exists an  $(\ell - 1)$ -form  $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$  such that

$$\int_{\mathbf{R}^N} \langle \psi, f \rangle = L_f(d^*\psi) = \tilde{L}_f(d^*\psi) = \int_{\mathbf{R}^N} \langle d^*\psi, \lambda \rangle = \int_{\mathbf{R}^N} \langle \psi, d\lambda \rangle$$

for all  $\ell$ -forms  $\psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell)$ .

This implies that  $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$  satisfies  $d\lambda = f$ .  $\square$

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