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On the representation as exterior differentials of closed forms with L^1 -coefficients

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Abstract

Let $N \geq 2$. If $g \in L_c^1(\mathbf{R}^N)$ has zero integral, then the equation $\operatorname{div} X = g$ need not have a solution $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$ (Wojciechowski 1999) or even $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$ (Bourgain and Brezis 2003). Using these results, we prove that, whenever $N \geq 3$ and $2 \leq \ell \leq N - 1$, there exists some ℓ -form $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$ such that $df = 0$ and the equation $d\lambda = f$ has no solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$. This provides a negative answer to a question raised by Baldi, Bruno and Pansu (2019).

Résumé

Sur la représentation comme différentielles extérieures des formes fermées à coefficients L^1

Soit $N \geq 2$. Si $g \in L_c^1(\mathbf{R}^N)$ est d'intégrale nulle, alors en général il n'est pas possible de résoudre l'équation $\operatorname{div} X = g$ avec $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$ (Wojciechowski 1999), ou même $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$ (Bourgain et Brezis 2003). En utilisant ces résultats, nous prouvons que, pour $N \geq 3$ et $2 \leq \ell \leq N - 1$, il existe une ℓ -forme $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$ avec $df = 0$ et telle que l'équation $d\lambda = f$ n'ait pas de solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$. Ceci donne une réponse négative à une question posée par Baldi, Bruno and Pansu (2019).

Version française abrégée

Répondant à une question posée par Baldi, Bruno et Pansu ([1]), nous montrons le résultat suivant :

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Théorème 1 *Soit N, ℓ entiers tels que $N \geq 3$ et $2 \leq \ell \leq N - 1$. Il existe une ℓ -forme différentielle fermée à coefficients L^1 , $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$, telle que l'équation $d\lambda = f$ n'ait pas de solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$.*

La preuve est faite par l'absurde et repose sur un résultat de non-existence bien connu pour l'équation $\operatorname{div} X = g$. Plus précisément, il existe des fonctions $g \in L^1_c((0, 1)^\ell; \mathbf{R})$ avec $\int g = 0$ et telles que l'équation $\operatorname{div} X = g$ n'ait pas de solution $X \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ (voir Bourgain et Brezis [2]). À partir d'une telle fonction g , nous construisons une forme différentielle explicite f , fermée, à support compact et à coefficients L^1 . En supposant le Théorème 1 faux, nous obtenons l'existence d'une fonction $G \in C^2_c((0, 1)^\ell; \mathbf{R})$ et d'un champ $Y \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ tels que $\operatorname{div} Y = g + G$, ce qui contredit les propriétés de la fonction g .

En combinant ce résultat avec les résultats de [5], [2], nous obtenons la conséquence suivante.

Corollaire 2 *Soient $N \geq 2$ et $1 \leq \ell \leq N$. Soit \mathcal{A} la classe des ℓ -formes $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$ satisfaisant la condition de compatibilité $df = 0$ (si $1 \leq \ell \leq N - 1$), respectivement $\int f = 0$ (si $\ell = N$). Alors nous avons l'équivalence $1 \iff 2$, où*

1. *l'équation $d\lambda = f$ a une solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ pour tout $f \in \mathcal{A}$.*
2. *$\ell = 1$.*

1 Introduction

We consider the Hodge system

$$d\lambda = f \text{ in } \mathbf{R}^N, \tag{1}$$

where f and λ are ℓ and $(\ell - 1)$ -forms respectively, f being given and satisfying the compatibility condition $df = 0$. We focus on the case where f has L^1 coefficients.

To start with, let us recall some known facts about the cases $\ell = N$ and $\ell = 1$.

In the case $\ell = N$, (1) reduces to the divergence equation. It was first shown by Wojciechowski [5] that there exists $g \in L^1_c(\mathbf{R}^N)$, with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W_{loc}^{1,1}(\mathbf{R}^N; \mathbf{R}^N)$. On the other hand, Bourgain and Brezis [2] proved, using a different method, the following: there exists $g \in L^1_c(\mathbf{R}^N)$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L_{loc}^{N/(N-1)}(\mathbf{R}^N; \mathbf{R}^N)$. In view of the embedding $W_{loc}^{1,1} \hookrightarrow L_{loc}^{N/(N-1)}$, this improves [5].

In the case $\ell = 1$, (1) reduces to the following “gradient” equation

$$\nabla \lambda = f, \tag{2}$$

where f is a vector field satisfying the compatibility condition $\nabla \times f = 0$ and λ is a function. Unlike the case $\ell = N$, this time (2) has a solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N)$. Actually, any solution of (2) belongs to $W_{loc}^{1,1}$ and, moreover, if f is compactly supported then we may choose $\lambda \in W^{1,1}$.

The question of the solvability in $W_{loc}^{1,1}$ of the system (1) with datum in L^1 in the remaining cases, i.e., $2 \leq \ell \leq N - 1$, has been recently raised by Baldi, Bruno and Pansu [1]. Our main result settles this problem.

Theorem 3 *Let $N \geq 3$. Let $2 \leq \ell \leq N - 1$. Then there exists some $f \in L_c^1(\mathbf{R}^N; \Lambda^\ell)$ such that $df = 0$ and the equation $d\lambda = f$ has no solution $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$.*

The proof of Theorem 3 we present is a simplification, communicated to the author by P. Mironescu, of the original one. This simplified version has the advantage of being relatively self-contained and elementary.

2 Proof of Theorem 3

We start with some auxiliary results.

Lemma 4 *Let $1 \leq \kappa \leq N - 1$ and $f \in L_c^1(\mathbf{R}^N; \Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in L_{loc}^q(\mathbf{R}^N; \Lambda^{\kappa-1})$, for all $1 \leq q < N/(N - 1)$, such that $d\omega = f$.*

Proof. Let E be “the” fundamental solution of Δ and set $\eta := E * f$. Let $\omega := d^*\eta$. First, $\eta \in W_{loc}^{1,q}(\mathbf{R}^N)$ (by elliptic regularity) and thus $\omega \in L_{loc}^q(\mathbf{R}^N)$, $1 \leq q < N/(N - 1)$. Next, $d\eta = E * df = 0$. Finally,

$$d\omega = dd^*\eta = (dd^* + d^*d)\eta = \Delta\eta = f.$$

Hence, ω has the required properties. □

A similar argument leads to the following.

Lemma 5 *Let $1 < r < \infty$, $k \in \mathbf{N}$. Let $1 \leq \kappa \leq N - 1$. Let $f \in W_c^{k,r}(\mathbf{R}^N; \Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in W_{loc}^{k+1,r}(\mathbf{R}^N; \Lambda^{\kappa-1})$ such that $d\omega = f$.*

We next recall the following “inversion of d with loss of regularity”. It is folklore, and one possible proof consists of using Bogovskii’s formula (see for example [4, Corollary 3.3 and Corollary 3.4] for related arguments).

Lemma 6 *Let $1 \leq \kappa \leq N-1$. Let Q be an open cube in \mathbf{R}^N . Then there exists some integer $m = m(N, \kappa)$ such that if $f \in C_c^k(Q; \Lambda^{\kappa-1})$, with $k \in \{m, m+1, \dots\} \cup \{\infty\}$, satisfies $df = 0$, then there exists some $\omega \in C_c^{k-m}(Q; \Lambda^{\kappa-1})$ such that $d\omega = f$.*

Combining Lemmas 4–6, we obtain the following

Proposition 7 *Let $1 \leq \kappa \leq N-1$. Let Q be an open cube in \mathbf{R}^N . Let $f \in L_c^1(Q; \Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in L_c^q(Q; \Lambda^{\kappa-1})$, for all $1 \leq q < N/(N-1)$, such that $d\omega = f$.*

Proof. Set $f_0 := f$. We consider a sequence $(\zeta_j)_{j \geq 0}$ in $C_c^\infty(Q; \mathbf{R})$ such that $\zeta_0 = 1$ on $\text{supp } f_0$ and, for $j \geq 1$, $\zeta_j = 1$ on $\text{supp } \zeta_{j-1}$. We let η_0 be a solution of $d\eta_0 = f_0$, constructed as in Lemma 4. We set $\omega_0 := \zeta_0 \eta_0$, so that $\omega_0 \in L_c^q(Q; \Lambda^{\kappa-1})$, $1 \leq q < N/(N-1)$ and

$$d\omega_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 d\eta_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 f_0 = \underbrace{d\zeta_0 \wedge \eta_0}_{-f_1} + f_0.$$

Let us note that $df_1 = -d^2\omega_0 + df_0 = 0$ and that $f_1 \in L_c^q(Q; \Lambda^\kappa)$, $1 \leq q < N/(N-1)$.

Fix some $1 < r < N/(N-1)$. By Lemma 5, there exists some $\eta_1 \in W_{loc}^{1,r}(\mathbf{R}^N; \Lambda^{\kappa-1})$ such that $d\eta_1 = f_1$. Set $\omega_1 := \zeta_1 \eta_1$. Then $\omega_1 \in W_c^{1,r}(Q; \Lambda^{\kappa-1})$ and, as above, $f_2 := f_1 - d\omega_1$ satisfies $df_2 = 0$ and $f_2 \in W_c^{1,r}(Q; \Lambda^\kappa)$. Applying again Lemma 5, we may find $\eta_2 \in W_{loc}^{2,r}(\mathbf{R}^N)$ such that $d\eta_2 = f_2$.

Iterating the above, we have

$$\begin{aligned} \omega_0 + \dots + \omega_j &\in L_c^q(Q; \Lambda^{\kappa-1}), \quad 1 \leq q < N/(N-1), \\ d(\omega_0 + \dots + \omega_j) &= f_0 - f_j, \quad \text{with } df_j = 0 \text{ and } f_j \in W_c^{j,r}(Q; \Lambda^\kappa). \end{aligned}$$

Let now j be such that $W^{j,q}(Q) \hookrightarrow C^m(Q)$, with m as in Lemma 6. Let $\xi \in C_c^0(Q; \Lambda^{\kappa-1})$ be such that $d\xi = -f_j$. Set $\omega := \omega_0 + \dots + \omega_j + \xi$. Then ω has all the required properties. \square

Let us note the following consequence of hypoellipticity of Δ and of the proofs of Proposition 7 and Lemmas 4 and 5 (but *not* of their statements).

Corollary 8 *Assume, in addition to the hypotheses of Proposition 7, that $f \in C^\infty(U)$ for some open set $U \subset Q$. Let $s \in \mathbf{N}$. Then we may choose ω such that, in addition, $\omega \in C^s(U)$.*

Proof of Theorem 3. We write the variables in \mathbf{R}^N as follows: $x = (x', x'')$, with $x' \in \mathbf{R}^\ell$ and $x'' \in \mathbf{R}^{N-\ell}$.

Pick some $g \in L_c^1((0, 1)^\ell; \mathbf{R})$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ (see [5], [2]). Clearly, for any $G \in C_c^2((0, 1)^\ell; \mathbf{R})$,

$$\text{the equation } \operatorname{div} Y = g + G \text{ has no solution } Y \in L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \mathbf{R}^\ell). \quad (3)$$

Let $\psi \in C_c^\infty((0, 1)^{N-\ell})$ be such that $\psi \equiv 1$ in some nonempty open set $V \subset (0, 1)^{N-\ell}$. Set $Q := (0, 1)^N$ and $\eta := g(x')\psi(x'')dx' \in L_c^1(Q; \Lambda^\ell)$. We note that $d\eta = g(x')d\psi(x'') \wedge dx' \in L_c^1(Q; \Lambda^{\ell+1})$. Let us also note that $d\eta = 0$ in $\mathbf{R}^\ell \times V$. By Corollary 8 with $U = (0, 1)^\ell \times V$, there exists some $\omega \in L_c^q(Q; \Lambda^\ell)$, $1 \leq q < N/(N-1)$, such that $d\omega = d\eta$ and $\omega \in C^2((0, 1)^\ell \times V)$.

Consider now the closed form $f := \eta - \omega \in L_c^1(Q; \Lambda^\ell)$. We claim that there exists no $\lambda \in W_{loc}^{1,1}(\mathbf{R}^N; \Lambda^{\ell-1})$ such that $d\lambda = f$. Argue by contradiction and let λ_i denote the coefficient, in λ , of $dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_\ell$, $1 \leq i \leq \ell$. Let ω_0 denote the coefficient of dx' in ω . Then, in $\mathbf{R}^\ell \times V$, we have

$$\sum_{i=1}^{\ell} (-1)^{i+1} \partial_i \lambda_i(x', x'') = g(x')\psi(x'') - \omega_0(x', x'') = g(x') - \omega_0(x', x''). \quad (4)$$

Hence, for a.e. $x'' \in V$, the following equation is satisfied in $\mathcal{D}'(\mathbf{R}^\ell)$:

$$\sum_{i=1}^{\ell} (-1)^{i+1} \partial_i \lambda'_i = g - \omega'_0, \quad (5)$$

with

$$\lambda'_i := \lambda_i(\cdot, x'') \in W_{loc}^{1,1}(\mathbf{R}^\ell) \text{ and } \omega'_0 = \omega_0(\cdot, x'') \in C_c^2((0, 1)^\ell). \quad (6)$$

The above properties (5) and (6), combined with the embedding $W_{loc}^{1,1}(\mathbf{R}^\ell) \hookrightarrow L^{\ell/(\ell-1)}(\mathbf{R}^\ell)$, contradict (3). \square

Remark 1 *We have actually proved the following improvement of Theorem 3. Let $N \geq 3$ and $2 \leq \ell \leq N-1$. Then there exists some $f \in L_c^1(\mathbf{R}^d; \Lambda^\ell)$*

satisfying $df = 0$ and such that the system $d\lambda = f$ has no solution

$$\lambda \in L_{loc}^1(\mathbf{R}^{(N-\ell)}; L_{loc}^{\ell/(\ell-1)}(\mathbf{R}^\ell; \Lambda^{\ell-1})).$$

3 Solution in $L^{N/(N-1)}$ when $1 \leq \ell \leq N - 1$

As mentioned in the introduction, when $\ell = N$, the system (1) with right-hand side $f \in L^1$ need not have a solution $\lambda \in L_{loc}^{N/(N-1)}$. In view of Theorem 3 and of Proposition 7, it is natural to ask whether, in the remaining cases $1 \leq \ell \leq N - 1$, given a closed ℓ -form $f \in L_c^1$, it is possible to solve (1) with $\lambda \in L_{loc}^{N/(N-1)}$. This is clearly the case when $\ell = 1$ (by the Sobolev embedding $W_{loc}^{1,1} \hookrightarrow L_{loc}^{N/(N-1)}$). Moreover, we may pick $\lambda \in W^{1,1}$. The remaining cases are settled by our next result. In what follows, we do not make any support assumption on f , and therefore the case where $\ell = 1$ is also of interest.

Proposition 9 *Let $N \geq 2$ and $1 \leq \ell \leq N - 1$. Then, for every $f \in L^1(\mathbf{R}^N; \Lambda^\ell)$ with $df = 0$, there exists some $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ such that $f = d\lambda$.*

Proof. Suppose $f \in L^1(\mathbf{R}^N; \Lambda^{\ell-1})$ with $df = 0$ as above. According to Bourgain and Brezis [3] (see Corollary 20 in [3] for a very similar statement), we have

$$\left| \int_{\mathbf{R}^d} \langle \psi, f \rangle \right| \lesssim \|f\|_{L^1} \|d^* \psi\|_{L^N}, \quad \forall \psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell). \quad (7)$$

Consider the functional

$$L_f : S = \{d^* \psi; \quad \psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell)\} \rightarrow \mathbf{R}, \quad L_f(d^* \psi) := \int_{\mathbf{R}^d} \langle \psi, f \rangle.$$

Here, S is endowed with the L^N -norm. The inequality (7) shows that L_f is well-defined and bounded. By the Hahn-Banach theorem, there exists an extension $\tilde{L}_f : L^N(\mathbf{R}^N; \Lambda^{\ell+1}) \rightarrow \mathbf{R}$ of L_f with $\|\tilde{L}_f\| = \|L_f\|$. Hence, there exists an $(\ell - 1)$ -form $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ such that

$$\int_{\mathbf{R}^N} \langle \psi, f \rangle = L_f(d^* \psi) = \tilde{L}_f(d^* \psi) = \int_{\mathbf{R}^N} \langle d^* \psi, \lambda \rangle = \int_{\mathbf{R}^N} \langle \psi, d\lambda \rangle$$

for all ℓ -forms $\psi \in C_c^\infty(\mathbf{R}^N; \Lambda^\ell)$.

This implies that $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ satisfies $d\lambda = f$. \square

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