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Monique Dauge, Michael A. Hall, San Vu Ngoc. Asymptotic Lattices, Good Labellings, and the Rotation Number for Quantum Integrable Systems. *Discrete and Continuous Dynamical Systems - Series A*, 2022, 10.3934/dcds.2022120 . hal-02104892v3

**HAL Id: hal-02104892**

**<https://hal.science/hal-02104892v3>**

Submitted on 4 Jul 2022

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# Asymptotic Lattices, Good Labellings, and the Rotation Number for Quantum Integrable Systems

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VŨ NGỌC SAN\*

## Abstract

This article introduces the notion of good labellings for asymptotic lattices in order to study joint spectra of quantum integrable systems from the point of view of inverse spectral theory. As an application, we consider a new spectral quantity for a quantum integrable system, the quantum rotation number. In the case of two degrees of freedom, we obtain a constructive algorithm for the detection of appropriate labellings for joint eigenvalues, which we use to prove that, in the semiclassical limit, the quantum rotation number can be calculated on a joint spectrum in a robust way, and converges to the well-known classical rotation number. The general results are applied to the semitoric case where formulas become particularly natural.

**Keywords :** Liouville integrable systems, rotation number, semitoric systems, quantization, pseudodifferential operators, semiclassical analysis, asymptotic lattice, good labelling, inverse problem, symplectic invariants, lattice detection.

**MS Classification :** 81S10, 81Q20, 58J40, 58J50, 65L09.

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# 1 Introduction

## 1.1 Motivations and aims

Our motivations mainly come from the connection between integrable two-degree of freedom Hamiltonian systems and the spectrum of related quantum systems. Let  $M$  be  $\mathbb{R}^4$  or, more generally, a 4-dimensional symplectic manifold. If a two-degree of freedom Hamiltonian system, given by a Hamiltonian  $H \in C^\infty(M)$ , is integrable in the classical Liouville sense, then it is well known that most of the dynamics takes place on invariant tori of dimension 2. On each such torus  $\Lambda$ , the motion is particularly simple: in suitable angle coordinates  $(\theta_1, \theta_2)$  on  $\Lambda$ , the Hamiltonian vector field  $\mathcal{X}_H$  is constant:

$$(\mathcal{X}_H)|_\Lambda = \alpha_1 \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_2}.$$

This is the content of the classical action-angle theorem (see for instance [26] and the references therein). The direction in  $\mathbb{RP}^1$  given by the frequency vector  $w := (\alpha_1, \alpha_2)$  is called the *rotation number*. If  $w$  is rational (by this we mean that  $\alpha_1$  and  $\alpha_2$  are linearly dependent over  $\mathbb{Q}$ ), then the trajectory of the Hamiltonian system on  $\Lambda$  is periodic. On the contrary, when  $w$  is irrational, this trajectory is dense on  $\Lambda$ . Thus, the knowledge of  $w$  gives important information on the nature of the dynamics. Understanding the variation of  $w$  is also crucial for the study of perturbations of  $H$ , via the various ‘‘KAM’’ theorems (see for instance the review article [58]).

The first goal of this article is to investigate the effect of this dynamical quantity on a *quantum* system. Assume for instance that we consider a Schrödinger operator  $P = -\hbar^2 \Delta + V$ , with a smooth potential  $V$  on a 2-dimensional Riemannian manifold  $X$ ; assume moreover that this operator is *quantum integrable* in the sense that there exists another selfadjoint operator  $\hat{J}$ , commuting with  $P$ , whose principal symbol  $J$  is an independent integral of motion for the

Hamiltonian  $H(x, \xi) = \|\xi\|^2 + V(x)$ . Therefore  $H$  is completely integrable, and we can ask the question: what is the manifestation of the underlying classical rotation number  $w$  on the spectrum of  $P$ ? It has been acknowledged for a long time that so-called non-degeneracy hypothesis, or diophantine hypothesis on the rotation number, à la KAM, can be crucial to obtain a good description of the spectrum of quantum integrable systems, see for instance [4, 5]. But, is there a direct, or recognizable, signature of the classical rotation number on the spectrum? A first answer was given by Hitrik and Sjöstrand in a series of papers [34, 35] where they study the case of weakly non-selfadjoint operators (*i.e.*,  $V$  is perturbed by a small imaginary term), and they proved that the asymptotics of the spectrum, in the semiclassical limit  $\hbar \rightarrow 0$ , exhibit very different behaviors depending on the rationality of the rotation number  $w$ .

In the purely selfadjoint case, the construction of quasi-modes in the various situations where  $w$  is strongly irrational or not was known for a long time, see for instance [41, 12]; however, perhaps surprisingly, only the non-selfadjoint case, where the spectrum, instead of being one-dimensional, is deployed in the complex plane, gives some hope to recognize useful geometric structures from the eigenvalues themselves. For quasi-periodic dynamics, this idea was exploited to recover the Birkhoff normal form from the complex spectrum in [30], and to define quantum monodromy in the non-selfadjoint case [57]. A strong motivation for our work is the recent paper [33], which naturally leads to this intriguing question: can you detect the rationality of the rotation number from the spectrum?

It is precisely this type of inverse spectral problem that we study in this paper. Here we stick to the simpler “normal” case, which means that instead of considering the spectrum of a truly non-selfadjoint operator, we consider the joint spectrum of a pair of commuting selfadjoint operators. The fully non-selfadjoint case is still largely open. We prove that one can define a *quantum rotation number*, in a very natural and concrete way, from the joint spectrum of two commuting operators as soon as we know appropriate labelling of the joint spectrum by pairs of integers, given for instance by the Bohr-Sommerfeld quantization rule that provides us “quantum numbers”; moreover we show that, in a suitable sense, this quantum rotation number converges to the classical rotation number in the semiclassical limit  $\hbar \rightarrow 0$ . This result, which we actually prove in any dimension, is however more delicate to use than one could think at first sight if one does not have an a priori knowledge of appropriate labellings of the joint spectra as  $\hbar \rightarrow 0$ . A fully relevant question is the possibility to construct such labellings with suitable regularity properties from the bare data of joint spectra.

Precisely because of this issue, the second goal of this article is to set a rigorous ground for the theory of labellings of joint spectra. Given a discrete set of points in  $\mathbb{R}^2$  (obtained as the joint spectrum of some unknown system), how can you label these points in a sensible manner, in a similar way as the usual Bohr-Sommerfeld rules would tell you to do so? This turns out to be a delicate question, even for systems with global action-angle variables. Such discrete sets are called asymptotics lattices; in this article, the theory is build up from scratch up to the point where it is finally able to provide the proof of the inverse spectral problem for the quantum rotation number. However, we expect further developments and generalizations in the future, for instance in relation to trace formulas.

After going through this constructive and combinatorial issue, we finally show that one can detect the classical rotation number from the joint spectrum of an associated quantum integrable system. We also believe that the quantum rotation number will prove to be a useful object in the study of quantum integrable (or near-integrable) systems, and we hope to apply this idea on concrete systems in a near future. The recent article [32] on asymmetric-top molecules nicely supports this idea.

The second motivation of this paper comes from the theory of semitoric systems, see [69, 52, 56, 54]. The general conjecture for quantum semitoric systems is that one can always recover the underlying classical system from the spectrum of a quantum semitoric system [53, Conjecture 9.1]. In [54], a sketch of proof was provided, which builds on a number of geometric and spectral invariants, like quantum monodromy [20, 65]. When we wrote this paper, the inverse result was known in the toric case [11], and recent advances showed that a general conjecture like this was not out of reach [44, 55, 43], although probably under suitable genericity assumptions. Thus, we were motivated to exhibit a new invariant that can be recovered from the spectrum. It now turns out that our work, and in particular the precise study of asymptotics lattice and their labellings (Sections 3.2 and 3.5), has been used crucially to finally completely solve the semitoric conjecture, in a constructive way [45].

While the rotation number can be defined for any completely integrable system, it bears a particular nice form in the case of semitoric systems, see Section 2.3. We believe that our semitoric theory and algorithms should find natural applications in the study of axisymmetric Schrödinger operators, a question that we hope to investigate in a future work.

Our work contributes to the general inverse spectral theory in the semiclassical limit. Of course, in such generality, this question has a long history, especially when restricted to the particular case of Laplace-Beltrami or Schrödinger operators; see for instance the survey [22] and the references therein. The semiclassical inverse problem for more general Hamiltonians, which is our concern here, was also considered, albeit by less numerous studies, see in particular [37] for a very general treatment. In the case of Liouville integrable systems, a line of program, closely following recent development in the classification of Lagrangian fibrations, was proposed in [64, 54, 61], but prior works have been produced in the Riemannian case, see for instance [71].

## 1.2 Outline and sketch of results

Our paper consists of two main sections, 2 and 3, devoted to classical and quantum rotation numbers, respectively. Correspondences between the two are proved in Section 3.

In Section 2 we recall and generalize, in the setting of completely integrable Hamiltonians  $H$  on  $2n$ -dimensional symplectic manifolds, the classical definition of the rotation number: for such  $H$ , associated with its constants of motions  $(f_1, \dots, f_{n-1})$ , there exist action-angle coordinates  $(I, \theta) \in T^*\mathbb{T}^n$ , in which  $H = g(I_1, \dots, I_n)$  and a local definition of the rotation number is, for each chosen Liouville torus  $\Lambda$

$$[w_I](\Lambda) = [\partial_1 g(I(\Lambda)) : \dots : \partial_n g(I(\Lambda))] \in \mathbb{RP}^{n-1}.$$

We explain under which conditions it is well defined (which seems to be an information that is difficult to locate in the literature), proving in particular that the rotation number  $[w_{I'}]$  associated with another set  $I'$  of action variables is deduced from  $[w_I]$  by a  $\mathrm{SL}(n, \mathbb{Z})$  transformation (Lemma 2.2). We also exhibit its relationship with Hamiltonian monodromy (Proposition 2.6). Then we restrict to the case  $n = 2$  to address the semitoric case in which the rotation number is well-defined modulo integers (Proposition 2.10).

In Section 3, we first recall known facts about the joint spectrum of semi-classical commuting  $\hbar$ -pseudodifferential operators  $(P_1, \dots, P_n)$ , arriving to the fundamental result that the joint spectrum  $\Sigma_{\hbar}$  is (locally) a deformed lattice modulo  $\mathcal{O}(\hbar^\infty)$ :

$$G_{\hbar}(\hbar k_1, \dots, \hbar k_n) + \mathcal{O}(\hbar^\infty), \quad (k_1, \dots, k_n) \in \mathbb{Z}^n$$

in which  $G_{\hbar}$  is a smooth map admitting an asymptotic expansion in integer powers of  $\hbar$  (Theorem 3.2). These Bohr-Sommerfeld quantization rules tell two things: first, the spectrum as a set resembles a lattice, and second, its elements can be numbered (or “labelled”) by multi-integers  $(k_1, \dots, k_n)$  in a coherent way as  $\hbar \rightarrow 0$ . This is what we call a “good labelling”.

In the special case when  $P_n = \hat{H}$  has the Hamiltonian  $H$  as principal symbol, and  $J_j$  is the principal symbol of  $P_j$ , so that the map  $(J_1, \dots, J_{n-1}, H)$  form a  $n$ -degree of freedom completely integrable system  $F$ , having at hands the joint spectra  $\Sigma_{\hbar}$  and their good labellings

$$\mathbb{Z}^n \ni (k_1, \dots, k_n) =: k \mapsto \lambda_k(\hbar) = (\lambda_k^{(1)}(\hbar), \dots, \lambda_k^{(n)}(\hbar)) \in \Sigma_{\hbar},$$

we define quantum rotation numbers as projectivized finite differences of the eigenvalues  $E_k(\hbar) = \lambda_k^{(n)}(\hbar)$  of  $\hat{H}$

$$[\hat{w}_{\hbar}](k) = [E_{k+e_1}(\hbar) - E_k(\hbar) : \dots : E_{k+e_n}(\hbar) - E_k(\hbar)] \in \mathbb{RP}^{n-1},$$

with  $e_1, \dots, e_n$  unit coordinate vectors in  $\mathbb{Z}^n$ . Then we prove (Theorem 3.30) that the quantum rotation number is a semiclassical deformation of the classical rotation number, namely: if  $c$  is a regular value of  $F$ , there exist action variables  $I$  so that

$$[\hat{w}_{\hbar}](k_{\hbar}) = [w_I](\Lambda) + \mathcal{O}(\hbar), \quad \hbar \rightarrow 0,$$

where the labels  $k_{\hbar}$  are such that  $\lambda_{k_{\hbar}}(\hbar) = c + \mathcal{O}(\hbar)$  and  $\Lambda$  is the Liouville torus  $F^{-1}(c)$ .

Though satisfying, this result is not the end of the story, but rather its starting point. The question that we address in the rest of the paper is, from the sole knowledge of the joint spectra  $\Sigma_{\hbar}$  in some window  $\Omega$  for some family of values of  $\hbar$  tending to 0, can we construct labellings of these sets so that a quantum rotation number can be computed? Before introducing an algorithm, we analyze in more detail the properties of different sorts of labellings of an “asymptotic lattice” (a family of sets  $\mathcal{L}_{\hbar}$  displaying the same structure as the joint spectra of commuting  $\hbar$ -pseudodifferential operators). We prove that good labellings are not unique, but can be deduced from each other by transformations in the affine group  $\mathrm{SL}(n, \mathbb{Z}) \times \mathbb{Z}^n$  (Proposition 3.22). We then face the issue that, in general, good labellings are not algorithmically accessible. Indeed,

the asymptotic lattice does not necessarily concentrate to a point in the observation window, but rather goes through. That is why we introduce a relaxed version of labelling, the *linear labellings*, which coincide with a good labelling up to a translation  $\varkappa_{\hbar} \in \mathbb{Z}^n$  (depending on  $\hbar$  in general). Fortunately, linear labellings retain enough information to allow for the determination of a rotation number.

The core of the paper is to obtain a robust and constructive determination of linear labellings for arbitrary 2-dimensional asymptotic lattices. To achieve this goal, we propose a two part algorithm. In the first part of the algorithm, for a chosen  $\hbar$ , small enough, we find a labelling of  $\mathcal{L}_{\hbar}$  by a set of bi-integers  $(m, n) \in \mathbb{Z}^2$  containing  $\mathbb{Z}^2 \cap B(0, \rho\hbar^{-1})$  for a  $\rho > 0$ . The label  $(0, 0)$  is associated with a closest element  $\lambda \in \mathcal{L}_{\hbar}$  to a distinguished point  $c \in \Omega$ . The conclusive statement of this part is Theorem 3.44 which, despite its apparent simplicity, necessitates an intricate proof. The most delicate part is to prove that all points of  $\mathcal{L}_{\hbar}$  (in the window  $\Omega$ ) are effectively labelled, and for this we use the dimension  $n = 2$  in a crucial way. In the second part of the algorithm, we combine this construction for any  $\hbar$  belonging to a sequence tending to 0, with appropriate  $\text{SL}(n, \mathbb{Z})$  transformations, in order to finally obtain a linear labelling (Theorem 3.48). The determination of a quantum rotation number from the sole knowledge of a joint spectrum, and its convergence to a classical rotation number come as natural consequences of this algorithm (Theorem 3.49).

In the semitoric case, the Hamiltonian  $J$  induces a periodic flow. This has remarkable consequences both on (quantum) rotation numbers and joint spectra. On the one hand, the classical rotation number can be identified with an angle with a natural dynamical interpretation (Proposition 2.10). On the other hand, the joint spectrum locally displays a nice structure of vertical bands (Proposition 3.33); this makes the process of detecting labellings easier. We show in Section 3.5.2 how to correct the first part of the general algorithm in order to obtain a labelling that respects the semitoric structure. As a consequence, we obtain that the semitoric version of the rotation number can be computed from the joint spectrum (Corollaries 3.52 and 3.51).

**Acknowledgements** We would like to thank the anonymous referees for their insights, which have substantially improved the presentation and clarified the geometric background. The authors are happy to acknowledge the support of the Centre Henri Lebesgue, program ANR-11-LABX-0020-0. M.H. benefited from ANR NOSEVOL (ANR 2011 BS01019 01) during the start of this project.

## 2 The Classical Rotation Number

In this section, we first recall the notion of rotation number for integrable Hamiltonian systems of any dimension. A general discussion for  $n = 2$  can also be found in [7]. Then, we show the relationship between the globalized version of rotation numbers with Hamiltonian monodromy (Section 2.2). Finally, we explain how the usual rotation number (which is an angle) fits within the framework of semitoric systems when  $n = 2$  (Section 2.3).



Let  $H$  be a completely integrable Hamiltonian on a symplectic manifold  $M$  of dimension  $2n$ : there exists smooth functions  $f_1, \dots, f_{n-1}$  on  $M$  such that, letting  $f_n := H$ , then for all  $j$  and  $k$ ,  $\{f_j, f_k\} = 0$  and the differentials  $df_1, \dots, df_n$  are almost everywhere independent. From the dynamical viewpoint, the Hamiltonian  $H$  defines a dynamical system through its Hamiltonian vector field  $\mathcal{X}_H$ , and the functions  $f_j$  are constants of motion. On the geometric side, we have a foliation of the phase space  $M$  by (possibly singular) Lagrangian leaves given by the common level sets of the map  $F = (f_1, \dots, f_n)$ .

## 2.1 Local rotation number

Let us assume that the map  $F := (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$  is proper, and let  $c$  be a regular value of  $F$ . By the action-angle theorem [49] (see [26] for a more modern proof), the level set  $F^{-1}(c)$  is a finite union of  $n$ -dimensional tori, called *Liouville tori*; and near each Liouville torus, there exist action-angle coordinates  $(I, \theta) := (I_1, \dots, I_n, \theta_1, \dots, \theta_n) \in \text{neigh}(\{0\} \times \mathbb{T}^n, \mathbb{R}^n \times \mathbb{T}^n)$  such that

$$H = g(I_1, \dots, I_n), \quad (1)$$

for some smooth map  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , and the symplectic form of  $M$  is  $dI_1 \wedge d\theta_1 + \dots + dI_n \wedge d\theta_n$ . More precisely, the action-angle theorem states that there is a local diffeomorphism:

$$G : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, c)$$

such that, in the new coordinates,  $F = G(I)$ . We shall always assume (as we may) that the actions  $I$  are *oriented* with respect to  $F$ , which means that  $\det dG(0) > 0$ . In particular,  $dg$  does not vanish near the origin, which enables the following definition:

**Definition 2.1** *The rotation number of  $H$  relative to the oriented action variables  $I := (I_1, \dots, I_n)$  on the Liouville torus  $\Lambda \subset M$  is the projective vector*

$$[w_I](\Lambda) := [\partial_1 g(I(\Lambda)) : \dots : \partial_n g(I(\Lambda))] \in \mathbb{RP}^{n-1}.$$

Probably the easiest dynamical interpretation of  $[w_I](\Lambda)$  is the following. One deduces from (1) that the Hamiltonian vector field of  $H$ , which is tangent to  $\Lambda$ , has the form

$$\mathcal{X}_H = \partial_1 g \frac{\partial}{\partial \theta_1} + \dots + \partial_n g \frac{\partial}{\partial \theta_n}.$$

Therefore, the flow of  $\mathcal{X}_H$  is a “straight line” winding quasi-periodically on the affine torus  $\Lambda$  in the coordinates  $(\theta_1, \dots, \theta_n)$ , and the rotation number  $[w_I]$  is simply the direction of the trajectory. The map  $\Lambda \mapsto \nabla g(\Lambda) := (\partial_1 g(I(\Lambda)), \dots, \partial_n g(I(\Lambda)))$  is generally referred to as the *frequency map* of the system. The rotation number is nothing but the projectivized version of the frequency map.



When  $n = 2$ , it is also convenient (and usual) to define the rotation number as the ratio:

$$w_I(\Lambda) := \frac{\partial_1 g(I(\Lambda))}{\partial_2 g(I(\Lambda))} \in \overline{\mathbb{R}} \quad (2)$$

with values in the 1-point compactification of the real line  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ . The diffeomorphism  $\overline{\mathbb{R}} \rightarrow \mathbb{R}\mathbb{P}^1, z \mapsto [z]$  is defined by  $[z] := [z : 1]$  if  $z \in \mathbb{R}$  and  $[z] := [1 : 0]$  if  $z = \infty$ , and  $[w_I](\Lambda) = [w_I(\Lambda)]$ .

Note that the rotation number is not well defined by  $H$  only: it depends on the choice of actions, which in turn might depend on the choice of constants of motion  $f_j$ . In this paper, we shall always assume that  $f_1, \dots, f_{n-1}$  are given, as part of the data of the integrable Hamiltonian  $H$ ; the question of the consequences of the choice of  $f_j$  is interesting and apparently not widely spread; we hope to return to this problem in a future paper.

Thus, we assume that  $F = (f_1, \dots, f_{n-1}, f_n = H)$  is fixed. We formalize the result on the change of action variables  $I \rightarrow I'$  in the next lemma for further reference.

**Lemma 2.2** *If  $I' = (I'_1, \dots, I'_n)$  is another set of action variables, then there is a matrix  $A \in \text{SL}(n, \mathbb{Z})$  such that*

$$dI := A dI', \quad (3)$$

and the new rotation number related to  $I'$  is

$$[w_{I'}] = {}^t A \circ [w_I], \quad (4)$$

where  $\circ$  denotes the natural action of  $\text{SL}(n, \mathbb{Z})$  on  $\mathbb{R}\mathbb{P}^{n-1}$  (elements of  $\mathbb{R}\mathbb{P}^{n-1}$  are viewed here as equivalence classes of vectors in  $\mathbb{R}^n$ ). When  $n = 2$ , if instead we write  $w_I$  as the quotient (2), then  $w_{I'}$  is obtained by the Möbius transformation:

$$w_{I'} = \frac{aw_I + c}{bw_I + d} \quad \text{in which} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (5)$$

An important fact is that the rank of the components of  $w_I$  over the *rationals* (when  $n = 2$ , this is simply the rationality of  $w_I$ ) is well defined (i.e. preserved by  $\text{SL}(n, \mathbb{Z})$  transformations). This is easy to understand from the dynamical viewpoint; when  $n = 2$ , a rational rotation number is equivalent to a periodic Hamiltonian flow for the vector field  $\mathcal{X}_H$  on the Liouville torus, which is of course independent of the choice of action-angle coordinates. In higher dimensions, the rank over  $\mathbb{Q}$  gives the dimension of the sub-torus where the dynamics takes place.

**Notation 2.3** *Let  $B_r$  be the set of regular Liouville tori.*

If  $F$  has connected level sets,  $B_r$  can be identified with the open subset in  $\mathbb{R}^n$  of regular values of  $F$ . In general, it follows from the action-angle theorem that  $B_r$  is a smooth covering above the open set of regular values of  $F$  in  $\mathbb{R}^2$ . In order to make our inverse statement precise, we introduce the following definition.

**Definition 2.4** A function  $[w] : W \rightarrow \mathbb{R}\mathbb{P}^{n-1}$  is called a rotation number for the system  $F = (f_1, \dots, f_{n-1}, f_n = H)$  on the open set  $W \subset B_r$  if for every  $\Lambda \in W$ , there exist a neighborhood  $U$  of  $\Lambda$  in  $W$  and a set of action variables  $I := (I_1, \dots, I_n)$  on  $U$  such that  $[w] = [w_I]$  on  $U$ .

(When  $n = 2$ , we will use the same terminology for a function  $w : W \rightarrow \overline{\mathbb{R}}$ ). As a consequence, a rotation number is always a smooth function.

**Remark 2.5** Let  $E \in \mathbb{R}$  be a regular value of the Hamiltonian  $H$ . The submanifold  $\Sigma_E := H^{-1}(E)$  is foliated by the level sets of  $f := (f_1, \dots, f_{n-1})$ . Regular level sets of  $f|_{\Sigma_E}$  correspond to Liouville tori; when  $n = 2$ , they form smooth one-dimensional families inside  $\Sigma_E$ . The restriction of the rotation number to this family is usually called the *rotation function*. It is important in many situations to know whether this function is a local diffeomorphism. This property (the so-called “isoenergetic KAM condition”, see for instance [1]) is invariant under Möbius transformations.  $\triangle$

## 2.2 Global rotation number and monodromy

We have seen that rotation numbers exist in a neighborhood of any regular Liouville torus. A natural question arises, whether it is possible to define a rotation number on the whole set of regular tori,  $B_r$ . In fact, since the rotation number depends on the choice of local action coordinates, it is naturally related to the so-called Hamiltonian monodromy of the system. We show in this section the relation between these two objects, which explains how to define a global rotation number. We were not able to locate this statement in the literature, however it is implicitly underlying Cushman’s argument for the non-triviality of the monodromy of the Spherical Pendulum reported in [26]; see also [19]. In the presence of global  $\mathbb{S}^1$  actions, this argument can be generalized to the co-called “fractional monodromy” [8].

In the realm of classical integrable systems (of any number of degrees of freedom  $n$ ), the Hamiltonian monodromy was discovered by Duistermaat in [26]. It is the obstruction to the existence of global action coordinates (modulo constants). In topological terms, it is merely the  $\mathrm{SL}(n, \mathbb{Z})$ -holonomy of the flat bundle over the set of regular tori of the system whose fiber is the homology of the corresponding Liouville torus. Since then, a number of references have explained this monodromy and its relationships with the many interesting geometric data involved. Nowadays, the prominent object which contains the Duistermaat monodromy is the integral affine structure on the set of regular tori. It shows up for instance in Mirror Symmetry questions, see [40]. The relationships with quantum integrable systems was pointed out in [20] and proved in [65]. See also [67] for an overview of various aspects of monodromy and semi-classical analysis.

Recall that, by the action-angle theorem, given a Liouville torus  $\Lambda$ , the oriented action variables  $I = (I_1, \dots, I_n)$  give local coordinates for  $B_r$ , and hence the differentials  $dI_1(\Lambda), \dots, dI_n(\Lambda)$  span the cotangent space of  $B_r$  at  $\Lambda$ , and Lemma 2.2 implies that  $T^*B_r$  can be seen as a flat bundle over  $B_r$  with structure group  $\mathrm{SL}(n, \mathbb{Z})$ . This defines the integral affine structure of  $B_r$ . The Hamiltonian monodromy is the holonomy of this flat bundle.

For our main algorithm in Section 3.5, it will be useful to follow action variables when we move along continuous paths in  $\mathbf{B}_r$ . The Hamiltonian monodromy gives a homomorphism:

$$\mu : \pi_1(\mathbf{B}_r) \rightarrow \mathrm{SL}(n, \mathbb{Z}),$$

defined up to a global conjugacy. Here  $\pi_1(\mathbf{B}_r)$  is the fundamental group of the set of regular tori  $\mathbf{B}_r$ . Since  $\mathbf{B}_r$  may not be simply connected, we introduce a simply connected covering

$$\pi : \tilde{\mathbf{B}}_r \rightarrow \mathbf{B}_r.$$

An element  $\tilde{\Lambda} \in \tilde{\mathbf{B}}_r$  can be seen as a homotopy class of a path from a fixed torus  $\Lambda_0$  to  $\pi(\Lambda)$ . Recall that a loop  $\gamma$  in  $\mathbf{B}_r$  acts fiberwise on the covering  $\tilde{\mathbf{B}}_r$  by concatenation of paths: given  $\tilde{\Lambda} \in \pi^{-1}(\gamma(0))$ , we define  $\gamma \cdot \tilde{\Lambda}$  to be (the homotopy class of)  $\tilde{\Lambda} \circ \gamma$ , *i.e.* the path corresponding to  $\tilde{\Lambda}$  followed by  $\gamma$ .

Let us now turn to the global definition of rotation numbers. Let  $g$  be as in (1), which we now view as a map on  $\mathbf{B}_r$ ; the rotation number  $[w_I(\Lambda)]$  can now be identified with the projectivized covector

$$[w_I(\Lambda)] := [d_\Lambda g] \in \mathbb{P}(T_\Lambda^* \mathbf{B}_r). \quad (6)$$

In view of (4), the collection of rotation numbers  $[w_I]$ , for all local action variables  $I$ , forms a (local) section of the projectivized bundle  $\mathbb{P}(T^* \mathbf{B}_r)$ . Because of monodromy, a global section may not exist. Rather, a global rotation number should be seen as a function on  $\tilde{\mathbf{B}}_r$ , *i.e.* a *multivalued* function on  $\mathbf{B}_r$ .

**Proposition 2.6** *Given a regular torus  $\Lambda \in \mathbf{B}_r$  for the integrable system  $F$ , a choice of action variables  $I$  near  $\Lambda$ , and  $\tilde{\Lambda} \in \pi^{-1}(\Lambda)$ , there exists a unique smooth function  $[\tilde{w}] : \tilde{\mathbf{B}}_r \rightarrow \mathbb{R}\mathbb{P}^{n-1}$  such that:*

1. *for any simply connected open set  $U \subset \mathbf{B}_r$ , the function  $[w] : U \rightarrow \mathbb{R}\mathbb{P}^{n-1}$  defined by*

$$[\tilde{w}] = [w] \circ \pi$$

*is a rotation number for the system  $F$  (Definition 2.4).*

2. *near  $\tilde{\Lambda}$ ,  $[\tilde{w}] = [w_I] \circ \pi$ .*

Moreover, for any loop  $\gamma \in \pi_1(W)$ ,

$$\gamma \cdot [\tilde{w}] = {}^t\mu(\gamma) \circ [\tilde{w}], \quad (7)$$

where  $\gamma \cdot [\tilde{w}] := (\tilde{\Lambda}' \mapsto [\tilde{w}](\gamma \cdot \tilde{\Lambda}'))$ .

**Proof.** The section of  $\mathbb{P}(T^* \mathbf{B}_r)$  given by the collection of all  $[w_I]$ 's can be lifted to  $\tilde{\mathbf{B}}_r$  by  $\pi^*$ . Since  $\tilde{\mathbf{B}}_r$  is simply connected, the lifted section is actually global. Given the local trivialization

determined by the choice of  $I$  near  $\Lambda$ , this global section gives rise to a unique map  $[\tilde{w}] : \tilde{\mathbb{B}}_r \rightarrow \mathbb{R}P^{n-1}$  satisfying 1. and 2, obtained by parallel transport; equation (7) follows from (4).  $\square$

Here is how this proposition spells out more concretely: we can find “global action variables” on  $\tilde{\mathbb{B}}_r$ , *i.e.* a map  $\tilde{I} : \tilde{\mathbb{B}}_r \rightarrow \mathbb{R}^n$  such that for each small open ball  $\tilde{U}$  on  $\tilde{\mathbb{B}}_r$ , the restriction of  $\tilde{I}$  to  $\tilde{U}$  descends to action variables  $I_{\tilde{U}}$  for the initial integrable system  $F$  on  $U = \pi(\tilde{U})$ . Moreover, we may assume that  $I_{\tilde{U}_0} = I$  if  $\tilde{U}_0$  is a neighborhood of the initial torus  $\tilde{\Lambda}$ . For each  $\tilde{U}$ ,  $[\tilde{w}]$  is defined on  $\tilde{U}$  by

$$[\tilde{w}] = [w_{I_{\tilde{U}}}] \circ \pi, \quad (8)$$

where  $[w_{I_{\tilde{U}}}]$  given by Definition 2.1. The fact that  $[\tilde{w}]$  is a smooth, single-valued function on  $\tilde{\mathbb{B}}_r$  follows from the transition formula (4), where by definition here all transition maps  $A$  are the identity matrix, because  $I_{\tilde{U}} = I_{\tilde{U}'}$  on  $\pi(\tilde{U}) \cap \pi(\tilde{U}')$ .

### 2.3 The semitoric case

A semitoric system, in a broad sense, is a two degree of freedom integrable Hamiltonian system with a global  $\mathbb{S}^1$  symmetry; more precisely, the Hamiltonian  $H$  commutes with a smooth function  $J$  whose Hamiltonian flow  $t \mapsto \varphi_J^t$  is  $2\pi$ -periodic, except at fixed points. Under some additional hypothesis (which ensures, for instance, that the level sets of the energy-momentum map  $(J, H)$  are connected), such systems have been introduced in [69] and classified in [52, 56]. Semitoric systems, in particular under their quantum version (see Section 3.4) are ubiquitous in physics, when investigating the coupling between spin, atoms, q-bits, small molecules, etc. For a discussion and more references on this topic, see [45]. The symplectic theory of semitoric systems borrows both from almost-toric systems [62, 46] and Hamiltonian  $\mathbb{S}^1$  spaces [39]. Recently, the initial definition has been generalized in several ways, allowing the inclusion of famous examples like the Spherical Pendulum, see [51, 36, 59].

In the present work, we don't need the classification of [52, 56], and hence we will use the terminology “semitoric” in a very general acceptance (which includes, for instance, lagrangian leaves with several focus-focus singularities [50]):

**Definition 2.7** *An integrable system  $F = (J, H)$  on a 4-dimensional symplectic manifold  $M$  will be called semitoric if the map  $F : M \rightarrow \mathbb{R}^2$  is proper, and the function  $J$  is a momentum map for an effective Hamiltonian  $\mathbb{S}^1$ -action on  $M$ .*

The existence of such a global symmetry has the interesting consequence that the rotation number for the system  $F = (J, H)$  is defined in a more natural way, and can be interpreted as an angle in  $\mathbb{R}/\mathbb{Z}$ :

**Definition 2.8** *If  $\Lambda$  is a Liouville torus for the semitoric system  $(J, H)$ , the (semitoric) rotation number  $w(\Lambda) \in \mathbb{R}/\mathbb{Z}$  is defined as follows. Take a point  $A \in \Lambda$  and let  $\mathbb{S}_A^1$  be the orbit of this point under the flow of  $J$ . Consider the  $H$ -flow of  $A$  and denote by  $A'$  the first return point of the*

trajectory starting from  $A$  on the orbit  $\mathbb{S}_A^1$ . (See Figure 1.) Then  $2\pi w(\Lambda)$  is the time necessary to flow, under the action of  $J$ , from  $A$  to  $A'$ :

$$\varphi_J^{2\pi w(\Lambda)}(A) = A'.$$

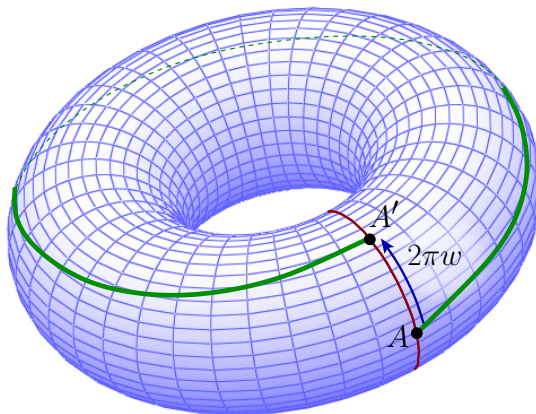


Figure 1: The semitoric rotation number.

This definition is of course not new. It was popularized in particular by Arnold in the treatment of perturbations of integrable systems, see for instance [2, Section 3 G]. In order to state in which sense this new rotation number coincides with the general rotation number from Definition 2.1 (more specifically Equation (2)), we recall the following basic fact from the theory of semitoric systems (see for instance [51, Lemma 5.1]):

**Lemma 2.9** *Let  $(J, H)$  be a semitoric system, and let  $\Lambda$  be a Liouville torus. Then there exists a set of oriented action integrals near  $\Lambda$  of the form  $(J, I_2)$ .*

It will be convenient to call such sets of action integrals “semitoric action variables”.

**Proposition 2.10** *Let  $(J, H)$  be a semitoric system, and let  $\Lambda$  be a Liouville torus. Let  $I$  be semitoric action variables near  $\Lambda$ . Let  $w(\Lambda)$  be the semitoric rotation number of Definition 2.8, and let  $w_I(\Lambda)$  be the rotation number from Equation (2). Then*

$$w(\Lambda) = (w_I(\Lambda) \pmod{\mathbb{Z}}).$$

**Proof.** Since  $I = (J, I_2)$ , we have  $\mathcal{X}_H = a_1\mathcal{X}_J + a_2\mathcal{X}_{I_2}$ , for some smooth functions  $a_j = a_j(\Lambda)$ , such that  $w_I(\Lambda) = a_1(\Lambda)/a_2(\Lambda)$ . Notice that, since  $\mathcal{X}_H$  and  $\mathcal{X}_J$  must be linearly independent on  $\Lambda$ , we must have  $a_2(\Lambda) \neq 0$ . Hence  $w_I$  is a standard real number.

From Definition 2.8, there exists a positive time  $\tau = \tau(\Lambda)$  such that the flow of  $\mathcal{X}_H$  at time  $\tau$  followed by the flow of  $\mathcal{X}_J$  at time  $-2\pi w(\Lambda)$  sends  $A$  to itself. In other words, the time-1 flow of

$$\mathcal{X}_{\text{per}} := \tau(\Lambda)\mathcal{X}_H - 2\pi w(\Lambda)\mathcal{X}_J \tag{9}$$

is the identity on  $\Lambda$ . By the theory of action-angle variables,  $\mathcal{X}_{\text{per}}$  is the Hamiltonian vector field of the Hamiltonian

$$H_{\text{per}} := \int_{\gamma} \alpha,$$

where  $\alpha$  is a primitive of the symplectic form in a neighborhood of  $\Lambda$ , and the cycle  $\gamma$  is the homology class of a periodic trajectory of  $H_{\text{per}}$  on  $\Lambda$ . Moreover, the action of  $\mathcal{X}_{\text{per}}$  must be effective, otherwise there would be a periodic trajectory of  $\mathcal{X}_{\text{per}}$  of period  $1/2$ , which would imply by (9) that the image of  $A$  by the flow of  $\mathcal{X}_H$  at time  $\tau(\Lambda)/2$  would return to the  $\mathbb{S}^1$ -orbit of  $A$ , contradicting the definition of  $\tau(\Lambda)$ . Hence the pair  $(J, H_{\text{per}})$  is a set of action variables near  $\Lambda$ , which implies  $H_{\text{per}} = \pm 2\pi I_2 + 2\pi kJ$  for some  $k \in \mathbb{Z}$ . From (9) we get

$$\tau(\Lambda)\mathcal{X}_H = \pm 2\pi\mathcal{X}_{I_2} + 2\pi(w(\Lambda) - k)\mathcal{X}_J.$$

The assumption that  $I = (J, I_2)$  is oriented means  $\frac{\partial I_2}{\partial H} > 0$ , which implies that only the sign  $+2\pi I_2$  can occur, and hence we get  $w_I(\Lambda) = w(\Lambda) - k$ .  $\square$

In concrete examples, computing the rotation number can be a hard task. But, even in the purely classical theory, it is an important piece of information about the system, related to the “tennis-racket effect” [19], and is intimately related to the symplectic invariants classifying semitoric systems, as demonstrated in the case of the spherical pendulum by [28].

### 3 The Quantum Rotation Number

The quantum version of the image of the map  $F = (f_1, \dots, f_n)$ , where  $f_j$  are commuting Hamiltonians, is the joint spectrum [13] of commuting operators  $P_1, \dots, P_n$ . With this in mind, it is very natural to try to define a “quantum rotation number” by replacing energies by “quantized energies”. That this is indeed possible will follow from the mathematical version of the semiclassical Bohr-Sommerfeld rules, which requires a rigorous framework establishing the relationship between quantum theory (functional analysis of operators on Hilbert spaces) and classical mechanics (symplectic geometry of Hamiltonian systems). In this paper, for this purpose, we use the well-established theory of semiclassical pseudodifferential quantization.

In Section 3.1, we recall the definition of the joint spectrum, and state the Bohr-Sommerfeld rules (Theorem 3.2). From this, in Section 3.2, we extract the crucial notion of asymptotic lattices and their good (and linear) labellings, which we believe is one of the main contributions of our paper. This leads, in Section 3.3, to a very natural definition of quantum rotation number, which converges to the classical rotation number as  $\hbar \rightarrow 0$ . From this point, we specialize to  $n = 2$ . The important case of semitoric systems is discussed in Section 3.4. Section 3.5 can be thought of as the technical core of the paper; we construct robust algorithms for detecting linear labellings of general 2-dimensional asymptotic lattices. These algorithms are applied in Section 3.6 to finally solve the inverse problem for the rotation number.

### 3.1 Bohr-Sommerfeld quantization rules for quantum integrable systems

Bohr-Sommerfeld rules bear several meanings in mathematics. We use here the original formulation coming from quantum theory, which describes the joint spectrum of commuting operators in terms of the underlying symplectic geometry of their symbols.

Let  $P$  be a (possibly unbounded) selfadjoint operator acting on a Hilbert space  $\mathcal{H}$ . The spectral theorem constructs from  $P$  its so-called spectral measure, which is a projector-valued measure on  $\mathbb{R}$ . The support of the spectral measure is the spectrum of  $P$ , denoted by  $\sigma(P)$ . The spectrum of  $P$  in an interval  $[E_1, E_2]$  is called discrete if any  $\lambda \in \sigma(P)$  is isolated in  $\sigma(P) \cap [E_1, E_2]$ , and for any  $\epsilon > 0$  small enough, the spectral projector associated with the interval  $[\lambda - \epsilon, \lambda + \epsilon]$  has finite rank.

The selfadjoint operators  $P_1, \dots, P_n$  are said to pairwise commute if their spectral measures commute, which we denote by  $[P_j, P_k] = 0$ . In this case, one can define the joint spectral measure, and by definition the *joint spectrum* of  $(P_1, \dots, P_n)$  is the support (in  $\mathbb{R}^n$ ) of this measure. We shall denote it by  $\Sigma(P_1, \dots, P_n)$ . The joint spectrum is called discrete if all  $\lambda \in \Sigma(P_1, \dots, P_n)$  are isolated and, for any small enough compact neighborhood  $K$  of  $\lambda$ , the joint spectral projector on  $K$  has finite rank.

Let  $P$  be a semiclassical  $\hbar$ -pseudodifferential operator on  $X = \mathbb{R}^n$ , or on a compact manifold  $X$  of dimension  $n$ , where the semiclassical parameter  $\hbar$  varies in an interval  $]0, \hbar_0]$  for some  $\hbar_0 > 0$ . More precisely, in the  $\mathbb{R}^n$  case, we will assume that  $P$  is the semiclassical Weyl quantization of a symbol in  $S(m)$  where  $m$  is an order function in the sense of [23, Definition 7.4]: there exist  $C > 0, N > 0$  such that

$$1 \leq m(X) \leq C \langle X - Y \rangle^N m(Y) \quad \forall X, Y \in \mathbb{R}^{2n},$$

where  $\langle X \rangle := (1 + \|X\|^2)^{1/2}$ . For instance, one can take  $m(X) := \langle X \rangle^d$ , for some  $d \in \mathbb{R}$ . A function  $a = a(x, \xi; \hbar) \in C^\infty(\mathbb{R}^{2n})$  belongs to the class  $S(m)$  if, by definition,

$$\forall \alpha \in \mathbb{N}^{2n}, \forall \hbar \in (0, \hbar_0], \quad |\partial^\alpha a(x, \xi; \hbar)| \leq C_\alpha m(x, \xi),$$

and, in this paper, we shall always assume that such a symbol  $a$  admits an asymptotic expansion  $a \sim \sum_{j \geq 0} \hbar^j a_j$  in non-negative integral powers of  $\hbar$  in this topology: for any  $N > 0$ ,

$$a(x, \xi; \hbar) - \sum_{j=0}^N \hbar^j a_j(x, \xi) \in \hbar^{N+1} S(m). \quad (10)$$

We then say that a pseudodifferential operator  $P$  belongs to  $S(m)$  (using the same notation, since there will be in general no ambiguity) if its Weyl symbol  $a$  belongs to  $S(m)$ . The first term  $a_0$  in (10) is called the principal symbol of  $P$ . Other classes of pseudodifferential operators can be used, see [9], but we stick to  $S(m)$  for its ease of use and ample documentation (see for instance [23, 47, 73]).

In the manifold case, we will assume that  $P$  belongs to the Kohn-Nirenberg class  $S^d(X)$ , ( $d \in \mathbb{R}$ ), which means that in local coordinates, after cut-off in the position variable  $x \in X$ ,



$P$  can be written as a pseudodifferential operator with a symbol  $a(x, \xi; \hbar)$  such that (see for instance [73, Chapter 14])

$$\forall (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \forall \hbar \in (0, \hbar_0], \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{d-|\beta|}.$$

In this case, to unify notations, we call  $m(x, \xi) := \langle \xi \rangle^d$  the corresponding order function, and denote  $S(m) := S^d(X)$ .

The operator  $P$  is said to be elliptic at infinity in  $S(m)$  if  $P \in S(m)$  and  $|a_0(x, \xi)| \geq cm(x, \xi)$  for some  $c > 0$  and for large  $(x, \xi)$  in the  $\mathbb{R}^n$  case; and similarly  $|a_0(x, \xi)| \geq c\langle \xi \rangle^d$  in the manifold case.

A *quantum integrable system* is the data of  $n$  commuting selfadjoint pseudodifferential operators  $P_1, \dots, P_n$  in  $S(m)$ , such that the map of the principal symbols  $F := (p_1, \dots, p_n)$  defines a classical integrable system in  $T^*X$ . For simplicity, we shall always assume that  $F$  is globally proper and that the operator  $Q := P_1^2 + \dots + P_n^2$  is elliptic at infinity in  $S(m^2)$ . This implies that  $m$  is unbounded. (We could also include the case of bounded  $m$  by imposing properness of  $F$  only onto some compact set, and considering the intersection of the joint spectrum with a strict compact subset of this set.) Note that, for pseudodifferential operators, the commutation property  $[P_j, P_k] = 0$  is equivalent to the weak commutation (see [9])

$$\forall u \in C_0^\infty(\mathbb{R}^n), \quad P_j P_k u = P_k P_j u.$$

In this case, for any  $f \in C_0^\infty(\mathbb{R}^n)$ , the operator  $f(P_1, \dots, P_n)$  (obtained via the joint spectral measure) is a pseudodifferential operator in  $S(m^{-N})$  for any  $N \in \mathbb{N}$  ([23, Chapter 8]).

Typical examples of quantum integrable systems (with unbounded symbols) are given by two degree of freedom Schrödinger operators with an axi-symmetric potential; these can occur either on  $\mathbb{R}^2$  [9], or on a Riemannian surface of revolution [13, 15], and can be used to obtain efficient numerical methods, see [3]. Other classical examples of quantum integrable systems, which do not feature a global  $\mathbb{S}^1$  symmetry, include the C-Neuman problem and the Laplace-Beltrami operator on an ellipsoid, see [63].

**Proposition 3.1 ([9])** *If  $P_1, \dots, P_n$  is a quantum integrable system, satisfying the above hypothesis, then its joint spectrum  $\Sigma(P_1, \dots, P_n)$  is discrete when  $\hbar$  is small enough.*

**Proof.** Let  $\lambda := (\lambda_1, \dots, \lambda_n) \in \Sigma(P_1, \dots, P_n)$ , and let  $K \subset \mathbb{R}$  be a compact neighborhood of  $\lambda$ . Since  $F$  is proper, the map  $f \circ F$  has compact support for any  $f \in C_0^\infty(\mathbb{R}^n)$ . Hence the pseudodifferential operator  $f(P_1, \dots, P_n)$  is compact for small  $\hbar$ . Choosing  $f \equiv 1$  on a neighborhood of  $K$ , we see that the spectral projector  $1_K(P_1, \dots, P_n) = f(P_1, \dots, P_n)1_K(P_1, \dots, P_n)$  is compact; hence its rank is finite.  $\square$

The following result, which is a mathematical justification of the old Bohr-Sommerfeld rule from quantum mechanics, states that the joint spectrum of a quantum integrable system takes the form of an approximate lattice in any neighborhood of a regular value of  $F$ . It was first

obtained by Colin de Verdière in the homogeneous setting [15], and then by Charbonnel in the semiclassical setting [9]. A purely microlocal approach was proposed in [66] (see also [68]). When  $n = 1$  and  $P$  is a Schrödinger operator, explicit methods for second order ODEs can be employed, see [70].

**Theorem 3.2 ([15, 9])** *Let  $P_1, \dots, P_n$  is a quantum integrable system satisfying the above hypothesis. Let  $c \in \mathbb{R}^n$  be a regular value of the principal symbol map  $F = (p_1, \dots, p_n)$ . Assume that the fiber  $F^{-1}(c)$  is connected. Then we can describe the joint spectrum around  $c$  as follows.*

1. **(joint eigenvalues have multiplicity one)** *There exists a non empty open ball  $B \subset \mathbb{R}^n$  (independent of  $\hbar$ ) around  $c$  and  $\hbar_0 > 0$  such that for any  $\lambda \in \Sigma(P_1, \dots, P_n) \cap B$  and for all  $\hbar < \hbar_0$ , the joint spectral projector of  $(P_1, \dots, P_n)$  onto the ball  $B(\lambda, \hbar^2)$  has rank 1.*
2. **(the joint spectrum is a deformed lattice)** *There exists a bounded open set  $U \subset \mathbb{R}^n$  and a smooth map  $G_\hbar : U \rightarrow \mathbb{R}^n$  admitting an asymptotic expansion in the  $C^\infty$  topology:*

$$G_\hbar = G_0 + \hbar G_1 + \hbar^2 G_2 + \dots$$

*such that the joint eigenvalues in  $\Sigma(P_1, \dots, P_n) \cap B$  are given by the quantities*

$$\lambda = G_\hbar(\hbar k_1, \dots, \hbar k_n) + \mathcal{O}(\hbar^\infty), \quad (11)$$

*where  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  is such that  $G_\hbar(\hbar k_1, \dots, \hbar k_n) \in B$ , and the  $\mathcal{O}(\hbar^\infty)$  remainder is uniform as  $\hbar \rightarrow 0$ . Moreover,  $G_0$  is a diffeomorphism from  $U$  onto a neighborhood of  $\bar{B}$  given by the action-angle theorem:*

$$F = G_0(I_1, \dots, I_n), \quad (12)$$

*where  $(I_1, \dots, I_n)$  is a set of action variables for the classical system, defined in a neighborhood of  $F^{-1}(c)$ , and  $U$  is a neighborhood of  $I_c = I(F^{-1}(c))$ .*

It is important to notice that, contrary to the classical case, one can not always assume that the actions  $(I_1, \dots, I_n)$  take values in a neighborhood of the origin (in other words,  $I_c$  is not necessarily zero). In fact  $I_c$  is given by the integrals of the Liouville 1-form of the cotangent bundle  $T^*X$  over a basis of cycles of the torus  $F^{-1}(c)$ .

**Remark 3.3** That this statement can be generalized to multiple connected components of  $F^{-1}(c)$  is “well known to specialists”; however to our knowledge that generalization cannot be found in the literature. We haven’t included it here because it will be important for us to make the connectedness hypothesis throughout the paper. However, we expect it to become necessary for the future study of integrable systems with hyperbolic singularities, which are very common in the spectroscopy of small molecules (eg. LiNC/NCLi, see [38]), and related to the so-called fractional monodromy, see for instance [29]. On the mathematical side, several authors have already attacked the semiclassical analysis near the hyperbolic separatrix, for instance, for 1D

Schrödinger operators [48, 6], Liouville surfaces [5], for general 1D pseudodifferential operators [16, 17], and transversally hyperbolic 2D systems [18].  $\triangle$

**Remark 3.4** The term  $G_1$  can be expressed in terms of Maslov indices and the subprincipal symbols of the operators  $P_1$ , see [66]; we won't need this particular formula here.  $\triangle$

**Remark 3.5** We have not attempted here to extend our analysis to commuting Berezin-Toeplitz operators on prequantizable symplectic manifolds (see [42] for a nice introduction to Berezin-Toeplitz quantization). However, the notion of asymptotic lattices that we develop in this paper is quantization-agnostic, and hence, thanks to the work of Charles [10], we believe that most of our results should be adaptable to that situation.  $\triangle$

## 3.2 Asymptotic lattices and good labellings

Sets of points in  $\mathbb{R}^n$  that are described by an equation of the type of (11) can be called 'asymptotic lattices', see [65]. Indeed, when  $\hbar$  is small enough, the map  $G_\hbar : U \rightarrow \mathbb{R}^n$  is a diffeomorphism onto its image, and hence it sends the local lattice  $\hbar\mathbb{Z}^n \cap U$  one-to-one into the joint spectrum  $\Sigma(P_1, \dots, P_n) \cap B \pmod{\mathcal{O}(\hbar^\infty)}$ . This also implies that for any strictly smaller ball  $\tilde{B} \Subset B$ , the map obtained by the restriction to lattice points

$$G_\hbar : \hbar\mathbb{Z}^n \cap G_\hbar^{-1}(\tilde{B}) \rightarrow \Sigma(P_1, \dots, P_n) \cap \tilde{B} \pmod{\mathcal{O}(\hbar^\infty)}$$

is onto. The goal of this section is to make these observations precise by introducing the formal definition of asymptotic lattices and deriving some of their properties; this analysis partly builds on and extends [65].

### 3.2.1 Asymptotic charts and asymptotic lattices

**Definition 3.6** Let  $B \subset \mathbb{R}^n$  be a simply connected bounded open set. Let  $\mathcal{L}_\hbar \subset B$  be a discrete subset of  $B$  depending on the small parameter  $\hbar \in \mathcal{I}$ , where  $\mathcal{I} \subset \mathbb{R}_+^*$  is a set of positive real numbers admitting 0 as an accumulation point. We say that  $(\mathcal{L}_\hbar, \mathcal{I}, B)$  is an asymptotic lattice if the following two statements hold:

1. there exist  $\hbar_0 > 0$ ,  $\epsilon_0 > 0$ , and  $N_0 \geq 1$  such that for all  $\hbar \in \mathcal{I} \cap ]0, \hbar_0]$ ,

$$\hbar^{-N_0} \min_{\substack{(\lambda_1, \lambda_2) \in \mathcal{L}_\hbar^2 \\ \lambda_1 \neq \lambda_2}} \|\lambda_1 - \lambda_2\| \geq \epsilon_0;$$

2. there exist a bounded open set  $U \subset \mathbb{R}^n$  and a family of smooth maps  $G_\hbar : U \rightarrow \mathbb{R}^n$ , for  $\hbar \in \mathcal{I}$ , admitting an asymptotic expansion in the  $C^\infty(U)$  topology:

$$G_\hbar = G_0 + \hbar G_1 + \hbar^2 G_2 + \dots \tag{13}$$

with  $G_j \in C^\infty(U; \mathbb{R}^n)$ , for all  $j \geq 0$ . Moreover,  $G_0$  is an orientation preserving diffeomorphism from  $U$  onto a neighborhood of  $\overline{B}$ , and

$$G_\hbar(\hbar\mathbb{Z}^n \cap U) = \mathcal{L}_\hbar + \mathcal{O}(\hbar^\infty) \quad \text{inside } B,$$

by which we mean: there exists a sequence of positive numbers  $(C_N)_{N \in \mathbb{N}}$ , such that for all  $\hbar \in \mathcal{I}$ , the following holds.

(a) For all  $\lambda \in \mathcal{L}_\hbar$  there exists  $k \in \mathbb{Z}^n$  such that  $\hbar k \in U$  and

$$\forall N \in \mathbb{N}, \quad \|\lambda - G_\hbar(\hbar k)\| \leq C_N \hbar^N. \quad (14)$$

(b) For every open set  $\tilde{U}_0 \Subset G_0^{-1}(B)$ , there exists  $\tilde{h}_0 > 0$  such that for all  $\hbar \in \mathcal{I} \cap ]0, \tilde{h}_0]$ , for all  $k \in \mathbb{Z}^n$  with  $\hbar k \in \tilde{U}_0$ , there exists  $\lambda \in \mathcal{L}_\hbar$  such that (14) holds.

Such a map  $G_\hbar$  will be called an asymptotic chart for  $\mathcal{L}_\hbar$ . We will also denote it by  $(G_\hbar, U)$ .

Notice that the orientation preserving requirement is not a true restriction; indeed, one may pick an element  $A \in \text{GL}(n, \mathbb{Z})$  with  $\det A = -1$ , and then  $(G_\hbar \circ A, A^{-1}U)$  is a good asymptotic chart if and only if  $(G_\hbar, U)$  is an ‘‘orientation reversing asymptotic chart’’.

For shortening notation, we shall often simply call  $\mathcal{L}_\hbar$  an asymptotic lattice, instead of the triple  $(\mathcal{L}_\hbar, \mathcal{I}, B)$ . Of course the simplest asymptotic lattice is the square lattice  $\hbar\mathbb{Z}^n \cap B$ , with chart  $G_\hbar = \text{Id}$ . In general, if  $G_\hbar$  is any map satisfying (13) with  $G_0$  a diffeomorphism onto a neighborhood of  $\overline{B}$ , then  $G_\hbar(\hbar\mathbb{Z}^n \cap U) \cap B$  is an asymptotic lattice.

With this definition, we may now rephrase Theorem 3.2 as follows.

**Theorem 3.7** *With the hypothesis of Theorem 3.2, near any regular value  $c \in \mathbb{R}^n$  with connected fiber, the joint spectrum of the quantum integral system  $(P_1, \dots, P_n)$ , with  $\hbar \in \mathcal{I} = ]0, \hbar_0]$  is an asymptotic lattice.*

Let us state some elementary properties of asymptotic charts, for further reference. For subsets  $A, B$  of  $\mathbb{R}^n$ , we use the notation  $A \Subset B$  to mean that  $\overline{A}$  is compact and contained in  $B$ . For a map  $G : U \rightarrow \mathbb{R}^n$ , where  $U \subset \mathbb{R}^d$  is an open set, we denote by  $G' : U \rightarrow \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)$  the full derivative (or linear tangent map) of  $G$ .

**Lemma 3.8** *Let  $(\mathcal{L}_\hbar, \mathcal{I}, B)$  be an asymptotic lattice, and let  $(G_\hbar, U)$  be an asymptotic chart for it. Let  $\tilde{h}_0 > 0$ . The following properties of  $G_\hbar$  hold.*

1. All derivatives of  $G_\hbar$  are bounded on any compact subset of  $U$ , uniformly for  $\hbar \leq \tilde{h}_0$ :

$$\forall \tilde{U} \Subset U, \forall \alpha \in \mathbb{N}^n, \exists C_{\tilde{U}, \alpha} > 0, \forall \hbar \in \mathcal{I} \cap ]0, \tilde{h}_0], \quad \sup_{\tilde{U}} \|\partial^\alpha G_\hbar\| \leq C_{\tilde{U}, \alpha}.$$

2. For any  $\tilde{U} \Subset U$ , there exist non-negative constants  $M, M'$  such that

$$\forall \hbar \in \mathcal{I} \cap ]0, \tilde{h}_0], \quad \sup_{\tilde{U}} \|G_\hbar - G_0\| \leq \hbar M, \quad \sup_{\tilde{U}} \|G'_\hbar - G'_0\| \leq \hbar M'. \quad (15)$$

3. For any  $\tilde{U} \Subset U$ , for any  $N \geq 0$ , there exists a non-negative constant  $R_N$  such that, for all  $\hbar_1, \hbar_2 \in \mathcal{I} \cap ]0, \hbar_0]$ ,

$$\sup_{\tilde{U}} \|G_{\hbar_2} - G_{\hbar_1}\| \leq R_N (|\hbar_2 - \hbar_1| + \hbar_1^N + \hbar_2^N).$$

4. For any  $\tilde{U} \Subset U$ , there exist non-negative constants  $L_0, L_1$  such that, for all  $\hbar \in \mathcal{I} \cap ]0, \hbar_0]$ ,  $\forall \xi_1, \xi_2 \in \tilde{U}$ , if the segment  $[\xi_1, \xi_2]$  is contained in  $\tilde{U}$  then

$$\|G_{\hbar}(\xi_2) - G_{\hbar}(\xi_1)\| \leq L_0 \|\xi_2 - \xi_1\| \quad (16)$$

$$\|G_{\hbar}(\xi_2) - G_{\hbar}(\xi_1) - G'_{\hbar}(\xi_1) \cdot (\xi_2 - \xi_1)\| \leq L_1 \|\xi_2 - \xi_1\|^2. \quad (17)$$

5. The map  $G_{\hbar}$  is formally invertible: there exists a smooth map  $F_{\hbar}$ , defined on a neighborhood of  $\bar{B}$ , that admits an asymptotic expansion

$$F_{\hbar} = F_0 + \hbar F_1 + \hbar^2 F_2 + \dots \quad (18)$$

such that

$$F_{\hbar} \circ G_{\hbar} = \text{Id} + \mathcal{O}(\hbar^{\infty}), \quad \text{and} \quad G_{\hbar} \circ F_{\hbar} = \text{Id} + \mathcal{O}(\hbar^{\infty}). \quad (19)$$

In particular,  $F_0 = G_0^{-1}$ .

6. For any  $\tilde{U} \Subset U$ , there exists  $\hbar_0 > 0$  such that for all  $\hbar \in \mathcal{I} \cap ]0, \hbar_0]$ , the restriction of  $G_{\hbar}$  to  $\tilde{U}$  is invertible onto its image. Moreover, its inverse  $G_{\hbar}^{-1}$  is smooth and admits an asymptotic expansion in powers of  $\hbar$  equal to that of  $F_{\hbar}$  in (18).
7. Given  $\xi \in U$ , the map  $\hbar \mapsto G_{\hbar}(\xi)$  is not necessarily continuous. However, there exists a smooth map  $\tilde{G} \in C^{\infty}([-\hbar_0, \hbar_0]) \times U$  such that, for all  $\xi \in U$ ,  $\forall \hbar \in \mathcal{I} \cap ]0, \hbar_0]$ ,  $G_{\hbar}(\xi) = \tilde{G}(\hbar, \xi) + R_{\hbar}(\xi)$ , and  $R_{\hbar} = \mathcal{O}(\hbar^{\infty})$  in the  $C^{\infty}(U)$  topology. In particular,  $\tilde{G}(0, \xi) = G_0(\xi)$ .
8. If  $\tilde{G}_{\hbar} = G_{\hbar} + \mathcal{O}(\hbar^{\infty})$  in  $C^{\infty}(U)$ , then  $\tilde{G}_{\hbar}$  is also an asymptotic chart for  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$ .

**Proof.** Items 1, 2 and 3 directly follow from the asymptotic expansion (13). Taylor's formula and Item 1 gives Item 4. Item 5 is a standard consequence of (13): the functions  $F_j$  can be defined by induction, using Taylor expansions, and  $F_{\hbar}$  is obtained by the Borel lemma. Item 7 also follows from a Borel summation of the formal series  $\sum \hbar^k G_k$ . To prove Item 6, observe that near any  $\xi \in U$ , if  $\hbar$  is small enough,  $G'_{\hbar}(\xi)$  is invertible due to the invertibility of  $G'_0$  and (15). Hence on a compact subset of  $U$ , we may find  $\hbar_0$  such the local inversion theorem applies if  $\hbar < \hbar_0$ , showing that  $G_{\hbar}$  is a local diffeomorphism for all  $\hbar \in \mathcal{I} \cap ]0, \hbar_0]$ . Thus, there exists  $\delta > 0$  such that for any  $\xi_0 \in \tilde{U}$ , the restriction of  $G_{\hbar}$  to the ball  $B(\xi_0, \delta)$  is a diffeomorphism onto its image. It follows from (19) that there exists  $\hbar_0 > 0$  depending only on  $G_{\hbar}$  and  $\tilde{U}$  such that

$$\forall \hbar \in \mathcal{I} \cap ]0, \hbar_0], \quad \forall \xi \in \tilde{U}, \quad \|F_{\hbar} \circ G_{\hbar}(\xi) - \xi\| \leq \delta/3.$$

Hence  $G_{\hbar}(\xi_1) = G_{\hbar}(\xi_2)$  implies  $\|\xi_1 - \xi_2\| \leq 2\delta/3$ , which in turn implies  $\xi_1 = \xi_2$ , proving the injectivity of  $G_{\hbar}$  on  $\tilde{U}$ , which gives the result. Notice that, by choosing  $\tilde{U}$  large enough, we can always ensure that  $G_{\hbar}(\tilde{U})$  contains a neighborhood of  $\bar{B}$ .  $\square$

**Remark 3.9** As usual in semiclassical analysis, equality of sets “modulo  $\mathcal{O}(\hbar^\infty)$ ” has to be taken with care. Given an asymptotic lattice  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$ , because of boundary effects, it is not true that the Hausdorff distance between  $\mathcal{L}_{\hbar}$  and  $B \cap G_{\hbar}(\hbar\mathbb{Z}^n \cap U)$  is  $\mathcal{O}(\hbar^\infty)$ . Consider the following example:  $n = 1$ ,  $B = ]0, 1[$ ,  $\mathcal{I} = ]0, 1[$ ,  $\mathcal{L}_{\hbar} = \hbar\mathbb{Z} \cap B$ . One can check that  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$  is an asymptotic lattice with chart  $G_{\hbar} = \text{Id} : U = ]-1, 2[ \rightarrow U$ . By virtue of Item 8 of Lemma 3.8, the map  $\tilde{G}_{\hbar}(\xi) = \xi + e^{-1/\hbar}$  is again an asymptotic chart for  $\mathcal{L}_{\hbar}$ . However, if  $\hbar$  is small enough,  $\min G_{\hbar}(\hbar\mathbb{Z} \cap U) \cap B = e^{-1/\hbar}$  while  $\min \mathcal{L}_{\hbar} = \hbar$ . So, in this case, the Hausdorff distance is  $\hbar - e^{-1/\hbar}$ , which is not  $\mathcal{O}(\hbar^\infty)$ .  $\triangle$

**Remark 3.10** It follows from Item 7 of Lemma 3.8 that, up to changing  $U$  for a smaller  $U' \Subset U$ , one can always choose an asymptotic chart  $G_{\hbar}$  that is smooth with respect to  $\hbar$ . This, in turns, improves Item 3 by suppressing the term  $\hbar_1^N + \hbar_2^N$ .  $\triangle$

### 3.2.2 Good labellings of an asymptotic lattice

We now turn to the description of the asymptotic lattices themselves. Using an asymptotic chart, we can put an integer label on each point of the lattice:

**Lemma 3.11** *If  $G_{\hbar}$  is an asymptotic chart for  $\mathcal{L}_{\hbar}$ , then there exists a family of maps  $k_{\hbar} : \mathcal{L}_{\hbar} \mapsto \mathbb{Z}^n$ ,  $\hbar \in \mathcal{I}$ , which is unique for  $\hbar$  small enough, and such that (14) holds with  $k = k_{\hbar}(\lambda)$ , i.e.  $\hbar k_{\hbar}(\lambda) \in U$  and*

$$\forall \lambda \in \mathcal{L}_{\hbar}, \quad \forall N \geq 0, \quad \|G_{\hbar}(\hbar k_{\hbar}(\lambda)) - \lambda\| \leq C_N \hbar^N. \quad (20)$$

*Moreover this map is injective.*

**Proof.** From Item 6 of Lemma 3.8, let  $\hbar_0 > 0$  and  $\tilde{U} \Subset U$  be such that, for all  $\hbar \leq \hbar_0$ ,  $G_{\hbar} : \tilde{U} \rightarrow B_1$  is invertible onto  $B_1$ , a fixed bounded open neighborhood of  $\bar{B}$ . Let  $\lambda \in \mathcal{L}_{\hbar}$ . If  $k$  and  $\tilde{k}$  in  $\mathbb{Z}^n$  satisfy (14) (for some arbitrary family  $(C_N)_{N \geq 0}$ ), then  $G_{\hbar}(\hbar k)$  and  $G_{\hbar}(\hbar \tilde{k})$  belong to  $B + B(0, C_N \hbar^N)$ . We can assume that  $\hbar_0$  is small enough so that the latter is contained in  $B_1$ . Thus,  $\hbar k$  and  $\hbar \tilde{k}$  belong to  $\tilde{U}$ . Moreover,  $\|G_{\hbar}(\hbar k) - G_{\hbar}(\hbar \tilde{k})\| \leq 2C_N \hbar^N$ , which entails  $\|\hbar k - \hbar \tilde{k}\| \leq 2L_F C_N \hbar^N$ , where  $L_F$  is a uniform upper bound for the Lipschitz norm of  $G_{\hbar}^{-1}$  on  $\tilde{U}$ . Choose any  $N_1 > 1$  and take  $\hbar_0$  small enough so that

$$2L_F C_{N_1} \hbar_0^{N_1-1} < 1; \quad (\hbar\text{-}21)$$

we obtain, for any  $\hbar \leq \hbar_0$ ,  $\|k - \tilde{k}\| < 1$  and hence  $k = \tilde{k}$ . This shows that, for any  $\hbar \leq \hbar_0$ , any  $\lambda \in \mathcal{L}_{\hbar}$  is associated with a unique  $k \in \mathbb{Z}^n$  such that (14) holds. We call this map  $k_{\hbar}$ . Notice that  $k_{\hbar} = \frac{1}{\hbar} G_{\hbar}^{-1} \big|_{\mathcal{L}_{\hbar}} + \mathcal{O}(\hbar^\infty)$ . In order to prove injectivity, assume  $k_{\hbar}(\lambda_1) = k_{\hbar}(\lambda_2)$ . From (14) again

we now get  $\|\lambda_1 - \lambda_2\| \leq 2C_N \hbar^N$  for all  $N$ . Let  $(\epsilon_0, N_0)$  be the constants defined by item 1 of Definition 3.6; if

$$2C_N \hbar^N < \epsilon_0 \hbar^{N_0}, \quad (\hbar-22)$$

which will happen if one chooses  $N > N_0$  and  $\hbar_0$  small enough, we conclude that  $\lambda_1 = \lambda_2$ .  $\square$

**Remark 3.12** Note that a consequence of the above lemma (and its proof) is that, once Condition 1 in the definition of asymptotic lattices (Definition 3.6) is fulfilled, it eventually holds with  $N_0 = 1$ , provided  $\hbar_0$  is small enough. Indeed, let  $\lambda_1, \lambda_2 \in \mathcal{L}_\hbar$  and let  $k_i = k_\hbar(\lambda_i)$ ,  $i = 1, 2$ : for all  $N \geq 0$ ,  $\|\lambda_i - G_\hbar(\hbar k_i)\| \leq C_N \hbar^N$ . Then if  $\|\lambda_1 - \lambda_2\| < \epsilon \hbar$  for some  $\epsilon > 0$ , we obtain

$$\|G_\hbar(\hbar k_1) - G_\hbar(\hbar k_2)\| < 2C_N \hbar^N + \epsilon \hbar,$$

and hence  $\|\hbar k_1 - \hbar k_2\| < L_F(2C_N \hbar^{N-1} + \epsilon) \hbar$ . Therefore, we may choose any  $N > 1$ , (and, as above, any  $N_1 > 1, N_2 > N_0$ ), and conclude by the injectivity of  $k_\hbar$  that  $\lambda_1 = \lambda_2$  as soon as  $\epsilon$  and  $\hbar_0$  verify

$$\hbar_0^{N_1-1} < \frac{1}{2L_F C_{N_1}}, \quad \hbar_0^{N_2-N_0} < \frac{\epsilon_0}{2C_{N_2}}, \quad \text{and} \quad \epsilon + 2C_N \hbar_0^{N-1} \leq \frac{1}{L_F}. \quad (\hbar-23)$$

In other words, if  $\epsilon$  and  $\hbar_0$  satisfy ( $\hbar-23$ ), then

$$\forall \hbar \in \mathcal{I} \cap ]0, \hbar_0] \quad \forall \lambda \in \mathcal{L}_\hbar, \quad B(\lambda, \epsilon \hbar) \cap \mathcal{L}_\hbar = \{\lambda\}. \quad (24)$$

In order to simplify the statement of our results, we could fix from now on  $N_1 = N = 2$  and  $N_2 = N_0 + 1$ , and ( $\hbar-23$ ) would be satisfied if we took  $\epsilon = \frac{1}{3L_F}$  and

$$\hbar_0 = \min \left( \frac{\epsilon_0}{4C_{N_0+1}} ; \frac{1}{6L_F C_2} \right).$$

However, we believe that in some situations, having the possibility to optimize such estimates by taking large  $N$ 's can be useful.  $\triangle$

**Remark 3.13** A completely similar argument shows that for all  $\lambda \in \mathcal{L}_\hbar$ , the integer vector  $k_\hbar(\lambda)$  defined in Lemma 3.11 is the unique element  $k \in \mathbb{Z}^n$  such that  $\hbar k \in U$  and

$$\|\lambda - G_\hbar(k \hbar)\| \leq \epsilon \hbar$$

as soon as  $L_F(C_N \hbar^{N-1} + \epsilon) < 1$  (which is implied by ( $\hbar-23$ )). Of course, this, in turn, implies the much better estimate (20).  $\triangle$

**Definition 3.14** The map  $k_\hbar$  from Lemma 3.11 will be called a good labelling of  $\mathcal{L}_\hbar$ .



Let  $\tilde{B} \Subset B$ . Then a good labelling is surjective onto  $(\frac{1}{\hbar}G_0^{-1}(\tilde{B})) \cap \mathbb{Z}^n$ , in a uniform way: by Item 2b of Definition 3.6 with  $\tilde{U}_0 = G_0^{-1}(\tilde{B})$ , there exists  $\tilde{\hbar}_0 > 0$  such that, for all  $\hbar \leq \tilde{\hbar}_0$  there exists  $\lambda \in \mathcal{L}_\hbar$  with  $\lambda = G_\hbar(\hbar k) + \mathcal{O}(\hbar^\infty)$ , and hence by Lemma 3.11,  $k_\hbar(\lambda) = k$ , as soon as  $\tilde{\hbar}_0$  satisfies (h-21).

It follows that the set  $\mathcal{L}_\hbar$  is always “dense in  $B$  as  $\hbar \rightarrow 0$ ”, by which we mean the following:

**Lemma 3.15** *For any  $c \in B$ , there exists a family  $(\lambda_\hbar)_{\hbar \in \mathcal{I}}$  with  $\lambda_\hbar \in \mathcal{L}_\hbar$  such that*

$$\lambda_\hbar = c + \mathcal{O}(\hbar).$$

**Proof.** If  $k(\hbar) := \lfloor \frac{\xi}{\hbar} \rfloor$  (the vector obtained by taking the integer part of all the components of  $\frac{\xi}{\hbar}$ ), where  $\xi := G_0^{-1}(c)$ , then  $\|\hbar k(\hbar) - \xi\| \leq \hbar$ . Thus, if  $\tilde{B} \Subset B$  is a neighborhood of  $c$ , then  $\hbar k(\hbar) \in G_0^{-1}(\tilde{B})$  for  $\hbar$  small enough, and we may define  $\lambda_\hbar$  to be the point in  $\mathcal{L}_\hbar$  associated with the label  $k(\hbar)$ . Since  $\|G_\hbar(\hbar k(\hbar)) - c\| \leq L_0 \|\hbar k(\hbar) - \xi\| + \|G_\hbar(\xi) - c\| \leq L_0 \hbar + M \hbar$ , where  $M$  is defined in (15), we get, for any  $N \geq 0$ ,  $\|\lambda_\hbar - c\| \leq (L_0 + M)\hbar + C_N \hbar^N$ .  $\square$

### 3.2.3 Linear labellings of an asymptotic lattice

Equipped with a good labelling (or, equivalently, with an asymptotic chart), an asymptotic lattice possesses the interesting feature that its individual points inherit a well-defined “smooth” evolution as  $\hbar$  varies, in the following sense. Let  $\lambda_\hbar \in \mathcal{L}_\hbar$ . If  $\tilde{\hbar}_0$  is small enough, then by the injectivity of Lemma 3.11 and (24), for any  $\hbar \leq \tilde{\hbar}_0$ ,  $\lambda_\hbar$  is the unique closest point to  $G_\hbar(\hbar k_\hbar(\lambda_\hbar))$  in  $\mathcal{L}_\hbar$ . Thus, we may now fix the integer  $k_0 = k_{\tilde{\hbar}_0}(\lambda_{\tilde{\hbar}_0})$  and consider the evolution of the corresponding point  $k_\hbar^{-1}(k_0) \in \mathcal{L}_\hbar$  as  $\hbar$  varies close to  $\tilde{\hbar}_0$ . Although this evolution may not be continuous, it is  $\mathcal{O}(\hbar^\infty)$ -close to the smooth map  $\hbar \mapsto \tilde{G}(\hbar, \hbar k_0)$ , where  $\tilde{G}$  is the smooth representative introduced in Lemma 3.8, Item 7. Notice that the typical behaviour of the point  $G_\hbar(\hbar k_0)$ , as  $\hbar \rightarrow 0$ , is to *leave* the neighborhood where the chart  $G_\hbar$  is meaningful. Indeed,  $\hbar k_0 \rightarrow 0 \in \mathbb{R}^n$ , and there is no reason why  $0$  should belong to  $U$ . This is analogous to the well-known phenomenon that occurs for eigenvalues of operators depending on a parameter, when we restrict our attention to a fixed spectral window: trying to follow a particular “mode” (eigenvector) as the parameter varies, the corresponding eigenvalue is likely to escape the window. Good labellings capture this phenomenon. For our purposes in this paper, it will be useful to disregard this “drift”, by defining the weaker notion of a good labelling “modulo a constant”, as follows.

**Definition 3.16** *A map  $\bar{k}_\hbar : \mathcal{L}_\hbar \rightarrow \mathbb{Z}^n$  is called a linear labelling for  $\mathcal{L}_\hbar$  if there exists a family  $(\varkappa_\hbar)_{\hbar \in ]0, \tilde{\hbar}_0]}$  of vectors in  $\mathbb{Z}^n$  such that  $\bar{k}_\hbar + \varkappa_\hbar$  is a good labelling for  $\mathcal{L}_\hbar$ .*

Of course any good labelling is a linear labelling; we shall see in Section 3.5 below that linear labellings may be easier to construct.

Another important property of asymptotic lattices is that they can be equipped with an “asymptotic  $\mathbb{Z}^n$ -action” in the following sense. Fix a good labelling  $k_\hbar$  for  $\mathcal{L}_\hbar$ . Let  $\tilde{B} \Subset B$  be open, and let  $\varkappa \in \mathbb{Z}^n$  be a fixed integral vector. Let  $\hat{B}$  be an open set such that  $\tilde{B} \Subset \hat{B} \Subset B$ . If  $\hbar$  is small

enough (depending on  $\tilde{B}$ ,  $\hat{B}$ , and  $\varkappa$ ), for any  $\lambda \in \mathcal{L}_{\tilde{h}} \cap \tilde{B}$ , we have  $\tilde{h}(k_{\tilde{h}}(\lambda) + \varkappa) \in G_0^{-1}(\hat{B})$ ; therefore, there exists a unique point in  $\mathcal{L}_{\tilde{h}}$ , denoted by  $\lambda + \varkappa$ , such that

$$k_{\tilde{h}}(\lambda + \varkappa) = k_{\tilde{h}}(\lambda) + \varkappa. \quad (25)$$

This property characterizes linear labellings, as follows.

**Proposition 3.17** *Let  $\mathcal{L}_{\tilde{h}}$  be an asymptotic lattice in the open set  $B$  with good labelling  $k_{\tilde{h}}$ . Let  $\tilde{B} \Subset B$  be open and connected. Let a map  $\tilde{k}_{\tilde{h}} : \mathcal{L}_{\tilde{h}} \rightarrow \mathbb{Z}^n$ , defined for  $\tilde{h} \in \mathcal{I}$ , commute with translations in the following sense: for any  $\varkappa \in \mathbb{Z}^n$ , there exists  $\tilde{h}_{\varkappa} > 0$  such that*

$$\forall \tilde{h} \leq \tilde{h}_{\varkappa}, \quad \forall \lambda \in \mathcal{L}_{\tilde{h}} \cap \tilde{B}, \quad \tilde{k}_{\tilde{h}}(\lambda + \varkappa) = \tilde{k}_{\tilde{h}}(\lambda) + \varkappa. \quad (26)$$

Then  $\tilde{k}_{\tilde{h}}$  is a linear labelling for  $(\mathcal{L}_{\tilde{h}} \cap \tilde{B}, \mathcal{I}, \tilde{B})$  (associated with the good labelling  $k_{\tilde{h}}$  restricted to  $\tilde{B}$ ).

In order to prove this proposition, we first show that  $\mathcal{L}_{\tilde{h}} \cap \tilde{B}$  is ‘connected by lattice paths’.

**Lemma 3.18** *Let  $\mathcal{L}_{\tilde{h}}$ ,  $\tilde{h} \in \mathcal{I}$ , be an asymptotic lattice in an open set  $B$ . Let  $\tilde{B} \Subset \hat{B} \Subset B$ , where  $\tilde{B}$  and  $\hat{B}$  are open and  $\tilde{B}$  is connected. Given a good labelling  $k_{\tilde{h}}$  for  $\mathcal{L}_{\tilde{h}}$ , there exists  $\tilde{h}_0 > 0$  such that, given any  $\tilde{h} \in ]0, \tilde{h}_0] \cap \mathcal{I}$ , and given any pair of points  $z_1, z_2 \in \mathcal{L}_{\tilde{h}} \cap \tilde{B}$ , there exists a finite sequence  $(\epsilon^{(j)})_{j \in \{1, \dots, N\}}$  in  $\{-1, 0, 1\}^n$ , and a finite sequence of points  $\lambda_j \in \mathcal{L}_{\tilde{h}} \cap \hat{B}$ ,  $j = 0, \dots, N$ , such that*

$$\lambda_0 = z_1, \quad \forall j = 0, \dots, N-1, \quad \lambda_{j+1} = \lambda_j + \epsilon^{(j+1)}, \quad \text{and} \quad \lambda_N = z_2. \quad (27)$$

**Proof.** Let  $(G_{\tilde{h}}, U)$  be an asymptotic chart for  $\mathcal{L}_{\tilde{h}}$  associated with  $k_{\tilde{h}}$  (see Lemma 3.11). Let  $c \in \tilde{B}$ . Let  $\tilde{V} \subset U$  be an open cube centered at  $\xi = G_0^{-1}(c)$ . Let us first prove the ‘lattice-connectedness’ property (27) for  $\tilde{B} = G_0(\tilde{V})$ . Let  $\hat{V} \subset U$  be another open cube containing  $\tilde{V}$ . It follows from the existence and the asymptotic expansion of  $G_{\tilde{h}}^{-1}$  (Item 6 of Lemma 3.8) that  $G_{\tilde{h}}^{-1}(\tilde{B}) \subset \hat{V}$ , if  $\tilde{h}$  is small enough. Therefore, if  $V \subset U$  is another open cube containing  $\hat{V}$ , Lemma 3.11 implies that, for  $\tilde{h}$  small enough,  $\tilde{h}k_{\tilde{h}}(\lambda) \subset V$  for any  $\lambda \in \mathcal{L}_{\tilde{h}} \cap \tilde{B}$ , and  $k_{\tilde{h}}^{-1}((\tilde{h}^{-1}V) \cap \mathbb{Z}^n) \subset \hat{B}$ . In  $V \cap \tilde{h}\mathbb{Z}^n$ , any two points  $\xi_1, \xi_2$  can obviously be joined by acting by the canonical basis of the lattice. Applying  $k_{\tilde{h}}^{-1}$ , this shows the required property (27). The number  $N$  of steps in (27) can be as large as  $\mathcal{O}(1/\tilde{h})$ , but the set of involved  $\varkappa$ ’s in  $\{-1, 0, 1\}^n$  being finite, we do get an  $\tilde{h}_0 > 0$  for which the result is uniform for all  $\tilde{h} \in \mathcal{I} \cap ]0, \tilde{h}_0]$ .

If  $\tilde{B} \subset B$  is a general connected open set, with compact closure contained in  $B$ , we can cover it by a finite number of deformed cubes of the form  $G_0(\tilde{V}) \Subset \hat{B}$ , as above. If  $\tilde{h}$  is small enough, the intersection of two such deformed cubes is either empty or contains a point in  $\mathcal{L}_{\tilde{h}}$ . Therefore the connectedness property holds for the union of two deformed cubes with a common point, and by induction for the union of all deformed cubes, hence for  $\tilde{B}$ .  $\square$

**Proof of Proposition 3.17.** Let  $\hbar_0 = \min_{\varkappa \in \{-1, 0, 1\}^n} \hbar_{\varkappa}$ . We may also assume that  $\hbar_0$  is small enough so that Lemma 3.18 holds. Hence, applying (26) to (27) we obtain  $\tilde{k}_{\hbar}(\lambda_{j+1}) = \tilde{k}_{\hbar}(\lambda_j) + \epsilon^{(j+1)}$ , and hence

$$\tilde{k}_{\hbar}(z_2) = \tilde{k}_{\hbar}(z_1) + \sum_{j=1}^N \epsilon^{(j)}.$$

On the other hand, if we consider the good labelling  $k_{\hbar}$ , the translation invariance property (25) gives similarly

$$k_{\hbar}(z_2) = k_{\hbar}(z_1) + \sum_{j=1}^N \epsilon^{(j)}.$$

Therefore,  $\tilde{k}_{\hbar}(z_2) - k_{\hbar}(z_2) = \tilde{k}_{\hbar}(z_1) - k_{\hbar}(z_1)$ , for all  $z_1, z_2 \in \tilde{B}$  and  $\hbar \leq \hbar_0$ ; so  $\tilde{k}_{\hbar} - k_{\hbar}$  is equal to a constant  $K(\hbar) \in \mathbb{Z}^n$  on  $\tilde{B}$ . Hence  $\tilde{k}_{\hbar} = k_{\hbar} - K(\hbar)$  is a linear labelling for  $\mathcal{L}_{\hbar}$  on  $\tilde{B}$ .  $\square$

Theorem 3.2 says that the joint spectrum of a quantum integrable system is an asymptotic lattice near any regular point of the momentum map; in other words, the joint spectrum possesses “good quantum numbers”  $(k_1, \dots, k_n)$ , which is formalized in Definition 3.14 as the existence of a good labelling. This property was used in [65] to define the notion of *quantum monodromy*.

In this paper, we shall need to go one step further, namely we will ask the question: can one construct a good labelling from the joint spectrum only? Indeed, a good labelling is not unique, according to the following, straightforward, result:

**Lemma 3.19** *If  $G_{\hbar}$  is an asymptotic chart for  $\mathcal{L}_{\hbar}$ , defined on an open set  $U$ , then for any orientation preserving linear transformation with integer coefficients  $A \in \mathrm{SL}(n, \mathbb{Z})$ , the map  $\tilde{G}_{\hbar} := G_{\hbar} \circ A$  is another asymptotic chart for  $\mathcal{L}_{\hbar}$ , defined on  $A^{-1}U$ , and corresponding to the new good labelling  $\tilde{k}_{\hbar} = A^{-1} \circ k_{\hbar}$ .*

**Proposition 3.20** *If  $\bar{k}_{\hbar}^{(1)}$  and  $\bar{k}_{\hbar}^{(2)}$  are two linear labellings for an asymptotic lattice  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$ , then for any connected open set  $\tilde{B} \Subset B$ , there exists a unique matrix  $A \in \mathrm{SL}(n, \mathbb{Z})$ , independent of  $\hbar$ , a family  $(\varkappa_{\hbar})_{\hbar \in \mathcal{I}}$  in  $\mathbb{Z}^n$ , and  $\hbar_0 > 0$  such that*

$$\forall \hbar \in \mathcal{I} \cap [0, \hbar_0], \quad \bar{k}_{\hbar}^{(2)} = A \circ \bar{k}_{\hbar}^{(1)} + \varkappa_{\hbar} \quad \text{on} \quad \mathcal{L}_{\hbar} \cap \tilde{B}.$$

**Proof. (a)** Let  $G_{\hbar}^{(1)}$  and  $G_{\hbar}^{(2)}$  be asymptotic charts associated with the linear labellings  $\bar{k}_{\hbar}^{(1)}$  and  $\bar{k}_{\hbar}^{(2)}$ , respectively, and let  $k_{\hbar}^{(1)}$  and  $k_{\hbar}^{(2)}$  be the corresponding good labellings. Let  $\lambda_0 \in \mathcal{L}_{\hbar}$  and choose it as an “origin” of the linear labellings: up to changing their constant term, one can assume that  $\bar{k}_{\hbar}^{(1)}(\lambda_0) = \bar{k}_{\hbar}^{(2)}(\lambda_0) = 0 \in \mathbb{Z}^n$ . Let  $\mathcal{B} = (e_1, \dots, e_n)$  be the canonical basis of  $\mathbb{Z}^n$ , and let  $(\lambda_1^{(j)}, \dots, \lambda_n^{(j)})$ ,  $j = 1, 2$ , be the corresponding points in  $\mathcal{L}_{\hbar}$  for the two labellings, namely:

$$\bar{k}_{\hbar}^{(j)}(\lambda_i^{(j)}) = e_i. \quad (28)$$

To make sure these points do exist, one can impose  $\lambda_0$  to stay in a  $\mathcal{O}(\hbar)$  neighborhood of a fixed point  $c \in \tilde{B}$ ; then by Item 6 of Lemma 3.8,  $(G_h^{(j)})^{-1}(\lambda_0)$  will stay in an  $\mathcal{O}(\hbar)$  neighborhood of  $\xi^{(j)} := (G_0^{(j)})^{-1}(c)$ . Hence, by (20), the same holds for  $\hbar k_h^{(j)}(\lambda_0)$ . Denote  $k_0^{(j)} := k_h^{(j)}(\lambda_0)$ . Let  $\tilde{U}_0^{(j)} \Subset (G^{(j)})^{-1}(B)$  containing  $\xi^{(j)}$ , and let  $\hbar$  be small enough so that  $\tilde{U}_0^{(j)}$ , for  $j = 1, 2$ , contains  $\hbar k_0^{(j)} + \hbar\mathcal{B}$ . By Item 2b of Definition 3.14, the points  $\lambda_i^{(j)}$  are now well-defined by (28), because  $\bar{k}_h^{(j)}(\lambda_i^{(j)}) = \bar{k}_h^{(j)}(\lambda_i^{(j)}) - \bar{k}_h^{(j)}(\lambda_0) = k_h^{(j)}(\lambda_i^{(j)}) - k_h^{(j)}(\lambda_0)$ , and hence  $\lambda_i^{(j)}$  is defined by

$$\hbar k_h^{(j)}(\lambda_i^{(j)}) = \hbar k_0^{(j)} + \hbar e_i \in \tilde{U}_0. \quad (29)$$

(b) Now, let  $v_i^{(j)} = (\lambda_i^{(j)} - \lambda_0)$ ,  $i = 1, \dots, n$ , be the corresponding ‘‘basis vectors’’ of the asymptotic lattice. By the uniform Taylor formula (17), and (15),

$$v_i^{(1)} = \hbar(G_h^{(1)})'(\hbar k_0^{(1)}) \cdot e_i + \mathcal{O}(\hbar^2) = \hbar(G_0^{(1)})'(\xi^{(1)}) \cdot e_i + \mathcal{O}(\hbar^2). \quad (30)$$

The point  $\lambda_i^{(1)}$  can also be labelled by  $k_h^{(2)}$ ; let

$$z_i := k_h^{(2)}(\lambda_i^{(1)}) - k_0^{(2)} \in \mathbb{Z}^n. \quad (31)$$

Remember that  $z_i$ , contrary to  $e_i$ , depends on  $\hbar$ . Similarly, we have

$$v_i^{(1)} = \hbar(G_0^{(2)})'(\xi^{(2)}) \cdot z_i + \mathcal{O}(\hbar^2),$$

and hence

$$e_i = [(G_0^{(1)})'(\xi^{(1)})]^{-1}(G_0^{(2)})'(\xi^{(2)}) \cdot z_i + \mathcal{O}(\hbar). \quad (32)$$

Let  $d_2 = d_2(\hbar) := \det(z_1, \dots, z_n) \in \mathbb{N}$ . Let  $\delta_{1,2} := \det\left([(G_0^{(1)})'(\xi^{(1)})]^{-1}(G_0^{(2)})'(\xi^{(2)})\right)$ . From (32),  $1 = \delta_{1,2}d_2 + \mathcal{O}(\hbar)$ . Now repeat the argument with exchanging the two labellings: we obtain a new integer  $d_1 = d_1(\hbar) \in \mathbb{N}$  such that  $1 = \delta_{2,1}d_1 + \mathcal{O}(\hbar)$ , with  $\delta_{2,1} = \delta_{1,2}^{-1}$ . Hence  $d_1d_2 = 1 + \mathcal{O}(\hbar)$ , which implies  $d_1d_2 = 1$  if  $\hbar$  is small enough, and hence  $d_2 = d_1 = 1$ . This shows that  $(z_1, \dots, z_n)$  is an oriented  $\mathbb{Z}$ -basis of  $\mathbb{Z}^n$ ; let  $A \in \mathrm{SL}(n, \mathbb{Z})$  be the corresponding matrix. Equation (32) gives  $[(G_0^{(2)})'(\xi^{(2)})]^{-1}(G_0^{(1)})'(\xi^{(1)}) = A + \mathcal{O}(\hbar)$ , and since the left-hand side does not depend on  $\hbar$ ,  $A$  must converge towards it as  $\hbar \rightarrow 0$ , but since  $A$  has integer coefficients,  $A$  must be constant for  $\hbar$  small enough, and we have

$$A = [(G_0^{(2)})'(\xi^{(2)})]^{-1}(G_0^{(1)})'(\xi^{(1)}) \in \mathrm{SL}(n, \mathbb{Z}). \quad (33)$$

From (31) and (29), we have

$$k_h^{(2)}(\lambda_i^{(1)}) = A \cdot e_i + k_0^{(2)} = A \cdot k_h^{(1)}(\lambda_i^{(1)}) - A \cdot k_0^{(1)} + k_0^{(2)},$$

which says that

$$k_h^{(2)}(\lambda) = A \circ k_h^{(1)}(\lambda) + \varkappa_h \quad (34)$$

with  $\varkappa_{\hbar} := k_0^{(2)} - A \cdot k_0^{(1)}$ , when  $\lambda$  is restricted to the set  $\{\lambda_1^{(1)}, \dots, \lambda_n^{(1)}\}$ . It remains to extend this equality to the whole set  $\mathcal{L}_{\hbar}$ .

(c) We cannot directly adapt the argument, replacing  $\lambda_i^{(1)}$  by any  $\lambda \in \mathcal{L}_{\hbar}$ , because the corresponding integral vector  $z_i$  from (31) would be unbounded (typically, of size  $1/\hbar$ ), which would make the Taylor formula unusable. Instead, we need a connectedness argument, as in Proposition 3.17. First, remark that we may now replace in (29) the vector  $e_i$  by a linear combination  $e := \sum n_i e_i$ , where  $n_i$  are bounded integers, defining a lattice point  $\mu$  by

$$\hbar k_{\hbar}^{(1)}(\mu) = \hbar k_0^{(1)} + \hbar e \in \tilde{U}_0.$$

Let  $v := \mu - \lambda_0$ . From

$$v = \hbar(G_0^{(1)})'(\xi^{(2)}) \cdot e + \mathcal{O}(\hbar^2),$$

and comparing with (30), we see that  $v = \sum n_i v_i^{(1)} + \mathcal{O}(\hbar)$ . Letting  $z := k_{\hbar}^{(2)}(\mu) - k_0^{(2)}$ , we see from

$$v = \hbar(G_0^{(2)})'(\xi^{(2)}) \cdot z + \mathcal{O}(\hbar^2),$$

that  $z = \sum n_i z_i + \mathcal{O}(\hbar)$ . Since the integers  $n_i$  are bounded, we must have  $z = \sum n_i z_i$  for  $\hbar$  small enough. This gives

$$k_{\hbar}^{(2)}(\mu) = z + k_0^{(2)} = A \cdot e + k_0^{(2)}.$$

Therefore, the equality (34) still holds for  $\mu$ , *i.e.* for all points of the form  $\lambda = \lambda_0 + \vec{v}$ , where  $\vec{v}$  is uniformly bounded in  $\mathbb{Z}^n$ . Let us restrict such  $\vec{v}$  to belong to  $\{-1, 0, 1\}^n$ . This means that the set of points satisfying (34) is invariant under the action of  $\{-1, 0, 1\}^n$ . From Lemma 3.18, this set must be  $\mathcal{L}_{\hbar} \cap \tilde{B}$ .  $\square$

### 3.2.4 $\hbar$ -continuity and drift for good labellings

If we want to fully understand the structure of asymptotic lattices, one should take into account the “constant term”  $\varkappa_{\hbar}$  that is discarded in linear labellings. In other words, do we have an analogue of Proposition 3.20 for good labellings? It turns out that answering this question leads to the introduction of two new interesting concepts attached to any asymptotic lattice  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$ : the  $\hbar$ -continuity and the drift. The former asks whether we have enough values of  $\hbar \in \mathcal{I}$  to continuously follow each individual point in the asymptotic lattice, as  $\hbar \rightarrow 0$ . When this is the case, we see that, generically, these points “pass through” the observation window  $B$  (like rain drops observed through a window) as if they wanted to converge to some unknown limit which may lie outside of  $B$ : this is what the drift will measure.

However, while this question is very natural, it is not necessary for the treatment of the rotation number, because, as we will see below (Section 3.3), the latter is defined in terms of differences of eigenvalues, which cancels the constant term. Therefore, the reader interested in our main results about rotation numbers can directly jump to Section 3.3).

**Definition 3.21** An asymptotic lattice is called  $\hbar$ -continuous if the set  $\mathcal{I}$ , which the small parameter  $\hbar$  belongs to, is such that the set  $\{\frac{1}{\hbar_1} - \frac{1}{\hbar_2}; \quad \hbar_1, \hbar_2 \in \mathcal{I}\}$  accumulates at zero, namely:

For all  $\epsilon > 0$ , there exists  $\hbar_0 > 0$  such that for all  $\hbar_1 \in \overline{\mathcal{I}} \setminus \{0\}$  with  $\hbar_1 < \hbar_0$ , there exists  $\hbar_2 \in \mathcal{I}$ ,  $\hbar_2 < \hbar_1$  such that

$$\frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \epsilon. \quad (35)$$

Of course, this property is satisfied if  $\mathcal{I} = (0, \hbar_0]$ , because the map  $\hbar \rightarrow \frac{1}{\hbar}$  is continuous. It also holds if the closure  $\overline{\mathcal{I}}$  contains  $[0, \hbar_0]$  for some  $\hbar_0 > 0$ . However, it is not satisfied if  $\mathcal{I} = \{1/k; \quad k \in \mathbb{N}^*\}$ , which is typical in geometric quantization. For  $\mathcal{I} = \{1/k^\beta; \quad k \in \mathbb{N}^*\}$ , the property is satisfied if and only if  $\beta \in ]0, 1[$ .

The following proposition can be seen as the quantum analogue of Lemma 2.2.

**Proposition 3.22** Let  $k_{\hbar}$  and  $\tilde{k}_{\hbar}$  be two good labellings for the  $\hbar$ -continuous asymptotic lattice  $\mathcal{L}_{\hbar}$ . Then there exists a unique orientation preserving transformation  $\tau \in \text{GA}_{\mathbb{Z}}^+(n, \mathbb{Z})$ , independent of  $\hbar$ , and  $\hbar_0 > 0$ , such that  $\tilde{k}_{\hbar} = \tau \circ k_{\hbar}$  for all  $\hbar \leq \hbar_0$ ,  $\hbar \in \mathcal{I}$ .

Here  $\text{GA}_{\mathbb{Z}}^+(n, \mathbb{Z})$  is the orientation preserving integral affine group  $\text{GA}_{\mathbb{Z}}^+(n, \mathbb{Z}) := \text{SL}(n, \mathbb{Z}) \times \mathbb{Z}^n$ .

**Proof.** Let  $G_{\hbar}$  and  $\tilde{G}_{\hbar}$  be the corresponding asymptotic charts (Equation (14)). Let  $\tilde{F}_{\hbar}$  be a formal inverse of  $\tilde{G}_{\hbar}$ , as in item 5 of Lemma 3.8, and define  $K_{\hbar} := \tilde{F}_{\hbar} \circ G_{\hbar}$ . Let  $c \in B$ , and let  $\xi = G_0^{-1}(c)$ . Let  $k = k(\hbar)$  be a family in  $\mathbb{Z}^n$  such that  $\hbar k(\hbar) = \xi + \mathcal{O}(\hbar)$  as  $\hbar \rightarrow 0$ . From (14), there exists another family  $\tilde{k} = \tilde{k}(\hbar) \in \mathbb{Z}^n$  such that

$$K_{\hbar}(\hbar k) = \hbar \tilde{k} + \mathcal{O}(\hbar^\infty). \quad (36)$$

Since  $K_{\hbar}$  admits a  $C^\infty$  asymptotic expansion  $K_{\hbar} = K_0 + \hbar K_1 + \dots$ , a uniform Taylor expansion gives, for any fixed integer vector  $v \in \mathbb{Z}^n$ ,

$$K_{\hbar}(\hbar(k+v)) - K_{\hbar}(\hbar k) = \hbar K'_0(\xi) \cdot v + \mathcal{O}(\hbar^2) = \hbar K'_0(\xi) \cdot v + \mathcal{O}(\hbar^2).$$

Since  $\hbar(k+v) = \xi + \mathcal{O}(\hbar)$  as well, we apply again (36) (with a possibly different integral vector  $\tilde{k}$ ) and obtain, letting  $\hbar \rightarrow 0$ ,

$$K'_0(\xi) \cdot v \in \mathbb{Z}.$$

This shows that the matrix  $K'_0(\xi)$  has integer entries. Swapping  $G_{\hbar}$  and  $\tilde{G}_{\hbar}$ , and repeating the argument, we obtain that the inverse of  $K'_0(\xi)$  has integer entries as well, meaning that  $K'_0(\xi) \in \text{GL}(n, \mathbb{Z})$ . In particular,  $\xi \mapsto K'_0(\xi)$  must be a constant matrix  $A \in \text{GL}(n, \mathbb{Z})$  in the connected open set  $G_0^{-1}(B)$ , which in turn implies that  $K_0(\xi) = A\xi + \alpha$ , for some  $\alpha \in \mathbb{R}^n$ . Thus, the maps  $G_0$  and  $\tilde{G}_0$  must differ by an affine transformation in  $\text{GA}(n, \mathbb{Z})$ :

$$\tilde{G}_0^{-1} \circ G_0(\xi) = A\xi + \alpha.$$

Let us assume, by contradiction, that  $\alpha \neq 0$ . We have

$$\begin{aligned}\tilde{k} &= \frac{1}{\hbar}K_0(\hbar k) + K_1(\hbar k) + \mathcal{O}(\hbar) \\ &= Ak + \frac{\alpha}{\hbar} + K_1(\xi) + R(\hbar),\end{aligned}\tag{37}$$

with  $R(\hbar) = \mathcal{O}(\hbar)$  (because  $\hbar k = \xi + \mathcal{O}(\hbar)$ ). Let  $\ell(\hbar) := \tilde{k}(\hbar) - Ak(\hbar)$ . Let  $\hbar_0 \in \mathcal{I}$  be small enough so that  $\sup_{\hbar \leq \hbar_0} \|R(\hbar)\| < \frac{1}{4}$  and so that the  $\hbar$ -continuity property (35) holds for  $\epsilon = \frac{1}{4|\alpha|}$ . Let

$$\mathcal{I}_0 := \{\hbar \in \mathcal{I} \mid \hbar \leq \hbar_0, \ell(\hbar) = \ell(\hbar_0)\},$$

and let  $\hbar_1 := \inf \mathcal{I}_0$ . Since  $\ell(\hbar) \sim \frac{\alpha}{\hbar}$ , it must be unbounded as  $\hbar \rightarrow 0$ , which implies that  $\hbar_1 > 0$ .

**Case 1:**  $\hbar_1 \in \mathcal{I}_0$ . Then, by  $\hbar$ -continuity, one can find  $\hbar_2 \in \mathcal{I}$ ,  $\hbar_2 < \hbar_1$ , such that  $\frac{1}{\hbar_2} - \frac{1}{\hbar_1} \leq \frac{1}{2|\alpha|}$ . Since  $\hbar_2 \notin \mathcal{I}_0$  and  $\ell(\hbar) \in \mathbb{Z}^n$ , we have

$$|\ell(\hbar_1) - \ell(\hbar_2)| \geq 1.$$

From (37) we get

$$|\ell(\hbar_1) - \ell(\hbar_2)| < |\alpha| \left( \frac{1}{\hbar_2} - \frac{1}{\hbar_1} \right) + \frac{1}{2} < 1,$$

a contradiction.

**Case 2:**  $\hbar_1 \notin \mathcal{I}_0$ . Let  $\eta := \frac{\hbar_1^2}{4|\alpha|}$ . By definition of  $\hbar_1$ , there exists  $\hbar_2 \in \mathcal{I}_0$  such that  $\hbar_1 < \hbar_2 \leq \hbar_1 + \eta$ . We have

$$\hbar_2 \leq \hbar_1 + \eta = \hbar_1 \left( 1 + \frac{\hbar_1}{4|\alpha|} \right) \leq \frac{\hbar_1}{1 - \frac{\hbar_1}{4|\alpha|}}.$$

Hence  $\frac{1}{\hbar_1} - \frac{1}{\hbar_2} \leq \frac{1}{4|\alpha|}$ . Since  $\hbar_1 \in \overline{\mathcal{I}}$ , by  $\hbar$ -continuity one can find  $\hbar_3 \in \mathcal{I}$ ,  $\hbar_3 < \hbar_1$ , such that  $\frac{1}{\hbar_3} - \frac{1}{\hbar_1} \leq \frac{1}{4|\alpha|}$ . Hence  $\frac{1}{\hbar_1} - \frac{1}{\hbar_3} \leq \frac{1}{2|\alpha|}$  and we may conclude as in Case 1 that

$$|\ell(\hbar_1) - \ell(\hbar_3)| < |\alpha| \left( \frac{1}{\hbar_1} - \frac{1}{\hbar_3} \right) + \frac{1}{2} < 1,$$

while  $|\ell(\hbar_1) - \ell(\hbar_3)| \geq 1$ , a contradiction.

Consequently we must have  $\alpha = 0$ . Thus we can write

$$\ell(\hbar) = K_1(\xi) + \mathcal{O}(\hbar),$$

which implies that  $\ell(\hbar)$  converges to  $K_1(\xi)$  as  $\hbar \rightarrow 0$  and hence that  $K_1(\xi) \in \mathbb{Z}^n$ , forcing it to be a constant  $\ell \in \mathbb{Z}^n$  on the connected open set  $G_0^{-1}(B)$ . We obtain

$$\tilde{k} = Ak + \ell + R(\hbar),$$



thus  $R(\hbar) \in \mathbb{Z}^n$ . Since  $R(\hbar) = \mathcal{O}(\hbar)$ , we must have, for  $\hbar < \hbar_0$  small enough,  $R(\hbar) = 0$ , which finishes the proof of the proposition.  $\square$

Proposition 3.22 shows that, given a set  $\mathcal{L}_\hbar \subset B$ , the set of good labellings on subsets  $\tilde{B} \subset B$ , *i.e.* for which  $(\mathcal{L}_\hbar \cap \tilde{B}, \mathcal{I}, \tilde{B})$  is an asymptotic lattice, form a flat sheaf over  $B$ . As a consequence, if  $B$  is simply connected and  $B$  is covered by open subsets on which  $\mathcal{L}_\hbar$  admit a good labelling, then  $\mathcal{L}_\hbar$  admits a good labelling on  $B$  itself.

The  $\hbar$ -continuity property has a natural interpretation in terms of the uniform continuity of individual points in  $\mathcal{L}_\hbar$ , as  $\hbar$  varies. Recall from the discussion before Definition 3.16 that the choice of a good labelling defines an  $\hbar$ -evolution of each individual point in  $\mathcal{L}_\hbar$ . If  $\mathcal{L}_\hbar$  is  $\hbar$ -continuous, Proposition 3.22 implies that this evolution is in fact intrinsic to  $\mathcal{L}_\hbar$  itself. However, this is not the case in general. The heuristics are very simple: given  $\hbar \in \mathcal{I}$ , if the closest element to  $\hbar$  in  $\mathcal{I}$  is of the form  $\hbar + \delta$ , then the lattice point corresponding to  $G_\hbar(k\hbar)$  gets displaced by a distance of order  $\mathcal{O}(k\delta)$ . If  $k\delta$  is of the same order as the mean spacing between points (*i.e.*  $\mathcal{O}(\hbar)$ ), then there will be a confusion between the evolution of this point with that of its closest neighbors on  $\mathcal{L}_\hbar$ . To avoid this confusion, we need  $k\delta \ll \hbar$ ; since  $k$  is in general of order  $1/\hbar$ , this means  $\delta \ll \hbar^2$ , which precisely gives the  $\hbar$ -continuity condition (35).

Let us now give a precise statement of this, which will be needed for the inverse problem in the next section.

**Proposition 3.23** *Let  $(\mathcal{L}_\hbar, \mathcal{I}, B)$  be an asymptotic lattice. There exists  $\hbar_0 > 0$ ,  $\delta > 0$ , and  $\epsilon > 0$  such that, if  $\hbar_1, \hbar_2 \in \mathcal{I} \cap ]0, \hbar_0]$  satisfy:*

$$\left| \frac{1}{\hbar_1} - \frac{1}{\hbar_2} \right| < \epsilon$$

*then the following holds. Fix  $\lambda_1 \in \mathcal{L}_{\hbar_1}$  and let  $\lambda_2 \in \mathcal{L}_{\hbar_2}$  be defined by*

$$\lambda_2 = k_{\hbar_2}^{-1}(k_{\hbar_1}(\lambda_1)),$$

*for some good labelling  $k_\hbar$ . Then*

$$\mathcal{L}_{\hbar_i} \cap B(\lambda_j, \delta\hbar_i) = \{\lambda_j\} \quad \forall (i, j) \in \{1, 2\}^2. \quad (38)$$

*Here  $B(\lambda_j, \delta\hbar_i)$  denotes the Euclidean ball centered at  $\lambda_j$ , of radius  $\delta\hbar_i$ .*

In other words, we fix  $\lambda_1 \in \mathcal{L}_{\hbar_1}$ , and we consider the evolution of  $\mathcal{L}_\hbar$  as  $\hbar$  moves from  $\hbar_1$  to  $\hbar_2$ ; then the closest element to  $\lambda_1$  in  $\mathcal{L}_{\hbar_2}$  is unique and is precisely the natural evolution of  $\lambda_1$  obtained by fixing its integer label  $k_{\hbar_1}(\lambda_1)$ .

**Proof.** First of all, by Remark 3.12 we may choose  $\delta_0 > 0$  such that  $\mathcal{L}_\hbar \cap B(\lambda, \delta_0\hbar) = \{\lambda\}$  for all  $\hbar < \hbar_0$  and all  $\lambda \in \mathcal{L}_\hbar$ . In particular, (38) holds when  $i = j$  for any  $\delta \leq \delta_0$ . Without loss of generality we may assume  $\hbar_2 < \hbar_1$ . Let  $(G_\hbar, U)$  be an asymptotic chart associated with the good labelling  $k_\hbar$ . Let  $k_1 = k_{\hbar_1}(\lambda_1)$ . We have

$$\lambda_2 = G_{\hbar_2}(h_2 k_1) + \mathcal{O}(\hbar_2^\infty) = G_{\hbar_2}(h_2 k_1) + \mathcal{O}(\hbar_1^\infty),$$

and hence  $\lambda_2 - \lambda_1 = G_{\hbar_2}(\hbar_2 k_1) - G_{\hbar_1}(\hbar_1 k_1) + \mathcal{O}(\hbar_1^\infty)$ . Let  $\tilde{U} \Subset U$  be such that  $G_0^{-1}(\tilde{U})$  contains  $\bar{B}$ . Taking  $\hbar_0$  small enough, we may assume that  $G_{\hbar}$  is invertible on  $\tilde{U}$  for all  $\hbar \in \mathcal{I} \cap ]0, \hbar_0]$ , see Lemma 3.8, Item 6. Therefore  $\hbar_1 k_1 \in B$  and hence is bounded. Thus, there exists  $M > 0$  such that

$$\|\hbar_2 k_1 - \hbar_1 k_1\| \leq \epsilon \hbar_1 \hbar_2 k_1 \leq \epsilon M \hbar_2.$$

Hence we may apply Item 4 of Lemma 3.8 and get  $\|G_{\hbar_2}(\hbar_2 k_1) - G_{\hbar_2}(\hbar_1 k_1)\| \leq L_0 \epsilon M \hbar_2$ . Using now Item 3 of the same lemma, we obtain

$$\begin{aligned} \|\lambda_2 - \lambda_1\| &\leq L_0 \epsilon M \hbar_2 + C(|\hbar_2 - \hbar_1| + \hbar_1^2 + \hbar_2^2) + \mathcal{O}(\hbar_1^\infty) \\ &\leq L_0 \epsilon M \hbar_2 + C(\epsilon + 2)\hbar_1^2 + \mathcal{O}(\hbar_1^\infty). \end{aligned} \quad (39)$$

Since  $\hbar_1 \leq \hbar_2(1 + \epsilon \hbar_1)$ , we see that, if  $\epsilon$  and  $\hbar_0$  are small enough, the right-hand side of (39) is less than  $\delta_0 \hbar_2/3$ . Choosing finally  $\delta = \delta_0/2$ , we see that (38) must hold. Indeed, if  $\mu_1 \in \mathcal{L}_{\hbar_1}$  and  $\mu_1 \neq \lambda_1$ , then  $\|\mu_1 - \lambda_1\| \geq \delta_0 \hbar_1$ , hence  $\|\mu_1 - \lambda_2\| \geq \|\mu_1 - \lambda_1\| - \|\lambda_2 - \lambda_1\| \geq \delta_0 \hbar_1 - \delta_0 \hbar_2/3 > \delta \hbar_1$ , and similarly if  $\mu_2 \in \mathcal{L}_{\hbar_2}$  and  $\mu_2 \neq \lambda_2$ , then  $\|\mu_2 - \lambda_1\| \geq \delta_0 \hbar_2 - \delta_0 \hbar_2/3 > \delta \hbar_2$ .  $\square$

We conclude this section by making precise the relationship between the natural evolution of  $\hbar$ -continuous asymptotic lattices given by Proposition 3.23 above, the ‘‘drift’’ mentioned before Definition 3.16, and the asymptotic expansion of the lattice points in  $\hbar$ .

**Proposition 3.24** *Let  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$  be an asymptotic lattice. Let  $c \in B$ , and let  $\lambda_{\hbar} \in \mathcal{L}_{\hbar}$  be such that  $\lambda_{\hbar} = c + \mathcal{O}(\hbar)$ . Let  $k_{\hbar}$  be a good labelling for this lattice. There there exists  $\alpha_c \in \mathbb{R}^n$ ,  $A_c \in \mathbb{M}_n(\mathbb{R})$ , and  $\beta_c \in \mathbb{R}^n$  such that*

$$\lambda_{\hbar} = \alpha_c + \hbar(A_c \cdot k_{\hbar}(\lambda_{\hbar}) + \beta_c) + \mathcal{O}(\hbar^2). \quad (40)$$

Moreover,

$$\alpha_c = c - G'_0(\xi_0) \cdot \xi_0, \quad A_c = G'_0(\xi_0), \quad \beta_c = G_1(\xi_0),$$

where  $G_{\hbar} = G_0 + \hbar G_1 + \dots$  is the asymptotic chart associated with  $k_{\hbar}$ , and  $\xi_0 = G_0^{-1}(c)$ .

**Proof.** Let  $k = k_{\hbar}(\lambda_{\hbar})$  for simplifying notation. Using a uniform Taylor formula (17), we have  $G_{\hbar}(\hbar k) = G_0(\xi_0) + G'_0(\xi_0) \cdot (\hbar k - \xi_0) + \hbar G_1(\xi_0) + \mathcal{O}(\|\hbar k - \xi_0\|^2) + \hbar \mathcal{O}(\|\hbar k - \xi_0\|) + \mathcal{O}(\hbar^2)$ .

Using the boundedness of  $G_{\hbar}^{-1}$  (Item 5 or 6 of Lemma 3.8), we have

$$\|\hbar k - \xi_0\| = \mathcal{O}(\|G_{\hbar}^{-1}(\hbar k) - c\|).$$

Since  $\lambda_{\hbar} = G_{\hbar}(\hbar k) + \mathcal{O}(\hbar^\infty)$ , this gives

$$\lambda_{\hbar} = \alpha_c + \hbar(A_c \cdot k_{\hbar}(\lambda_{\hbar}) + \beta_c + \mathcal{O}(\|\lambda_{\hbar} - c\|)) + \mathcal{O}(\|\lambda_{\hbar} - c\|^2) + \mathcal{O}(\hbar^2), \quad (41)$$

which establishes the result.  $\square$

Assume now that the asymptotic lattice  $\mathcal{L}_{\hbar}$  is  $\hbar$ -continuous; we see from (40) that the natural evolution of a point  $\lambda \in \mathcal{L}_{\hbar}$ , as  $\hbar$  varies slightly, which corresponds to freezing the integer  $k$ , is to move ‘‘as if it wanted to tend to  $\alpha_c$ ’’; and, in general  $\alpha_c \neq c$ .

**Definition 3.25** Let  $(\mathcal{L}_\hbar, \mathcal{I}, B)$  be an  $\hbar$ -continuous asymptotic lattice. Let  $c \in B$  and  $\xi_0 := G_0^{-1}(c)$ . We call the quantity

$$\delta_c := G'_0(\xi_0) \cdot \xi_0 \in \mathbb{R}^n$$

the drift of the asymptotic lattice at  $c$ . It does not depend on the choice of a good labelling.

**Proof.** Let show that the drift is indeed well defined. Let  $\tilde{G}_\hbar$  be another asymptotic chart of  $\mathcal{L}_\hbar$ . Since  $\mathcal{L}_\hbar$  is  $\hbar$ -continuous, we may apply Proposition 3.22, and obtain  $\tau \in \text{GA}_{\mathbb{Z}}^+(n, \mathbb{Z})$  such that

$$G_\hbar(\hbar k_\hbar) = \tilde{G}_\hbar(\hbar \tau(k_\hbar)) + \mathcal{O}(\hbar^\infty)$$

for any indices  $k_\hbar$  such that  $\hbar k_\hbar$  belongs to  $G_0^{-1}(B)$ . Whence (with  $\tau(k) = Ak + \ell$ , i.e.  $A = \tau'$ )

$$\begin{aligned} G_0(\hbar k_\hbar) &= \tilde{G}_0(\hbar \tau(k_\hbar)) + \mathcal{O}(\hbar) \\ &= \tilde{G}_0(\hbar(Ak_\hbar + \ell)) + \mathcal{O}(\hbar) \\ &= \tilde{G}_0(\hbar Ak_\hbar) + \mathcal{O}(\hbar) \\ &= \tilde{G}_0 \circ A(\hbar k_\hbar) + \mathcal{O}(\hbar). \end{aligned}$$

We deduce that  $G_0(\xi) = \tilde{G}_0 \circ A(\xi)$  for any  $\xi \in G_0^{-1}(B)$ , which means  $G_0 = \tilde{G}_0 \circ A$ . Let  $\tilde{\xi}_0 = \tilde{G}_0^{-1}(c)$ , we have  $\tilde{\xi}_0 = A(\xi_0)$  and  $G'_0(\xi_0) \cdot v = \tilde{G}'_0(A(\xi_0)) \cdot A(v)$ , hence  $G'_0(\xi_0) \cdot \xi_0 = \tilde{G}'_0(\tilde{\xi}_0) \cdot \tilde{\xi}_0$ .  $\square$

**Remark 3.26** When the asymptotic lattice  $(\mathcal{L}_\hbar, \mathcal{I}, B)$  is the intersection of  $B$  with a joint spectrum of a quantum integrable system of commuting pseudodifferential operators, we may take  $\mathcal{I} = ]0, \hbar_0]$ , which is  $\hbar$ -continuous, and the drift is well-defined. In this case, the drift has a natural geometric definition. Since  $M = T^*X$  is a cotangent bundle, there is a globally defined primitive for the symplectic form, namely the Liouville 1-form  $\alpha$ . The formula  $I_j(\Lambda) = \frac{1}{2\pi} \int_{\gamma_j} \alpha$ , where  $(\gamma_1, \dots, \gamma_n)$  is a basis of loops on the Liouville torus  $\Lambda$ , defines action variables that depend only on this choice of basis. Hence, the integral affine manifold  $B_r$  is “exact” in the sense that its structure group can be reduced to the linear group  $\text{GL}(n, \mathbb{Z})$  (instead of the affine group). (This fact was used in [65] to show that quantum monodromy is linear instead of affine.) Therefore, the tangent bundle  $TB_r$  has a natural Euler vector field  $E = \sum_j I_j \frac{\partial}{\partial I_j}$ . Now, consider the momentum map  $F : M \rightarrow \mathbb{R}^n$ . By the action-angle theorem, we have a smooth local diffeomorphism  $G_0$  such that  $F = G_0(I)$ ; in other words,  $G_0 : B_r \rightarrow \mathbb{R}^n$  is the natural map induced by  $F$  on the leaf space. The push-forward of  $E$  by this map is  $d_I G_0 \cdot E = G'_0(I) \cdot I$ , i.e. the drift of Definition 3.25.  $\triangle$

Since the drift is well-defined, we can expect to find a way to recover it directly from the asymptotic lattice, without choosing a particular good labelling. This is the content of the following lemma.

**Proposition 3.27** *Let  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$  be an  $\hbar$ -continuous asymptotic lattice. Let  $c \in B$ , and let  $\lambda_{\hbar} \in \mathcal{L}_{\hbar}$  be such that  $\lambda_{\hbar} = c + \mathcal{O}(\hbar)$ . For  $\hbar_1, \hbar_2 \in \mathcal{I}$ , let  $\lambda(\hbar_1, \hbar_2)$  be a closest element to  $\lambda_{\hbar_1}$  in  $\mathcal{L}_{\hbar_2}$ . We define the divided difference*

$$\delta(\hbar_1, \hbar_2) := \hbar_1 \frac{\lambda_{\hbar_1} - \lambda(\hbar_1, \hbar_2)}{\hbar_1 - \hbar_2}. \quad (42)$$

*For any  $N > 2$ , there exists positive numbers  $\hbar_0, \varepsilon_0$ , and  $C > 0$  such that for all  $\varepsilon \leq \varepsilon_0$ ,  $\hbar_1 \in \mathcal{I}$ ,  $\hbar_1 \leq \hbar_0$  and  $\hbar_2 \in \mathcal{I}$  such that*

$$\hbar_2 < \hbar_1, \quad \hbar_1 - \hbar_2 \geq \hbar_1^N, \quad \text{and} \quad \frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \varepsilon, \quad (43)$$

*then  $\lambda(\hbar_1, \hbar_2)$  is unique, and*

$$|\delta(\hbar_1, \hbar_2) - \delta_c| \leq C\hbar_1,$$

*where  $\delta_c$  is the drift of  $\mathcal{L}_{\hbar}$  at  $c$ .*

As we shall see below (Lemma 3.28), for an  $\hbar$ -continuous asymptotic lattice, it is always possible to find couples  $(\hbar_1, \hbar_2)$  satisfying the requirements (43) while being arbitrarily small. Hence, the conclusion of this lemma is that we can recover the drift  $\delta_c$  from the limit of the divided difference (42), as  $\hbar_1 \rightarrow 0$ .

**Proof of Proposition 3.27.** Let  $k_{\hbar}$  be a good labelling for  $\mathcal{L}_{\hbar}$ , and  $G_{\hbar}$  an associated asymptotic chart. From Proposition 3.23 there exist  $\delta > 0$ ,  $\varepsilon_0 > 0$  and  $\hbar_0 > 0$  such that for all  $\hbar_1, \hbar_2 \in \mathcal{I} \cap ]0, \hbar_0]$  with  $|\hbar_1^{-1} - \hbar_2^{-1}| < \varepsilon_0$ , the element  $\lambda(\hbar_1, \hbar_2)$  is uniquely defined, and  $\|\lambda(\hbar_1, \hbar_2) - \lambda_{\hbar_1}\| < \delta\hbar_2$ . Moreover,  $k_{\hbar_1}(\lambda_{\hbar_1}) = k_{\hbar_2}(\lambda(\hbar_1, \hbar_2))$ . For ease of notation, let us denote by  $k$  this integer. We have

$$\lambda_{\hbar_1} = G_{\hbar_1}(\hbar_1 k) + \mathcal{O}(\hbar_1^{\infty}), \quad \lambda(\hbar_1, \hbar_2) = G_{\hbar_2}(\hbar_2 k) + \mathcal{O}(\hbar_2^{\infty}). \quad (44)$$

It follows from Item 2 of Lemma 3.8 that  $c = G_0(\hbar_1 k) + \mathcal{O}(\hbar_1)$ , and hence, because  $\hbar_2 \in (\frac{\hbar_1}{1+\hbar_1\varepsilon}, \hbar_1)$ , there exists a constant  $C > 0$  such that

$$\|\hbar_1 k - \xi_0\| \leq C\hbar_1, \quad \text{and} \quad \|\hbar_2 k - \xi_0\| \leq C\hbar_1,$$

where  $\xi_0 = G_0^{-1}(c)$ . On the other hand, from Item 3 of Lemma 3.8, there exists  $C_N > 0$  such that

$$\|G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_1 k)\| \leq C_N(|\hbar_1 - \hbar_2| + \hbar_1^N + \hbar_2^N).$$

(Note that we could have got rid of the terms  $\hbar_1^N + \hbar_2^N$  by choosing an asymptotic chart that is smooth in  $\hbar$ , see Remark 3.10). Hence, since  $\hbar_1 - \hbar_2 \geq \hbar_1^N$ ,

$$\hbar_1 \frac{\|G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_1 k)\|}{\hbar_1 - \hbar_2} \leq 3C_N \hbar_1 \quad (45)$$

From Item 4 of Lemma 3.8, we can write

$$G_{\hbar_2}(\hbar_1 k) - G_{\hbar_2}(\hbar_2 k) = (\hbar_1 - \hbar_2) G'_{\hbar_2}(\hbar_2 k) \cdot (k) + \mathcal{O}(|\hbar_1 - \hbar_2|^2 \|k\|^2),$$

which, in view of (45), and using that  $\hbar_j k = \xi_0 + \mathcal{O}(\hbar_1)$  for  $j = 1, 2$ , we obtain

$$\begin{aligned} \hbar_1 \frac{G_{\hbar_1}(\hbar_1 k) - G_{\hbar_2}(\hbar_2 k)}{\hbar_1 - \hbar_2} &= G'_{\hbar_2}(\hbar_2 k) \cdot (\hbar_1 k) + \mathcal{O}(\hbar_1 |\hbar_1 - \hbar_2| \|k\|^2) \\ &= G'_{\hbar_2}(\xi_0) \cdot (\xi_0) + \mathcal{O}(\hbar_1) + \mathcal{O}(\hbar_1 \epsilon \hbar_1 \hbar_2 \|k\|^2) \\ &= \delta_c + \mathcal{O}(\hbar_2) + \mathcal{O}(\hbar_1) + \mathcal{O}(\epsilon \hbar_2 \|\xi_0\|^2) \\ &= \delta_c + \mathcal{O}(\hbar_1). \end{aligned}$$

Finally, using (44) and the fact that  $\hbar_1 - \hbar_2 \geq \hbar_1^N$ , this yields the desired estimate.  $\square$

The following lemma shows that the requirement (43) can always be met.

**Lemma 3.28** *Assume  $\mathcal{I}$  is  $\hbar$ -continuous. For any  $\epsilon > 0$ , for any  $N > 2$ , there exists  $\hbar_0 > 0$  such that for all  $\hbar_1 \in \mathcal{I} \cap ]0, \hbar_0]$ , there exists  $\hbar_2 \in \mathcal{I}$  such that*

$$\hbar_1 < \hbar_2, \quad \frac{1}{\hbar_2} - \frac{1}{\hbar_1} < \epsilon \quad \text{and} \quad \hbar_1 - \hbar_2 \geq \hbar_1^N.$$

**Proof.** By contradiction, assume that the statement of the lemma does not hold. Let  $\hbar_3 := \inf\{\hbar \in \mathcal{I}; \hbar < \hbar_1, \frac{1}{\hbar} - \frac{1}{\hbar_1} < \epsilon\}$ . By Definition 3.21 the set in question is not empty, hence  $\hbar_3 \geq 0$ , and  $\hbar_3 \in [\frac{\hbar_1}{1+\hbar_1\epsilon}, \hbar_1]$ . By the negation of the lemma, one must have  $\hbar_1 - \hbar_3 < \hbar_1^N$ , and hence  $\frac{1}{\hbar_3} - \frac{1}{\hbar_1} < \hbar_1^{N-2}(1 + \hbar_0\epsilon)$ . Applying Definition 3.21 (with  $\epsilon$  replaced by  $\frac{\epsilon}{2}$ ) to this  $\hbar_3 \in \overline{\mathcal{I}} \setminus \{0\}$ , we obtain  $\hbar_4 \in \mathcal{I}$  with  $\hbar_4 < \hbar_3$  and  $\frac{1}{\hbar_4} - \frac{1}{\hbar_3} < \frac{\epsilon}{2}$ . Hence  $\frac{1}{\hbar_4} - \frac{1}{\hbar_1} < \frac{\epsilon}{2} + \hbar_0^{N-2}(1 + \hbar_0\epsilon)$ . If  $\hbar_0$  was small enough, the right-hand side is less than  $\epsilon$ , thus  $\hbar_4$  contradicts the definition of  $\hbar_3$ .  $\square$

### 3.3 Quantum rotation number

In this section, we define a spectral quantity from the joint spectrum of a quantum integrable system, which will be the natural analogue of the rotation number  $w_I(\Lambda)$ . As we shall see, making this quantity a purely spectral invariant is not obvious, because, in the same way as the classical rotation number  $w_I(\Lambda)$  depended on the choice of the action variables  $I$  (Definition 2.1), the quantum rotation number will depend on the choice of a good labelling.

Let us consider a quantum Hamiltonian  $\hat{H}$ , and assume that it is completely integrable, in the sense that there exists a quantum integrable system  $(P_1, \dots, P_n)$  (see Section 3.1) with  $P_n = \hat{H}$ , with proper classical momentum map  $F = (f_1, \dots, f_{n-1}, f_n = H) : M \rightarrow \mathbb{R}^n$  (here  $f_j$  is the principal symbol of  $P_j$ ). Let  $c \in F(M)$  be a regular value of  $F$ , and assume that  $F^{-1}(c)$  is connected. From Theorem 3.7, we know that the joint spectrum  $\Sigma_{\hbar}$  is an asymptotic lattice near  $c$ .

**Definition 3.29** Let  $\lambda \mapsto k_{\hbar}(\lambda)$  be a good labelling for the joint spectrum  $\Sigma_{\hbar}$  in a ball  $B \subset \mathbb{R}^n$ . Let  $\tilde{B} \Subset B$  be an open set. Let  $\lambda \in \Sigma \cap \tilde{B}$ , and let  $k = k_{\hbar}(\lambda) \in \mathbb{Z}^n$  be the corresponding label. Denote

$$\lambda =: (\lambda_k^{(1)}(\hbar), \dots, \lambda_k^{(n)}(\hbar)),$$

and let  $E_k(\hbar) := \lambda_k^{(n)}(\hbar)$  be the corresponding eigenvalues of  $\hat{H}$ . We define the quantum rotation number, for the good labelling  $k_{\hbar}$ , to be

$$[\hat{w}_{\hbar}](k) := [E_{k+e_1}(\hbar) - E_k(\hbar) : \dots : E_{k+e_n}(\hbar) - E_k(\hbar)] \in \mathbb{RP}^{n-1}, \quad (46)$$

where  $(e_1, \dots, e_n)$  is the canonical basis of  $\mathbb{Z}^n$ .

The restriction to the smaller open subset  $\tilde{B}$  ensures that the labels  $(k + e_j)$  do correspond to joint eigenvalues in  $B$ , when  $\hbar < \hbar_0$  and  $\hbar_0$  is small enough. When  $n = 2$ , as in the classical case (Equation (2)), it is often more convenient to think of the rotation number as an element of the 1-point compactification  $\overline{\mathbb{R}}$ ,

$$\hat{w}_{\hbar}(k) := \frac{E_{k+e_1}(\hbar) - E_k(\hbar)}{E_{k+e_2}(\hbar) - E_k(\hbar)} \in \overline{\mathbb{R}}. \quad (47)$$

The following result shows that, once a good labelling is known, the classical rotation number can be recovered from the quantum rotation number, in the semiclassical limit.

**Theorem 3.30** Let  $c \in \mathbb{R}^n$  be a regular value of  $F$ , and assume that the Liouville torus  $\Lambda := F^{-1}(c)$  is connected. Let  $\lambda_{\hbar}$  be a joint eigenvalue in  $\Sigma_{\hbar}$  such that

$$\lambda_{\hbar} = c + \mathcal{O}(\hbar).$$

Let  $G_{\hbar}$  be an asymptotic chart for  $\Sigma_{\hbar}$  in a neighborhood of  $c$ , let  $k = k(\hbar) \in \mathbb{Z}^n$  be the corresponding label for  $\lambda_{\hbar}$  and let  $I := G_0^{-1} \circ F$  be the associated action variables (see Equation (12)). Then

$$[\hat{w}_{\hbar}](k) = [w_I](\Lambda) + \mathcal{O}(\hbar).$$

The term  $\mathcal{O}(\hbar)$  is relative to the topology of  $\mathbb{RP}^{n-1}$ , and is uniform if  $c$  varies in a compact subset. The existence of  $\lambda_{\hbar}$  in the statement of the theorem is guaranteed by Lemma 3.15.

**Proof.** Let  $g_{\hbar}$  be the last component of  $G_{\hbar}$ , so that  $E_k(\hbar) = g_{\hbar}(\hbar k) + \mathcal{O}(\hbar^{\infty})$ . We have the  $C^{\infty}$  asymptotic expansion

$$g_{\hbar}(\xi) = g_0(\xi) + \hbar g_1(\xi) + \dots$$

where  $g_0$  is the last component of  $G_0$ ; thus,  $H = g_0(I)$ . By definition

$$[w_I](\Lambda) = [\partial_1 g_0(I(\Lambda)) : \dots : \partial_n g_0(I(\Lambda))].$$

A Taylor formula (Item 4 of Lemma 3.8) gives

$$\begin{aligned} E_{k+e_j}(\hbar) - E_k(\hbar) &= g_\hbar(\hbar(k + e_j)) - g_\hbar(\hbar k) + \mathcal{O}(\hbar^\infty) \\ &= \hbar \partial_j g_\hbar(\hbar k) + \mathcal{O}(\hbar^2) \\ &= \hbar \partial_j g_0(\hbar k) + \mathcal{O}(\hbar^2), \end{aligned}$$

because  $\hbar k$  is bounded. In fact,  $\hbar k = G_0^{-1}(c) + \mathcal{O}(\hbar) = I(\Lambda) + \mathcal{O}(\hbar)$ ; hence  $E_{k+e_j}(\hbar) - E_k(\hbar) = \hbar \partial_j g_0(I(\Lambda)) + \mathcal{O}(\hbar^2)$ , which yields

$$[\hat{w}_\hbar](k) = [\partial_1 g_0(I(\Lambda)) + \mathcal{O}(\hbar) : \dots : \partial_n g_0(I(\Lambda)) + \mathcal{O}(\hbar)],$$

which gives the result, because  $(\partial_1 g_0(I(\Lambda)), \dots, \partial_n g_0(I(\Lambda))) \neq 0$ .  $\square$

### 3.4 Quantum rotation number for semitoric systems

If  $F = (J, H)$  is a classical semitoric system, we have shown in Section 2.3 that the rotation number is well defined as an angle in  $\mathbb{R}/\mathbb{Z}$ . We will show the quantum analogue here, namely that for a quantum semitoric system, the rotation number can be defined with no ambiguity on the choice of a good labelling.

**Definition 3.31 ([53])** *A quantum integrable system  $(\hat{J}, \hat{H})$  is called semitoric if the corresponding classical system  $(J, H)$  given by the principal symbols is semitoric.*

Examples include Laplace-Beltrami operators on surfaces of revolution, Schrödinger operators with axi-symmetric potential (like the quantum spherical pendulum), or toric Laplacians [25]. On the Toeplitz side (which we do not consider explicitly here, see Remark 3.5), many spin-coupling systems (or Jaynes-Cummings) that are used in quantum optic or chemistry are semitoric systems, see for instance [60, 21, 43] and references therein.

**Remark 3.32** It would be very interesting to have a purely spectral characterization of a quantum semitoric system. The quantum analogue of the Hamiltonian  $\mathbb{S}^1$  action should be reflected in the fact that the spectrum of  $\hat{J}$  coming from a bounded region of the joint spectrum is close to an arithmetic sequence of the type  $\alpha + \hbar(j + \mu)$ ,  $j \in \mathbb{Z}$  (see Proposition 3.33 below).

However the fact that a point  $c \in \mathbb{R}^2$  is a regular value of  $F$  seems more delicate to obtain in a purely spectral way. Moreover, the semitoric hypothesis also impacts the singularity types at the boundary of the image of  $F = (J, H)$  (see Proposition 3.37). We don't address these issues here. Instead, we assume the semitoric nature of the system, and from this we try and recover the rotation number.  $\triangle$

In the semitoric case, Proposition 3.24 can be improved.

**Proposition 3.33** *Let  $(\hat{J}, \hat{H})$  be a semitoric quantum integrable system defined for  $\hbar \in ]0, \hbar_0]$ . Let  $c \in \mathbb{R}^2$  be a regular value of  $F$  and assume that  $F^{-1}(c)$  is connected, where  $F = (J, H)$  is*



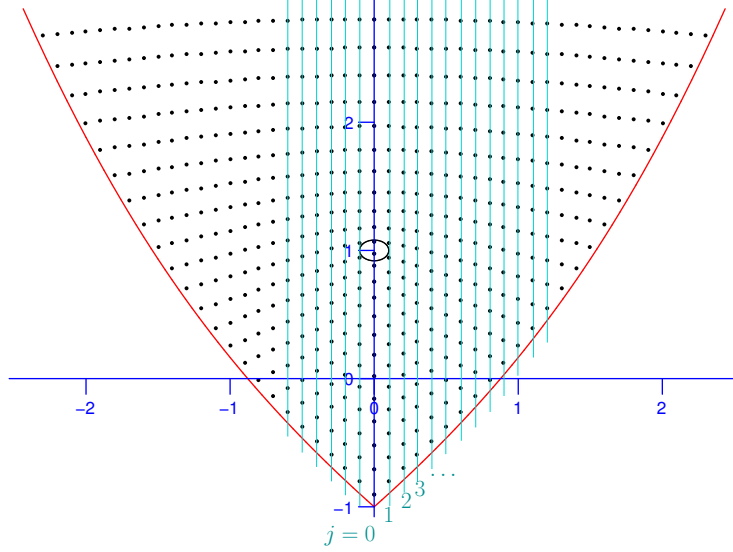


Figure 2: Joint spectrum of the Spherical Pendulum. Joint eigenvalues are organized along vertical lines indexed by  $j$ .

the classical momentum map. Then there exist a ball  $B$  around  $c$ ,  $\alpha \in \mathbb{R}$ ,  $\mu \in \mathbb{R}$ , and a good labelling  $\lambda \mapsto (j, k)$  of the joint spectrum  $\Sigma_{\hbar}$  in  $B$  such that, uniformly for  $\lambda = (J_{j,k}(\hbar), E_{j,k}(\hbar)) \in \Sigma_{\hbar} \cap B$ ,

$$J_{j,k}(\hbar) = \alpha + \hbar(j + \mu + \mathcal{O}(\lambda - c)) + \mathcal{O}(\hbar^2). \quad (48)$$

Loosely speaking, this proposition says that, in a small ball around  $c$ , the joint spectrum of a semitoric system is organized along regularly spaced vertical lines  $J = \alpha + \hbar(j + \mu)$ , and the quantum number  $j \in \mathbb{Z}$  labels these lines. See Figure 2. If  $J$  is proper (and  $\hat{J}$  has no subprincipal symbol), the result follows from the cluster structure of the spectrum of pseudodifferential operators with periodic characteristics, see for instance [27, 24, 14, 31]. In our case, we do not impose the properness of  $J$  and there is no restriction on subprincipal symbols; the result is still valid because we restrict the joint spectrum to a small  $B$  (it would not hold for the usual spectrum of  $\hat{J}$  alone).

**Lemma 3.34** *Under the assumptions of Proposition 3.33, let  $B$  be a bounded, simply connected open subset of regular values around  $c$ , such that  $(\Sigma_{\hbar}, ]0, \hbar_0], B)$  is an asymptotic lattice. Then this asymptotic lattice admits a semitoric asymptotic chart, i.e. an asymptotic chart  $G_{\hbar} \sim G_0 + \hbar G_1 + \dots$  such that the first component  $G_0^{(1)}$  of  $G_0 = (G_0^{(1)}(\xi_1, \xi_2), G_0^{(1)}(\xi_1, \xi_2)) : U \rightarrow \mathbb{R}^2$  satisfies, for all  $(\xi_1, \xi_2) \in U$ ,*

$$dG_0^{(1)}(\xi_1, \xi_2) = d\xi_1. \quad (49)$$

**Proof.** Let  $G_{\hbar}$  be an asymptotic chart for  $\Sigma_{\hbar}$  near  $c$ , see (11) and Definition 3.6, and let  $I = (I_1, I_2)$  be the corresponding action coordinates:  $F = G_0 \circ I$ . Since  $(J, H)$  is semitoric, there

exist oriented action coordinates near  $\Lambda := F^{-1}(c)$  of the form  $(J, J_2)$ . Hence by Lemma 2.2 there is an affine transformation  $\tau \in \text{GA}(n, \mathbb{Z})$  such that  $(I_1, I_2) = \tau(J, J_2) = A(J, J_2) + z$ , where  $A \in \text{SL}(2, \mathbb{Z})$  and  $z \in \mathbb{R}^2$ . Then the map  $\hat{G}_h(\xi) := G_h(A\xi)$  is an asymptotic chart by Lemma 3.19 and we have:

$$d\hat{G}_0 = dG_0 \circ A = dG_0 \circ d\tau = d(G_0 \circ \tau).$$

Since  $G_0(\tau(J, J_2)) = (J, H)$ , this implies  $d(G_0 \circ \tau)^{(1)} = d\xi_1$ , and gives the result.  $\square$

Equation (49) means that, up to a constant, the chart  $G_h$  is associated with semitoric action variables (Lemma 2.9).

**Proof of Proposition 3.33.** Let us consider the proof of Proposition 3.24. In view of Lemma 3.34, if we project (41) on the first component, the term  $\mathcal{O}(\|\lambda_h - c\|^2)$  disappears, because the first component of  $G_0$  is an affine map. This gives the required estimate.  $\square$

**Remark 3.35** Comparing with Proposition 3.24, we see that the number  $\alpha$  in (50) is equal to  $c^{(1)} - \xi_0^{(1)}$ , i.e. the first component of  $c - \xi_0$ . Thus  $\xi_0^{(1)}$  is the first component of the *drift* of the joint spectrum (Definition 3.25). Recall that  $\xi_0^{(1)}$  is also the value of the action integral along the  $\mathbb{S}^1$ -cycle on the torus  $\Lambda_c$ . This is consistent with Remark 3.26, since  $J$  is an action variable and can be completed into a set of local affine coordinates  $(J = I_1, I_2)$  of  $B_r$ .  $\triangle$

**Theorem 3.36** *Let  $(\hat{J}, \hat{H})$  be a semitoric quantum integrable system, with momentum map  $F = (J, H)$ . Let  $c \in \mathbb{R}^2$  be a regular value of  $F$ , and assume that the Liouville torus  $\Lambda := F^{-1}(c)$  is connected. Let  $G_h = G_0 + \hbar G_1 + \dots$  be an asymptotic chart for  $\Sigma_h$  near  $c$ , such that the associated good labelling  $\lambda \mapsto (j, k)$  satisfies Equation (48). Then (49) holds, i.e. the first component of  $G_0$  satisfies, for all  $(\xi_1, \xi_2)$  near  $G_0^{-1}(c)$ :*

$$dG_0^{(1)}(\xi_1, \xi_2) = d\xi_1.$$

Moreover, let  $\lambda_h$  be a joint eigenvalue in  $\Sigma_h$  such that

$$\lambda_h = c + \mathcal{O}(\hbar).$$

Then the corresponding quantum rotation number (Equation (47)) satisfies:

$$\hat{w}_h(j, k) = w(\Lambda) + \mathcal{O}(\hbar) \pmod{\mathbb{Z}},$$

where  $w(\Lambda)$  is the semitoric rotation number in the sense of Definition 2.8.

**Proof.** By assumption, the first component of  $\lambda_h$  has the asymptotic expansion:

$$J_{j,k}(\hbar) = \alpha + \hbar(j + \mu + \mathcal{O}(\lambda - c)) + \mathcal{O}(\hbar^2), \quad (50)$$

where  $\alpha, \mu \in \mathbb{R}$  do not depend on  $\hbar$ . From Theorem 3.30, we have

$$[\hat{w}_\hbar](j, k) = [w_I](\Lambda) + \mathcal{O}(\hbar), \quad (51)$$

where  $I = G_0^{-1} \circ F$ . From Lemma 3.34, we introduce a semitoric asymptotic chart  $\hat{G}_\hbar := G_\hbar \circ A$ , where  $A \in \text{SL}(2, \mathbb{Z})$  is such that  $\hat{G}_\hbar$  satisfies

$$d\hat{G}_0^{(1)}(\xi_1, \xi_2) = d\xi_1. \quad (52)$$

From the proof of Proposition 3.33,  $\hat{G}_\hbar$  satisfies Equation (48): there exists  $\alpha', \mu'$  in  $\mathbb{R}$  such that,  $\forall \hbar < \hbar_0$ ,  $J_{j,k}(\hbar) = \alpha' + \hbar(j' + \mu' + \mathcal{O}(\lambda - c)) + \mathcal{O}(\hbar^2)$ ; here  $j' = aj + bk$ , where  $a, b$  are the integers forming the first line of  $A^{-1}$ . Comparing with Equation (50), we obtain, since  $\lambda_\hbar - c = \mathcal{O}(\hbar)$ ,

$$\alpha' + \hbar(aj + bk + \mu') = \alpha + \hbar(j + \mu) + \mathcal{O}(\hbar^2),$$

hence there exists a constant  $C > 0$  such that, for all  $\hbar \in \mathcal{I}$ ,

$$|\alpha' - \alpha + \hbar((a-1)j + bk + \mu' - \mu)| \leq C\hbar^2. \quad (53)$$

This holds for all  $\hbar$ -dependent couples  $(j, k) = (j(\hbar), k(\hbar))$  that label a joint eigenvalue  $\lambda_\hbar \in \Sigma_\hbar$  near  $c$ . If  $u, v$  are given integers, independent of  $\hbar$ , then the joint eigenvalue labelled by  $(j(\hbar) + u, k(\hbar) + v)$  differs from  $\lambda_\hbar$  by  $\mathcal{O}(\hbar)$ . Hence (53) must hold for this new label as well, and writing the triangle inequality we obtain, for all fixed  $(u, v) \in \mathbb{Z}^2$ ,

$$\hbar |(a-1)u + bv| \leq 2C_{u,v}\hbar^2.$$

Choosing  $(u, v) = (0, 1)$  or  $(1, 0)$ , as soon as  $2C_{u,v}\hbar < 1$ , this implies  $a = 1$  and  $b = 0$  (and hence  $\alpha = \alpha'$  and  $\mu' = \mu$  from (53)). This entails that the matrix  $A$  takes the form  $A = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ . Hence Equation (49) follows from (52). This proves that, up to a constant, the action variables  $I$  are in fact semitoric. Thus, from Proposition 2.10 we obtain

$$w(\Lambda) = w_I(\Lambda) \pmod{\mathbb{Z}},$$

which, together with (51) finishes the proof of the Theorem (recall that in the semitoric case, the direction  $[w_I(\Lambda)]$  can never be vertical).  $\square$

In order to illustrate this result, we have produced a numerical comparison between classical and quantum rotation numbers, in the case of the spherical pendulum (an axisymmetric Schrödinger operator on the sphere  $\mathbb{S}^2$ ), see Figure 3.

Theorem 3.36 will be used in Section 3.5.2 to obtain, algorithmically, a good labelling for the joint spectrum of a quantum semitoric system, based on the fact that the “global label  $j$ ” corresponding to the integer in (48) is easy to detect. However, even for a semitoric system, assigning another quantum number  $k$  to each joint eigenvalue can be more delicate. While it is in principle always possible near any regular value  $c$  of the moment map  $(J, H)$ , thanks to

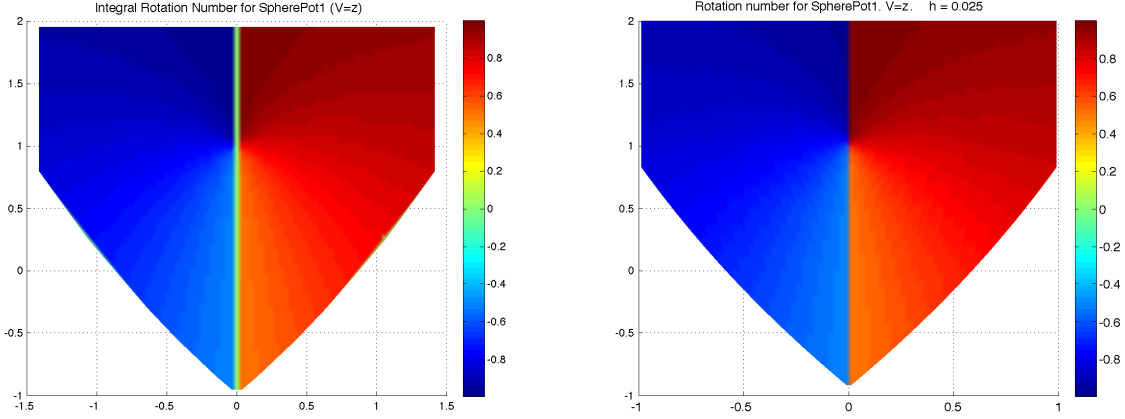


Figure 3: Classical (left) and quantum (right) rotation numbers for an axisymmetric Schrödinger operator on the sphere  $\mathbb{S}^2$  with potential  $V = z$  (*quantum spherical pendulum*).

Theorem 3.7, there is no global recipe: due to a possibly non-trivial monodromy, a good labelling for the joint spectrum may simply not globally exist [65]. There is, however, a common situation where a natural second quantum number shows up: consider, for a fixed  $j$ , the corresponding vertical spectral band  $V_j$ . If the spectrum inside  $V_j$  is either bounded from above or from below, and if this bound corresponds to an elliptic singularity — which is generically the case —, then labelling  $k$  in increasing order from the bounded side will provide a good labelling. To understand this, let  $c = (x, y)$  be a rank-one elliptic singularity of  $F$ , where  $x$  is a regular value of  $J$ . This means that the restriction of  $H$  to the submanifold  $J^{-1}(x)$  admits a Morse-Bott non-degenerate critical point  $m$  with  $H(m) = y$ . Let us assume that  $F^{-1}(c)$  is connected; by the theory of non-degenerate singularities of integrable systems [72], this fiber must then be a circle of critical points (the corresponding point in the reduced space  $J^{-1}(x)/\mathbb{S}^1$  is a standard Morse non-degenerate critical point). Letting  $x$  vary in a small neighborhood, we thus obtain a smooth cylinder of critical points in  $M$ , whose critical values form a smooth curve through  $c$  in  $\mathbb{R}^2$ , which belongs to the boundary of  $F(M)$ . This situation is called a *simple  $J$ -transversal elliptic singularity*. From the viewpoint of the energy  $H$ , there are two situations, where  $c$  is either a local maximum or minimum of  $H$  restricted to  $J^{-1}(x)$ . For simplicity we shall only deal with the minimum case (which we call ‘positive’ in the statement below). This is the case of the Spherical Pendulum, Figure 2; of course the maximum case is completely similar.

**Proposition 3.37** *Let  $(\hat{J}, \hat{H})$  be a semitoric quantum integrable system with principal symbols  $(J, H)$ . Let  $c \in \mathbb{R}^2$  be a simple positive  $J$ -transversal elliptic critical value of  $F$ . Then, for  $\hbar$  small enough, the joint eigenvalues  $\lambda \in \Sigma_\hbar$  in a neighborhood of  $c$  belong to the union of disjoint vertical bands  $V_j$  given by the equation*

$$x = \alpha + \hbar(j + \mu + \mathcal{O}(\lambda - c)) + \mathcal{O}(\hbar^2), \quad j \in \mathbb{Z} \quad (54)$$

where  $\alpha, \mu \in \mathbb{R}$  are fixed. In each vertical band  $V_j$ , the  $y$  coordinates of the joint eigenvalues are distinct and bounded from below. We label them in increasing  $y$ -order by a non-negative integer  $k \in \mathbb{N}$ . Then, for any regular value  $c'$  of  $F$ , close to  $c$ , the labels  $(j, k)$  form a good labelling of  $\Sigma_{\hbar}$  near  $c'$ .

Contrary to the rest of the article, for this result the standard action-angle theorem (and its semi-classical version) is not enough. We need to resort to the microlocal study of the spectrum near a simple transversally elliptic singularity, which was done in [68, Theorem 5.2.4].

**Theorem 3.38 ([68])** *Let  $(\hat{J}, \hat{H})$  be a quantum integrable system, with momentum map  $F = (J, H)$ , and let  $c$  be a simple transversally elliptic critical value of  $F$ . Then the joint spectrum  $\Sigma_{\hbar}$  in a neighborhood of  $c$  (independent of  $\hbar$ ) can be described as follows:*

1. joint eigenvalues have multiplicity one, in the sense of item 1. of Theorem 3.2;
2.  $\Sigma_{\hbar}$  is an ‘‘asymptotic half lattice’’ in the sense that item 2. of Theorem 3.2 holds:

$$\lambda = G_{\hbar}(\hbar k_1, \hbar k_2) + \mathcal{O}(\hbar^{\infty}), \quad (55)$$

when replacing  $(k_1, k_2) \in \mathbb{Z}^2$  by  $(k_1, k_2) \in \mathbb{Z} \times \mathbb{N}$ , and replacing Equation (12) by

$$F = G_0(\xi_1, q_2) \quad (56)$$

where  $q_2(x, \xi) = (x_2^2 + \xi_2^2)/2$ . Here the local coordinates  $(x_1, \xi_1, x_2, \xi_2) \in T^*\mathbb{S}^1 \times \mathbb{R}^2$  near  $\mathbb{S}^1 \times \{\beta\} \times (0, 0)$  for some  $\beta \in \mathbb{R}$ , describing a neighborhood of the circle  $F^{-1}(c)$ , are symplectic, and  $G_0$  is a local diffeomorphism from  $(\mathbb{R}^2, (\beta, 0))$  to a neighborhood of  $\bar{B}$ .

**Proof of Proposition 3.37.** It follows from (56) that  $J = g(\xi_1, q_2)$  for a smooth  $g$ . Since  $J$ ,  $\xi_1$  and  $q_2$  all have  $2\pi$ -periodic flows, there must exist integers  $a, b$  such that  $dJ = ad\xi_1 + bdq_2$ . Since the  $J$ -action is effective,  $a$  and  $b$  must be co-prime. The hypothesis of  $J$ -transversality implies that  $a \neq 0$ . Thus  $\xi_1 = a^{-1}J - ba^{-1}q_2 + \text{const}$ , and the same argument implies that  $a^{-1} \in \mathbb{Z}$  and hence  $a = \pm 1$ . Up to composing by the symplectomorphism  $(x_1, \xi_1) \mapsto (-\xi_1, x_1)$  (and replacing  $k_1$  by  $-k_1$  in (55)), one may assume that  $a = 1$ . Thus  $(\xi_1, q_2) = \tau(J, q_2)$ , where  $\tau \in \text{GA}(n, \mathbb{Z})$  with linear part  $A = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$ . Arguing as in the proof of Proposition 3.33, we let  $\hat{G}_{\hbar}(x, y) := G_{\hbar}(A(x, y))$ , and the joint spectrum is described as

$$\lambda = G_{\hbar}(\hbar k_1, \hbar k_2) + \mathcal{O}(\hbar^{\infty}) = \hat{G}_{\hbar}(\hbar(j, k_2)) + \mathcal{O}(\hbar^{\infty}),$$

with  $k_1 = j - bk_2$ . Then  $\hat{G}_{\hbar} = \hat{G}_0 + \hbar\hat{G}_1 + \dots$ , with

$$\hat{G}_0(x, y) = G_0 \circ A(x, y) = G_0(x - by + \alpha, y),$$

for some constant  $\alpha \in \mathbb{R}$ . But since the first component of  $G_0$  is  $g$  and  $J = g(\xi_1, q_2) = g(J - bq_2, q_2)$ , we deduce that for all  $(x, y)$  near  $A^{-1}(\beta, 0) = (\beta, 0)$ ,

$$\hat{G}_0(x, y) = (x + \alpha, f(x, y))$$

for some smooth function  $f$  with  $\partial_y f \neq 0$ . We conclude as in the proof of Proposition 3.33 that the first component of  $\lambda$  takes the form

$$J_{j,k_2}(\hbar) = \alpha + \hbar j + \hbar \mu + \hbar \mathcal{O}(\lambda - c) + \mathcal{O}(\hbar^2).$$

Hence we obtain the description of the vertical bands  $B_j$  in (54), which are disjoint if  $|\lambda - c|$  is small enough (independently of  $\hbar$ ) and  $\hbar$  itself is small enough. Finally, if  $j$  is fixed, the second component of the joint eigenvalues is given by

$$E_{j,k_2}(\hbar) = f(\hbar j, \hbar k_2) + \hbar \mathcal{O}(\lambda - c) + \mathcal{O}(\hbar^2).$$

Thus,  $E_{j,k_2+1} - E_{j,k_2} = \hbar \partial_y f(c) + \hbar \mathcal{O}(\lambda - c) + \mathcal{O}(\hbar^2)$ . For a fixed value of  $J$ ,  $H$  is assumed to be minimal at the elliptic critical point. Since  $q_2 \geq 0$ , we must have  $\partial_{q_2} H \geq 0$  when  $q_2 = 0$ , which implies  $\partial_y f > 0$ . Hence  $E_{j,k_2+1} - E_{j,k_2} \geq \hbar/C$  for some constant  $C > 0$ .

It remains to prove that  $(j, k_2)$  is a good labelling away from the critical value. Since  $(k_1, k_2) = A(j, k_2)$ , this is equivalent to showing that  $(k_1, k_2)$  is a good labelling. Since item 1. in Definition 3.6 is already given by Theorem 3.38, we must only prove item 2. Let  $c'$  be a regular value of  $F$ . Let  $B$  be a ball around  $c$  in which the description of the joint spectrum by Theorem 3.38 is valid:

$$\lambda = G_{\hbar}(\hbar k_1, \hbar k_2) + \mathcal{O}(\hbar^{\infty}),$$

where  $(k_1, k_2) \in \mathbb{Z} \times \mathbb{N}$ , and  $G_{\hbar}$  is defined in a neighborhood of  $(\beta, 0)$ . Let  $B' \subset B$  be a ball of regular values around  $c'$ . We claim that  $G_{\hbar}|_{G_0^{-1}(B')}$  is an asymptotic chart for  $\Sigma_{\hbar}$  in  $B'$ . Indeed, let  $(\beta', \gamma') := G_0^{(-1)}(c')$ ; necessarily  $\gamma' > 0$  because  $c'$  is regular. Now assume that  $(k_1, k_2) \in \mathbb{Z}^2$  is such that  $\hbar(k_1, k_2) \in G_0^{-1}(B')$ ; then  $\hbar k_2$  is close to  $\gamma'$ , and in particular,  $k_2 > 0$ . Hence, by (55), there exists a joint eigenvalue in  $\lambda \in B'$  such that  $\lambda = G_{\hbar}(\hbar k_1, \hbar k_2) + \mathcal{O}(\hbar^{\infty})$ . Therefore, according to Definition 3.6 (item 2.),  $G_{\hbar}$  is an asymptotic chart for  $\Sigma_{\hbar}$  in  $B'$ .  $\square$

### 3.5 Labelling algorithms

In the inverse spectral problem, we are given a (portion of a) joint spectrum  $\Sigma_{\hbar}$ ; we know that  $\Sigma_{\hbar}$  is an asymptotic lattice, but we don't have access, *a priori*, to any asymptotic chart. In Theorem 3.30, we showed that the classical rotation number can be obtained from the joint spectrum, *provided* eigenvalues are properly labelled. Hence, in order to recover the rotation number from  $\Sigma_{\hbar}$  itself, as a bare discrete set of points, we have to find a way to construct an appropriate labelling. In view of formula (46), a *linear labelling* (Definition 3.16) is enough for this purpose.

In this section, we show how to obtain, in an algorithmic and robust way, a linear labelling from a bare 2-dimensional asymptotic lattice  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$ .

We proceed in two steps: first, we construct an algorithm that, for any fixed value of  $\hbar$ , assigns integer labels to the points of  $\mathcal{L}_{\hbar}$ . However, these labels may not have the required continuity with respect to  $\hbar$  that would entitle them to be linear labellings. To remedy this, in a second

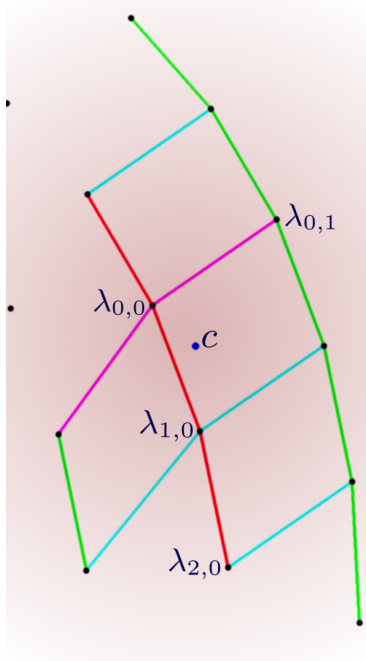


Figure 4: The labelling algorithm

step, we exhibit another algorithm that, given any sequence  $(\hbar_i)$  in  $\mathcal{I}$  tending to 0, correct the previously obtained labels to make them a true linear labelling.

In the semitoric case, the simple structure of the joint spectrum, organized in vertical bands (Theorem 3.36), makes it easier. In particular, the second step is not necessary, see Section 3.5.2.

We wrote our algorithms in the two-dimensional case, which corresponds to our initial motivations. We believe that the general case could be dealt with in a similar way for most parts, although some steps like Theorem 3.44 would require a different approach.

### 3.5.1 An algorithm for fixed $\hbar$

The core part of our labelling algorithm can be executed for any fixed value of  $\hbar \in \mathcal{I}$ . Actually, it can be applied to any finite set  $\mathcal{L}_\hbar \subset B$ . If  $c \in B$ , we will use the expression “choose a closest point to  $c$ ” to indicate that we have to choose a point in  $\mathcal{L}_\hbar$  that minimizes the distance to  $c$ . After such a point has been picked up and labelled, we remove it from  $\mathcal{L}_\hbar$ ; thus, subsequent calls to “choose a closest point to...” implicitly indicate that this point should be chosen within  $\mathcal{L}_\hbar$  minus all the already labelled points. This is of course always possible as long as this new set is not empty.

If  $(n, m)$  is a label for a point  $\lambda \in \mathcal{L}_\hbar$ , we shall denote this point  $\lambda = \lambda_{n,m}$ . See Figure 4. The complete algorithm consists in the twelve following steps.



1. Choose an open subset  $B_0 \Subset B$ , and fix  $c \in B_0$ .
2. Choose a closest point to  $c$ . Label it as  $(0, 0)$ .
3. Choose a closest point to  $\lambda_{0,0}$  (in the set  $\mathcal{L}_h \setminus \{\lambda_{0,0}\}$ ). Label it as  $(1, 0)$ .
4. Choose a closest point to  $2\lambda_{1,0} - \lambda_{0,0} = \lambda_{1,0} + (\lambda_{1,0} - \lambda_{0,0})$  and label it as  $(2, 0)$ .  
Continuing in this fashion (if  $\lambda_{n-1,0}$  is chosen, take  $\lambda_{n,0}$  to be a closest point to  $\lambda_{n-1,0} + (\lambda_{n-1,0} - \lambda_{n-2,0})$ ), label points  $\lambda_{n,0}$ ,  $n > 0$ , until the next point lies outside of  $B_0$ .  
Label  $\lambda_{n,0}$  for negative  $n$  in the same way, starting from the closest point to  $2\lambda_{0,0} - \lambda_{1,0}$ .
5. Choose a closest point to  $\lambda_{0,0}$  not already labeled and label it as  $(0, 1)$ .
6. Label a closest point to  $\lambda_{0,1} + (\lambda_{1,0} - \lambda_{0,0})$  as  $(1, 1)$ .
7. Use the points  $\lambda_{0,1}, \lambda_{1,1}$  to repeat the process in step 4, labelling as many points  $\lambda_{n,1}$  as possible (if  $\lambda_{n-1,1}$  is chosen, take  $\lambda_{n,1}$  to be a closest point to  $\lambda_{n-1,1} + (\lambda_{n-1,1} - \lambda_{n-2,1})$ ).
8. Label a closest point to  $2\lambda_{0,1} - \lambda_{0,0}$  as  $\lambda_{0,2}$ . Repeat steps 6-7 to label all points  $\lambda_{n,2}$ .
9. Continuing as above, label all points  $\lambda_{n,m}$ ,  $m > 0$  which lie in the given neighborhood.
10. Label a closest point to  $2\lambda_{0,0} - \lambda_{0,1}$  as  $(0, -1)$ .
11. Repeat steps 6,7,8,9 with negative  $m$  indices.
12. Finally, if the determinant of the vectors  $(\lambda_{1,0} - \lambda_{0,0}, \lambda_{0,1} - \lambda_{0,0})$  is negative, switch the labelling  $\lambda_{n,m} \mapsto \lambda_{n,-m}$  (in order to make it oriented).

Let us prove now some properties of this algorithm when  $(\mathcal{L}_h, \mathcal{I}, B)$  is a given asymptotic lattice. We use the notation of Definition 3.6; in particular,  $G_h : U \rightarrow \mathbb{R}^n$  is an asymptotic chart, and  $k_h$  is the corresponding good labelling. Of course, both of them are *a priori* unknown. Let  $B_0 \Subset B$  be an open subset containing  $c$ , and let  $\tilde{B}_0$  be an open subset such that  $B_0 \Subset \tilde{B}_0 \Subset B$ . Let  $\tilde{h}_0$  be given by Item 2b of Definition 3.6 with  $\tilde{U}_0 := G_0^{-1}(\tilde{B}_0)$ . Because the results of this section are made to be directly implementable on a computer, we shall try to write all estimates as explicitly as possible. Let  $\tilde{U} \Subset U$  and  $\tilde{h}_0 \in ]0, \tilde{h}_0]$  be such that  $(G_h)|_{\tilde{U}}$  is invertible onto a neighborhood of  $\bar{B}$  for all  $h \in \mathcal{I} \cap ]0, \tilde{h}_0]$ , see Item 6 of Lemma 3.8. From now on in this section, every  $h$  is tacitly assumed to belong to  $\mathcal{I} \cap ]0, \tilde{h}_0]$ .

Let  $U_0 = G_0^{-1}(B_0)$  and  $U_h = G_h^{-1}(B_0)$ . Then  $U_0 \Subset G_0^{-1}(B)$  and for all  $h \leq \tilde{h}_0$ ,  $U_h \Subset \tilde{U}$ . It follows from the asymptotic expansion of  $G_h^{-1}$  that there exists  $C > 0$  such that

$$\forall h \leq \tilde{h}_0, \quad U_h \subset U_0 + B(0, Ch),$$

and hence for any  $R > 0$ , there exists  $\tilde{h}_0^{[0]} \in ]0, \tilde{h}_0]$  such that

$$\forall h \leq \tilde{h}_0^{[0]}, \quad U_h + B(0, Rh) \subset \tilde{U}_0 \quad \text{and} \quad U_h + B(0, Rh) \subset \tilde{U}. \quad (h-57)$$

As in Lemma 3.11, let  $L_F$  be an upper bound on the Lipschitz constant of  $G_h^{-1}$  on a neighborhood of  $\bar{B}$ . Choose  $\epsilon \in (0, \frac{1}{L_F})$ , and let  $\hbar_0^{[1]} \in ]0, \hbar_0]$  be small enough to verify ( $\hbar$ -23), so that (24) holds.

Note that, with the exception of the three points  $\lambda_{0,0}$ ,  $\lambda_{1,0}$ , and  $\lambda_{0,1}$ , all points are constructed by the following process:

- (i) Choose a point  $\mu_1 \in B_0 \cap \mathcal{L}_\hbar$ .
- (ii) Choose a “vector”  $\vec{v} = \mu_0 - \mu_{-1}$  (the difference between two previously constructed points  $\mu_0$  and  $\mu_{-1} \in B_0 \cap \mathcal{L}_\hbar$ ).
- (iii) Identify a closest point  $\mu_2 \in \mathcal{L}_\hbar$  to  $\mu_1 + \vec{v}$ .

The following lemma shows that this process is uniformly well-defined if these points lie in ball of size  $\mathcal{O}(\hbar)$  and  $\hbar_0$  is small enough. Then, Step (iii) amounts to picking up the natural parallel transport defined in (25):

$$\mu_2 = \mu_1 + \underset{k_h}{(k_h(\mu_0) - k_h(\mu_{-1}))}.$$

**Lemma 3.39** *Given  $\epsilon \in (0, \frac{1}{L_F})$ , there exists  $L > 0$  such that the following holds. Choose  $\hbar \leq \hbar_0^{[1]}$  and four points  $\mu_i \in \mathcal{L}_\hbar$  satisfying (i)-(iii) above, and let  $k_i := k_h(\mu_i)$  be the corresponding multi-integers in  $\mathbb{Z}^2$ , for  $i \in \{-1, 0, 1, 2\}$ . Let  $\rho > 0$  be such that  $\mu_1$  and  $\mu_{-1}$  belong to the ball  $B(\mu_0, \rho\hbar)$ . If  $\hbar \leq \hbar_0^{[0]}$  defined by ( $\hbar$ -57) with some  $R > 2L_F\rho$ , and if for some  $N > 2$ ,  $\hbar$  satisfies the inequalities*

$$3C_N\hbar^{N-1} < \frac{R}{L_F} - 2\rho \quad \text{and} \quad \hbar(L(\rho + C_N\hbar^{N-1})^2 + 4C_N\hbar^{N-2}) \leq \epsilon, \quad (\hbar\text{-58})$$

where  $C_N$  is defined in (20), then  $\mu_2$  is unique and  $k_2 = k_1 + k_0 - k_{-1}$ .

**Proof.** We wish to consider the point  $\lambda_2 \in \mathcal{L}_\hbar$  whose label is  $k_1 + k_0 - k_{-1}$ . First of all we show that  $\hbar(k_1 + k_0 - k_{-1}) \in \tilde{U}_0$ . For all  $\lambda \in \mathcal{L}_\hbar$ , we get from (20)

$$\|\hbar k - G_h^{-1}(\lambda)\| \leq L_F C_N \hbar^N.$$

Therefore,  $\hbar k_j$ , for  $j \in \{-1, 0, 1\}$ , belongs to  $U_\hbar + B(0, L_F C_N \hbar^N) \subset \tilde{U}$  provided  $L_F C_N \hbar^{N-1} < R$ , which is a consequence of the first inequality in ( $\hbar$ -58). Thus, we may apply  $G_h^{-1}$  and use (20) to obtain, for  $j \in \{-1, 1\}$ ,

$$\|\hbar k_0 - \hbar k_j\| \leq L_F \|\mu_0 - \mu_j\| + 2L_F C_N \hbar^N < 2L_F(\hbar\rho + C_N \hbar^N). \quad (59)$$

Using the inequality for  $j = -1$  we get  $\hbar(k_1 + k_0 - k_{-1}) \in U_\hbar + B(0, L_F(3C_N\hbar^N + 2\hbar\rho))$ . From ( $\hbar$ -57) and the first inequality in ( $\hbar$ -58), we get  $\hbar(k_1 + k_0 - k_{-1}) \in \tilde{U}_0$ . Since  $U_0 \Subset G_0^{-1}(B)$ , by Item 2b of Definition 3.6 we may define  $\lambda_2 := k_h^{(-1)}(k_1 + k_0 - k_{-1}) \in \mathcal{L}_\hbar$ , i.e., for all  $N \geq 0$ ,

$$\|\lambda_2 - G_h(\hbar(k_1 + k_0 - k_{-1}))\| \leq C_N \hbar^N.$$

We have, using a Taylor expansion at  $\hbar k_1$ , see (17),

$$\begin{aligned} \|\lambda_2 - (\mu_1 + \vec{v})\| &\leq \|G_{\hbar}(\hbar(k_1 + k_0 - k_{-1})) - (\mu_1 + \vec{v})\| + C_N \hbar^N \\ &\leq \|G_{\hbar}(\hbar k_1) - \mu_1 + G'_{\hbar}(\hbar k_1) \cdot (\hbar k_0 - \hbar k_{-1}) - \vec{v}\| + L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + C_N \hbar^N \\ &\leq \|G'_{\hbar}(\hbar k_1) \cdot (\hbar k_0 - \hbar k_{-1}) - \vec{v}\| + L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + 2C_N \hbar^N . \end{aligned}$$

We can also write  $\|\vec{v} - G_{\hbar}(\hbar k_0) + G_{\hbar}(\hbar k_{-1})\| \leq 2C_N \hbar^N$ . By Taylor expanding at  $\hbar k_0$ ,

$$\|G_{\hbar}(\hbar k_0) - G_{\hbar}(\hbar k_{-1}) - G'_{\hbar}(\hbar k_0) \cdot (\hbar k_0 - \hbar k_{-1})\| \leq L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 .$$

Hence

$$\|\lambda_2 - (\mu_1 + \vec{v})\| \leq \|(G'_{\hbar}(\hbar k_0) - G'_{\hbar}(\hbar k_1)) \cdot (\hbar k_0 - \hbar k_{-1})\| + 2L_1 \|\hbar k_0 - \hbar k_{-1}\|^2 + 4C_N \hbar^N .$$

It follows from Item 1 of Lemma 3.8 that there exists a constant  $L_2 > 0$  such that

$$\forall [\xi_1, \xi_2] \subset \tilde{U}, \forall v \in \mathbb{R}^n, \quad \|G'_{\hbar}(\xi_2) \cdot v - G'_{\hbar}(\xi_1) \cdot v\| \leq L_2 \|\xi_2 - \xi_1\| \|v\| .$$

This gives

$$\|(G'_{\hbar}(\hbar k_0) - G'_{\hbar}(\hbar k_1)) \cdot (\hbar k_0 - \hbar k_{-1})\| \leq L_2 \|\hbar k_0 - \hbar k_1\| \|\hbar k_0 - \hbar k_{-1}\| ,$$

and hence

$$\|\lambda_2 - (\mu_1 + \vec{v})\| \leq \|\hbar k_0 - \hbar k_{-1}\| (2L_1 \|\hbar k_0 - \hbar k_{-1}\| + L_2 \|\hbar k_0 - \hbar k_1\|) + 4C_N \hbar^N .$$

Using (59) again,

$$\|\lambda_2 - (\mu_1 + \vec{v})\| < \hbar^2 (4(2L_1 + L_2)L_F^2(\rho + C_N \hbar^{N-1})^2 + 4C_N \hbar^{N-2}) .$$

Hence, for  $\hbar$  small enough: precisely, as soon as

$$\hbar(4(2L_1 + L_2)L_F^2(\rho + C_N \hbar^{N-1})^2 + 4C_N \hbar^{N-2}) \leq \epsilon ,$$

we may apply (24) to see that the closest point to  $(\mu_1 + \vec{v})$  in  $\mathcal{L}_{\hbar}$  must be  $\lambda_2$ , which proves the lemma with  $L = 4(2L_1 + L_2)L_F^2$ .  $\square$

Of course, as in the end of Remark 3.12, one can simplify estimates by choosing specific values, for instance  $\epsilon = \frac{1}{3L_F}$ ,  $R = 3L_F\rho$ , and then we may replace (h-58) by the stronger assumption that there exist  $\tilde{N} > 1$  such that

$$C_N \hbar^{N-1} \leq \min\left(\frac{\rho}{3}; \frac{\epsilon}{8}\right) \quad \text{and} \quad \hbar \leq \frac{\epsilon}{4L\rho^2} .$$

**Corollary 3.40** *With  $\lambda_{m,n}$  the collection of points constructed in the algorithm without Step 12, let  $k_{n,m} = k_{\hbar}(\lambda_{m,n})$ . Set*

$$\vec{z}_1 = k_{1,0} - k_{0,0} \quad \text{and} \quad \vec{z}_2 = k_{0,1} - k_{0,0}. \quad (60)$$

*There exists  $\hbar_0^{[2]} > 0$  (see Remark 3.41 below) such that for all  $\hbar \leq \hbar_0^{[2]}$ , we have*

$$k_{n,m} - k_{0,0} = n \vec{z}_1 + m \vec{z}_2. \quad (61)$$

*Moreover, for all  $N \geq 1$ ,*

$$\|\lambda_{1,0} - \lambda_{0,0}\| \leq \hbar \rho_0, \quad \|\lambda_{0,1} - \lambda_{0,0}\| \leq \hbar \rho_0, \quad \text{with} \quad \rho_0 := L_0 + 2C_N \hbar^{N-1}, \quad (62)$$

*where  $L_0$  is an upper bound on the Lipschitz constant of  $G_{\hbar}$  on  $\tilde{U}$ , see (16), and*

$$\|\vec{z}_1\| \leq \tilde{\rho}, \quad \|\vec{z}_2\| \leq \tilde{\rho}, \quad \text{with} \quad \tilde{\rho} := L_F(L_0 + 4C_N \hbar^{N-1}). \quad (63)$$

*After Step 12,  $\vec{z}_1$  and  $\vec{z}_2$  are possibly swapped.*

**Proof.** Let  $\lambda_1 = k_{\hbar}^{(-1)}(k_{0,0} + \vec{e}_1)$ , where  $\vec{e}_1 = (1, 0)$ . Necessarily,  $\lambda_1 \neq \lambda_{0,0}$ ; hence, by Step 3,  $\|\lambda_1 - \lambda_{0,0}\| \geq \|\lambda_{1,0} - \lambda_{0,0}\|$ . Since (using (14)),

$$\|\lambda_1 - \lambda_{0,0}\| \leq \|G_{\hbar}(\hbar(k_{0,0} + \vec{e}_1)) - G_{\hbar}(\hbar k_{0,0})\| + 2C_N \hbar^N \leq L_0 \hbar + 2C_N \hbar^N; \quad (64)$$

this shows the first inequality in (62). We now proceed with Step 4 of the algorithm by applying Lemma 3.39 to the triple  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{1,0})$ , with  $\rho = \rho_0$ , and thus obtain a unique  $\lambda_{2,0} = \lambda_{0,0} + \vec{z}_1$ .

From (64) we immediately obtain

$$\begin{aligned} \hbar \|\vec{z}_1\| &\leq L_F \|G_{\hbar}(\hbar k_{1,0}) - G_{\hbar}(\hbar k_{0,0})\| \leq L_F \|\lambda_{1,0} - \lambda_{0,0}\| + 2L_F C_N \hbar^N \\ &\leq L_F L_0 \hbar + 4L_F C_N \hbar^N = \hbar \tilde{\rho}. \end{aligned}$$

This proves the first inequality of (63). In order to repeat the application of Lemma 3.39 to the triple  $(\lambda_{1,0}, \lambda_{2,0}, \lambda_{2,0})$  we need to estimate  $\|\lambda_{2,0} - \lambda_{1,0}\|$ . Since  $k_{2,0} = k_{1,0} + \vec{z}_1$ , we get

$$\|\lambda_{2,0} - \lambda_{1,0}\| \leq L_0 \hbar \|\vec{z}_1\| + 2C_N \hbar^N \leq \hbar \rho_1 \quad (65)$$

with  $\rho_1 := L_0 \tilde{\rho} + 2C_N \hbar^{N-1}$ . Thus we may apply Lemma 3.39 with  $\rho = \rho_1$ , and obtain that  $\lambda_{3,0}$  is labelled by  $k_{0,3} = k_{0,0} + 2\vec{z}_1$ . Therefore, we can estimate  $\|\lambda_{3,0} - \lambda_{2,0}\| \leq \hbar \rho_1$  exactly as in (65). Repeating this process with the same  $\rho = \rho_1$ , we complete Step 4 to obtain all  $\lambda_{n,0}$  as long as they belong to  $B_0$ , and obtain, for all  $n$ ,  $\|\lambda_{n,0} - \lambda_{n-1,0}\| \leq \hbar \rho_1$  and

$$k_{n,0} - k_{0,0} = n \vec{z}_1, \quad \text{i.e.} \quad \lambda_{n,0} = \lambda_{0,0} + \underset{k_{\hbar}}{n \vec{z}_1}. \quad (66)$$

Next, we consider  $\lambda_{(0,1)}$  from Step 5. If  $\lambda_1$  was not labelled in Step 4, we have  $\|\lambda_1 - \lambda_{0,0}\| \geq \|\lambda_{0,1} - \lambda_{0,0}\|$ , which leads to the same estimates of as above, namely (62) and (63) hold. If, on the contrary,  $\lambda_1$  was labelled in Step 4, then by (66) there exists a non-zero integer  $n$  such that  $\vec{e}_1 = n\vec{z}_1$ . Thus the point  $\lambda_2 = k_{\hbar}(k_{0,0} + \vec{e}_2)$  was not labelled in Step 4. Therefore  $\|\lambda_2 - \lambda_{0,0}\| \geq \|\lambda_{0,1} - \lambda_{0,0}\|$  and we the above estimates still hold, with the same  $\rho$ , which proves (62) and (63). Therefore, we may continue to follow the algorithm and use Lemma 3.39 at each step, proving the corollary.  $\square$

**Remark 3.41** Let us investigate how small  $\hbar_0^{[2]}$  should be for Corollary 3.40 to hold. We do not try to have an optimal bound, but rather to check what the geometric constraints are. First, we have the upper bound  $\hbar_0^{[1]}$  defined by ( $\hbar$ -23), which ensures that  $k_{\hbar}$  is well defined and one-to-one. This one is quite weak in principle, because we are free to increase the exponent of  $\hbar$  to make it smaller. Then we apply Lemma 3.39 with  $\rho = \rho_0$ , which gives another upper bound given by ( $\hbar$ -58), where we are free to choose another  $N > 2$ . Essentially, this means  $\hbar \lesssim \frac{\epsilon}{L\rho_0} \sim \frac{1}{L_0(L_1+L_2)L_F^3}$ . Another application of Lemma 3.39 with  $\rho = \rho_1 \sim L_FL_0^2$  gives a new bound ( $\hbar$ -58), which, roughly speaking, imposes  $\hbar \lesssim \frac{1}{L_0^2(L_1+L_2)L_F^4}$ , which is *a priori* stronger than the previous one, at least if  $L_F, L_0 \geq 1$  and we neglect the term  $C_N\hbar^N$ . However, at each application of the lemma we are free to optimize by choosing a different  $N$ . In both cases one needs to select  $R > 2L_F\rho$ , which gives yet another bound by adjusting  $\hbar_0^{[0]} \sim o(1/R)$ , see ( $\hbar$ -57). Given  $R$ , this last bound only depends on the size of the domain  $B_0$  within  $B$ ; we can improve it if necessary by choosing a smaller  $B_0$ .  $\triangle$

We now consider in more details the construction of the first three points  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$  near  $c$ . Naturally, we assume that  $\hbar$  is small enough so that  $\mathcal{L}_{\hbar}$  contains at least three points. Our aim is to prove that  $\vec{z}_1$  and  $\vec{z}_2$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^2$ . After  $\lambda_{0,0}$  is chosen,  $\vec{v}_1 = \lambda_{1,0} - \lambda_{0,0}$  and  $\vec{v}_2 = \lambda_{0,1} - \lambda_{0,0}$  are chosen by a minimization process. We have to be careful with the fact that the vectors  $\vec{z}_i$  do not satisfy the same minimization properties because  $G'_{\hbar}(\hbar k_{0,0})$  is not orthogonal, in general.

**Lemma 3.42** *There exists  $\hbar_0^{[3]} > 0$  (given by ( $\hbar$ -67) and ( $\hbar$ -70) below) such that if  $\hbar \leq \hbar_0^{[3]}$ , then the entries of  $\vec{z}_1$  are co-prime integers.*

**Proof.** Assume that there exists  $\vec{z}_0 \in \mathbb{Z}^2 \setminus \{0\}$  and  $n \in \mathbb{N}^*$  such that  $\vec{z}_1 = n\vec{z}_0$ , where  $n$  may depend on  $\hbar$ . Let us first show that  $\hbar(k_{0,0} + \vec{z}_0) \in \tilde{U}_0$  (notice that  $\tilde{U}_0$  is not assumed to be convex). From Lemma 3.15, for any  $N$  we have

$$\|c - \lambda_{0,0}\| \leq (M + L_0)\hbar + C_N\hbar^N.$$

Likewise, if we let  $k_{0,0} := k_{\hbar}(\lambda_{0,0})$  and  $\xi := G_0^{-1}(c)$ , then  $\xi \in \tilde{B}_0$  and we have

$$\begin{aligned} \|\xi - \hbar k_{0,0}\| &\leq L_F (\|G_{\hbar}(\xi) - \lambda_{0,0}\| + \|\lambda_{0,0} - G_{\hbar}(\hbar k_{0,0})\|) \\ &\leq L_F (M\hbar + \|c - \lambda_{0,0}\| + C_N\hbar^N) \\ &\leq L_F(2M + L_0 + 2C_N\hbar^{N-1})\hbar. \end{aligned}$$

Therefore, using (63),

$$\|\hbar(k_{0,0} + \vec{z}_0) - \xi\| \leq \|\hbar k_{0,0} - \xi\| + \hbar \|\vec{z}_0\| \leq L_F(2M + L_0 + 2C_N \hbar^{N-1})\hbar + \hbar \tilde{\rho}/n$$

Since  $\tilde{B}_0$  is open and independent of  $\hbar$ , there is  $r_0 > 0$  such that  $B(\xi, r_0) \subset \tilde{U}_0$ . Assume  $\hbar(L_F(2M + L_0 + 2C_N \hbar^{N-1}) + \tilde{\rho}) < r_0$ , i.e.

$$\hbar L_F(2M + 2L_0 + 6C_N \hbar^{N-1}) < r_0. \quad (\hbar-67)$$

Then  $\hbar(k_{0,0} + \vec{z}_0) \in B(\xi, r_0) \subset \tilde{U}_0$ . We may now let  $\mu := k_h^{-1}(\hbar(k_{0,0} + \vec{z}_0))$  be the corresponding element in  $\mathcal{L}_h$ . By (17), for any  $\vec{z} \in \mathbb{R}^2$  such that  $\|\vec{z}\| \leq \|\vec{z}_1\|$ ,

$$\|G_h(\hbar k_{0,0} + \hbar \vec{z}) - G_h(\hbar k_{0,0}) - G'_h(\hbar k_{0,0}) \cdot (\hbar \vec{z})\| \leq L_1 \|\hbar \vec{z}\|^2 \quad (68)$$

and hence

$$\begin{aligned} \|\lambda_{1,0} - \lambda_{0,0}\| &= \|G'_h(\hbar k_{0,0}) \cdot (\hbar \vec{z}_1)\| + \sigma_1 \\ \|\mu - \lambda_{0,0}\| &= \|G'_h(\hbar k_{0,0}) \cdot (\hbar \vec{z}_0)\| + \sigma_0, \end{aligned} \quad (69)$$

with  $|\sigma_j| \leq L_1 \|\hbar \vec{z}_1\|^2 + 2C_N \hbar^N$ , for  $j = 0, 1$ . From the algorithm we know that  $\|\lambda_{1,0} - \lambda_{0,0}\| \leq \|\mu - \lambda_{0,0}\|$  and therefore

$$(n-1) \|G'_h(\hbar k_{0,0}) \cdot (\hbar \vec{z}_0)\| \leq \sigma_0 - \sigma_1.$$

From Lemma 3.8, there exists  $\Gamma > 0$  independent of  $\hbar$  such that  $\|G_h'^{-1}\| \geq \Gamma$  on  $\tilde{U}$ , and hence

$$(n-1) \leq \frac{\sigma_0 - \sigma_1}{\Gamma \|\hbar \vec{z}_0\|} \leq \frac{2L_1 \hbar^2 \tilde{\rho}^2 + 4C_N \hbar^N}{\Gamma \hbar}.$$

Thus, if

$$\hbar(2L_1 \tilde{\rho}^2 + 4C_N \hbar^{N-2}) < \Gamma, \quad (\hbar-70)$$

then we must have  $n = 1$ . □

**Lemma 3.43** For all  $\hbar \leq \hbar_0^{[3]}$ , the vectors  $\vec{z}_1$  and  $\vec{z}_2$  (60) defined in Corollary 3.40 are linearly independent.

**Proof.** If  $\vec{z}_1$  and  $\vec{z}_2$  are colinear, there exists  $\sigma \in \mathbb{Q}$  such that  $\vec{z}_2 = \sigma \vec{z}_1$ . Since the coefficients of  $\vec{z}_1$  are co-prime, we must have  $\sigma = n \in \mathbb{Z}$ . Writing,  $k_{0,1} = k_{0,0} + \vec{z}_2 = k_{0,0} + n\vec{z}_1$ , we obtain from Corollary 3.40 that  $\lambda_{0,1} = \lambda_{n,0}$ , which contradicts Step 5 of the algorithm. □

We arrive at the main result of this section.

**Theorem 3.44** There exists  $\hbar_0^{[4]} > 0$  (defined in (3.5.1)) such that  $\vec{z}_1$  and  $\vec{z}_2$  defined in Corollary 3.40 form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^2$  for any  $\hbar \leq \hbar_0^{[4]}$ .

**Proof.** Let  $D = D(\hbar) := \det(\vec{z}_1, \vec{z}_2)$ . We know that  $D \neq 0$  by the previous lemma. We want now to prove that  $D = \pm 1$ . By Pick's formula, the number of integral points in the closed convex hull of the three points  $k_{0,0}$ ,  $k_{1,0}$  and  $k_{0,1}$  (other than these vertices), where points in the boundary count half, is  $\frac{|D|-1}{2}$ . Hence if  $|D| \geq 2$ , there is at least one such integral point, call it  $\ell \in \mathbb{Z}^2$ , set  $\vec{z}_3 = \ell - k_{0,0}$ , and let  $\mu = k_h^{-1}(\hbar\ell)$  be the corresponding element of  $\mathcal{L}_\hbar$ .

(i) Since the components of  $\vec{z}_1$  are co-prime by Lemma 3.42,  $\vec{z}_3$  cannot be colinear to  $\vec{z}_1$ , because it then would be equal to it, and  $\ell$  would be a vertex of the triangle. Therefore, by Corollary 3.40,  $\mu$  cannot be one of the  $\lambda_{n,0}$ . This implies

$$\|\lambda_{0,1} - \lambda_{0,0}\| \leq \|\mu - \lambda_{0,0}\|. \quad (71)$$

(ii) Set  $\vec{u}_i = G'_h(\hbar k_{0,0}) \cdot \vec{z}_i$ ,  $i = 1, 2, 3$ . As  $\vec{z}_3$  is a convex linear combination of  $\vec{z}_1$  and  $\vec{z}_2$ , the same holds between  $\vec{u}_3$ , and  $\vec{u}_1$ ,  $\vec{u}_2$ . Using (68), we see that there exists a constant  $C > 0$  that can be made explicit (namely  $C \geq 2L_1\tilde{\rho} + 2C_N\hbar^{N-2}$ ) such that the ball  $B(\mu, \hbar^2 C)$  intersects the interior of the triangle  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$ . Let  $\tilde{\mu}$  be a point of this intersection. Since  $\|\vec{z}_j - \vec{z}_3\| \geq 1$  for  $j = 1, 2$ , because these are non zero integer vectors, the point  $\tilde{\mu}$  must stay away from a ball of size  $\delta\hbar$  of the vertices, for some  $\delta \in ]0, 1/L_F[$  independent of  $\hbar$  and which can also be made explicit.

(iii) We now need some elementary triangle estimates in order to bound from below the distance  $\|\lambda_{1,0} - \lambda_{0,0}\| - \|\tilde{\mu} - \lambda_{0,0}\|$ . By construction of the algorithm,  $\|\lambda_{1,0} - \lambda_{0,0}\| \leq \|\lambda_{0,1} - \lambda_{0,0}\|$ . Hence the angle at the vertex  $\lambda_{0,1}$  is strictly less than  $\frac{\pi}{2}$ , which implies that the orthogonal projection  $H$  of  $\lambda_{0,0}$  onto the line  $(\lambda_{1,0}\lambda_{0,1})$  is located on the strict half line starting at  $\lambda_{0,1}$  and containing  $\lambda_{1,0}$ . Thus there exists  $r > 0$  be such that the ball  $B(\lambda_{0,1}, r)$  does not contain  $H$  (see Figure 3.5.1). In

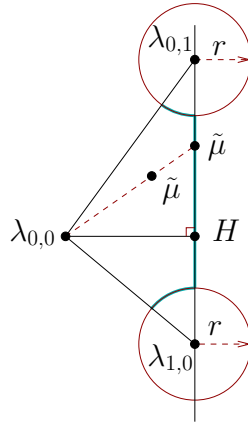


Figure 5: The point  $\tilde{\mu}$  is pushed onto the thickened path.

fact any  $r \leq \|\lambda_{1,0} - \lambda_{0,1}\| / 2$  fulfills this requirement. We choose  $r = \min(\|\lambda_{1,0} - \lambda_{0,1}\| / 2, \delta\hbar)$  so that, in addition,  $\tilde{\mu} \notin B(\lambda_{0,1}, r) \cup B(\lambda_{1,0}, r)$ . Let us push  $\tilde{\mu}$  along the ray  $(\lambda_{0,0}\mu)$  until it



reaches either the boundary of one of the balls  $B(\lambda_{0,1}, r)$  or  $B(\lambda_{1,0}, r)$ , or the opposite edge of the triangle,  $E := (\lambda_{1,0}\lambda_{0,1})$ . We may call  $\tilde{\mu}$  again this new point. Doing this, the distance  $d := \|\tilde{\mu} - \lambda_{0,0}\|$  can not decrease. When  $\tilde{\mu}$  is on the boundary of a ball, it is clear that  $d$  increases as we move  $\tilde{\mu}$  on the circle towards the edge  $E$ . So, for any configuration of  $\tilde{\mu}$ , there is a position of  $\tilde{\mu}$  on  $E$  that produces a larger or equal distance  $d$ . Now it is easy to study the variation of  $d = d(x)$  as a function of the abscissa  $x$  of  $\tilde{\mu}$  along  $E$ . Taking  $x = 0$  for the point  $H$  and  $x =: b > 0$  at  $\lambda_{0,1}$ , and using  $d(x) = \sqrt{x^2 + h^2}$ , with  $h = \|H - \lambda_{0,0}\|$ , we obtain for  $x \in (0, b - r)$  that  $d$  is increasing and

$$d(b) - d(x) \geq \frac{rb}{2d(b)} \geq \frac{r^2}{2d(b)}.$$

If the point  $\lambda_{1,0}$  has a negative abscissa  $c < 0$ , the distance  $d$  will have a local maximum also at  $c$ , and we can repeat the argument above to get, when  $|c| > r$ ,

$$\forall x \in (c + r, 0), \quad d(c) - d(x) \geq \frac{r|c|}{2d(c)} \geq \frac{r^2}{2d(c)}.$$

Recall that  $d(c) = \|\lambda_{1,0} - \lambda_{0,0}\| \leq \|\lambda_{0,1} - \lambda_{0,0}\| = d(b)$ . Thus,

$$\forall x \in (c + r, b - r), \quad d(b) - d(x) \geq \frac{r^2}{2d(b)}.$$

This finally gives

$$\|\lambda_{0,1} - \lambda_{0,0}\| - \|\tilde{\mu} - \lambda_{0,0}\| \geq \frac{r^2}{2\|\lambda_{0,1} - \lambda_{0,0}\|}. \quad (72)$$

(iv) From (71) we can write

$$\begin{aligned} \|\lambda_{0,1} - \lambda_{0,0}\| &\leq \|\mu - \lambda_{0,0}\| \leq \|\tilde{\mu} - \lambda_{0,0}\| + C\hbar^2 \\ &\leq \|\lambda_{0,1} - \lambda_{0,0}\| - \frac{r^2}{2\|\lambda_{0,1} - \lambda_{0,0}\|} + C\hbar^2 \quad \text{by (72)}. \end{aligned}$$

Therefore

$$\frac{r^2}{2\|\lambda_{0,1} - \lambda_{0,0}\|} \leq C\hbar^2.$$

Since  $\|k_{0,1} - k_{1,0}\| \geq 1$ , we have  $\|\lambda_{0,1} - \lambda_{1,0}\| \geq \hbar/L_F - 2C_N\hbar^N$ , while (62) says  $\|\lambda_{0,1} - \lambda_{0,0}\| \leq L_0\hbar + 2C_N\hbar^N$ . We obtain

$$\frac{\min((\frac{1}{L_F} - 2C_N\hbar^{N-1})^2, \delta^2)}{2(L_0 + 2C_N\hbar^{N-1})} \leq C\hbar$$

which is of course impossible if  $\hbar$  is small enough, namely if one takes  $\hbar \leq \hbar_0^{[4]}$  with

$$\begin{aligned} 2C_N(\hbar_0^{[4]})^{N-1} &< \frac{1}{L_F} \quad \text{and} \quad \delta < \frac{1}{L_F} - 2C_N(\hbar_0^{[4]})^{N-1} \\ \text{and} \quad \hbar_0^{[4]} &< \frac{\delta^2}{2C(L_0 + 2C_N(\hbar_0^{[4]})^{N-1})}. \end{aligned} \quad (\hbar-73)$$

For all  $\hbar \leq \hbar_0^{[4]}$ , we conclude that  $|D| = 1$ . This concludes the proof of the theorem.  $\square$

**Remark 3.45** In order to have a rough idea of the size of the various bounds on  $\hbar$ , following up Remark 3.41, we may neglect the terms  $C_N \hbar^N$  and get from ( $\hbar$ -67) and ( $\hbar$ -70) and the approximation  $\Gamma \sim 1/L_0$  that  $\hbar_0^{[3]} \lesssim \min\left(\hbar_0^{[2]}; \frac{r_0}{2L_F(M+L_0)}; \frac{1}{2L_0^3 L_1 L_F^2}\right)$ . Then, from (3.5.1) and  $C \sim 2L_0^2 L_1 L_F^2$ , we get  $\hbar_0^{[4]} \lesssim \min\left(\hbar_0^{[3]}; \frac{1}{4L_0^3 L_1 L_F^4}\right)$ .  $\triangle$

**Definition 3.46** When the vectors  $(\vec{z}_1, \vec{z}_2)$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^2$ , the triple  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$  will be called an affine basis of  $\mathcal{L}_\hbar$  at  $\lambda_{0,0}$ .

The algorithm does not necessarily label *all* points of  $\mathcal{L}_\hbar \cap B_0$ . Indeed, by construction, the set of produced labels  $(n, m)$  is of the form

$$\mathcal{E} := \{(n, m) \in \mathbb{Z}^2; \quad m_{\min} \leq m \leq m_{\max}; \quad n_{\min}(m) \leq n \leq n_{\max}(m)\},$$

where  $m_{\min}$ ,  $m_{\max}$ , and  $n_{\min}(m)$  and  $n_{\max}(m)$ , for  $m \in \{m_{\min}, \dots, m_{\max}\}$ , may depend on  $\hbar$ . For a given  $m$ ,  $n_{\max}(m)$  is the smallest positive integer produced by Step 4 such that  $\lambda_{n,m} \in B_0$  and  $\lambda_{n+1,m} \notin B_0$ . The integer  $n_{\min}(m)$  is defined in a similar way,  $m_{\max}$  is the smallest positive integer produced by Step 8 such that  $\lambda_{0,m} \in B_0$  but  $\lambda_{0,m+1} \notin B_0$ , and  $m_{\min}$  is defined in a similar way. Thus, there is no reason why a set of the form  $\mathcal{E}$  would fill up  $\mathcal{L}_\hbar \cap B_0$  entirely. However, since  $\mathcal{E}$  will fill up all integral points of any convex set  $V \subset \tilde{U}_0$ , the algorithm is guaranteed to label all points of  $\mathcal{L}_\hbar$  in *some*  $\hbar$ -independent open subset containing  $c$ . It would be interesting to improve the algorithm in order to make sure that it explores the whole connected component of  $B_0$ .

Note that, after the last step (orientation test) of the algorithm, the basis  $\vec{v}_1, \vec{v}_2$  is made direct, so is the case for the basis  $\vec{z}_1, \vec{z}_2$  because by convention  $\det G'_0(\xi) > 0$ . It follows from Theorem 3.44 and Corollary 3.40 that for each  $\hbar$  small enough, we can define a matrix  $Z_\hbar \in \text{SL}(2, \mathbb{Z})$  such that  $Z_\hbar(\vec{e}_1, \vec{e}_2) = (\vec{z}_1, \vec{z}_2)$ , and the ‘‘labelling’’  $\lambda_{n,m} \mapsto (n, m)$  of the algorithm is such that

$$\lambda_{n,m} = G_\hbar(\hbar(k_{0,0} + n\vec{z}_1 + m\vec{z}_2)) + \mathcal{O}(\hbar^\infty) = G_\hbar \circ Z_\hbar(\hbar(\varkappa + n\vec{e}_1 + m\vec{e}_2)) + \mathcal{O}(\hbar^\infty)$$

where  $\varkappa = Z_\hbar^{-1}k_{0,0} \in \mathbb{Z}^2$ . However, this does not produce a linear labelling for  $\mathcal{L}_\hbar$  because in general the map  $\hbar \rightarrow Z_\hbar$  will not be constant (and hence, not continuous), even for arbitrary small values of  $\hbar$ . In order to produce a linear labelling, we should find a way to ‘‘detect’’ the matrix  $Z_\hbar$ , which will allow to correct the initial algorithm and make it smooth in  $\hbar$ . This is the aim of the following sections.

### 3.5.2 A semitoric algorithm

If a quantum system is known to be semitoric, the joint spectrum is an asymptotic lattice with a special property given by Proposition 3.33. In this case, we expect a well designed algorithm not to provide *any* labelling, but rather a *semitoric labelling*, *i.e.* associated with a semitoric chart, see Lemma 3.34.

Let  $(j, k)$  be the semitoric labelling of Theorem 3.36. From (48), we know that for any  $\epsilon \in (0, \frac{1}{2})$ , there exists a ball  $B'$  around  $c$  such that, if  $\hbar_0$  is small enough,  $\Sigma_\hbar \cap B'$  is contained in a union of disjoint vertical strips of width  $2\epsilon\hbar$ :

$$\Sigma_\hbar \cap B' \subset \bigcup_{j \in \mathbb{Z}} V_j(\hbar), \quad (74)$$

where

$$V_j(\hbar) = [\alpha + \hbar(j + \mu) - \hbar\epsilon, \alpha + \hbar(j + \mu) + \hbar\epsilon] \times \mathbb{R} \subset \mathbb{R}^2. \quad (75)$$

The precise size of  $B'$  depends on the variations of the subprincipal term  $G_1$  of a semitoric asymptotic chart  $G_\hbar \sim G_0 + \hbar G_1 + \dots$ .

Now, from the data of  $\mathcal{L}_\hbar$ , perform the generic algorithm of the previous section, in order to obtain an oriented affine basis  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$  of  $\mathcal{L}_\hbar$  (Theorem 3.44, Definition 3.46, and Step 12). Let  $V$  be the vertical strip of width  $\hbar^{3/2}$ , vertically centered at  $\lambda_{0,0}$ , and define  $\mu \in \mathcal{L}_\hbar$  as the nearest point to  $\lambda_{0,0}$  located in  $V$  and *above*  $\lambda_{0,0}$ . The existence of a semitoric chart (Lemma 3.34) and the decomposition (74)-(75) imply that  $\mu$  exists in an  $\mathcal{O}(\hbar)$  neighborhood of  $\lambda_{0,0}$ , belongs to the strip  $V_{j_0}$  that contains  $\lambda_{0,0}$ , and is unique if  $\hbar$  is small enough. Therefore, we know from the analysis of the general algorithm (Corollary 3.40) that there exists bounded co-prime integers  $(n, m) \in \mathbb{Z}^2$  such that

$$k_\hbar(\mu) - k_{0,0} = n \vec{z}_1 + m \vec{z}_2.$$

In practice, the integers  $(n, m)$  can be found by expressing  $(\mu - \lambda_{0,0})/\hbar$  on the basis  $(v_1, v_2)$  and then rounding its coefficients to their nearest integers. Finally, we choose  $(n', m') \in \mathbb{Z}^2$  such that  $n'm - m'n = 1$ , and define  $\mu' \in \mathcal{L}_\hbar$  by

$$k_\hbar(\mu') = n' \vec{z}_1 + m' \vec{z}_2.$$

We obtain in this way a new oriented affine basis of  $\mathcal{L}_\hbar$  given by

$$(\mu' - \lambda_{0,0}, \mu - \lambda_{0,0}). \quad (76)$$

**Proposition 3.47** *The affine basis (76) is associated with a semitoric asymptotic chart.*

**Proof.** The uniqueness of  $\mu$  ensures that its label, in a semitoric chart  $\hat{G}_{\hbar}$  associated with a labelling  $\hat{k}_{\hbar}$ , is equal to  $\hat{k}_{\hbar}(\lambda_{0,0}) + (0, 1)$ . Hence the label of  $\mu'$  must be of the form  $\hat{k}_{\hbar}(\lambda_{0,0}) + (1, \ell)$ , for some  $\ell \in \mathbb{Z}$ . Hence the affine basis (76) is obtained from the one associated with the chart  $\hat{G}_{\hbar}$  by composition with the matrix  $A = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}$ . This matrix  $A$  preserves the semitoric property of the chart.  $\square$

### 3.5.3 An algorithm for a sequence of values of $\hbar$

In our way to reconstructing a linear labelling from the data of the sets  $\mathcal{L}_{\hbar}$  and of a window  $B$  with a distinguished point  $c \in B$ , we introduce a “algorithm with uniform labelling” on a decreasing sequence  $\hbar_j \in \mathcal{I}$ ,  $j \geq 1$ , tending to 0. The algorithm is inductive: once the labelling  $k_{\hbar_j} : \Lambda_{\hbar_j} \ni \lambda^{(j)} \mapsto (n^{(j)}, m^{(j)}) \in \mathbb{Z}^2$  is known, together with a matrix  $S_j \in \mathbf{M}_2(\mathbb{Z})$ , it will produce the labelling  $k_{\hbar_{j+1}} : \Lambda_{\hbar_{j+1}} \rightarrow \mathbb{Z}^2$  (and the matrix  $S_{j+1}$ ). Thus, it theoretically defines  $k_{\hbar_j}$  for all  $j$ . Of course, in practice, if one wants to obtain the labelling  $k_{\hbar_j}$  for a specific  $j$ , it is enough to stop at the step  $j$ .

The algorithm works by running the previous algorithm of Section 3.5.1 with all values  $\hbar_j$ , for all  $j = 1, \dots$ , and self-adjusting the resulting labelling for each  $j$ . In order to have a more efficient implementation, if we know in advance that we want to stop at a specific step  $j = j_{\text{stop}}$ , it is in fact not necessary to compute the full labellings for the values of  $j$  less than  $j_{\text{stop}}$ ; for these values, it is enough to find the correct “affine basis”, which corresponds to Steps 1 to 5 of the algorithm of Section 3.5.1. Thus, the new algorithm, with exit test at  $j = j_{\text{stop}}$ , works as follows.

- a) Choose an open subset  $B_0 \Subset B$ , and fix  $c \in B_0$ . Let  $S_0 = \text{Id} \in \mathbf{M}_2(\mathbb{R})$ . Let  $j = 1$ .
- b) Apply steps 1 to 5 of the algorithm of Section 3.5.1 with  $\hbar = \hbar_j$ . This defines points  $\lambda_{n,m}^{(j)}$ . In particular, we have an origin  $\lambda_{0,0}^{(j)}$ , and the first generating vectors  $v_1^{(j)} = \lambda_{1,0}^{(j)} - \lambda_{0,0}^{(j)}$  and  $v_2^{(j)} = \lambda_{0,1}^{(j)} - \lambda_{0,0}^{(j)}$ .
- c) Define  $T_j \in \mathbf{M}_2(\mathbb{R})$  to be the matrix formed by the column vectors:

$$T_j := (\hbar_j^{-1} v_1^{(j)}, \hbar_j^{-1} v_2^{(j)}).$$

If  $T_j$  is not invertible or  $j = 1$ , increase  $j$  by one, let  $S_j := S_{j-1}$ , and go back to Step b). If  $T_j$  is invertible, make it oriented as in Step 12 of the previous algorithm.

- d) Define  $A_j := T_j^{-1} T_{j-1}$ , and let  $A_j^{\sharp} \in \mathbf{M}_2(\mathbb{Z})$  be the matrix obtained by rounding the entries of  $A_j$  to their “nearest integer” (in the usual, unique way). If  $\det A_j^{\sharp} \neq 1$ , define  $S_j := S_{j-1}$ . If  $\det A_j^{\sharp} = 1$ , define

$$S_j := S_{j-1} (A_j^{\sharp})^{-1}.$$

- e) In case of a concrete implementation, if  $j < j_{\text{stop}}$ , increase  $j$  by one and go back to Step **b**). Otherwise, finish the previous algorithm, *i.e.* perform Steps **6** to **12**. Let  $\lambda \mapsto (n, m)$  be the resulting labelling.
- f) The new labelling of  $\mathcal{L}_{\hbar_j}$  is the map  $\lambda^{(j)} \mapsto (\tilde{n}, \tilde{m})$  given by a linear transformation acting on the labelling  $\lambda$  of the previous step according to

$$\begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = S_j \begin{pmatrix} n \\ m \end{pmatrix}.$$

In other words,  $\lambda_{n,m} = \lambda_{S_j(m,n)}^{(j)} = \lambda_{\tilde{m},\tilde{n}}^{(j)}$ .

**Theorem 3.48** *Let  $(\mathcal{L}_{\hbar}, \mathcal{I}, B)$  be an asymptotic lattice, where  $B \subset \mathbb{R}^2$ . Let  $\hbar_j \in \mathcal{I}$ ,  $j \geq 1$ , be a decreasing sequence tending to 0. Then the previously described algorithm produces a linear labelling of the asymptotic lattice  $(\mathcal{L}_{\hbar}, \mathcal{I}', B)$ , where  $\mathcal{I}' = \{\hbar_j, j \in \mathbb{N}^*\}$ .*

**Proof.** We interpret the newly introduced objects with respect to an asymptotic chart  $G_{\hbar}$  (which is known to exist, but is unknown). Let us denote by  $Z_j$  the matrix formed by the column vectors  $z_1^{(j)} = \vec{z}_1$  and  $z_2^{(j)} = \vec{z}_2$  defined in Corollary 3.40(60) for  $\hbar = \hbar_j$  (initial algorithm):

$$Z_j := (z_1^{(j)}, z_2^{(j)}).$$

Formula (17), in view of (63), gives

$$T_j = G'_0(\xi) Z_j + \mathcal{O}(\hbar_j).$$

By Theorem 3.44,  $Z_j$  is unimodular and hence  $T_j$  is invertible (with positive determinant) if  $\hbar_j$  is small enough, which happens for all  $j \geq j_0$ , for some  $j_0$ . Thus,

$$A_j := T_j^{-1} T_{j-1} = Z_j^{-1} Z_{j-1} + \mathcal{O}(\hbar_{j-1}).$$

Therefore, if  $j_0$  is large enough, we obtain for all  $j \geq j_0$ ,

$$A_j^{\sharp} = Z_j^{-1} Z_{j-1} \quad \text{and} \quad \det A_j^{\sharp} = 1.$$

Set  $\tilde{Z}_j = Z_j S_j^{-1}$ . The matrix  $\tilde{Z}_j$  has integer coefficients and satisfies by definition of the new labelling for  $\hbar = \hbar_j$

$$\tilde{Z}_j \begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = Z_j \begin{pmatrix} n \\ m \end{pmatrix}.$$

We check that the sequence  $\tilde{Z}_j$  is stationary as  $j \rightarrow \infty$ :

$$\tilde{Z}_j = Z_j S_j^{-1} = Z_j A_j^{\sharp} S_{j-1}^{-1} = Z_j Z_j^{-1} Z_{j-1} S_{j-1}^{-1} = Z_{j-1} S_{j-1}^{-1} = \dots = \tilde{Z}_{j_0}.$$

By (61), in terms of the good labelling  $k_{\hbar}$  associated with  $G_{\hbar}$

$$k_{\hbar}(\tilde{\lambda}_{\tilde{n},\tilde{m}}^{(j)}) = \tilde{n} \tilde{z}_1 + \tilde{m} \tilde{z}_2 + k_{0,0}^{(j)},$$

i.e.

$$\tilde{Z}_{j_0}^{-1} k_{\hbar}(\tilde{\lambda}_{n,m}^{(j)}) = n \tilde{e}_1 + m \tilde{e}_2 + \tilde{Z}_{j_0}^{-1} k_{0,0}^{(j)}.$$

Thus we have constructed a linear labelling  $\lambda \rightarrow (n, m)$  for  $(\mathcal{L}_{\hbar}, \mathcal{I}', B)$ , where  $(n, m)$  is such that  $\lambda = \tilde{\lambda}_{n,m}^{(j)}$ , associated with the asymptotic chart  $\tilde{G}_{\hbar} := G_{\hbar} \circ \tilde{Z}_{j_0}$ .  $\square$

### 3.6 The inverse problem for the rotation number

We can finally apply our algorithms to the initial question, because having a linear labelling is actually enough for recovering the rotation number. In Theorem 3.30 we have obtained the classical rotation number as an  $\mathcal{O}(\hbar)$  limit of the quantum rotation number; this suggests that one can actually expect the recovery process to be robust with respect to smaller (namely,  $\mathcal{O}(\hbar^2)$ ) perturbations of the asymptotic lattice. Previous results for special cases of semiclassical operators show that this expectation is very natural [64, 44].

**Theorem 3.49** *Let  $(\hat{J}, \hat{H})$  be a quantum integrable system with principal symbols  $F := (J, H)$ . Let  $\Sigma_{\hbar} := \Sigma_{\hbar}(\hat{J}, \hat{H})$  be the joint spectrum of  $(\hat{J}, \hat{H})$ . Let  $c \in \mathbb{R}^2$  be a regular value of  $F$ , such that the fiber  $F^{-1}(c)$  is compact and connected. Then there exists a small ball  $B$  around  $c$  for which the following statements hold.*

- *From the knowledge of the family of discrete sets  $B \cap \Sigma_{\hbar} \subset \mathbb{R}^2$  where  $\hbar$  varies in a sequence  $\mathcal{I}' = (\hbar_j)_{j \in \mathbb{N}^*}$  accumulating at zero, one can recover the classical rotation number  $[w]$  in  $B$  modulo the natural action of Möbius transformations (Lemma 2.2).*
- *This recovery is stable under  $\mathcal{O}(\hbar^2)$  perturbations of the joint spectrum, and is algorithmic; precisely, there exists a constructive algorithm with the following property: given any such data  $(\Sigma_{\hbar}, \mathcal{I}', B)$ , there exists a choice of action coordinates  $I$  in  $B$  such that for any  $b \in B$ , the algorithm produces a sequence  $[w_{\hbar_i}](b) \in \mathbb{RP}^1$ , where  $\hbar_i \in \mathcal{I}'$ , such that*

$$[w_{\hbar_i}](b) = [w_I](\Lambda) + \mathcal{O}(\hbar_i) \tag{77}$$

where  $\Lambda = F^{-1}(b)$ .

**Proof.** We first assume that  $(\Sigma_{\hbar} := \Sigma_{\hbar}(\hat{J}, \hat{H}))_{\hbar \in \mathcal{I}'}$  is given exactly, without any  $\mathcal{O}(\hbar^2)$  error term. By Theorem 3.7, there exists a ball  $B$  around  $c$ , contained in the set of regular values of  $F$ , and  $\hbar_0 > 0$ , such that  $(\Sigma_{\hbar} \cap B, ]0, \hbar_0[, B)$  is an asymptotic lattice.

Let  $\tilde{k}_{\hbar}$  be a linear labelling of a neighborhood  $\tilde{B} \Subset B$  of  $c$  constructed by the algorithm of Theorem 3.48. Let  $b \in \tilde{B}$  and let  $\lambda \in \mathcal{L}_{\hbar}$  be such that  $\lambda = b + \mathcal{O}(\hbar)$  (for instance, choose a

closest point to  $b$ ). Let  $(\bar{n}, \bar{m}) = \bar{k}_\hbar(\lambda)$  be the corresponding label obtained from the algorithm, *i.e.*  $\lambda = \lambda_{\bar{n}, \bar{m}}$ .

Let  $k_\hbar$  be the good labelling associated with  $\bar{k}_\hbar$  (it is not known from the algorithm). There exist  $\hbar$ -dependent integers  $n_0, m_0$  such that for any  $\lambda \in \Sigma_\hbar$ , if we let  $(n, m) = k_\hbar(\lambda)$  and  $(\bar{n}, \bar{m}) = \bar{k}_\hbar(\lambda)$ , then

$$(n, m) = (\bar{n}, \bar{m}) + (n_0, m_0). \quad (78)$$

Recall from (47) that the quantum rotation number  $\hat{w}_\hbar(n, m)$  is by definition

$$\hat{w}_\hbar(n, m) = \frac{E_{n+1, m}(\hbar) - E_{n, m}(\hbar)}{E_{n, m+1}(\hbar) - E_{n, m}(\hbar)},$$

where for all  $n, m$ , we denote by  $E_{n, m}(\hbar)$  the second component of  $\lambda = k_\hbar^{-1}(n, m) \in \Sigma_\hbar$ . Hence

$$\hat{w}_\hbar(n, m) = \frac{\pi_2 \lambda_{\bar{n}+1, \bar{m}} - \pi_2 \lambda_{\bar{n}, \bar{m}}}{\pi_2 \lambda_{\bar{n}, \bar{m}+1} - \pi_2 \lambda_{\bar{n}, \bar{m}}},$$

where  $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the projection onto the second factor; note that this quantity can be computed directly from the algorithm. We denote it by  $[w_{\hbar_i}](b)$ . It just remains to apply Theorem 3.30 using an asymptotic chart  $G_\hbar$  associated with the good labelling  $k_\hbar$ : for the action variables  $I = G_0^{-1}(F)$ , defined in a fixed neighborhood of the torus  $\Lambda_c = F^{-1}(c)$ , containing  $\Lambda_b = F^{-1}(b)$ , the classical rotation number is

$$[w_I(\Lambda_b)] = [\hat{w}_\hbar(n, m)] + \mathcal{O}(\hbar)$$

which gives (77), and hence can be recovered in the limit as  $\hbar \rightarrow 0$ ,  $\hbar \in \mathcal{I}'$ .

It remains to investigate the effect of the error term  $\mathcal{O}(\hbar^2)$  in the knowledge of the joint spectrum. By (24), this will not affect, if  $\hbar$  is small enough, the choice of all points of the form  $\mu_2$  in Lemma 3.39, that is, once the affine basis  $(\lambda_{0,0}, \lambda_{1,0}, \lambda_{0,1})$  is chosen (Definition 3.46). In contrast, the choice of this affine basis *can* depend on the error term, because we are minimizing distances of order  $\hbar$ . Fortunately, this error won't affect estimates like (68), provided we accept to make the constant  $L_1$  larger, which is harmless. Thus, the perturbed triple  $(\lambda'_{0,0}, \lambda'_{1,0}, \lambda'_{0,1})$  is still an affine basis for  $\hbar$  small enough, and the rest of the algorithm (Lemma 3.39) goes through, leading to a labelling  $\lambda'_{\bar{n}, \bar{m}}$  of the perturbed joint spectrum near  $c$ . Let  $\lambda_{\bar{n}, \bar{m}}$  be the unique point in  $\Sigma_\hbar$  that is  $\mathcal{O}(\hbar^2)$ -close to  $\lambda'_{\bar{n}, \bar{m}}$ , using (24) again. We have  $\lambda_{\bar{n}, \bar{m}} = b + \mathcal{O}(\hbar)$ , and because of that uniqueness,  $\lambda_{\bar{n}, \bar{m}} \mapsto (\bar{n}, \bar{m})$  is a linear labelling of  $\Sigma_\hbar$ . As above, we can now introduce the good labelling for which (78) holds. The resulting ‘‘perturbed quantum rotation number’’ will be computed as

$$\hat{w}'_\hbar(n, m) = \frac{\pi_2 \lambda'_{\bar{n}+1, \bar{m}} - \pi_2 \lambda'_{\bar{n}, \bar{m}}}{\pi_2 \lambda'_{\bar{n}, \bar{m}+1} - \pi_2 \lambda'_{\bar{n}, \bar{m}}} = \frac{\pi_2 \lambda_{\bar{n}+1, \bar{m}} - \pi_2 \lambda_{\bar{n}, \bar{m}}}{\pi_2 \lambda_{\bar{n}, \bar{m}+1} - \pi_2 \lambda_{\bar{n}, \bar{m}}} + \mathcal{O}(\hbar).$$

Hence, as above, we introduce the corresponding asymptotic chart  $G_\hbar$  and conclude that

$$[w_I(\Lambda_b)] = [\hat{w}'_\hbar(n, m)] + \mathcal{O}(\hbar),$$



with  $I = G_0^{-1} \circ F$ . □

We may now turn to the global problem which, for simplicity, we state without the  $\mathcal{O}(\hbar^2)$  perturbation. The notation follows Proposition 2.6.

**Theorem 3.50** *Let  $(\hat{J}, \hat{H})$  be a quantum integrable system with principal symbols  $F := (J, H)$ . Let  $\mathbf{B}_r$  be the set of regular Liouville tori of  $F$ , and let  $B_r = F(\mathbf{B}_r)$ ; we assume that all fibers  $F^{-1}(c)$  for  $c \in B_r$  are compact and connected. Then, from the joint spectrum  $\Sigma_{\hbar}$  of  $(\hat{J}, \hat{H})$ , one can construct a map  $\hat{\omega} : \tilde{\mathbf{B}}_r \rightarrow \mathbb{RP}^1$  such that, if  $[\tilde{\omega}]$  is the globalized rotation number defined in Proposition 2.6, then*

$$\hat{\omega} = A \circ [\tilde{\omega}] + \mathcal{O}(\hbar),$$

for some fixed  $A \in \mathrm{SL}(2, \mathbb{Z})$ , and the remainder is locally uniform. In particular, the globalized rotation number  $[\tilde{\omega}]$  can be recovered from the joint spectrum, modulo a global Möbius transformation.

**Proof.** Because of the connectedness of fibers of  $F$ , we may identify  $\mathbf{B}_r$  with its image  $B_r$ , and endow the latter with the integral affine structure given by action coordinates. We construct the map  $\hat{\omega}$  by a Čech cohomology argument. Fix  $c_0 \in B_r$ , and let  $\gamma : [0, 1] \rightarrow B_r$  be a path starting at  $\gamma(0) = c_0$ . Let  $c = \gamma(1)$ . Let  $\lambda \in \Sigma_{\hbar}$  be a nearest point to  $c$ , so that, by Lemma 3.15,

$$\lambda = c + \mathcal{O}(\hbar).$$

Applying Theorem 3.48 we get a neighborhood  $B_0$  of  $c_0$  equipped with a linear labelling  $\bar{k}_{0, \hbar}$  of  $(\Sigma_{\hbar} \cap B_0, \mathcal{I}', B_0)$ . Cover the image  $\gamma([0, 1])$  by a finite union of small balls  $B_1, \dots, B_N$ , such that  $c \in B_N$ ,  $B_i \cap B_{i+1} \neq \emptyset$ , for all  $i = 0, \dots, N$ , and such that on each  $B_i$ , the algorithm of Theorem 3.48 produces a linear labelling  $\bar{k}_{i, \hbar}$  of  $(\Sigma_{\hbar} \cap B_i, \mathcal{I}', B_i)$ . From Proposition 3.20 applied to the restrictions of  $\bar{k}_{i, \hbar}$  and  $\bar{k}_{i+1, \hbar}$  on the asymptotic lattice  $\Sigma_{\hbar} \cap B_i \cap B_{i+1}$ , there exists a unique matrix  $A_i \in \mathrm{SL}(2, \mathbb{Z})$  and a family  $(\varkappa_{i, \hbar})_{\hbar \in \mathcal{I}'}$  in  $\mathbb{Z}^2$  such that, for  $\hbar$  small enough,

$$\bar{k}_{i+1, \hbar} = A_i \circ \bar{k}_{i, \hbar} + \varkappa_{i, \hbar} \quad \text{on} \quad \Sigma_{\hbar} \cap B_i \cap B_{i+1}. \quad (79)$$

We define

$$\hat{\omega}(\gamma) = [\hat{\omega}_N](A_0^{-1} \circ A_1^{-1} \circ \dots \circ A_{N-1}^{-1} \circ \bar{k}_{N, \hbar}(\lambda)), \quad (80)$$

where  $[\hat{\omega}_N]$  is the quantum rotation number associated with  $\bar{k}_{N, \hbar}$ . It is constructed from the joint spectrum as in Theorem 3.49. Note that the transition matrices  $A_i$  can be detected from the algorithm by comparing the affine basis described in Theorem 3.44.

By Lemma 3.20, the sheaf that assigns to a point  $c \in B_r$  a linear labelling (given by the algorithm) on a small neighborhood of  $c$  has constant transition functions, when  $\hbar$  is small enough, modulo the addition of  $\hbar$ -families in  $\mathbb{Z}^2$ . This ensures that the cocycle condition for the linear part of the transition functions is satisfied, and hence that the definition in (80) is invariant by homotopy transformation of the path  $\gamma$  with fixed endpoints, provided the initial labelling  $\bar{k}_{0, \hbar}$  is fixed. Thus, it defines a map

$$\hat{\omega} : \tilde{\mathbf{B}}_r \rightarrow \mathbb{RP}^1.$$

This map will be modified by a global  $\mathrm{SL}(2, \mathbb{Z})$  transformation if one changes  $\bar{k}_{0, \hbar}$ . *A priori*, the smallness of  $\hbar$  for which (80) is defined depends on  $\gamma$ ; however, if  $\gamma$  stays in a compact region, one can use a fixed, finite covering by small balls on which the algorithm applies, and hence obtain a uniform  $\hbar_0 > 0$  for which (80) holds for all  $\hbar \leq \hbar_0$ .

From (79) we get, using (33),

$$A_i = [(G'_{i+1,0}(\xi_{i+1}))^{-1}(G_{i,0})'(\xi_i)]. \quad (81)$$

with  $\xi_i := G_{i,0}^{-1}(c_i)$ .

Let  $\tilde{I} : \tilde{\mathbf{B}}_r \rightarrow \mathbb{R}^2$  be the global action variable used in Proposition 2.6 to define  $[\tilde{w}]$ . Along the path  $\gamma$ , the balls  $B_i$  can be lifted to open sets  $\tilde{B}_i \subset \tilde{\mathbf{B}}_r$  on which  $\pi_i$ , the restriction of  $\pi : \tilde{\mathbf{B}}_r \rightarrow \mathbf{B}_r$  is a diffeomorphism. Since both  $\tilde{I} \circ \pi_i^{-1}$  and  $I_i := G_{i,0}^{-1} \circ F$  are action variables above  $B_i$ , we must have

$$\tilde{I} \circ \pi_i^{-1} = Z_i \circ I_i,$$

for some  $Z_i \in \mathrm{SL}(2, \mathbb{Z})$ , which is of course independent of  $\hbar$ . Using (81), we obtain  $A_i = Z_{i+1}^{-1} \circ Z_i$ , for all  $i = 0, \dots, N-1$ . Thus,  $A_{N-1} \cdots A_1 A_0 = Z_N^{-1} Z_0$ . For  $\hbar$  small enough,  $Z_0^{-1} Z_N \bar{k}_{N, \hbar}(\lambda)$  stays in the image of the linear labelling  $\bar{k}_{N, \hbar}$ ; hence we can write

$$\begin{aligned} [\tilde{w}](\gamma) &= [\hat{w}_N](Z_0^{-1} Z_N \bar{k}_{N, \hbar}(\lambda)) \\ &= [w_{Z_0^{-1} Z_N I_N}(\Lambda)] + \mathcal{O}(\hbar) \quad \text{by Theorem 3.30} \\ &= {}^t Z_0 \circ [w_{Z_N I_N}(\Lambda)] + \mathcal{O}(\hbar) \quad \text{by Lemma 2.2} \\ &= {}^t Z_0 \circ [\tilde{w}](\gamma) + \mathcal{O}(\hbar) \end{aligned}$$

because by definition,  $[\tilde{w}]|_{\tilde{B}_N} = [w_{\tilde{I} \circ \pi_N^{-1}}] = [w_{Z_N I_N}]$ .  $\square$

Finally, let us consider the semitoric case. Combining Theorem 3.36 with the semitoric algorithm of Section 3.5.2 (Proposition 3.47), we obtain:

**Corollary 3.51** *Let  $(\hat{J}, \hat{H})$  be a semitoric quantum integrable system with principal symbols  $F = (J, H)$ . Let  $c \in \mathbb{R}^2$  be regular value of  $F$ . Then from the joint spectrum  $(\Sigma_{\hbar})_{\hbar \in \mathcal{I}}$  in a neighborhood  $B$  of  $c$ , where  $\mathcal{I}$  is a set accumulating at zero, one can algorithmically compute the quantum rotation number (47), and this quantum rotation number is an  $\mathcal{O}(\hbar)$  deformation of the classical semitoric rotation number  $w(\Lambda)$ , uniformly for all Liouville tori  $\Lambda \subset F^{-1}(B)$ .*

In the presence of an elliptic singularity we may apply Proposition 3.37, instead of the general algorithm, to obtain the corresponding statement.

**Corollary 3.52** *Let  $(\hat{J}, \hat{H})$  be a semitoric quantum integrable system with principal symbols  $F = (J, H)$ . Let  $c \in \mathbb{R}^2$  be a simple  $J$ -transversal elliptic critical value of  $F$ . Then from the joint spectrum  $(\Sigma_{\hbar})_{\hbar \in \mathcal{I}}$  in a neighborhood  $B$  of  $c$ , where  $\mathcal{I}$  is a set accumulating at zero, using the labelling of Proposition 3.37, one can compute the quantum rotation number (47), and this quantum rotation number is an  $\mathcal{O}(\hbar)$  deformation of the classical semitoric rotation number  $w(\Lambda)$ , uniformly for all Liouville tori  $\Lambda \subset F^{-1}(B)$ .*

In both cases, the detection of the rotation number is robust with respect to an  $\mathcal{O}(\hbar^2)$  perturbation of the joint spectrum.

**Remark 3.53** We would like to stress the difference between recovering the rotation number directly from the spectrum, in a constructive way (Theorem 3.49), and simply proving the abstract injectivity result that the joint spectrum determines the rotation number (which means that if two systems have the same joint spectrum, then they must share the same rotation number, modulo Möbius transformations). The latter, which is of course a consequence of the former, is actually quite easier (it doesn't require any algorithm); in view of Proposition 2.6, it is a direct consequence of Proposition 3.22 which implies that the integral affine structure of  $B_r$  is determined by the joint spectrum. This fact was already exploited in [65] to prove that classical monodromy determines quantum monodromy.  $\triangle$

**Remark 3.54** It would be quite interesting (and challenging) to extend our results to the case where the fibers of  $F$  could have several connected components. The microlocal derivation of Bohr-Sommerfeld rules should continue to hold, giving rise, on the spectral side, to a superposition of several asymptotic lattices. In the one-dimensional case, it was possible to separate each contribution by a Fourier analysis [64]; however, in that case, because of dimension 1, the labelling problem was obvious; the multidimensional case is certainly much more involved and, to our knowledge, completely open.  $\triangle$

**Remark 3.55** Quantum integrable systems, as defined in Section 3, come from pseudo-differential operators, and hence are defined for an interval of values of  $\hbar \in ]0, \hbar_0]$ . For this reason, they are obviously  $\hbar$ -continuous, and one can apply Proposition 3.27 to detect their drift. However, the detection of the rotation number in Theorem 3.49 does not use the  $\hbar$ -continuity. As a consequence, it can in principle be applied to quantum integrable systems defined by Berezin-Toeplitz operators on compact, prequantizable symplectic manifolds, using the Bohr-Sommerfeld theory developed in [10].  $\triangle$

We hope that the formalism of asymptotic lattices, that we have tried to develop here in a precise way, independently of any particular quantization scheme, should help attacking other inverse problems for quantum integrable systems. For instance, our analysis is already used in [45] to solve the inverse spectral conjecture for semitoric systems; it may also prove useful for the geometric quantization of Lagrangian fibrations, extending the first order Bohr-Sommerfeld quantization that is often used instead. However, in order to get a more complete picture of asymptotic lattices vs. joint spectra, one should include singularities of the moment map into the picture.

## References

- [1] V. I. Arnol'd. Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. *Uspehi Mat. Nauk*, 18(5

(113):13–40, 1963.

- [2] V. I. Arnol'd. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, second edition, 1988.
- [3] C. Bernardi, M. Dauge, and Y. Maday. *Spectral methods for axisymmetric domains*, volume 3 of *Series in Applied Mathematics (Paris)*. Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris; North-Holland, Amsterdam, 1999.
- [4] P. Bleher. Distribution of energy levels of a quantum free particle on a surface of revolution. *Duke Math. J.*, 74(1):45–93, 1994.
- [5] P. Bleher, D. Kosygin, and Y. Sinai. Distribution of energy levels of quantum free particle on the Liouville surface and trace formulae. *Commun. Math. Phys.*, 170(2):375–403, 1995.
- [6] P. M. Bleher. Semiclassical quantization rules near separatrices. *Comm. Math. Phys.*, 165(3):621–640, 1994.
- [7] A. V. Bolsinov and A. T. Fomenko. *Integrable Hamiltonian systems; Geometry, topology, classification*. Chapman & Hall, 2004. Translated from the 1999 Russian original.
- [8] H. Broer, K. Efsthathiou, and O. Lukina. A geometric fractional monodromy theorem. *Discrete Contin. Dyn. Syst. Ser. S*, 3(4):517–532, 2010.
- [9] A.-M. Charbonnel. Comportement semi-classique du spectre conjoint d'opérateurs pseudo-différentiels qui commutent. *Asymptotic Analysis*, 1:227–261, 1988.
- [10] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10):1527–1566, 2003.
- [11] L. Charles, Á. Pelayo, and S. Vũ Ngọc. Isospectrality for quantum toric integrable systems. *Ann. Sci. École Norm. Sup.*, 43:815–849, 2013.
- [12] Y. Colin de Verdière. Quasi-modes sur les variétés riemanniennes. *Invent. Math.*, 43:15–52, 1977.
- [13] Y. Colin de Verdière. Spectre conjoint d'opérateurs pseudo-différentiels qui commutent I. *Duke Math. J.*, 46(1):169–182, 1979.
- [14] Y. Colin de Verdière. Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. *Comment. Math. Helv.*, 54:508–522, 1979.
- [15] Y. Colin de Verdière. Spectre conjoint d'opérateurs pseudo-différentiels qui commutent II. *Math. Z.*, 171:51–73, 1980.

- [16] Y. Colin de Verdière and B. Parisse. Équilibre instable en régime semi-classique I : Concentration microlocale. *Comm. Partial Differential Equations*, 19(9–10):1535–1563, 1994.
- [17] Y. Colin de Verdière and B. Parisse. Équilibre instable en régime semi-classique II : Conditions de Bohr-Sommerfeld. *Ann. Inst. H. Poincaré. Phys. Théor.*, 61(3):347–367, 1994.
- [18] Y. Colin de Verdière and S. Vũ Ngọc. Singular Bohr-Sommerfeld rules for 2D integrable systems. *Ann. Sci. École Norm. Sup. (4)*, 36:1–55, 2003.
- [19] R. Cushman and L. Bates. *Global aspects of classical integrable systems*. Birkhäuser, 1997.
- [20] R. Cushman and J. J. Duistermaat. The quantum spherical pendulum. *Bull. Amer. Math. Soc. (N.S.)*, 19:475–479, 1988.
- [21] R. Cushman, H. Dullin, A. Giacobbe, D. Holm, M. joyeux, P. Lynch, D. Sadovskii, and B. Zhilinskii. The CO<sub>2</sub> molecule as a quantum realization of the 1 : 1 : 2 resonant swing-spring with monodromy. *Phys. Rev. Lett.*, 93(24302), 2004.
- [22] K. Datchev and H. Hezari. Inverse problems in spectral geometry. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 455–485. Cambridge Univ. Press, Cambridge, 2013.
- [23] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.
- [24] S. Dozias. Clustering for the spectrum of  $h$ -pseudodifferential operators with periodic flow on an energy surface. *J. Funct. Anal.*, 145(2):296–311, 1997.
- [25] E. B. Dryden, V. Guillemin, and R. Sena-Dias. Equivariant inverse spectral theory and toric orbifolds. *Advances in Mathematics*, 231(3):1271–1290, 2012.
- [26] J. J. Duistermaat. On global action-angle variables. *Comm. Pure Appl. Math.*, 33:687–706, 1980.
- [27] J. J. Duistermaat and V. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29:39–79, 1975.
- [28] H. R. Dullin. Semi-global symplectic invariants of the spherical pendulum. *J. Differential Equations*, 254(7):2942–2963, 2013.
- [29] A. Giacobbe. Fractional monodromy: parallel transport of homology cycles. *Differential Geometry and its Applications*, 26(2):140–150, 2008.
- [30] M. A. Hall. Diophantine tori and non-selfadjoint inverse spectral problems. *Math. Res. Lett.*, 20(2):255–271, 2013.

- [31] M. A. Hall, M. Hitrik, and J. Sjöstrand. Spectra for semiclassical operators with periodic bicharacteristics in dimension two. *Int. Math. Res. Not.*, 20:10243–10277, 2015.
- [32] K. Hamraoui, L. Van Damme, P. Mardešić, and D. Sugny. Classical and quantum rotation numbers of asymmetric-top molecules. *Phys. Rev. A*, 97:032118, Mar 2018.
- [33] M. Hitrik and Sjöstrand. Rational invariant tori and band edge spectra for non-selfadjoint operators. *J. Eur. Math. Soc.*, 20(2):391–457, 2018.
- [34] M. Hitrik and J. Sjöstrand. Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions. I. *Ann. Henri Poincaré*, 5(1):1–73, 2004.
- [35] M. Hitrik, J. Sjöstrand, and S. Vũ Ngoc. Diophantine tori and spectral asymptotics for non-selfadjoint operators. *Amer. J. Math.*, 169(1):105–182, 2007.
- [36] S. Hohloch, S. Sabatini, D. Sepe, and M. Symington. Faithful semitoric systems. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 084, 66, 2018.
- [37] A. Iantchenko, J. Sjöstrand, and M. Zworski. Birkhoff normal forms in semi-classical inverse problems. *Math. Res. Lett.*, 9(2-3):337–362, 2002.
- [38] M. Joyeux, D. A. Sadovskii, and J. Tennyson. Monodromy of the LiNC/NCLi molecule. *Chemical Physics Letters*, 382:439–442, 2003.
- [39] Y. Karshon. Periodic hamiltonian flows on four dimensional manifolds. *Mem. Amer. Math. Soc.*, 672, 1999.
- [40] M. Kontsevich and Y. Soibelman. Affine structures and non-Archimedean analytic spaces. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 321–385. Birkhäuser Boston, Boston, MA, 2006.
- [41] V. F. Lazutkin. *KAM theory and semiclassical approximations to eigenfunctions*, volume 24 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1993. With an addendum by A. I. Shnirelman.
- [42] Y. Le Floch. *A brief introduction to Berezin-Toeplitz operators on compact Kähler manifolds*. CRM Short Courses. Springer, Cham, 2018.
- [43] Y. Le Floch and A. Pelayo. Symplectic geometry and spectral properties of classical and quantum coupled angular momenta. *J. Nonlinear Sci.*, 29(2):655–708, 2019.
- [44] Y. Le Floch, Á. Pelayo, and S. Vũ Ngoc. Inverse spectral theory for semiclassical Jaynes-Cummings systems. *Math. Ann.*, 364(3):1393–1413, 2016.
- [45] Y. Le Floch and S. Vũ Ngoc. The inverse spectral problem for quantum semitoric systems. Preprint arXiv 2104.06704 (105p), 2021.

- [46] N. C. Leung and M. Symington. Almost toric symplectic four-manifolds. *J. Symplectic Geom.*, 8:143–187, 2010.
- [47] A. Martinez. *An introduction to semiclassical and microlocal analysis*. Universitext. Springer-Verlag, New York, 2002.
- [48] C. März. Spectral asymptotics for Hill’s equation near the potential maximum. *Asymptotic Analysis*, 5:221–267, 1992.
- [49] H. Mineur. Sur les systèmes mécaniques dans lesquels figurent des paramètres fonctions du temps. Étude des systèmes admettant  $n$  intégrales premières uniformes en involution. Extension à ces systèmes des conditions de quantification de Bohr-Sommerfeld. *J. Ecole Polytechn.*, III(Cahier 1, Fasc. 2 et 3):173–191, 237–270, 1937.
- [50] J. Palmer, Á. Pelayo, and X. Tang. Semitoric systems of non-simple type. arXiv:1909.03501, 2019.
- [51] Á. Pelayo, T. Ratiu, and S. Vũ Ngọc. The affine invariant of generalized semitoric systems. *Nonlinearity*, 30(11):3993–4028, 2017.
- [52] Á. Pelayo and S. Vũ Ngọc. Semitoric integrable systems on symplectic 4-manifolds. *Invent. Math.*, 177(3):571–597, 2009.
- [53] Á. Pelayo and S. Vũ Ngọc. Symplectic theory of completely integrable Hamiltonian systems. *Bull. Amer. Math. Soc. (N.S.)*, 48(3):409–455, 2011.
- [54] Á. Pelayo and S. Vũ Ngọc. First steps in symplectic and spectral theory of integrable systems. *Discrete Contin. Dyn. Syst.*, 32(10):3325–3377, 2012.
- [55] Á. Pelayo and S. Vũ Ngọc. Semiclassical inverse spectral theory for singularities of focus-focus type. *Commun. Math. Phys.*, 329(2):809–820, 2014.
- [56] Á. Pelayo and S. Vũ Ngọc. Constructing integrable systems of semitoric type. *Acta Math.*, 206:93–125, 2011.
- [57] Q.-S. Phan. Spectral monodromy of non-self-adjoint operators. *J. Math. Phys.*, 55:013504, 2014.
- [58] J. Pöschel. A lecture on the classical KAM theorem. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 707–732. Amer. Math. Soc., Providence, RI, 2001.
- [59] T. Ratiu, C. Wacheux, and N. T. Zung. Convexity of singular affine structures and toric-focus integrable hamiltonian systems. arXiv:1706.01093.



- [60] D. Sadovskii and B. Zhilinskiĭ. Monodromy, diaboloic points, and angular momentum coupling. *Phys. Lett. A*, 256(4):235–244, 1999.
- [61] D. Sepe and S. Vũ Ngọc. Integrable systems, symmetries, and quantization. *Lett. Math. Phys.*, 108(3):499–571, 2017.
- [62] M. Symington. Four dimensions from two in symplectic topology. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 153–208. Amer. Math. Soc., Providence, RI, 2003.
- [63] J. Toth. Various quantum mechanical aspects of quadratic forms. *Jour. Funct. Analysis*, 130:1–42, 1995.
- [64] S. Vũ Ngọc. Symplectic inverse spectral theory for pseudodifferential operators. In *Geometric aspects of analysis and mechanics*, volume 292 of *Progr. Math.*, pages 353–372. Birkhäuser/Springer, New York, 2011.
- [65] S. Vũ Ngọc. Quantum monodromy in integrable systems. *Commun. Math. Phys.*, 203(2):465–479, 1999.
- [66] S. Vũ Ngọc. Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. *Comm. Pure Appl. Math.*, 53(2):143–217, 2000.
- [67] S. Vũ Ngọc. Quantum monodromy and bohr–sommerfeld rules. *Letters in Mathematical Physics*, 55(3):205–217, 2001.
- [68] S. Vũ Ngọc. *Systèmes intégrables semi-classiques: du local au global*. Number 22 in *Panoramas et Synthèses*. SMF, 2006.
- [69] S. Vũ Ngọc. Moment polytopes for symplectic manifolds with monodromy. *Adv. in Math.*, 208:909–934, 2007.
- [70] D. R. Yafaev. Passage through a potential barrier and multiple wells. *Algebra i Analiz*, 29(2):242–273, 2017. Reprinted in *St. Petersburg Math. J.* **29** (2018), no. 2, 399–422.
- [71] S. Zelditch. The inverse spectral problem for surfaces of revolution. *J. Differential Geom.*, 49:207–264, 1998.
- [72] N. T. Zung. Symplectic topology of integrable hamiltonian systems, I: Arnold-Liouville with singularities. *Compositio Math.*, 101:179–215, 1996.
- [73] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.