Unbiased truncated quadratic variation for volatility estimation in jump diffusion processes

Chiara Amorino, Arnaud Gloter

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April 19, 2019

Abstract

The problem of integrated volatility estimation for the solution X of a stochastic differential equation with Lévy-type jumps is considered under discrete high-frequency observations in both short and long time horizon. We provide an asymptotic expansion for the integrated volatility that gives us, in detail, the contribution deriving from the jump part. The knowledge of such a contribution allows us to build an unbiased version of the truncated quadratic variation, in which the bias is visibly reduced. In earlier results the condition $\beta > \frac{1}{2(2-\alpha)}$ on β (that is such that $(\frac{1}{n})^\beta$ is the threshold of the truncated quadratic variation) and on the degree of jump activity α was needed to have the original truncated realized volatility well-performed (see [22], [13]). In this paper we theoretically relax this condition and we show that our unbiased estimator achieves excellent numerical results for any couple (α, β).

Lévy-driven SDE, integrated variance, threshold estimator, convergence speed, high frequency data.

1 Introduction

The class of solutions of Lévy-driven stochastic differential equations has many applications in various area such as neuroscience, physics and finance. Indeed, it includes the stochastic Morris-Lecar neuron model [10] as well as important examples taken from finance such as the Barndorff-Nielsen-Shephard model [4], the Kou model [19] and the Merton model [24]; to name just a few.
In this work we aim at estimating the integrated volatility in short and long time based on discrete observations $X_{t_0}, ..., X_{t_n}$; with $t_0 = 0 \leq t_1 \leq ... \leq t_n = T_n$, of the process X given by

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz), \quad t \in \mathbb{R}_+,$$

where $W = (W_t)_{t\geq 0}$ is a one dimensional Brownian motion and $\tilde{\mu}$ is a compensated Poisson random measure, with a possible infinity jump activity.

We consider here the setting of high frequency observations , i.e. $\Delta_n = \sup_{i=0,...,n-1} \Delta_{n,i} \to 0$ as $n \to \infty$, with $\Delta_{n,i} = (t_{i+1} - t_i)$. Both cases $T_n \in [0, \infty[$ fixed and $\lim_{n\to\infty} T_n = \infty$ are dealt and so we want to estimate, respectively, $IV_1 := \frac{1}{T}\int_0^T a^2(X_s)f(X_s)ds$ and $IV_2 := \int_{\mathbb{R}} a^2(x)f(x)\pi(dx)$, where π is an invariant measure and f a polynomial growth function. If on one side the estimation of IV_2, to our knowledge, has never been considered before, on the other the estimation of IV_1 has been widely studied because of its great importance in finance. Indeed, taking $f \equiv 1$, IV_1 turns out to be the so called integrated volatility that has particular relevance in measuring and forecasting the asset risks; its estimation on the basis of discrete observations of X is one of the long-standing problems.
In the sequel we will present some known results denoting by IV_1 the classical integrated volatility, that is we are assuming that f equals to 1.

When X is continuous, the canonical way for estimating the integrated volatility is to use the realized volatility or approximate quadratic variation at time T:

$$[X, X]_T^n := \sum_{i=0}^{n-1} (\Delta X_i)^2, \qquad \text{where } \Delta X_i = X_{t_{i+1}} - X_{t_i}.$$

Under very weak assumptions on b and a (namely when $\int_0^T b^2(X_s)ds$ and $\int_0^T a^4(X_s)ds$ are finite for all $t \in (0, T]$), we have a central limit theorem (CLT) with rate $\sqrt{n}$: the processes $\sqrt{n}([X, X]_T^n - IV_1)$ converge in the sense of stable convergence in law for processes, to a limit Z which is defined on an

*Laboratoire de Mathématiques et Modélisation d'Evry, CNRS, Univ Evry, Université Paris-Saclay, 91037, Evry, France.

extension of the space and which conditionally is a centered Gaussian variable whose conditional law is characterized by its (conditional) variance $V_T := 2\int_0^T a^4(X_s)ds$.

When X has jumps, the variable $[X,X]_T^n$ no longer converges to IV_1. However, there are other known methods to estimate the integrated volatility.

The first type of jump-robust volatility estimators are the *Multipower variations* (cf [5], [6], [14]), which we do not explicitly recall here. These estimators satisfy a CLT with rate $\sqrt{n}$ but with a conditional variance bigger than V_T (so they are rate-efficient but not variance-efficient).

The second type of volatility estimators, introduced by Jacod and Todorov in [16], is based on estimating locally the volatility from the empirical characteristic function of the increments of the process over blocks of decreasing length but containing an increasing number of observations, and then summing the local volatility estimates.

Another method to estimate the integrated volatility in jump diffusion processes, introduced by Mancini in [21], is the use of the *truncated realized volatility* or *truncated quadratic variance* (see [14], [22]):

$$I\hat{V}_T^n := \sum_{i=0}^{n-1}(\Delta X_i)^2 1_{\{|\Delta X_i|\leq v_n\}},$$

where v_n is a sequence of positive truncation levels, typically of the form $(\frac{1}{n})^\beta$ for some $\beta \in (0,\frac{1}{2})$.

Below we focus on the estimation of IV_1 through the implementation of the truncated quadratic variation, that is based on the idea of summing only the squared increments of X whose absolute value is smaller than some threshold v_n.

It is shown in [13] that $I\hat{V}_T^n$ has exactly the same limiting properties as $[X,X]_T^n$ does for some $\alpha \in [0,1)$ and $\beta \in [\frac{1}{2(2-\alpha)},\frac{1}{2})$, where α is the degree of jump activity or Blumenthal-Getoor index, that is the supremum of r for which $\int_{\mathbb{R}}(|z|^r \wedge 1)F(z)dz$ is almost surely finite; F is a Lévy measure which accounts for the jumps of the process and it is such that the compensator $\bar{\mu}$ has the form $\bar{\mu}(dt,dz) = F(z)dzdt$.

Mancini has proved in [22] that, when the jumps of X are those of a stable process with index $\alpha \geq 1$, the truncated quadratic variation is such that

$$(I\hat{V}_T^n - IV_1) \overset{\mathbb{P}}{\sim} (\frac{1}{n})^{\beta(2-\alpha)}. \tag{1}$$

This rate is less than $\sqrt{n}$ and no proper CLT is available in this case.

In this paper, in order to estimate $IV_1 := \frac{1}{T}\int_0^T a^2(X_s)f(X_s)ds$ and $IV_2 := \int_{\mathbb{R}} a^2(x)f(x)\pi(dx)$, we consider in particular the truncated quadratic variation defined in the following way:

$$Q_n := \frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(X_{t_{i+1}} - X_{t_i})^2\varphi_{\Delta_{n,i}^\beta}(X_{t_{i+1}} - X_{t_i}),$$

where φ is a C^∞ function that vanishes when the increments of the data are too large compared to the typical increments of a continuous diffusion process, and thus can be used to filter the contribution of the jumps.

We aim to extend the results proved in short time in [22] characterising precisely the noise introduced by the presence of jumps in both short and long time and finding consequently some corrections to reduce such a noise.

The main result of our paper is the asymptotic expansion for the integrated volatility in short and long time. Compared to earlier results, which exists only in short time case, our asymptotic expansion provides us precisely the limit to which $\frac{1}{\Delta_n^{\beta(2-\alpha)}}(Q_n - IV_1)$ converges when $\Delta_n^{\beta(2-\alpha)} > \sqrt{n}$, that in case of uniform discretization steps (for which $\Delta_n = \frac{T}{n}$) matches with the condition $\beta < \frac{1}{2(2-\alpha)}$.

In thhe case where the discretization step is uniform our work extends [22]. Indeed, we find

$$Q_n - IV_1 = \frac{Z_n}{\sqrt{n}} + (\frac{1}{n})^{\beta(2-\alpha)}c_\alpha\int_{\mathbb{R}}\varphi(u)|u|^{1-\alpha}du\int_0^T|\gamma|^\alpha(X_s)f(X_s)ds + o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)}),$$

where $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a^4(X_s)f^2(X_s)ds)$ stably with respect to X. In Theorem 3 and 4 below the result is extended to non uniform sampling step as well. The asymptotic expansion here above allows us to deduce the behaviour of the truncated quadratic variation for each couple (α,β), that is a plus compared to (1).

Furthermore, providing we know α (and if we don't it is enough to estimate it previously, see for example [28]), we can improve the performance of the truncated quadratic variation subtracting the noise due to the presence of jumps to the original estimator or taking particular functions φ that make the bias derived from the jump part equal to zero. Using the asymptotic expansion of the integrated volatility we also provide the rate of the error left after having applied the corrections.

Moreover, in the case where the volatility is constant, we show numerically that the corrections gained by the knowledge of the asymptotic expansion for the integrated volatility in short time allows us to reduce visibly the noise for any $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$. It is a clear improvement because, if the original truncated quadratic variation was a well-performed estimator only if $\beta > \frac{1}{2(2-\alpha)}$ (condition that never holds for $\alpha \geq 1$), the unbiased truncated quadratic variation achieves excellent results for any couple (α, β).

The outline of the paper is the following. In Section 2 we present the assumptions on the process X. In Section 3.1 we define the truncated quadratic variation, while Section 3.2 contains the main results of the paper. In Section 4 we show the numerical performance of the unbiased estimator. The Section 5 is devoted to the statement of propositions useful for the proof of the main results, that is given in Section 6. In Section 7 we give some technical tools about Malliavin calculus, required for the proof of some propositions, while other proofs and some technical results are presented in the Appendix.

2 Model, assumptions

Let X be a solution to

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz), \quad t \in \mathbb{R}_+, \tag{2}$$

where $W = (W_t)_{t\geq 0}$ is a one dimensional Brownian motion and μ is a Poisson random measure on $[0, \infty) \times \mathbb{R}$ associated to the Lévy process $L = (L_t)_{t\geq 0}$, with $L_t := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}(ds, dz)$. The compensated measure is $\tilde{\mu} = \mu - \bar{\mu}$; we suppose that the compensator has the following form: $\bar{\mu}(dt, dz) := F(dz)dt$, where conditions on the Levy measure F will be given later.
We denote $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which W and μ are defined. The initial condition X_0, W and L are independent.

2.1 Assumptions

We suppose that the functions $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

ASSUMPTION 1: *The functions $b(x)$, $\gamma(x)$ and $a(x)$ are globally Lipschitz.*

Under Assumption 1 the equation (2) admits a unique non-explosive càdlàg adapted solution possessing the strong Markov property, cf [3] (Theorems 6.2.9. and 6.4.6.).

ASSUMPTION 2: *There exists a constant $t > 0$ such that X_t admits a density $p_t(x, y)$ with respect to the Lebesgue measure on $\mathbb{R}$; bounded in $y \in \mathbb{R}$ and in $x \in K$ for every compact $K \subset \mathbb{R}$. Moreover, for every $x \in \mathbb{R}$ and every open ball $U \in \mathbb{R}$, there exists a point $z = z(x, U) \in supp(F)$ such that $\gamma(x)z \in U$.*

The last assumption was used in [23] to prove the irreducibility of the process X. Other sets of conditions, sufficient for irreducibility, can be found in the same source.

ASSUMPTION 3 (Ergodicity):

1. *For all $q > 0$, $\int_{|z|>1} |z|^q F(z)dz < \infty$.*
2. *There exists $C > 0$ such that $xb(x) \leq -C|x|^2$, if $|x| \to \infty$.*
3. *$|a(x)|/|x| \to 0$ as $|x| \to \infty$.*
4. *$\forall q > 0$ we have $\mathbb{E}|X_0|^q < \infty$.*

The points 2 - 3 and 4 of the Assumption 3 here above are required only in the case of long time observation.
Assumption 2 ensures, together with the Assumption 3 and the fifth point of Assumption 4 below, the existence of unique invariant distribution π, as well as the ergodicity of the process X, as stated in the Lemma 2 below.

ASSUMPTION 4 (Jumps):

1. *The jump coefficient γ is bounded from below, that is*

$$\inf_{x\in\mathbb{R}} |\gamma(x)| := \gamma_{min} > 0$$

2. *The Lévy measure F is absolutely continuous with respect to the Lebesgue measure and we denote $F(z) = \frac{F(dz)}{dz}$.*

3. *The Lévy measure F satisfies $F(dz) = \frac{g(z)}{|z|^{1+\alpha}}dz$, where $\alpha \in (0,2)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous symmetric nonnegative bounded function with $g(0) = 1$.*

4. *The function g is differentiable on $\{0 < |z| \leq \eta\}$ for some $\eta > 0$ with continuous derivative such that $\sup_{0<|z|\leq\eta} |\frac{g'}{g}| < \infty$.*

5. *The jump coefficient γ is upper bounded, i.e. $\sup_{x\in\mathbb{R}} |\gamma(x)| := \gamma_{max} < \infty$.*

Assumptions 4.1 and 4.5 are useful to compare size of jumps of X and L. Assumption 4.4 is satisfied by a large class of processes: α- stable process ($g = 1$), truncated α-stable processes ($g = \tau$, a truncation function), tempered stable process ($g(z) = e^{-\lambda|z|}$, $\lambda > 0$).

We will use some moment inequalities for jump diffusions, gathered in the following lemma:

Lemma 1. *Let X satisfies Assumptions 1-4. Let $L_t := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}(ds, dz)$ and let $\mathcal{F}_s := \sigma\{(W_u)_{0<u\leq s}, (L_u)_{0<u\leq s}, X_0\}$. Then, for all $t > s$,*
1) for all $p \geq 2$, $\mathbb{E}[|X_t - X_s|^p]^{\frac{1}{p}} \leq c|t-s|^{\frac{1}{p}}$,
2) for all $p \geq 2$, $p \in \mathbb{N}$, $\mathbb{E}[|X_t - X_s|^p|\mathcal{F}_s] \leq c|t-s|(1 + |X_s|^p)$.
3) for all $p \geq 2$, $p \in \mathbb{N}$, $\sup_{h\in[0,1]} \mathbb{E}[|X_{s+h}|^p|\mathcal{F}_s] \leq c(1 + |X_s|^p)$.
4) for all $p > 1$, $\mathbb{E}[|X_t^c - X_s^c|^p]^{\frac{1}{p}} \leq |t-s|^{\frac{1}{2}}$ and $\mathbb{E}[|X_t^c - X_s^c|^p|\mathcal{F}_s]^{\frac{1}{p}} \leq c|t-s|^{\frac{1}{2}}(1 + |X_s|^p)$,
where we have denoted by X^c the continuous part of the process X.

The first two points follow from Theorem 66 of [26] and Proposition 3.1 in [27]. The third point is showed in [2], below Lemma 1, and the last one in Section 8 of [12].

The following Lemma states that Assumptions $1-4$ are sufficient for the existence of an invariant measure π such that an ergodic theorem holds and moments of all order exist.

Lemma 2. *Under assumptions 1 to 4, the process X admits a unique invariant distribution π and the ergodic theorem holds:*

1. *For every measurable function $h : \mathbb{R} \to \mathbb{R}$ satisfying $\pi(h) < \infty$, we have a.s.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t h(X_s)ds = \pi(h).$$

2. *For all $q > 0$, $\pi(|x|^q) < \infty$.*

3. *For all $q > 0$, $\sup_{t\geq 0} \mathbb{E}[|X_t|^q] < \infty$.*

A proof is in [12] (Section 8 of Supplement) in the case $\alpha \in (0,1)$ and the proof relies on [23]. The case $\alpha \in (0,2)$ is dealt in [2].

3 Setting and main results

Let X be the solution to (2). Suppose that we observe a finite sample

$$X_{t_0}, ..., X_{t_n}; \qquad 0 = t_0 \leq t_1 \leq ... \leq t_n = T.$$

Every observation time point depends also on n, but to simplify the notation we suppress this index. We will be working in a high-frequency setting, i.e.

$$\Delta_n := \sup_{i=0,\dots,n-1} \Delta_{n,i} \longrightarrow 0, \quad n \to \infty,$$

with $\Delta_{n,i} := (t_{i+1} - t_i)$.
We study both the cases $T \in \mathbb{R}$ fixed and $\lim_{n\to\infty} T = \infty$.

We denote by IV_1 the quantity $\frac{1}{T}\int_0^T a^2(X_s)f(X_s)ds$ and by IV_2 $\int_{\mathbb{R}} a^2(x)f(x)\pi(dx)$, where π is the invariant measure introduced in Lemma 2 and f a polynomial growth function.

In order to estimate IV_1 and IV_2 we introduce Q_n, based on the idea of summing only some of the squared increments of X, those whose absolute value is smaller than $2\Delta_{n,i}^{\beta}$, with $\beta \in (0, \frac{1}{2})$. Indeed, we set

$$Q_n := \frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}}(X_{t_{i+1}} - X_{t_i})^2 \varphi_{\Delta_{n,i}^{\beta}}(X_{t_{i+1}} - X_{t_i}), \tag{3}$$

where

$$\varphi_{\Delta_{n,i}^{\beta}}(X_{t_{i+1}} - X_{t_i}) = \varphi(\frac{X_{t_{i+1}} - X_{t_i}}{\Delta_{n,i}^{\beta}}),$$

with φ a smooth version of the indicator function, such that $\varphi(\zeta) = 0$ for each ζ, with $|\zeta| \geq 2$ and $\varphi(\zeta) = 1$ for each ζ, with $|\zeta| \leq 1$.
It is worth noting that, if we consider an additional constant k in φ (that becomes $\varphi_{k\Delta_{n,i}^{\beta}}(X_{t_{i+1}} - X_{t_i}) = \varphi(\frac{X_{t_{i+1}} - X_{t_i}}{k\Delta_{n,i}^{\beta}})$), the only difference is the interval on which the function is 1 or 0: it will be 1 for $|X_{t_{i+1}} - X_{t_i}| \leq k\Delta_{n,i}^{\beta}$; 0 for $|X_{t_{i+1}} - X_{t_i}| \geq 2k\Delta_{n,i}^{\beta}$. Hence, for shortness in notations, we restrict the theoretical analysis to the situation where $k = 1$ while, for applications, we may take the threshold level as $k\Delta_{n,i}^{\beta}$ with $k \neq 1$.

3.1 Conditions on the step discretization

In this paragraph we introduce all the assumptions on the step discretization that we will need and we will use, a little at a time, in the proofs of the main results.
We consider both the cases T fixed and $\lim_{n\to\infty} T = \infty$.

ASSUMPTION S1 (Step Discretization, T fixed):

1. There exists a measurable function $s \mapsto H(s, 0)$ such that for all function h continuous and bounded,

$$\eta_{sampling}^{(n)}(h) = \frac{1}{n}\sum_{i=0}^{n-1} h(X_{t_i}) \to \eta(h) = \int_0^T h(X_s)H(s, 0)ds.$$

2. For $\delta \in [0, 1)$, there exists a measurable function $s \mapsto H(s, \delta)$ such that, for every continuous function $h : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\Delta_n^{\delta}}\frac{1}{n}\sum_{i=0}^{n-1} h(X_{t_i})\Delta_{n,i}^{\delta} \to \int_0^T h(X_s)H(s, \delta)ds \tag{4}$$

3. $\exists \delta_0 > 0 : |\frac{1}{n} - \frac{\Delta_{n,i}}{\sum_{i=0}^{n-1}\Delta_{n,i}}| \leq \frac{\Delta_n^{\beta(2-\alpha)+\delta_0}}{n}$, $\forall i \in \{0, ..., n-1\}$, for $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$.

We observe that, for $\delta = 0$, the point 2 fall back into point 1. It is therefore a condition stronger than the first one, but it is not always required. Conditions on the sampling step analogous to those stated in first and second points are introduced in Section 2.6 of Mykland and Zhang [25], related to the existence of quadratic variation in time (see also Example 2.24 in [17]).
We remark that, considering a uniform steps discretization, the three conditions here above clearly hold.

ASSUMPTION S2 (Step Discretization, $T \to \infty$):

1. $\exists\ c_1, c_2 > 0$ such that $c_1 \leq \frac{\max_{i\in\{0,...,n-1\}}\Delta_{n,i}}{\min_{i\in\{0,...,n-1\}}\Delta_{n,i}} \leq c_2$.

2. For $\delta \in [0, 1)$ there exists c_δ such that $\forall n$, $(\min_{i\in\{0,...,n-1\}}\Delta_{n,i})^{1-\delta}\sum_{j=1}^{n}|\frac{1}{\Delta_{n,j}^{1-\delta}} - \frac{1}{\Delta_{n,j-1}^{1-\delta}}| < c_\delta$.

3. $\exists \delta_0 > 0 : |\frac{1}{n} - \frac{\Delta_{n,i}}{\sum_{i=0}^{n-1}\Delta_{n,i}}| \leq \frac{\Delta_n^{\beta(2-\alpha)+\delta_0}}{n}$, $\forall i \in \{0, ..., n-1\}$, for $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$.

Again, if we consider a uniform discretization, the three conditions here above hold.
The second point is an assumption of regularity on the function $j \mapsto \Delta_{n,j}^{\delta-1}$. It comes naturally from the proof of the lemma below.
We observe that, when $T \to \infty$, it doesn't make sens to add a condition as (4) because its left hand side always converges to the same quantity for all $\delta \in [0, 1)$, as consequence of the following lemma, that we will prove in the appendix:

Lemma 3. *Suppose that Assumptions 1 to 4 and the points 1 and 2 of S2 hold. Then, for every measurable function $h : \mathbb{R} \to \mathbb{R}$ with bounded derivative such that $\pi(h) < \infty$ and for $\delta \in [0, 1)$ we have the following convergence in probability:*

$$\frac{1}{\sum_{i=0}^{n-1} \Delta_{n,i}^{\delta}} \sum_{i=0}^{n-1} \Delta_{n,i}^{\delta} h(X_{t_i}) \xrightarrow{\mathbb{P}} \int_{\mathbb{R}} h(x)\pi(dx). \tag{5}$$

3.2 Main results

3.2.1 Decomposition of the truncated quadratic variation

In this section we enunciate theorems that explain the asymptotic behavior of Q_n. First of all we define

$$\tilde{Q}_n := \frac{1}{n\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} \Big(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz) \Big)^2 \frac{f(X_{t_i})}{\Delta_{n,i}} \varphi_{\Delta_{n,i}^{\beta}}(X_{t_{i+1}} - X_{t_i}). \tag{6}$$

in order to write the decomposition of the truncated quadratic variation into two parts: the continuous quadratic variation and the quadratic variation deriving from jumps.

Theorem 1. *Suppose that Assumptions 1 to 4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in definition (3) and in the third point of Assumption 4, respectively. Then, as $\Delta_n \to 0$,*

$$Q_n = \frac{1}{n} \sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (X_{t_{i+1}}^c - X_{t_i}^c)^2 + \Delta_n^{\beta(2-\alpha)} \tilde{Q}_n + \mathcal{E}_n = \tag{7}$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} \Big(\int_{t_i}^{t_{i+1}} a(X_s) dW_s \Big)^2 + \Delta_n^{\beta(2-\alpha)} \tilde{Q}_n + \mathcal{E}_n, \tag{8}$$

where X_s^c is the continuous part of the process X_s, $\mathcal{E}_n$ is both $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and, for each $\tilde{\epsilon} > 0$, $o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$; with $o_{\mathbb{P}}(\Delta_n^k)$ such that $\frac{o_{\mathbb{P}}(\Delta_n^k)}{\Delta_n^k} \xrightarrow{\mathbb{P}} 0$.

We now consider the difference between the truncated quadratic variation and the discretized volatility and we make explicit its decomposition into the statistical error and the noise term due to the jumps. To do that, we introduce

$$\hat{Q}_n := \frac{1}{n\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} f(X_{t_i}) \gamma^2(X_{t_i}) \Delta_{n,i}^{\frac{2}{\alpha}-1} d(\gamma(X_{t_i}) \Delta_{n,i}^{\frac{1}{\alpha}-\beta}), \tag{9}$$

where $d(\zeta) := \mathbb{E}[(S_1^{\alpha})^2 \varphi(S_1^{\alpha}\zeta)]$; $(S_t^{\alpha})_{t\geq 0}$ is an α-stable process.

Theorem 2. *Suppose that Assumptions 1 to 4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in Definition 3 and in the third point of Assumption 4, respectively. Then, as $\Delta_n \to 0$,*

$$Q_n - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i}) a^2(X_{t_i}) = \frac{Z_n}{\sqrt{n}} + \Delta_n^{\beta(2-\alpha)} \hat{Q}_n + \mathcal{E}_n, \tag{10}$$

where $\mathcal{E}_n$ is always $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and, adding the condition $\beta > \frac{1}{4-\alpha}$, it is also $o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$. Moreover,

1. *If T is fixed we suppose moreover that point 1 of Assumption S1 holds, then Z_n here above is such that $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a^4(X_s) f^2(X_s) H(s, 0) ds)$ stably with respect to X.*
2. *If otherwise we are in the case $\lim_{n\to\infty} T = \infty$, we suppose that points 1 and 2 of Assumption S2 hold. In this case $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_{\mathbb{R}} a^4(x) f^2(x) \pi(dx))$.*

We recognize in the expansion (10) the statistical error of model without jumps given by Z_n, whose variance is equal to the so called quadricity. The term $\hat{Q}_n$ is a bias term arising from the presence of jumps and given by (9). From this explicit expression it is possible to remove the bias term (see Section 4).
The term $\mathcal{E}_n$ is an additional error term that is always negligible compared to the bias deriving from the jump part $\Delta_n^{\beta(2-\alpha)}\hat{Q}_n$ (that is of order $\Delta_n^{\beta(2-\alpha)}$ by Theorems 3 and 4 below). It also gives us an upper bound to the order of the error we get after having removed the bias. In particular, if $\alpha\beta$ is small enough (that is $\alpha\beta < \frac{1}{2}$), we get that the error term $\mathcal{E}_n$ is $o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}})$ and so it is upper bounded by a term whose order is roughly the same as the statistical error's one.
The bias term admits a first order expansion that does not require the knowledge of the density of S^{α}.

Proposition 1. *Suppose that Assumptions 1 to 4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in Definition 3 and in the third point of Assumption 4, respectively. Then*

$$\hat{Q}_n = \frac{1}{n\Delta_n^{\beta(2-\alpha)}} c_\alpha \sum_{i=0}^{n-1} f(X_{t_i})|\gamma|^\alpha(X_{t_i})\Delta_{n,i}^{\beta(2-\alpha)}(\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du) + \tilde{\mathcal{E}}_n, \tag{11}$$

with

$$c_\alpha = \begin{cases} \frac{\alpha(1-\alpha)}{4\Gamma(2-\alpha)\cos(\frac{\alpha\pi}{2})} & \text{if } \alpha \neq 1,\ \alpha < 2 \\ \frac{1}{2\pi} & \text{if } \alpha = 1. \end{cases} \tag{12}$$

$\tilde{\mathcal{E}}_n = o_{\mathbb{P}}(1)$ *and, if* $\alpha < \frac{4}{3}$, *it is also* $\frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}) = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})})$.

We underline that we have not replaced directly the right hand side of (11) in (10), observing that $\Delta_n^{\beta(2-\alpha)}\tilde{\mathcal{E}}_n = \mathcal{E}_n$, because $\Delta_n^{\beta(2-\alpha)}\tilde{\mathcal{E}}_n$ is always $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ but to get it is also $o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$ the additional condition $\alpha < \frac{4}{3}$ is required.
In the case $\alpha < \frac{4}{3}$ we get the following corollary:

Corollary 1. *Suppose that Assumptions 1 to 4 and point 1 of Assumption S1 (or points 1 and 2 of Assumption S2, if* $\lim_{n\to\infty} T = \infty$*) hold and that* $\alpha \in (0, \frac{4}{3})$, $\beta \in (\frac{1}{4-\alpha}, (\frac{1}{2\alpha} \wedge \frac{1}{2}))$. *If* φ *is such that* $\int_{\mathbb{R}} |u|^{1-\alpha}\varphi(u)du = 0$ *then,* $\forall \tilde{\epsilon} > 0$,

$$Q_n - \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) = \frac{Z_n}{\sqrt{n}} + o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}), \tag{13}$$

with Z_n *defined as in Theorem 2 here above.*

It is always possible to build a function φ for which the condition here above is respected (see Section 4).
We observe that, if $\alpha \geq \frac{4}{3}$ but $\gamma = k \in \mathbb{R}$, the result still holds choosing φ such that $\int_{\mathbb{R}} u^2\varphi(u)\, f_\alpha(\frac{1}{k}u\Delta_{n,i}^{\beta-\frac{1}{\alpha}})du$ is equal to 0, where f_α is the density of the α-stable process. Indeed, starting from (10), we have that $\hat{Q}_n$ is now zero: by its definition (9) it is equal to $\frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1} f(X_{t_i})k^\alpha\Delta_{n,i}^{\frac{2}{\alpha}-1}\int_{\mathbb{R}} z^2\varphi(zk\Delta_{n,i}^{\frac{1}{\alpha}-\beta})f_\alpha(z)dz =$
$= \frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1} f(X_{t_i})k^{\alpha-3}\Delta_{n,i}^{3\beta-\frac{1}{\alpha}-1}\int_{\mathbb{R}} u^2\varphi(u)f_\alpha(\frac{1}{k}u\Delta_{n,i}^{\beta-\frac{1}{\alpha}})du = 0$, where we have used a change of variable.
Equation (13) gives us the behaviour of the unbiased estimator, that is the truncated quadratic variation after having removed the noise derived from the presence of jumps. Taking α and β as in the corollary here above we also have reduced the error term $\mathcal{E}_n$ to be $o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}})$, which means that after having applied the corrections we get an error that is upper bounded by a term whose order is, in the case of finite time horizon, roughly the same as the statistical error's one.

3.2.2 Asymptotic expansion for the integrated volatility in short and long time

The limits of $\hat{Q}_n$ are given below in both cases T fixed and $T \to \infty$.
When T is fixed we have the following result:

Theorem 3. *Suppose that Assumptions 1 2, 4 and points 1 and 5 of Assumption 3 hold. Moreover we suppose that* T *is fixed and that points 1 and 2 of Assumption S1 hold. Then, as* $\Delta_n \to 0$,

$$\hat{Q}_n \xrightarrow{\mathbb{P}} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du \int_0^T |\gamma(X_s)|^\alpha f(X_s)H(s, \beta(2-\alpha))ds. \tag{14}$$

Moreover, if we add the third point of Assumption S1, we have

$$Q_n - IV_1 = \frac{Z_n}{\sqrt{n}} + \Delta_n^{\beta(2-\alpha)} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du \int_0^T |\gamma(X_s)|^\alpha f(X_s)H(s, \beta(2-\alpha))ds + o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)}), \tag{15}$$

where $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a^4(X_s)f^2(X_s)H(s,0)ds)$ *stably with respect to* X.

It is worth noting that, in both [15] and [22], the integrated volatility estimation in short time is dealt and they show that the truncated quadratic variation has rate $\sqrt{n}$ if $\beta > \frac{1}{2(2-\alpha)}$.
We remark that the jump part is negligible compared to the statistic error if $\Delta_n^{\beta(2-\alpha)} < n^{-\frac{1}{2}}$, it follows the condition $\Delta_n < n^{-\frac{1}{2\beta(2-\alpha)}}$ on the discretization step. If we use, in particular, an uniform step discretization such that $\forall i \in \{0, ..., n-1\}$ $\Delta_{n,i} = \Delta_n = \frac{1}{n}$, then the condition becomes $n^{-1} < n^{-\frac{1}{2\beta(2-\alpha)}}$

and so $\beta > \frac{1}{2(2-\alpha)}$, that is the same condition given in [15].
However, if we take (α, β) for which such a condition doesn't hold, we can still use that we know in detail the noise deriving from jumps to implement corrections that still make the unbiased estimator well-performed (see Section 4).

We also study the asymptotic expansion for the integrated volatility in long time that, to our knowledge, hasn't never been dealt before. We have the following result:

Theorem 4. *Suppose that Assumptions 1 to 4 and points 1 and 2 of Assumption S2 hold. We assume moreover that* $\lim_{n\to\infty} T = \infty$ *and* $n\Delta_n = O(T)$. *Then, as* $\Delta_n \to 0$,

$$\hat{Q}_n \xrightarrow{\mathbb{P}} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du \int_{\mathbb{R}} |\gamma(x)|^\alpha f(x)\pi(dx). \tag{16}$$

Moreover, if we add the third condition of Assumption S2 we have

$$Q_n - \frac{1}{T}\int_0^T f(X_s)a^2(X_s)ds = \frac{Z_n}{\sqrt{n}} + \Delta_n^{\beta(2-\alpha)} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du \int_{\mathbb{R}} |\gamma(x)|^\alpha f(x)\pi(dx) + o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)}), \tag{17}$$

where $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_{\mathbb{R}} a^4(x)f^2(x)\pi(dx))$.

Because of the ergodic theorem, $\frac{1}{T}\int_0^T f(X_s)a^2(X_s)ds$ converges to IV_2, but slowly (with rate $\sqrt{T}$). Anyway for the applications the convergence to IV_2 is not required.

We observe that, if we take a discretization step that is $\Delta_n = n^{-\rho}$, with $\rho \in (0,1)$, the jump part is negligible compared to the statistical error if $n^{-\rho\beta(2-\alpha)} < n^{-\frac{1}{2}}$ and so if $\rho > \frac{1}{2\beta(2-\alpha)}$. Since β is always less than $\frac{1}{2}$ it means that ρ must be more than $\frac{1}{2-\alpha}$ or, equivalently, $\alpha < 2 - \frac{1}{\rho}$.
It is worth noting that smaller is ρ and less choice we have on α. In particular for $\rho < \frac{1}{2}$ there is no α for which the condition here above holds. On the other side, for ρ close to 1, we fall back on the condition $\alpha < 1$.

4 Unbiased estimation in the case of constant volatility

In this section we consider a concrete application of the unbiased volatility estimator in a jump diffusion model and we investigate its numerical performance.
We consider our model (2) in which we assume, in addition, that the functions a and γ are both constants. Suppose that we are given a discrete sample $X_{t_0}, ..., X_{t_n}$ with $t_i = i\Delta_n = \frac{i}{n}$ for $i = 0, ..., n$. We remark that, with such a discretization step, all the points of Assumption S1 and S2 clearly hold.
We now want to analyze the estimation improvement; to do it we compare the classical error committed using the truncated quadratic variation with the unbiased estimation derived by our main results.
We define the estimator we are going to use, in which we have clearly taken $f \equiv 1$ and we have introduced a threshold k in the function φ, so it is

$$Q_n = \frac{1}{n}\sum_{i=0}^{n-1} \frac{(X_{t_{i+1}} - X_{t_i})^2}{\Delta_{n,i}} \varphi_{k\Delta_{n,i}^\beta}(X_{t_{i+1}} - X_{t_i}) = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \varphi_{k\Delta_{n,i}^\beta}(X_{t_{i+1}} - X_{t_i}). \tag{18}$$

If normalized, the error committed estimating the volatility is $E_1 := (Q_n - \sigma^2)\sqrt{n}$.
We start from (11) that in our case, taking into account the presence of k, is

$$\hat{Q}_n = c_\alpha \gamma^\alpha k^{2-\alpha} \left(\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du\right) + \tilde{\mathcal{E}}_n. \tag{19}$$

We now get different methods to make the error smaller.
First of all we can replace (19) in (10) and so we can reduce the error by subtracting a correction term, building the new estimator $Q_n^c := Q_n - \Delta_n^{\beta(2-\alpha)} c_\alpha \gamma^\alpha k^{2-\alpha} (\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du)$. The error committed estimating the volatility with such a corrected estimator is $E_2 := (Q_n^c - \sigma^2)\sqrt{n}$.
Another approach consists of taking a particular function $\tilde{\varphi}$ that makes the main contribution of $\hat{Q}_n$ equal to 0. We define $\tilde{\varphi}(\zeta) = \varphi(\zeta) + c\psi(\zeta)$, with ψ a $\mathcal{C}^\infty$ function such that $\psi(\zeta) = 0$ for each ζ, $|\zeta| \geq 2$ or $|\zeta| \leq 1$. In this way, for any $c \in \mathbb{R} \setminus \{0\}$, $\tilde{\varphi}$ is still a smooth version of the indicator function such that $\tilde{\varphi}(\zeta) = 0$ for each ζ, $|\zeta| \geq 2$ and $\tilde{\varphi}(\zeta) = 1$ for each ζ, $|\zeta| \leq 1$. We can therefore leverage the arbitrariness in c to make the main contribution of $\hat{Q}_n$ equal to zero, choosing $\tilde{c} := \frac{\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du}{\int_{\mathbb{R}} \psi(u)|u|^{1-\alpha} du}$, which is such that $\int_{\mathbb{R}} (\varphi + \tilde{c}\psi(u))|u|^{1-\alpha} du = 0$.

Hence, it is possible to achieve an improved estimation of the volatility by used the truncated quadratic variation $Q_{n,c} := \sum_{i=0}^{n-1}(X_{t_{i+1}} - X_{t_i})^2(\varphi + \tilde{c}\psi)(\frac{X_{t_{i+1}} - X_{t_i}}{k\Delta_{n,i}^{\beta}})$. To make it clear we will analyze the quantity $E_3 := (Q_{n,c} - \sigma^2)\sqrt{n}$.
Another method widely used in numerical analysis to improve the rate of convergence of a sequence is the so-called Richardson extrapolation. We observe that the first term on the right hand side of (19) does not depend on n and so we can just write $\hat{Q}_n = \hat{Q} + \tilde{\mathcal{E}}_n$. Replacing it in (10) we get

$$Q_n = \sigma^2 + \frac{Z_n}{\sqrt{n}} + \frac{1}{n^{\beta(2-\alpha)}}\hat{Q} + \mathcal{E}_n \qquad \text{and}$$

$$Q_{2n} = \sigma^2 + \frac{Z_{2n}}{\sqrt{2n}} + \frac{1}{2^{\beta(2-\alpha)}}\frac{1}{n^{\beta(2-\alpha)}}\hat{Q} + \mathcal{E}_{2n},$$

where we have also used that $\Delta_n^{\beta(2-\alpha)}\tilde{\mathcal{E}}_n = \mathcal{E}_n$. We can therefore use $\frac{Q_n - 2^{\beta(2-\alpha)}Q_{2n}}{1-2^{\beta(2-\alpha)}}$ as improved estimator of σ^2.
We give simulation results for E_1, E_2 and E_3 in the situation where $\sigma = 1$. The given mean and the deviation standard are each based on 500 Monte Carlo samples. We choose to simulate a tempered stable process (that is F satisfies $F(dz) = \frac{e^{-|z|}}{|z|^{1+\alpha}}$) in the case $\alpha < 1$ while, in the interest of computational efficiency, we will exhibit results gained from the simulation of a stable Lévy process in the case $\alpha \geq 1$ ($F(dz) = \frac{1}{|z|^{1+\alpha}}$).
We have taken the smooth functions φ and ψ as below:

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ e^{\frac{1}{3} + \frac{1}{|x|^2 - 4}} & \text{if } 1 \leq |x| < 2 \\ 0 & \text{if } |x| \geq 2 \end{cases} \tag{20}$$

$$\psi_M(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \text{ or } |x| \geq M \\ e^{\frac{1}{3} + \frac{1}{|3-x|^2 - 4}} & \text{if } 1 < |x| \leq \frac{3}{2} \\ e^{\frac{1}{|x|^2 - M} - \frac{5}{21} + \frac{4}{4M^2 - 9}} & \text{if } \frac{3}{2} < |x| < M; \end{cases} \tag{21}$$

choosing opportunely the constant M in the definition of ψ_M we can make its decay slower or faster. We observe that the theoretical results still hold even if the support of $\tilde{\varphi}$ changes as M changes and so it is $[-M, M]$ instead of [-2, 2].
Concerning the constant k in the definition of φ, we fix it equal to 3 in the simulation of the tempered stable process, while its value is 2 in the case $\alpha > 1$, $\beta = 0.2$ and, in the case $\alpha > 1$ and $\beta = 0.49$, it increases as α and γ increase.
The results of the simulations are given in columns 3-6 of Table 1a for $\beta = 0.2$ and in columns 3-6 of Table 1b for $\beta = 0.49$.

α	γ	**Mean** E_1	**Rms** E_1	**Mean** E_2	**Mean** E_3
0.1	1	3.820	3.177	0.831	0.189
	3	5.289	3.388	1.953	-0.013
0.5	1	15.168	9.411	0.955	1.706
	3	14.445	5.726	2.971	0.080
0.9	1	13.717	4.573	4.597	0.311
	3	42.419	6.980	13.664	-0.711
1.2	1	32.507	11.573	0.069	2.137
	3	112.648	21.279	-0.915	0.800
1.5	1	50.305	12.680	0.195	0.923
	3	250.832	27.170	-5.749	3.557
1.9	1	261.066	20.729	-0.530	9.139
	3	2311.521	155.950	-0.304	-35.177

(a) $\beta = 0.2$

α	γ	**Mean** E_1	**Rms** E_1	**Mean** E_2	**Mean** E_3
0.1	1	1.092	1.535	0.307	-0.402
	3	1.254	1.627	0.378	-0.372
0.5	1	2.503	1.690	0.754	-0.753
	3	4.680	2.146	1.651	-0.824
0.9	1	2.909	1.548	0.217	0.416
	3	8.042	1.767	0.620	-0.404
1.2	1	7.649	1.992	-0.944	-0.185
	3	64.937	9.918	-1.692	-2.275
1.5	1	25.713	3.653	-1.697	3.653
	3	218.591	21.871	-4.566	-13.027
1.9	1	238.379	14.860	-6.826	16.330
	3	2357.553	189.231	3.827	-87.353

(b) $\beta = 0.49$

Table 1: Monte Carlo estimates of E_1, E_2 and E_3 from 500 samples. We have here fixed $n = 700$; $\beta = 0.2$ in the first table and $\beta = 0.49$ in the second one.

It appears that the estimation we get using the truncated quadratic variation performs worse as soon as α and γ become bigger (see column 3 in both Tables 1a and 1b). However, after having applied

the corrections, the error seems visibly reduced. A proof of which lies, for example, in the comparison between the error and the root mean square: before the adjustment in both Tables 1a and 1b the third column dominates the fourth one, showing that the bias of the original estimator dominates the standard deviation while, after the implementation of our main results, we get E_2 and E_3 for which the bias is much smaller.
We observe that for $\alpha < 1$, in both cases $\beta = 0.2$ and $\beta = 0.49$, it is possible to choose opportunely M (on which ψ's decay depends) to make the error E_3 smaller than E_2. On the other hand, for $\alpha > 1$, the approach who consists of subtracting the jump part to the error results better than the other, since E_3 is in this case generally bigger than E_2, but to use this method the knowledge of γ is required. It is worth noting that both the approaches used, that lead us respectively to E_2 and E_3, work well for any $\beta \in (0, \frac{1}{2})$.
We recall that, in [15], the condition found on β to get a well-performed estimator was

$$\beta > \frac{1}{2(2-\alpha)}, \tag{22}$$

that is not respected in the case $\beta = 0.2$. Our results match the ones in [15], since the third column in Table 1b (where $\beta = 0.49$) is generally smaller than the third one in Table 1a (where $\beta = 0.2$). We emphasise nevertheless that, comparing columns 5 and 6 in the two tables, there is no evidence of a dependence on β of E_2 and E_3.
The price you pay is that, to implement our corrections, the knowledge of α is request. Such corrections turn out to be a clear improvement also because for α that is less than 1 the original estimator (18) is well-performed only for those values of the couple (α, β) which respect the condition (22) while, for $\alpha \geq 1$, there is no $\beta \in (0, \frac{1}{2})$ for which such a condition can hold. That's the reason why, in the lower part of both Tables 1a and 1b, E_1 is so big.
Using our main results, instead, we get E_2 and E_3 that are always small and so we obtain two corrections which make the unbiased estimator always well-performed without adding any requirement on α or β.

5 Developments in small time

In order to prove our main results we need some developments in small time.

In the sequel, for $\delta \geq 0$, we will denote $R(\Delta_{n,i}^{\delta}, x)$ for any function $R(\Delta_{n,i}^{\delta}, x) = R_{i,n}(x)$, where $R_{i,n} : \mathbb{R} \longrightarrow \mathbb{R}$, $x \mapsto R_{i,n}(x)$ is such that

$$\exists c > 0 \qquad |R_{i,n}(x)| \leq c(1 + |x|^c)\Delta_{n,i}^{\delta} \tag{23}$$

with c independent of i, n.
The functions R represent the term of rest and have the following useful property, consequence of the just given definition:

$$R(\Delta_{n,i}^{\delta}, x) = \Delta_{n,i}^{\delta} R(\Delta_{n,i}^{0}, x). \tag{24}$$

We point out that it does not involve the linearity of R, since the functions R on the left and on the right side are not necessarily the same but only two functions on which the control (23) holds with $\Delta_{n,i}^{\delta}$ and $\Delta_{n,i}^{0}$, respectively.

We now state a proposition in which we prove a bound for the total variation distance between the conditional law of the rescaled Levy process and the α-stable distribution. It will be shown in Section 7.

Proposition 2. *Suppose that Assumptions 1 to 4 hold. Let $(S_t^{\alpha})_{t\geq 0}$ be an α-stable process. Let h be a measurable bounded function such that $\|h\|_{pol} := \sup_{x\in\mathbb{R}}(\frac{|h(x)|}{1+|x|^p}) < \infty$, for some $p \geq 1$, $p \geq \alpha$ hence*

$$|h(x)| \leq \|h\|_{pol} (|x|^p + 1). \tag{25}$$

Moreover we denote $\|h\|_{\infty} := \sup_{x\in\mathbb{R}} |h(x)|$. Then, for any $\epsilon > 0$,

$$|\mathbb{E}[h(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n})] - \mathbb{E}[h(S_1^{\alpha})]| \leq C_\epsilon \Delta_n log(\Delta_n^{-\frac{1}{\alpha}}) \|h\|_{\infty} + C_\epsilon \Delta_n^{\frac{1}{\alpha}} \|h\|_{\infty}^{1-\frac{\alpha}{p}-\epsilon} \|h\|_{pol}^{\frac{\alpha}{p}+\epsilon} log(\Delta_n^{-\frac{1}{\alpha}}) + \tag{26}$$

$$+ C_\epsilon \Delta_n^{\frac{1}{\alpha}} \|h\|_{\infty}^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon} \|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon} log(\Delta_n^{-\frac{1}{\alpha}}) 1_{\{\alpha>1\}},$$

where C_ϵ is a constant independent of n.

Remark 1. *The previous theorem is an extension of Theorem 4.2 in [9], it will be useful when $\|h\|_\infty$ is large, compared to $\|h\|_{pol}$. For instance, it is the case if consider a function $h(x) := |x|^2 1_{|x|\leq M}$ for M large.*

The next proposition will be useful for the proof of main results. It will be shown in the appendix.

Proposition 3. *Suppose that Assumptions 1 to 4 hold. We define, for $i \in \{0, ..., n-1\}$,*

$$\Delta X_i^J := \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz) \qquad \text{and} \qquad \Delta \tilde{X}_i^J := \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{t_i})\, z\, \tilde{\mu}(ds, dz).$$

1. *Then we have*

$$(\Delta X_i^J)^2 \varphi_{\Delta_{n,i}^\beta}(\Delta X_i) = (\Delta \tilde{X}_i^J)^2 \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J) + o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1)}), \tag{27}$$

where $o_{L^1}(\Delta_{n,i}^k)$ is such that $\mathbb{E}_i[|o_{L^1}(\Delta_{n,i}^k)|] = R(\Delta_{n,i}^k, x)$, with the notation $\mathbb{E}_i[.] = \mathbb{E}[.|\mathcal{F}_{t_i}]$, $(\mathcal{F}_{t_i})_{t\in[0,T]}$ has been defined in Lemma 1. Moreover, for each $\tilde{\epsilon} > 0$ and f the function introduced in the definition of Q_n,

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (\Delta X_i^J)^2 \varphi_{\Delta_{n,i}^\beta}(\Delta X_i) = \frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (\Delta \tilde{X}_i^J)^2 \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J) + o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}). \tag{28}$$

2. *We also have*

$$(\int_{t_i}^{t_{i+1}} a(X_s)dW_s)\Delta X_i^J \varphi_{\Delta_{n,i}^\beta}(\Delta X_i) = (\int_{t_i}^{t_{i+1}} a(X_s)dW_s)\Delta \tilde{X}_i^J \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J) + o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1)}) \tag{29}$$

and

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (\int_{t_i}^{t_{i+1}} a(X_s)dW_s)\Delta X_i^J \varphi_{\Delta_{n,i}^\beta}(\Delta X_i) =$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (\int_{t_i}^{t_{i+1}} a(X_s)dW_s)\Delta \tilde{X}_i^J \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J) + o_{L^1}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}). \tag{30}$$

6 Proof of main results

In our proofs, the following lemma will be useful:

Lemma 4. *Let us denote by $\Delta X_i^J := \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz)$ and let $\mathcal{F}_s$ be the filtration defined in Lemma 1. Then*

1. *For each $q \geq 2$ $\exists \epsilon > 0$ such that*

$$\mathbb{E}[|\Delta X_i^J 1_{\{|\Delta X_i^J| \leq 4\Delta_{n,i}^\beta\}}|^q |\mathcal{F}_{t_i}] = R(\Delta_{n,i}^{1+\beta(q-\alpha)}, X_{t_i}) = R(\Delta_{n,i}^{1+\epsilon}, X_{t_i}). \tag{31}$$

$$\mathbb{E}[|\Delta \tilde{X}_i^J 1_{\{|\Delta \tilde{X}_i^J| \leq 4\Delta_{n,i}^\beta\}}|^q |\mathcal{F}_{t_i}] = R(\Delta_{n,i}^{1+\beta(q-\alpha)}, X_{t_i}) = R(\Delta_{n,i}^{1+\epsilon}, X_{t_i}). \tag{32}$$

2. *For each $q \geq 1$ we have*

$$\mathbb{E}[|\Delta X_i^J 1_{\left\{\frac{\Delta_{n,i}^\beta}{4} \leq |\Delta X_i^J| \leq 4\Delta_{n,i}^\beta\right\}}|^q |\mathcal{F}_{t_i}] = R(\Delta_{n,i}^{1+\beta(q-\alpha)}, X_{t_i}). \tag{33}$$

Proof. Reasoning as in Lemma 10 in [2] we easily get (31). Observing that $\Delta \tilde{X}_i^J$ is a particular case of ΔX_i^J where γ is fixed, evaluated in X_{t_i}, it follows that (32) can be obtained in the same way of (31). Using the bound on ΔX_i^J obtained from the indicator function we get that the left hand side of (33) is upper bounded by

$$c\Delta_{n,i}^{\beta q} \mathbb{E}[1_{\left\{\frac{\Delta_{n,i}^\beta}{4} \leq |\Delta X_i^J| \leq 4\Delta_{n,i}^\beta\right\}} |\mathcal{F}_{t_i}] \leq \Delta_{n,i}^{\beta q} R(\Delta_{n,i}^{1-\alpha\beta}, X_{t_i}),$$

where in the last inequality we have used Lemma 11 in [2] on the interval $[t_i, t_{i+1}]$ instead of on $[0, h]$. From property (24) of R we get (33). □

6.1 Proof of Theorem 1.

We observe that, using the dynamic (2) of X and the definition of the continuous part X^c, we have that

$$X_{t_{i+1}} - X_{t_i} = (X^c_{t_{i+1}} - X^c_{t_i}) + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz). \tag{34}$$

Replacing (34) in definition (3) of Q_n we have

$$\begin{aligned} Q_n = &\frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (X^c_{t_{i+1}} - X^c_{t_i})^2 + \frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (X^c_{t_{i+1}} - X^c_{t_i})^2 (\varphi_{\Delta^\beta_{n,i}}(\Delta X_i) - 1) + \\ &+ \frac{2}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (X^c_{t_{i+1}} - X^c_{t_i}) \Big(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz)\Big) \varphi_{\Delta^\beta_{n,i}}(\Delta X_i) + \\ &+ \frac{1}{n}\sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} \Big(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz)\Big)^2 \varphi_{\Delta^\beta_{n,i}}(\Delta X_i) =: \sum_{j=1}^{4} I^n_j. \end{aligned} \tag{35}$$

Comparing (35) with (7), using also definition (6) of $\tilde{Q}_n$, it follows that our goal is to show that $I^n_2 + I^n_3 = \mathcal{E}_n$, that is both $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and $o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$. In the sequel the constant c may change value from line to line. By the definition of X^c we have

$$|I^n_2| \leq \frac{c}{n}\sum_{i=0}^{n-1} |\frac{f(X_{t_i})}{\Delta_{n,i}}| [|\int_{t_i}^{t_{i+1}} a(X_s) dW_s|^2 + |\int_{t_i}^{t_{i+1}} b(X_s) ds|^2] |\varphi_{\Delta^\beta_{n,i}}(\Delta X_i) - 1| =: |I^n_{2,1}| + |I^n_{2,2}|.$$

Concerning $I^n_{2,1}$, using Holder inequality we have

$$\mathbb{E}[|I^n_{2,1}|] \leq \frac{c}{n}\sum_{i=0}^{n-1} \mathbb{E}[|\frac{f(X_{t_i})}{\Delta_{n,i}}| \mathbb{E}_i[|\int_{t_i}^{t_{i+1}} a(X_s) dW_s|^{2p}]^{\frac{1}{p}} \mathbb{E}_i[|\varphi_{\Delta^\beta_{n,i}}(\Delta X_i) - 1|^q]^{\frac{1}{q}}], \tag{36}$$

where $\mathbb{E}_i$ is the conditional expectation wit respect to $\mathcal{F}_{t_i}$.
We now use Burkholder-Davis-Gundy inequality to get, for $p_1 \geq 2$,

$$\mathbb{E}_i[|\int_{t_i}^{t_{i+1}} a(X_s) dW_s|^{p_1}]^{\frac{1}{p_1}} \leq \mathbb{E}_i[|\int_{t_i}^{t_{i+1}} a^2(X_s) ds|^{\frac{p_1}{2}}]^{\frac{1}{p_1}} \leq R(\Delta^{\frac{p_1}{2}}_{n,i}, X_{t_i})^{\frac{1}{p_1}} = R(\Delta^{\frac{1}{2}}_{n,i}, X_{t_i}), \tag{37}$$

where in the last inequality we have used the polynomial growth of a and third point of Lemma 1. We now observe that, from the definition of φ we know that $\varphi_{\Delta^\beta_{n,i}}(\Delta X_i) - 1$ is different from 0 only if $|\Delta X_i| > \Delta^\beta_{n,i}$. We consider two different sets: $|\Delta X^J_i| < \frac{1}{2}\Delta^\beta_{n,i}$ and $|\Delta X^J_i| \geq \frac{1}{2}\Delta^\beta_{n,i}$. We recall that $\Delta X_i = \Delta X^c_i + \Delta X^J_i$ and so, if $|\Delta X_i| > \Delta^\beta_{n,i}$ and $|\Delta X^J_i| < \frac{1}{2}\Delta^\beta_{n,i}$, then it means that $|\Delta X^c_i|$ must be more than $\frac{1}{2}\Delta^\beta_{n,i}$. Using a conditional version of Tchebychev inequality we have that, $\forall r > 1$,

$$\mathbb{P}_i(|\Delta X^c_i| \geq \frac{1}{2}\Delta^\beta_{n,i}) \leq c\frac{\mathbb{E}_i[|\Delta X^c_i|^r]}{\Delta^{\beta r}_{n,i}} \leq R(\Delta^{(\frac{1}{2}-\beta)r}_{n,i}, X_{t_i}), \tag{38}$$

where $\mathbb{P}_i$ is the conditional probability with respect to $\mathcal{F}_{t_i}$; the last inequality follows from the fourth point of Lemma 1. If otherwise $|\Delta X^J_i| \geq \frac{1}{2}\Delta^\beta_{n,i}$, then we introduce the set $N_{i,n} := \Big\{|\Delta L_s| \leq \frac{2\Delta^\beta_{n,i}}{\gamma_{min}}; \forall s \in (t_i, t_{i+1}]\Big\}$. We have $\mathbb{P}_i(\Big\{|\Delta X^J_i| \geq \frac{1}{2}\Delta^\beta_{n,i}\Big\} \cap (N_{i,n})^c) \leq \mathbb{P}_i((N_{i,n})^c)$, with

$$\mathbb{P}_i((N_{i,n})^c) = \mathbb{P}_i(\exists s \in (t_i, t_{i+1}] : |\Delta L_s| > \frac{\Delta^\beta_{n,i}}{2\gamma_{min}}) \leq c\int_{t_i}^{t_{i+1}} \int_{\frac{\Delta^\beta_{n,i}}{2\gamma_{min}}}^{\infty} F(z) dz ds \leq c\Delta^{1-\alpha\beta}_{n,i}, \tag{39}$$

where we have used the third point of Assumption 4. Furthermore, using Markov inequality,

$$\mathbb{P}_i(\Big\{|\Delta X^J_i| \geq \frac{1}{2}\Delta^\beta_{n,i}\Big\} \cap N_{i,n}) \leq c\mathbb{E}_i[|\Delta X^J_i|^r 1_{N_{i,n}}] R(\Delta^{-\beta r}_{n,i}, X_{t_i}) \leq R(\Delta^{-\beta r+1+\beta(r-\alpha)}_{n,i}, X_{t_i}) = R(\Delta^{1-\beta\alpha}_{n,i}, X_{t_i}), \tag{40}$$

where in the last equality we have used the first point of Lemma 4, observing that $1_{N_{i,n}}$ acts like the indicator function in (31) (see also (219) in [2]). Now using (38), (39), (40) and the arbitrariness of r we have

$$\mathbb{P}_i(|\Delta X_i| > \Delta^\beta_{n,i}) = \mathbb{P}_i(|\Delta X_i| > \Delta^\beta_{n,i}, |\Delta X^J_i| < \frac{1}{2}\Delta^\beta_{n,i}) + \mathbb{P}_i(|\Delta X_i| > \Delta^\beta_{n,i}, |\Delta X^J_i| \geq \frac{1}{2}\Delta^\beta_{n,i}) \leq$$

$$\leq R(\Delta_{n,i}^{[(\frac{1}{2}-\beta)r]\wedge[1-\alpha\beta]}, X_{t_i}) = R(\Delta_{n,i}^{1-\alpha\beta}, X_{t_i}). \tag{41}$$

Taking p big and q next to 1 in (36) and replacing there (37) with $p_1 = 2p$ and (41) we get, $\forall \epsilon > 0$,

$$\mathbb{E}[|I_{2,1}^n|] \leq \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}[|\frac{f(X_{t_i})}{\Delta_{n,i}}|R(\Delta_{n,i}, X_{t_i})R(\Delta_{n,i}^{1-\alpha\beta-\epsilon}, X_{t_i})] \leq \Delta_n^{1-\alpha\beta-\epsilon}\frac{c}{n}\sum_{i=1}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1, X_{t_i})].$$

Now, for each $\tilde{\epsilon} > 0$, we can always find an ϵ smaller than it, that is enough to get $I_{2,1}^n = o_{\mathbb{P}}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}})$ and so

$$I_{2,1}^n = o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}). \tag{42}$$

Moreover $I_{2,1}^n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$, indeed

$$\frac{\mathbb{E}[|I_{2,1}^n|]}{\Delta_n^{\beta(2-\alpha)}} \leq \Delta_n^{1-\alpha\beta-\beta(2-\alpha)-\epsilon}\frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1, X_{t_i})]. \tag{43}$$

Since we can always find an $\epsilon > 0$ such that $\beta < \frac{1}{2} - \epsilon$, we observe that the exponent on Δ_n is positive. Using the polynomial growth of both f and R and the third point of Lemma 2 we get that (43) goes to zero in norm 1 and so in probability.
Let us now consider $I_{2,2}^n$. We observe that $|\varphi_{\Delta_{n,i}^{\beta}}(\Delta X_i) - 1| \leq c$. Moreover, by adding and subtracting $b(X_{t_i})$ in the integral we get

$$(\int_{t_i}^{t_{i+1}} b(X_s)ds)^2 \leq c\Delta_{n,i}^2 b^2(X_{t_i}) + c(\int_{t_i}^{t_{i+1}}[b(X_s) - b(X_{t_i})]ds)^2.$$

Using Jensen inequality and the regularity of b we get

$$\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} b(X_s)ds)^2] \leq R(\Delta_{n,i}^2, X_{t_i}) + \int_{t_i}^{t_{i+1}} \|b'\|_\infty^2 \mathbb{E}_i[|X_s - X_{t_i}|^2]ds \leq$$

$$\leq R(\Delta_{n,i}^2, X_{t_i}) + c\int_{t_i}^{t_{i+1}} \Delta_{n,i}(1 + |X_{t_i}|^2)ds = R(\Delta_{n,i}^2, X_{t_i}), \tag{44}$$

where in the last inequality we have used the second point of Lemma 1. Using (44) we get

$$\mathbb{E}[|I_{2,2}^n|] \leq \Delta_n\frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1, X_{t_i})] \tag{45}$$

and so $I_{2,2}^n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ since

$$\frac{\mathbb{E}[|I_{2,2}^n|]}{\Delta_n^{\beta(2-\alpha)}} \leq \Delta_n^{1-\beta(2-\alpha)}\frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1, X_{t_i})], \tag{46}$$

that goes to 0 because the exponent on Δ_n is always more than zero, f and R have polynomial growth and we can use the third point of Lemma 2. Moreover, using (45), we have that $I_{2,2}^n = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}})$ and so it is $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$. From (42), (43), (45) and (46) we get $I_2^n = \mathcal{E}_n$.
Let us now consider I_3^n. From the definition of the process (X_t^c) it is

$$\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}[\int_{t_i}^{t_{i+1}} b(X_s)ds + \int_{t_i}^{t_{i+1}} a(X_s)dW_s]\Delta X_i^J \varphi_{\Delta_{n,i}^{\beta}}(\Delta X_i) =: I_{3,1}^n + I_{3,2}^n.$$

We use on $I_{3,1}^n$ Cauchy-Schwartz inequality, (44) and Lemma 10 in [2], getting

$$\mathbb{E}[|I_{3,1}^n|] \leq \frac{2}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}R(\Delta_{n,i}^{1+\beta(2-\alpha)}, X_{t_i})^{\frac{1}{2}}R(\Delta_{n,i}^2, X_{t_i})^{\frac{1}{2}}] \leq \Delta_n^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1, X_{t_i})],$$

where we have also used property (24) on R. We observe it is $\frac{1}{2} + \beta - \frac{\alpha\beta}{2} > \frac{1}{2}$ if and only if $\beta(1 - \frac{\alpha}{2}) > 0$, that is always true. We can therefore say that $I_{3,1}^n = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}})$ and so

$$I_{3,1}^n = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{47}$$

Moreover,

$$\frac{\mathbb{E}[|I^n_{3,1}|]}{\Delta_n^{\beta(2-\alpha)}} \leq \Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}} \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1,X_{t_i})], \tag{48}$$

that goes to zero using the polynomial growth of both f and R and the third point of Lemma 2 and observing that the exponent on Δ_n is positive for $\beta < \frac{1}{2}\frac{1}{(1-\frac{\alpha}{2})}$, that is always true.

Concerning $I^n_{3,2}$, we start proving that $I^n_{3,2} = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. From (29) in Proposition 3 we have

$$\frac{I^n_{3,2}}{\Delta_n^{\beta(2-\alpha)}} = \frac{1}{\Delta_n^{\beta(2-\alpha)}}\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}[\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}}a(X_s)dW_s + o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1})]. \tag{49}$$

By the definition of o_{L^1} the last term here above goes to zero in norm 1 and so in probability. The first term of (49) can be seen as

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}}\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)[\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s + \int_{t_i}^{t_{i+1}}(a(X_s)-a(X_{t_i}))dW_s]. \tag{50}$$

On the first term of (50) here above we want to use Lemma 9 of [11] in order to get that it converges to zero in probability, so we have to show the following:

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}}\frac{2}{n}\sum_{i=0}^{n-1}\mathbb{E}_i[\frac{f(X_{t_i})}{\Delta_{n,i}}\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s] \xrightarrow{\mathbb{P}} 0, \tag{51}$$

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}}\frac{4}{n^2}\sum_{i=0}^{n-1}\mathbb{E}_i[\frac{f^2(X_{t_i})}{\Delta^2_{n,i}}(\Delta\tilde{X}_i^J)^2\varphi^2_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)(\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s)^2] \xrightarrow{\mathbb{P}} 0, \tag{52}$$

where $\mathbb{E}_i[.] = \mathbb{E}[.|\mathcal{F}_{t_i}]$.
Using the independence between W and L we have that the left hand side of (51) is

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}}\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\mathbb{E}_i[\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)]\mathbb{E}_i[\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s] = 0. \tag{53}$$

Now, in order to prove (52), we use Holder inequality with p big and q next to 1 on its left hand side, getting it is upper bounded by

$$\Delta_n^{1-2\beta(2-\alpha)}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}\frac{f^2(X_{t_i})}{\Delta^2_{n,i}}\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s)^{2p}]^{\frac{1}{p}}\mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)|^{2q}]^{\frac{1}{q}} \leq$$

$$\leq \Delta_n^{1-2\beta(2-\alpha)}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}\frac{f^2(X_{t_i})}{\Delta^2_{n,i}}R(\Delta_{n,i},X_{t_i})R(\Delta_{n,i}^{\frac{1}{q}+\frac{\beta}{q}(2q-\alpha)},X_{t_i}) \leq$$

$$\leq \Delta_n^{1-2\beta(2-\alpha)+2\beta-\alpha\beta-\epsilon}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})R(1,X_{t_i}), \tag{54}$$

where we have used (37), (32) and property (24) of R. We observe that the exponent on Δ_n is positive if $\beta < \frac{1}{2-\alpha} - \epsilon$ and we can always find an $\epsilon > 0$ such that it is true. Hence, using also that $\frac{1}{n\Delta_n}$ is bounded, the polynomial growth of both f and R and the third point of Lemma 2, we get that (54) goes to zero in norm 1 and so in probability.
Concerning the second term of (50), using Cauchy-Schwartz inequality, (37) and (32) we have

$$\mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)||\int_{t_i}^{t_{i+1}}[a(X_s)-a(X_{t_i})]dW_s|] \leq \mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J)|^2]^{\frac{1}{2}}\mathbb{E}_i[|\int_{t_i}^{t_{i+1}}[a(X_s)-a(X_{t_i})]dW_s|^2]^{\frac{1}{2}} \leq$$

$$\leq R(\Delta_{n,i}^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)},X_{t_i})\mathbb{E}_i[\int_{t_i}^{t_{i+1}}\|a'\|_\infty|X_s-X_{t_i}|^2ds]^{\frac{1}{2}} \leq \Delta_{n,i}^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)}R(1,X_{t_i})(\int_{t_i}^{t_{i+1}}\Delta_{n,i}(1+|X_{t_i}|^2)ds)^{\frac{1}{2}} \leq$$

$$\leq \Delta_{n,i}^{\frac{3}{2}+\frac{\beta}{2}(2-\alpha)}R(1,X_{t_i}), \tag{55}$$

where we have also used the third point of Lemma 1 and the property (24) of R. Replacing (55) in the second term of (50) we get it is upper bounded in norm 1 by

$$\Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1,X_{t_i})], \tag{56}$$

that goes to zero since the exponent on Δ_n is more than 0 for $\beta < \frac{1}{2}\frac{1}{(1-\frac{\alpha}{2})}$, that is always true. Using (49) - (52) and (56) we get

$$\frac{I^n_{3,2}}{\Delta_n^{\beta(2-\alpha)}} \xrightarrow{\mathbb{P}} 0. \tag{57}$$

We now want to show that $I^n_{3,2}$ is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.
Using (30) in Proposition 3 we get it is enough to prove that

$$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}[\Delta\tilde{X}_i^J\varphi_{\Delta^{\beta}_{n,i}}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}}a(X_s)dW_s] \xrightarrow{\mathbb{P}} 0, \tag{58}$$

where the left hand side here above can be seen as (50), with the only difference that now we have $\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}$ instead of $\Delta_n^{\beta(2-\alpha)}$. We have again, acting like we did in (53) and (54),

$$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\mathbb{E}_i[\Delta\tilde{X}_i^J\varphi_{\Delta^{\beta}_{n,i}}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s] \xrightarrow{\mathbb{P}} 0 \tag{59}$$

and

$$\frac{1}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}}\frac{4}{n^2}\sum_{i=0}^{n-1}\mathbb{E}_i[\frac{f^2(X_{t_i})}{\Delta^2_{n,i}}(\Delta\tilde{X}_i^J)^2\varphi^2_{\Delta^{\beta}_{n,i}}(\Delta\tilde{X}_i^J)(\int_{t_i}^{t_{i+1}}a(X_{t_i})dW_s)^2] \leq \Delta_n^{2\tilde{\epsilon}+2\beta-\alpha\beta-\epsilon}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})R(1,X_{t_i}), \tag{60}$$

that goes to zero in norm 1 and so in probability. Using also (55) we have that
$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\Delta\tilde{X}_i^J\varphi_{\Delta^{\beta}_{n,i}}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}}[a(X_s)-a(X_{t_i})]dW_s$ is upper bounded in norm 1 by

$$\Delta_n^{\frac{\beta}{2}(2-\alpha)+\tilde{\epsilon}}\frac{1}{n}\mathbb{E}[|f(X_{t_i})||R(1,X_{t_i})|], \tag{61}$$

that goes to zero since the exponent on Δ_n is always positive. Using (58) - (61) we get $I^n_{3,2} = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}})$ and so

$$I^n_{3,2} = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{62}$$

From (42), (43), (45), (46), (47), (48), (57) and (62) it follows (7).
Now, in order to prove (8), we recall the definition of X^c_t:

$$X^c_{t_{i+1}} - X^c_{t_i} = \int_{t_i}^{t_{i+1}}b(X_s)ds + \int_{t_i}^{t_{i+1}}a(X_s)dW_s. \tag{63}$$

Replacing (63) in (7) and comparing it with (8) it follows that our goal is to show that

$$A^n_1 + A^n_2 := \frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\int_{t_i}^{t_{i+1}}b(X_s)ds)^2 + \frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\int_{t_i}^{t_{i+1}}b(X_s)ds)(\int_{t_i}^{t_{i+1}}a(X_s)dW_s) = \mathcal{E}_n.$$

Using (44) and property (24) of R we know that

$$\frac{\mathbb{E}[|A^n_1|]}{\Delta_n^{\beta(2-\alpha)}} \leq \frac{1}{\Delta_n^{\beta(2-\alpha)}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}R(\Delta^2_{n,i},X_{t_i})] \leq \Delta_n^{1-\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1,X_{t_i})] \tag{64}$$

and

$$\frac{\mathbb{E}[|A^n_1|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \leq \Delta_n^{\frac{1}{2}+\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R(1,X_{t_i})], \tag{65}$$

that go to zero since the exponent on Δ_n is always more than 0, f and R have both polynomial growth and we can use the third point of Lemma 2.
Let us now consider A^n_2. By adding and subtracting $b(X_{t_i})$ in the first integral, as we have already done, we get that

$$A^n_2 = \sum_{i=0}^{n-1}\zeta_{n,i} + A^n_{2,2} := \frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\int_{t_i}^{t_{i+1}}b(X_{t_i})ds)(\int_{t_i}^{t_{i+1}}a(X_s)dW_s) +$$

$$+\frac{2}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\int_{t_i}^{t_{i+1}}[b(X_s)-b(X_{t_i})]ds)(\int_{t_i}^{t_{i+1}}a(X_s)dW_s).$$

Using Lemma 9 in [11], we want to show that

$$\sum_{i=0}^{n-1} \zeta_{n,i} = \mathcal{E}_n \tag{66}$$

and so that the following convergences hold:

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} \zeta_{n,i} \xrightarrow{\mathbb{P}} 0 \qquad \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \sum_{i=0}^{n-1} \zeta_{n,i} \xrightarrow{\mathbb{P}} 0; \tag{67}$$

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}} \sum_{i=0}^{n-1} \zeta_{n,i}^2 \xrightarrow{\mathbb{P}} 0 \qquad \frac{1}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}} \sum_{i=0}^{n-1} \zeta_{n,i}^2 \xrightarrow{\mathbb{P}} 0. \tag{68}$$

We have

$$\sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] = \frac{2}{\Delta_n^{\beta(2-\alpha)}} \frac{1}{n} \sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} \Delta_{n,i} b(X_{t_i}) \mathbb{E}_i[\int_{t_i}^{t_{i+1}} a(X_s) dW_s] = 0$$

and so the two convergences in (67) both hold. Concerning (68), using (37) we have

$$\Delta_n^{1-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b^2(X_{t_i}) \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} a(X_s) dW_s)^2] \leq \Delta_n^{2-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b^2(X_{t_i}) R(1, X_{t_i})$$

and

$$\Delta_n^{1-2(\frac{1}{2}-\tilde{\epsilon})} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b^2(X_{t_i}) \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} a(X_s) dW_s)^2] \leq \Delta_n^{1+2\tilde{\epsilon}} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b^2(X_{t_i}) R(1, X_{t_i}),$$

that go to zero in norm 1 and so in probability since $\frac{1}{n\Delta_n}$ is bounded and the fact that the exponent on Δ_n is always positive. It follows (68) and so (66). Concerning $A^n_{2,2}$, using Holder inequality, (37), the regularity of b and Jensen inequality it is

$$\mathbb{E}[|A^n_{2,2}|] \leq c \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}} \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} \|b'\|_\infty |X_s - X_{t_i}| ds)^q]^{\frac{1}{q}} R(\Delta_{n,i}^{\frac{1}{2}}, X_{t_i})] \leq$$

$$\leq c \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}} (\Delta_{n,i}^{q-1} \int_{t_i}^{t_{i+1}} \mathbb{E}_i[|X_s - X_{t_i}|^q] ds)^{\frac{1}{q}} R(\Delta_{n,i}^{\frac{1}{2}}, X_{t_i})] \leq$$

$$\leq c \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}} (\Delta_{n,i}^{q-1} \int_{t_i}^{t_{i+1}} \Delta_{n,i} (1 + |X_{t_i}|^q) ds)^{\frac{1}{q}} R(\Delta_{n,i}^{\frac{1}{2}}, X_{t_i})].$$

So we get

$$\frac{\mathbb{E}[|A^n_{2,2}|]}{\Delta_n^{\beta(2-\alpha)}} \leq \Delta_n^{\frac{1}{q}+\frac{1}{2}-\beta(2-\alpha)} \frac{c}{n} \sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})| R(1, X_{t_i})] \qquad \text{and} \tag{69}$$

$$\frac{\mathbb{E}[|A^n_{2,2}|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \leq \Delta_n^{\frac{1}{q}+\tilde{\epsilon}} \frac{c}{n} \sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})| R(1, X_{t_i})]. \tag{70}$$

Since it holds for $q \geq 2$, the best choice is to take $q = 2$, in this way we get that (69) and (70) go to 0 in norm 1, using the polynomial growth of both f and R, the third point of Lemma 2 and the fact that the exponent on Δ_n is in both cases more than zero, because of $\beta < \frac{1}{2-\alpha}$.
From (64), (65), (67), (69) and (70) it follows (8).

6.2 Proof of Theorem 2

Proof. We want to prove

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{f(X_{t_i})}{\Delta_{n,i}} (\int_{t_i}^{t_{i+1}} a(X_s) dW_s)^2 - \frac{1}{n} \sum_{i=0}^{n-1} f(X_{t_i}) a^2(X_{t_i}) = \frac{Z_n}{\sqrt{n}} + \mathcal{E}_n, \tag{71}$$

and

$$\tilde{Q}_n = \hat{Q}_n + \frac{1}{\Delta_n^{\beta(2-\alpha)}} \mathcal{E}_n,$$

where $\mathcal{E}_n$ is always $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and, if $\beta > \frac{1}{4-\alpha}$, then it is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$. We can rewrite the last equation here above as

$$\tilde{Q}_n = \hat{Q}_n + o_{\mathbb{P}}(1) \tag{72}$$

and, for $\beta > \frac{1}{4-\alpha}$,

$$\tilde{Q}_n = \hat{Q}_n + \frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{73}$$

Using them and (8) it follows (10). Hence we are now left to prove (71) - (73).

Proof of (71).
We can see the left hand side of (71) as

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}[(\int_{t_i}^{t_{i+1}} a(X_s)dW_s)^2 - \int_{t_i}^{t_{i+1}} a^2(X_s)ds] + \frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\int_{t_i}^{t_{i+1}}[a^2(X_s) - a^2(X_{t_i})]ds =: M_n^Q + B_n.$$

We want to show that $B_n = \mathcal{E}_n$, it means that it is both $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$. Considering the development up to second order of the function a^2 we get

$$a^2(X_s) - a^2(X_{t_i}) = 2aa'(X_{t_i})(X_s - X_{t_i}) + ((a')^2 + aa'')(\tilde{X}_{t_i})(X_s - X_{t_i})^2, \tag{74}$$

where $\tilde{X}_{t_i} \in [X_{t_i}, X_s]$. Replacing (74) in the definition of B_n it is

$$\frac{2}{n}\sum_{i=0}^{n-1}\frac{(faa')(X_{t_i})}{\Delta_{n,i}}\int_{t_i}^{t_{i+1}}(X_s - X_{t_i})ds + \frac{1}{n}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\int_{t_i}^{t_{i+1}}((a')^2 + aa'')(\tilde{X}_{t_i})(X_s - X_{t_i})^2 ds =: B_1^n + B_2^n.$$

We start by proving that $B_2^n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. Indeed, using Holder inequality taking p big and q next to 1, it is

$$\mathbb{E}[|B_2^n|] \le \frac{c}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}\int_{t_i}^{t_{i+1}}\mathbb{E}_i[|(a')^2 + aa''|^p(\tilde{X}_{t_i})]^{\frac{1}{p}}\mathbb{E}_i[|X_s - X_{t_i}|^{2q}]^{\frac{1}{q}}ds] \le$$

$$\le \frac{c}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}\int_{t_i}^{t_{i+1}}(1 + |X_{t_i}|^p)^{\frac{1}{p}}(1 + |X_{t_i}|^{2q})^{\frac{1}{q}}|s - t_i|^{\frac{1}{q}}ds] \le$$

$$\le \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}R(\Delta_{n,i}^{1+\frac{1}{q}}, X_{t_i})] \le \frac{\Delta_n^{\frac{1}{q}}}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f((a')^2 + aa'')|(X_{t_i})R(1, X_{t_i})],$$

where we have used the third point of Lemma 1 for the first expected value and the second point on the second one. It follows

$$\frac{\mathbb{E}[|B_2^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \Delta_n^{1-\epsilon-\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f|(X_{t_i})R(1, X_{t_i})] \qquad \text{and} \tag{75}$$

$$\frac{\mathbb{E}[|B_2^n|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \le \Delta_n^{\frac{1}{2}-\epsilon+\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f|(X_{t_i})R(1, X_{t_i})], \tag{76}$$

that go to zero using the polynomial growth of f and R . We have also used the third point of Lemma 2 and observed that the exponent on Δ_n is always more than 0.
Concerning B_1^n, we recall that from (2) it follows

$$X_s - X_{t_i} = \int_{t_i}^{s} b(X_u)du + \int_{t_i}^{s} a(X_u)dW_u + \int_{t_i}^{s}\int_{\mathbb{R}}\gamma(X_{u^-})z\tilde{\mu}(du, dz)$$

and so, replacing it in the definition of B_1^n, we get three terms: $B_1^n := I_1^n + I_2^n + I_3^n$.
We start considering I_1^n on which we use polynomial growth of b and the third point of Lemma 1 to get

$$\mathbb{E}[|I_1^n|] \le \frac{2}{n}\sum_{i=0}^{n-1}\mathbb{E}[\frac{|f(X_{t_i})|}{\Delta_{n,i}}|aa'|(X_{t_i})\int_{t_i}^{t_{i+1}}(\int_{t_i}^{s}R(1, X_{t_i})du)ds] \le \Delta_n\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|faa'|(X_{t_i})R(1, X_{t_i})].$$

It follows

$$\frac{\mathbb{E}[|I_1^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \Delta_n^{1-\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|faa'|(X_{t_i})R(1, X_{t_i})] \qquad \text{and} \tag{77}$$

$$\frac{\mathbb{E}[|I_1^n|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \leq \Delta_n^{\frac{1}{2}+\tilde{\epsilon}} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[|faa'|(X_{t_i})R(1, X_{t_i})], \tag{78}$$

that go to zero because of the polynomial growth of f, a, a', a'' and R and the fact that $1-\beta(2-\alpha) > 0$. Considering I_2^n, we define $\zeta_{n,i} := \frac{2}{n}\frac{(faa')(X_{t_i})}{\Delta_{n,i}} \int_{t_i}^{t_{i+1}} (\int_{t_i}^{s} a(X_u)dW_u)ds$. We want to use Lemma 9 in [11] to get that

$$\frac{I_2^n}{\Delta_n^{\beta(2-\alpha)}} \xrightarrow{\mathbb{P}} 0 \qquad \text{and} \qquad \frac{I_2^n}{\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}} \xrightarrow{\mathbb{P}} 0 \tag{79}$$

and so we have to show the following :

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0, \qquad \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0; \tag{80}$$

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}^2] \xrightarrow{\mathbb{P}} 0, \tag{81}$$

$$\frac{1}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}^2] \xrightarrow{\mathbb{P}} 0. \tag{82}$$

By the definition of $\zeta_{n,i}$ it is $\mathbb{E}_i[\zeta_{n,i}] = 0$ and so (80) is clearly true. The left hand side of (81) is

$$\Delta_n^{1-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{4}{n} \sum_{i=0}^{n-1} \frac{(faa')^2(X_{t_i})}{\Delta_{n,i}^2} \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} (\int_{t_i}^{s} a(X_u)dW_u)ds)^2]. \tag{83}$$

Using Fubini theorem and Ito isometry we have

$$\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} (\int_{t_i}^{s} a(X_u)dW_u)ds)^2] = \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} (t_{i+1}-s)a(X_s)dW_s)^2] = \mathbb{E}_i[\int_{t_i}^{t_{i+1}} (t_{i+1}-s^2)a^2(X_s)ds] \leq R(\Delta_{n,i}^3, X_{t_i}), \tag{84}$$

where in the last inequality we have used polynomial growth of a and the third point of Lemma 1. Because of (84), we get that (83) is upper bounded by

$$\Delta_n^{1-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} (faa')^2(X_{t_i})R(\Delta_{n,i}, X_{t_i}) \leq \Delta_n^{2-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{1}{n} \sum_{i=0}^{n-1} (faa')^2(X_{t_i})R(1, X_{t_i}),$$

that converges to zero in norm 1 and so (81) follows, since $2-2\beta(2-\alpha) > 0$ for $\beta < \frac{1}{2-\alpha}$, that is always true. Moreover we have used that $n\Delta_n$ is bounded, the polynomial growth of f, a, a' and R and the third point of Lemma 1.
Acting in the same way we get that the left hand side of (82) is upper bounded by

$$\Delta_n^{1-(1-2\tilde{\epsilon})} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} (faa')^2(X_{t_i})R(\Delta_{n,i}, X_{t_i}) \leq \Delta_n^{1+2\tilde{\epsilon}} \frac{1}{n\Delta_n} \frac{1}{n} \sum_{i=0}^{n-1} (faa')^2(X_{t_i})R(1, X_{t_i}),$$

that goes to zero in norm 1. In order to show also

$$\frac{I_3^n}{\Delta_n^{\beta(2-\alpha)}} \xrightarrow{\mathbb{P}} 0 \qquad \text{and} \qquad \frac{I_3^n}{\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}} \xrightarrow{\mathbb{P}} 0, \tag{85}$$

we define $\tilde{\zeta}_{n,i} := \frac{2}{n}\frac{(faa')(X_{t_i})}{\Delta_{n,i}} \int_{t_i}^{t_{i+1}} (\int_{t_i}^{s} \int_{\mathbb{R}} \gamma(X_{u^-})z\tilde{\mu}(du, dz))ds$. We have again $\mathbb{E}_i[\tilde{\zeta}_{n,i}] = 0$ and so (80) holds with $\tilde{\zeta}_{n,i}$ in place of $\zeta_{n,i}$. We now act like we did in (84), using Fubini theorem and Ito isometry. It follows

$$\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} (\int_{t_i}^{s} \int_{\mathbb{R}} \gamma(X_{u^-})z\tilde{\mu}(du, dz)ds)^2] = \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} (t_{i+1}-s)\gamma(X_{s^-})z\tilde{\mu}(ds, dz))^2] =$$

$$= \mathbb{E}_i[\int_{t_i}^{t_{i+1}} (t_{i+1}-s)^2\gamma^2(X_{s^-})ds(\int_{\mathbb{R}} z^2F(z)dz)] \leq R(\Delta_{n,i}^3, X_{t_i}), \tag{86}$$

having used in the last inequality the definition of $\bar{\mu}(ds, dz)$, the fact that $\int_{\mathbb{R}} z^2F(z)dz < \infty$, the polynomial growth of γ and the third point of Lemma 1. Replacing (86) in the left hand side of (81) and (82), with $\tilde{\zeta}_{n,i}$ in place of $\zeta_{n,i}$, we have

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}_i[\tilde{\zeta}_{n,i}^2] \leq \Delta_n^{1-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{c}{n} \sum_{i=0}^{n-1} \frac{(faa')^2(X_{t_i})}{\Delta_{n,i}^2} R(\Delta_{n,i}^3, X_{t_i}) \leq \Delta_n^{2-2\beta(2-\alpha)} \frac{1}{n\Delta_n} \frac{1}{n} \sum_{i=0}^{n-1} (faa')^2(X_{t_i})R(1, X_{t_i})$$

$$\text{and } \frac{1}{\Delta_n^{1-2\bar{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[\tilde{\zeta}_{n,i}^2] \leq \Delta_n^{1+2\bar{\epsilon}}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}(faa')^2(X_{t_i})R(1,X_{t_i}).$$

Again, they converge to zero in norm 1 and thus in probability since $2-2\beta(2-\alpha)>0$ always holds, using also the polynomial growth of a, a', f and R and the third point of Lemma 2. Therefore, we get (85).
From (75), (76), (77), (78), (79) and (85) it follows that

$$B_n = \mathcal{E}_n. \tag{87}$$

Concerning $M_n^Q := \sum_{i=0}^{n-1}\hat{\zeta}_{n,i}$, we have to act differently for T fixed and $\lim_{n\to\infty}T=\infty$.
Case 1: T fixed
Genon - Catalot and Jacod have proved in [11] that, in the continuous framework, the following conditions are enough to get $\sqrt{n}M_n^Q \to N(0,2\int_0^T(a^4f^2)(X_s)H(s,0)ds)$ stably with respect to X:

- $\mathbb{E}_i[\hat{\zeta}_{n,i}]=0$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}^2] \xrightarrow{\mathbb{P}} 2\int_0^T(a^4f^2)(X_s)H(s,0)ds$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}^4] \xrightarrow{\mathbb{P}} 0$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}(W_{t_{i+1}}-W_{t_i})] \xrightarrow{\mathbb{P}} 0$.

Theorem 2.2.15 in [14] adapts the previous theorem to our framework, in which there is the presence of jumps.
We observe that the conditions here above are respected, hence

$$M_n^Q = \frac{Z_n}{\sqrt{n}}, \text{ where } Z_n \xrightarrow{n} N(0,2\int_0^T(a^4f^2)(X_s)H(s,0)ds), \tag{88}$$

stably with respect to X.
Case 2: $\lim_{n\to\infty}T=\infty$.
In order to show the asymptotic normality we have to prove that $\hat{\zeta}_{n,i}$ is a martingal difference array such that $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}^2] \xrightarrow{\mathbb{P}} 2\int_{\mathbb{R}}a^4(x)f^2(x)\pi(dx)$ and that $\sum_{i=0}^{n-1}\mathbb{E}_i[|\hat{\zeta}|_{n,i}^{2+\delta}] \xrightarrow{\mathbb{P}} 0$, for a constant $\delta>0$. The previous conditions are true as a consequence of the the building of our sequence $\hat{\zeta}_{n,i}$ and using Lemma 3 with $\delta=0$. So we get

$$M_n^Q = \frac{Z_n}{\sqrt{n}}, \text{ where } Z_n \xrightarrow{n} N(0,2\int_{\mathbb{R}}a^4(x)f^2(x)\pi(dx)). \tag{89}$$

From (87), (88) and (89), it follows (71).

Proof of (72).
We use Proposition 3 replacing (27) in the definition (6) of $\tilde{Q}_n$. Recalling that the convergence in norm 1 implies the convergence in probability it is clear that we have to prove the result on

$$\frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\Delta\tilde{X}_i^J)^2\varphi_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J) =$$

$$= \frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\gamma^2(X_{t_i})\Delta_{n,i}^{\frac{2}{\alpha}}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}})^2\varphi_{\Delta_{n,i}^\beta}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}}\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}), \tag{90}$$

where we have also rescaled the process in order to apply Proposition 2. We now define

$$g_{i,n}(y) := y^2\varphi_{\Delta_{n,i}^\beta}(y\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}), \tag{91}$$

hence we can rewrite (90) as

$$\frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\gamma^2(X_{t_i})\Delta_{n,i}^{\frac{2}{\alpha}}[g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)]]+$$

$$+\frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}\gamma^2(X_{t_i})\Delta_{n,i}^{\frac{2}{\alpha}}\mathbb{E}[g_{i,n}(S_1^\alpha)] =: \sum_{i=0}^{n-1}A_{1,i}^n + \hat{Q}_n, \tag{92}$$

where S_1^α is the α-stable process at time $t=1$. We want to show that $\sum_{i=0}^{n-1} A_{1,i}^n$ converges to zero in probability. With this purpose in mind, we take the conditional expectation of $A_{1,i}^n$ and we apply Proposition 2 on the interval $[t_i, t_{i+1}]$ instead of on $[0, \Delta_n]$, observing that property (25) holds on $g_{i,n}$ for $p=2$. By the definition (91) of $g_{i,n}$, we have $\|g_{i,n}\|_\infty = R(\Delta_{n,i}^{2(\beta-\frac{1}{\alpha})}, X_{t_i})$ and $\|g_{i,n}\|_{pol} = R(1, X_{t_i})$. Replacing them in (26) we have that

$$|\mathbb{E}_i[g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}})] - \mathbb{E}[g_{i,n}(S_1^\alpha)]| \leq c_{\epsilon,\alpha}\Delta_{n,i}|\log(\Delta_{n,i})|R(\Delta_{n,i}^{2(\beta-\frac{1}{\alpha})}, X_{t_i}) +$$

$$+c_{\epsilon,\alpha}\Delta_{n,i}^{\frac{1}{\alpha}}|\log(\Delta_{n,i})|R(\Delta_{n,i}^{2(\beta-\frac{1}{\alpha})(1-\frac{\alpha}{2}-\epsilon)}, X_{t_i}) + c_{\epsilon,\alpha}\Delta_{n,i}^{\frac{1}{\alpha}}|\log(\Delta_{n,i})|R(\Delta_{n,i}^{2(\beta-\frac{1}{\alpha})(\frac{3}{2}-\frac{\alpha}{2}-\epsilon)}, X_{t_i})1_{\alpha>1}.$$

To get $\sum_{i=0}^{n-1} A_{1,i}^n := o_{\mathbb{P}}(1)$, we want to use Lemma 9 of [11]. We have

$$\sum_{i=0}^{n-1}|\mathbb{E}_i[A_{1,i}^n]| \leq \frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})||\log(\Delta_{n,i})|[\Delta_{n,i}^{\frac{2}{\alpha}+2(\beta-\frac{1}{\alpha})}R(1,X_{t_i}) + \Delta_{n,i}^{\frac{3}{\alpha}-1+(2-\alpha-\epsilon)(\beta-\frac{1}{\alpha})}R(1,X_{t_i}) +$$

$$+\Delta_{n,i}^{\frac{3}{\alpha}-1+(3-\alpha-\epsilon)(\beta-\frac{1}{\alpha})}R(1,X_{t_i})1_{\alpha>1}] \leq \Delta_n^{\alpha\beta}|\log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}) +$$

$$+\Delta_n^{\frac{1}{\alpha}-\epsilon}|\log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}) + \Delta_n^{\beta-\epsilon}|\log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i})1_{\alpha>1}, \tag{93}$$

where we have used property (24) and the monotony of the logarithmic function in order to say that $\log(\Delta_{n,i}) \leq \log(\Delta_n)$. Using the polynomial growth of f and R, the fifth point of Assumption 4 in order to bound γ and the third point of Lemma 2, (93) converges to 0 in norm 1 and so in probability since $\Delta_n^{\alpha\beta}\log(\Delta_n) \to 0$ for $n \to \infty$ and we can always find an $\epsilon > 0$ such that $\Delta_n^{\frac{1}{\alpha}-\epsilon}$ does the same.
To use Lemma 9 of [11] we have also to show that

$$\Delta_n^{1-2\beta(2-\alpha)}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}\frac{f^2(X_{t_i})}{\Delta_{n,i}^2}\gamma^4(X_{t_i})\Delta_{n,i}^{\frac{4}{\alpha}}\mathbb{E}_i[(g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)])^2] \xrightarrow{P} 0. \tag{94}$$

We observe that $\mathbb{E}_i[(g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)])^2] \leq c\mathbb{E}_i[g_{i,n}^2(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}})] + c\mathbb{E}_i[\mathbb{E}[g_{i,n}(S_1^\alpha)]^2]$. Now, using equation (32) of Lemma 4, we observe it is

$$\mathbb{E}_i[g_{i,n}^2(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}}})] = \frac{\Delta_{n,i}^{-\frac{4}{\alpha}}}{\gamma^4(X_{t_i})}\mathbb{E}_i[(\Delta\tilde{X}_i^J)^4\varphi^2_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)] = \frac{\Delta_{n,i}^{-\frac{4}{\alpha}}}{\gamma^4(X_{t_i})}R(\Delta_{n,i}^{1+\beta(4-\alpha)}, X_{t_i}), \tag{95}$$

where φ acts as the indicator function. Moreover we observe that

$$\mathbb{E}[g_{i,n}(S_1^\alpha)] = \int_{\mathbb{R}} z^2\varphi(\Delta_{n,i}^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})z)f_\alpha(z)dz = d(\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}-\beta}), \tag{96}$$

with $f_\alpha(z)$ the density of the stable process. We now introduce the following lemma, that will be shown in the Appendix:

Lemma 5. *Suppose that Assumptions 1-4 hold. Then, for each ζ_n such that $\zeta_n \to 0$ and for each $\hat{\epsilon} > 0$,*

$$d(\zeta_n) = |\zeta_n|^{\alpha-2}c_\alpha\int_{\mathbb{R}}|u|^{1-\alpha}\varphi(u)du + o_{\mathbb{P}}(|\zeta_n|^{-\hat{\epsilon}} + |\zeta_n|^{2\alpha-2-\hat{\epsilon}}), \tag{97}$$

where c_α has been defined in (12).

Since $\frac{1}{\alpha} - \beta > 0$, $\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}-\beta}$ goes to zero for $n \to \infty$ and so we can take ζ_n as $\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}-\beta}$, getting that

$$\mathbb{E}[g_{i,n}(S_1^\alpha)] = d(\gamma(X_{t_i})\Delta_{n,i}^{\frac{1}{\alpha}-\beta}) = R(\Delta_{n,i}^{(\frac{1}{\alpha}-\beta)(\alpha-2)}, X_{t_i}). \tag{98}$$

Replacing (95) and (98) in the left hand side of (94) we get it is upper bounded by

$$\sum_{i=0}^{n-1}\mathbb{E}_i[(A_{1,i}^n)^2] = \Delta_n^{1-2\beta(2-\alpha)}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})\gamma^4(X_{t_i})\Delta_{n,i}^{\frac{4}{\alpha}-2}(R(\Delta_{n,i}^{1+\beta(4-\alpha)-\frac{4}{\alpha}}, X_{t_i}) + R(\Delta_{n,i}^{4\beta-\frac{4}{\alpha}+2-2\alpha\beta}, X_{t_i})) \leq$$

$$\leq \Delta_n^{\alpha\beta\wedge 1}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i})R(1,X_{t_i}), \tag{99}$$

that converges to zero in norm 1 and so in probability in both cases T fixed and $T\to\infty$, using the polynomial growth of f and R and the fact that the exponent on Δ_n is always positive. From (93) and (99) it follows

$$\sum_{i=0}^{n-1} A^n_{1,i} = o_{\mathbb{P}}(1). \tag{100}$$

and so (72).

Proof of (73).
We use Proposition 3 replacing (28) in definition (6) of $\tilde{Q}_n$. Our goal is to prove that

$$\frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\frac{f(X_{t_i})}{\Delta_{n,i}}(\Delta\tilde{X}_i^J)^2\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}_i^J) = \hat{Q}_n + o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})}).$$

On the left hand side of the equation here above we can act like we did in (90) - (92). To get (73) we are therefore left to show that , if $\beta > \frac{1}{4-\alpha}$, then $\sum_{i=0}^{n-1} A^n_{1,i}$ is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})})$. To prove it, we want to use Lemma 9 of [11], hence we want to prove the following:

$$\frac{1}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[A^n_{1,i}]\xrightarrow{\mathbb{P}}0 \quad \text{and} \tag{101}$$

$$\frac{1}{\Delta_n^{2(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})}}\sum_{i=0}^{n-1}\mathbb{E}_i[(A^n_{1,i})^2]\xrightarrow{\mathbb{P}}0. \tag{102}$$

Using (93) we have that, if $\alpha>1$, then the left hand side of (101) is in module upper bounded by

$$\frac{\Delta_n^{\beta-\epsilon}|log(\Delta_n)|}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}) = \Delta_n^{3\beta-\alpha\beta-\frac{1}{2}+\tilde{\epsilon}-\epsilon}|log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}),$$

that goes to zero since we have choosen $\beta>\frac{1}{4-\alpha}>\frac{1}{2(3-\alpha)}$. Otherwise, if $\alpha\leq 1$, then (93) gives us that the left hand side of (101) is in module upper bounded by

$$\frac{\Delta_n^{\alpha\beta}|log(\Delta_n)|}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}) = \Delta_n^{2\beta-\frac{1}{2}+\tilde{\epsilon}}|log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R(1,X_{t_i}),$$

that goes to zero because $\beta>\frac{1}{4-\alpha}>\frac{1}{4}$.
Using also (99), the left hand side of (102) turns out to be upper bounded by
$\Delta_n^{-1+4\beta-2\alpha\beta+2\tilde{\epsilon}}\Delta_n^{\alpha\beta\wedge 1}\frac{1}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i})R(1,X_{t_i})$, that goes to zero in norm 1 and so in probability since we have choosen $\beta>\frac{1}{4-\alpha}$. It follows (102) and so (10).

□

6.3 Proof of Proposition 1

Proof. To prove the proposition we replace (97) in the definition of $\hat{Q}_n$. It follows that our goal is to show that

$$I_1^n + I_2^n := \frac{1}{n\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1} f(X_{t_i})\gamma^2(X_{t_i})\Delta_{n,i}^{\frac{2}{\alpha}-1}(o_{\mathbb{P}}(|\Delta_{n,i}^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})|^{-\hat{\epsilon}} + |\Delta_{n,i}^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})|^{2\alpha-2-\hat{\epsilon}})) = \tilde{\mathcal{E}}_n,$$

where $\tilde{\mathcal{E}}_n$ is always $o_{\mathbb{P}}(1)$ and, if $\alpha<\frac{4}{3}$, it is also $\frac{1}{\Delta_n^{\beta(2-\alpha)}}o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.
We have that $I_1^n = o_{\mathbb{P}}(1)$ since it is upper bounded by

$$\Delta_n^{\frac{2}{\alpha}-1-2\beta+\alpha\beta-\hat{\epsilon}(\frac{1}{\alpha}-\beta)}\frac{1}{n}\sum_{i=0}^{n-1}R(1,X_{t_i})o_{\mathbb{P}}(1),$$

that goes to zero in norm 1 and so in probability since we can always find an $\hat{\epsilon}>0$ such that the exponent on Δ_n is positive.
Also I_2^n is $o_{\mathbb{P}}(1)$. Indeed it is upper bounded by

$$\Delta_n^{\frac{2}{\alpha}-1-2\beta+\alpha\beta-2(\frac{1}{\alpha}-\beta)+2(1-\alpha\beta)-\hat{\epsilon}(\frac{1}{\alpha}-\beta)}\frac{1}{n}\sum_{i=0}^{n-1}R(1,X_{t_i})o_{\mathbb{P}}(1). \tag{103}$$

We observe that the exponent on Δ_n is $1-\alpha\beta-\hat{\epsilon}(\frac{1}{\alpha}-\beta)$ and we can always find $\hat{\epsilon}$ such that it is more than zero, hence (103) converges in norm 1 and so in probability.
In order to show that $I_1^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}-\beta(2-\alpha)})$ we observe that

$$\frac{I_1^n}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}-\beta(2-\alpha)}} \leq \Delta_n^{\frac{2}{\alpha}-1-\frac{1}{2}+\tilde{\epsilon}-\hat{\epsilon}(\frac{1}{\alpha}-\beta)} \frac{1}{n}\sum_{i=0}^{n-1} R(1, X_{t_i}) o_{\mathbb{P}}(1).$$

If $\alpha < \frac{4}{3}$ we can always find $\tilde{\epsilon}$ and $\hat{\epsilon}$ such that the exponent on Δ_n is more than zero, getting the convergence wanted. It follows $I_1^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.
To conclude, $I_2^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}) = o_{\mathbb{P}}(\Delta_n^{1-2\beta-\tilde{\epsilon}})$. Indeed,

$$\frac{I_2^n}{\Delta_n^{1-2\beta-\tilde{\epsilon}}} \leq \Delta_n^{\frac{2}{\alpha}-1-1+\alpha\beta+\tilde{\epsilon}-2(\frac{1}{\alpha}-\beta)+2(1-\alpha\beta)-\hat{\epsilon}(\frac{1}{\alpha}-\beta)} \frac{1}{n}\sum_{i=0}^{n-1} R(1, X_{t_i}) o_{\mathbb{P}}(1). \tag{104}$$

The exponent on Δ_n is $2\beta-\alpha\beta+\tilde{\epsilon}-\hat{\epsilon}(\frac{1}{\alpha}-\beta)$ and so we can always find $\tilde{\epsilon}$ and $\hat{\epsilon}$ such that it is positive. It follows the convergence in norm 1 and so in probability of (104). The proposition is therefore proved. □

6.4 Proof of Corollary 1

Proof. We observe that (13) is a consequence of (11) in the case where $\hat{Q}_n = 0$. Moreover, $\beta < \frac{1}{2\alpha}$ implies that $\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}$ is negligible compared to $\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}$. It follows (13). □

6.5 Proof of Theorem 3.

Proof. The convergence (14) clearly follows from (11) and the second point of Assumption S1 with $\delta = \beta(2-\alpha)$.
Concerning the proof of (15), we can see its left hand side as

$$Q_n - \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) + \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) - IV_1$$

and so, using (10), it turns out that our goal is to show that

$$\frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) - IV_1 = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)}). \tag{105}$$

It is

$$\frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) - \frac{1}{T}\int_0^T (fa^2)(X_s)ds = \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i}) - \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (fa^2)(X_s)ds.$$

We now act as we did in (74), considering this time the development up to second order of the function fa^2 instead of a^2. Replacing it in the equation here above we get

$$\sum_{i=0}^{n-1} f(X_{t_i})a^2(X_{t_i})(\frac{1}{n} - \frac{\Delta_{n,i}}{\sum_{i=0}^{n-1}\Delta_{n,i}}) - \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}}\sum_{i=0}^{n-1}(fa^2)'(X_{t_i})\int_{t_i}^{t_{i+1}}(X_s - X_{t_i})ds +$$

$$-\frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}}\sum_{i=0}^{n-1}\frac{1}{2}\int_{t_i}^{t_{i+1}}(fa^2)''(\tilde{X}_{t_i})(X_s - X_{t_i})^2 ds =: I_1^n + I_2^n + I_3^n,$$

where $\tilde{X}_{t_i} \in [X_{t_i}, X_s]$. Now, using the third point of the Assumption S1, we have

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}}\mathbb{E}[|I_1^n|] \leq \Delta_n^{\delta_0}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[(fa^2)(X_{t_i})], \tag{106}$$

that goes to zero because of the polynomial growth of both f and a and the third point of Lemma 2.
Concerning I_2^n, we act like we did in the proof of Theorem 2 to get that B_1^n defined below equation (74) was $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. We have used the dynamic of the process X to get $\mathbb{E}_i[|\int_{t_i}^{t_{i+1}}(X_s - X_{t_i})ds|] \leq R(\Delta_{n,i}^2, X_{t_i})$.
We observe moreover that, as a consequence of the third point of Assumption S1, we have

$$\frac{\Delta_{n,i}}{\sum_{i=0}^{n-1}\Delta_{n,i}} \leq \frac{1}{n} + \frac{\Delta_n^{\beta(2-\alpha)+\delta_0}}{n} \leq \frac{c}{n}. \tag{107}$$

It follows

$$\frac{\mathbb{E}[|I_2^n|]}{\Delta_n^{\beta(2-\alpha)}} \leq \Delta_n^{1-\beta(2-\alpha)} \sum_{i=0}^{n-1} \mathbb{E}[(fa^2)'(X_{t_i}) \frac{\Delta_{n,i}}{\sum_{i=0}^{n-1} \Delta_{n,i}} R(1, X_{t_i})] \leq \Delta_n^{1-\beta(2-\alpha)} \frac{c}{n} \sum_{i=0}^{n-1} \mathbb{E}[(fa^2)'(X_{t_i}) R(1, X_{t_i})], \tag{108}$$

that goes to zero since $1-\beta(2-\alpha)$ is always more than 0.
Also on I_3^n we act like we did on B_2^n in the proof of theorem 2 to get $\mathbb{E}_i[|\int_{t_i}^{t_{i+1}} (fa^2)''(\tilde{X}_{t_i})(X_s - X_{t_i})^2 ds|] \leq R(\Delta_{n,i}^{2-\epsilon}, X_{t_i})$, (see above equation (75)). Using also (107) it follows

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}} \mathbb{E}[|I_3^n|] \leq \Delta_n^{1-\beta(2-\alpha)-\epsilon} \frac{c}{n} \sum_{i=0}^{n-1} \mathbb{E}[(fa^2)''(X_{t_i}) R(1, X_{t_i})]. \tag{109}$$

Again, it goes to zero in norm 1 and so in probability. From (106), (108) and (109) it follows (105) and so the theorem is proved. □

6.6 Proof of Theorem 4.

Proof. The convergence (16) is a consequence of Lemma 3, that we can apply since we have assumed that points 1 and 2 of Assumption S2 hold.
Concerning the proof of (17), we can again add and subtract $\frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i}) a^2(X_{t_i})$ and so our goal is to show (105), with T that now goes to ∞ for $n \to \infty$. We observe that we can act like we did in the previous theorem because, having assumed the third point of the Assumption S2, the proof here above still hold. □

7 Proof of developments in small time: Proposition 2.

This section is dedicate to the proof of Proposition 2. Proposition 3 will be proved in the appendix.
To prove Proposition 2, it is convenient to introduce an adequate truncation function and to consider a rescaled process, as explained in the next subsections. Moreover, the proof of Proposition 2 requires some Malliavin calculus; we recall in what follows all the technical tools to make easier the understanding of the paper.

7.1 Localization and rescaling

We introduce a truncation function in order to suppress the big jumps of (L_t). Let $\tau : \mathbb{R} \to [0,1]$ be a symmetric function, continuous with continuous derivative, such that $\tau = 1$ on $\left\{|z| \leq \frac{1}{4}\eta\right\}$ and $\tau = 0$ on $\left\{|z| \geq \frac{1}{2}\eta\right\}$, with η defined in the fourth point of Assumption 4.
On the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ we consider the Lévy process (L_t) defined below (2) which measure is $F(dz) = \frac{g(z)}{|z|^{1+\alpha}} 1_{\mathbb{R}\setminus\{0\}}(z) dz$, according with the third point of Assumption 4, and the truncated Lévy process (L_t^τ) with measure $F^\tau(dz)$ given by $F^\tau(dz) = \frac{g(z)\tau(z)}{|z|^{1+\alpha}} 1_{\mathbb{R}\setminus\{0\}}(z) dz$. This can be done by setting $L_t := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}(ds, dz)$, as we have already done, and $L_t^\tau := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}^\tau(ds, dz)$, where $\tilde{\mu}$ and $\tilde{\mu}^\tau$ are the compensated Poisson random measures associated respectively to

$$\mu(A) := \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,T]} 1_A(t,z) \mu^g(dt, dz, du), \qquad A \subset [0,T] \times \mathbb{R},$$

$$\mu^\tau(A) := \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,T]} 1_A(t,z) 1_{u \leq \tau(z)} \mu^g(dt, dz, du), \qquad A \subset [0,T] \times \mathbb{R},$$

for μ^g a Poisson random measure on $[0,T] \times \mathbb{R} \times [0,1]$ with compensator $\bar{\mu}^g(dt, dz, du) = dt \frac{g(z)}{|z|^{1+\alpha}} 1_{\mathbb{R}\setminus\{0\}}(z) dz du$.
By construction, the restrictions of the measures μ and μ^τ to $[0, \Delta_n] \times \mathbb{R}$ coincide on the set $\{(u,z)$ such that $u \leq \tau(z)\}$, and thus coincide on the event

$$\Omega_n := \left\{\omega \in \Omega; \mu^g([0, \Delta_n] \times \left\{z \in \mathbb{R} : |z| \geq \frac{\eta}{4}\right\} \times [0,1]) = 0\right\}.$$

Since $\mu^g([0, \Delta_n] \times \left\{z \in \mathbb{R} : |z| \geq \frac{\eta}{4}\right\} \times [0,1])$ has a Poisson distribution with parameter

$$\lambda_n := \int_0^{\Delta_n} \int_{|z| \geq \frac{\eta}{4}} \int_0^1 \frac{g(z)}{|z|^{1+\alpha}} du\, dz\, dt \leq c\Delta_n;$$

we deduce that

$$\mathbb{P}(\Omega_n^c) \leq c\Delta_n. \tag{110}$$

Then we have

$$\mathbb{P}((L_t)_{t\leq\Delta_n} \neq (L_t^\tau)_{t\leq\Delta_n}) \leq \mathbb{P}(\Omega_n^c) \leq c\Delta_n. \tag{111}$$

To prove Proposition 2 we have to rescale the process $(L_t)_{t\in[0,1]}$, we therefore introduce an auxiliary Lévy process $(L_t^n)_{t\in[0,1]}$ defined possibly on another filtered space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ and admitting the decomposition $L_t^n := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}^n(dt, dz)$, with $t \in [0,1]$; where $\tilde{\mu}^n$ is a compensated Poisson random measure $\tilde{\mu}^n = \mu^n - \bar{\mu}^n$, with compensator

$$\bar{\mu}^n(dt, dz) = dt\frac{g(z\Delta_n^{\frac{1}{\alpha}})}{|z|^{1+\alpha}}\tau(z\Delta_n^{\frac{1}{\alpha}})1_{\mathbb{R}\setminus\{0\}}(z)dz. \tag{112}$$

By construction, the process $(L_t^n)_{t\in[0,1]}$ is equal in law to the rescaled truncated process $(\Delta_n^{-\frac{1}{\alpha}} L^\tau_{\Delta_n t})_{t\in[0,1]}$ that coincides with $(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n t})_{t\in[0,1]}$ on Ω_n.

7.2 Malliavin calculus

In this section, we recall some results on Malliavin calculus for jump processes. We refer to [8] for a complete presentation and to [9] for the adaptation to our framework. We will work on the Poisson space associated to the measure μ^n defining the process $(L_t^n)_{t\in[0,1]}$ of the previous section, assuming that n is fixed. By construction, the support of μ^n is contained in $[0,1] \times E_n$, where $E_n := \left\{ z \in \mathbb{R} : |z| < \frac{\eta}{2}\frac{1}{\Delta_n^{\frac{1}{\alpha}}} \right\}$, with η defined in the fourth point of Assumption 4. We recall that the measure μ^n has compensator

$$\bar{\mu}^n(dt, dz) = dt\frac{g(z\Delta_n^{\frac{1}{\alpha}})}{|z|^{1+\alpha}}\tau(z\Delta_n^{\frac{1}{\alpha}})1_{\mathbb{R}\setminus\{0\}}(z)dz := dtF_n(z)dz. \tag{113}$$

In this section we assume that the truncation function τ satisfies the additional assumption

$$\int_{\mathbb{R}} |\frac{\tau'(z)}{\tau(z)}|^p \tau(z)dz < \infty, \qquad \forall p \geq 1.$$

We now define the Malliavin operators L and Γ (omitting their dependence in n) and their basic properties (see [8] Chapter IV, sections 8-9-10). For a test function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ measurable, $\mathcal{C}^2$ with respect the second variable, with bounded derivative and such that $f \in \cap_{p\geq 1} L^p(\bar{\mu}^n(dt, dz))$, we set $\mu^n(f) = \int_0^1 \int_{\mathbb{R}} f(t,z)\mu^n(dt, dz)$. As auxiliary function, we consider $\rho : \mathbb{R} \to [0, \infty)$ such that ρ is symmetric, two times differentiable and such that $\rho(z) = z^4$ if $z \in [0, \frac{1}{2}]$ and $\rho(z) = z^2$ if $z \geq 1$. Thanks to the truncation τ, we do not need that ρ vanishes at infinity. Assuming the fourth point of Assumption 4, we check that ρ, ρ' and $\rho\frac{F_n'}{F_n}$ belong to $\cap_{p\geq 1} L^p(F_n(z)dz)$. With these notations, we define the Malliavin operator L on the functional $\mu^n(f)$ as follows:

$$L(\mu^n(f)) := \frac{1}{2}\mu^n(\rho' f' + \rho\frac{F_n'}{F_n}f' + \rho f''),$$

where f' and f'' are derivative with respect to the second variable. This definition permits to construct a linear operator on the space $D \subset \cap_{p\geq 1} L^p(F_n(z)dz)$ which is self-adjoint: $\forall \Phi, \Psi \in D$, $\mathbb{E}\Phi L\Psi = \mathbb{E}L\Phi\Psi$ (see Section 8 in [8] for the details on the construction of D).
We associate to L the symmetric bilinear operator Γ:

$$\Gamma(\Phi, \Psi) = L(\Phi, \Psi) - \Phi L(\Psi) - \Psi L(\Phi).$$

If f and h are two test functions, we have

$$\Gamma(\mu^n(f), \mu^n(h)) = \mu^n(\rho f' h'). \tag{114}$$

The operators L and Γ satisfy the chain rule property:

$$LF(\Phi) = F'(\Phi)L\Phi + \frac{1}{2}F''(\Phi)\Gamma(\Phi, \Phi), \qquad \Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi).$$

These operators permit to establish the following integration by parts formula (see [8] Theorem 8-10 p.103).

Theorem 5. *Let Φ and Ψ be random variable in D and f be a bounded function with bounded derivatives up to order two. If $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p\geq 1} L^p$, then we have*

$$\mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi), \tag{115}$$

with

$$\mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)). \tag{116}$$

The random variable L_1^n belongs to the domain of the operators L and Γ. Computing $L(L_1^n)$, $\Gamma(L_1^n, L_1^n)$ and $\mathcal{H}_{L_1^n}(1)$ it is possible to deduce the following useful inequalities, proved in Lemma 4.3 in [9].

Lemma 6. *We have*

$$\sup_n \mathbb{E}|\mathcal{H}_{L_1^n}(1)|^p \leq C_p \qquad \forall p \geq 1,$$

$$\sup_n \mathbb{E}|\int_0^1 \int_{|z|>1} |z|\mu^n(ds,dz)\mathcal{H}_{L_1^n}(1)|^p \leq C_p \qquad \forall p \geq 1.$$

With this background we can proceed to the proof of Proposition 2.

7.3 Proof of Proposition 2

Proof. The first step is to construct on the same probability space two random variables whose laws are close to the laws of $\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n}$ and S_1^α. We recall briefly the notation of Section 7.1: μ^n is a Poisson random measure with compensator $\bar{\mu}^n(dt,dz)$ defined in (112) and the process L_t^n is defined by

$$L_t^n = \int_0^t \int_{\mathbb{R}} z\tilde{\mu}^n(ds,dz) = \int_0^t \int_{|z|\leq \Delta_n^{-\frac{1}{\alpha}}\frac{\eta}{2}} z\tilde{\mu}^n(ds,dz) \tag{117}$$

with $\tilde{\mu}^n = \mu^n - \bar{\mu}^n$. Using triangle inequality we have

$$|\mathbb{E}[h(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n})] - \mathbb{E}[h(S_1^\alpha)]| \leq |\mathbb{E}[h(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n})] - \mathbb{E}[h(L_1^n)]| + |\mathbb{E}[h(L_1^n) - h(S_1^\alpha)]|. \tag{118}$$

By the definition of L_1^n it is

$$|\mathbb{E}[h(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n})] - \mathbb{E}[h(L_1^n)]| = |\mathbb{E}[h(\Delta_n^{-\frac{1}{\alpha}} L_{\Delta_n}) - h(\Delta_n^{-\frac{1}{\alpha}} L^\tau_{\Delta_n})]| \leq 2\|h\|_\infty \mathbb{P}(\Omega_n^c) \leq c\|h\|_\infty \Delta_n, \tag{119}$$

where in the last inequality we have used (111). In order to get an estimation to the second term of (118) we now construct a variable approximating the law of S_1^α and based on the Poisson measure μ^n :

$$L_t^{\alpha,n} := \int_0^t \int_{|z|\leq \Delta_n^{-\frac{1}{\alpha}}\frac{\eta}{2}} h_n(z)\tilde{\mu}^n(ds,dz), \tag{120}$$

where h_n is an odd function built in the proof of Theorem 4.1 in [9] for which the following lemma holds:

Lemma 7. *1. For each test function f, defined as in Section 7.2, we have*

$$\int_0^1 \int_{|z|\leq \frac{\eta}{2}\Delta_n^{-\frac{1}{\alpha}}} f(t,h_n(z))\bar{\mu}^n(dt,dz) = \int_0^1 \int_{|\omega|\leq \frac{\eta}{2}\Delta_n^{-\frac{1}{\alpha}}} f(t,\omega)\bar{\mu}^{\alpha,n}(dt,d\omega), \tag{121}$$

where $\bar{\mu}^n(dt,dz)$ is the compensator defined in (112) and

$$\bar{\mu}^{\alpha,n}(dt,d\omega) = dt\frac{\tau(\omega\Delta_n^{\frac{1}{\alpha}})}{|\omega|^{1+\alpha}}d\omega$$

is the compensator of a measure associated to an α- stable process whose jumps are truncated with the function τ.

2. There exists $\epsilon_0 > 0$ such that, for $|z| \leq \epsilon_0\Delta_n^{-\frac{1}{\alpha}}$,

$$|h_n(z) - z| \leq cz^2\Delta_n^{\frac{1}{\alpha}} + c|z|^{1+\alpha}\Delta_n \qquad \text{if } \alpha \neq 1,$$

$$|h_n(z) - z| \leq cz^2\Delta_n|log(|z|\Delta_n)| \qquad \text{if } \alpha = 1.$$

3. The function h_n is $\mathcal{C}^1$ on $(-\epsilon_0\Delta_n^{-\frac{1}{\alpha}}, \epsilon_0\Delta_n^{-\frac{1}{\alpha}})$ and for $|z| < \epsilon_0\Delta_n^{-\frac{1}{\alpha}}$,

$$|h_n'(z) - 1| \leq c|z|\Delta_n^{\frac{1}{\alpha}} + c|z|^\alpha\Delta_n \qquad \text{if } \alpha \neq 1,$$

$$|h_n'(z) - 1| \leq c|z|\Delta_n|log(|z|\Delta_n)| \qquad \text{if } \alpha = 1.$$

The second and the third point of the lemma here above are proved in Lemma 4.5 of [9], while the first point is proved in Theorem 4.1 [9] and it shows us, using the exponential formula for Poisson measure, that h_n is the function that turns our measure μ^n into the measure associated to an α-stable process truncated with the function τ. Thus $(L_t^{\alpha,n})_{t\in[0,1]}$ is a Lévy process with jump intensity $\omega \mapsto \frac{\tau(\omega\Delta_n^{\frac{1}{\alpha}})}{|\omega|^{1+\alpha}}$ and we recognize the law of an α-stable truncated process. We deduce, similarly to (119),

$$|\mathbb{E}[h(L_1^{\alpha,n})] - \mathbb{E}[h(S_1^{\alpha})]| \leq c \left\|h\right\|_{\infty} \Delta_n. \tag{122}$$

Proposition 2 is a consequence of (118), (119), (122) and the following lemma:

Lemma 8. *Suppose that Assumptions 1 to 4 hold. Let h be as in Proposition 2. Then, for any $\epsilon > 0$ and for $p \geq \alpha$,*

$$|\mathbb{E}[h(L_1^n) - h(L_1^{\alpha,n})]| \leq C_\epsilon \Delta_n |log(\Delta_n)| \left\|h\right\|_{\infty} + C_\epsilon \Delta_n^{\frac{1}{\alpha}} \left\|h\right\|_{\infty}^{1-\frac{\alpha}{p}+\epsilon} \left\|h\right\|_{pol}^{\frac{\alpha}{p}-\epsilon} |log(\Delta_n)| +$$

$$+ C_\epsilon \Delta_n^{\frac{1}{\alpha}} \left\|h\right\|_{\infty}^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon} \left\|h\right\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon} |log(\Delta_n)| 1_{\alpha>1}.$$

Proof. The proof is based of the comparison of the representation of (117) and (120). Since in Lemma 7 the difference $h_n(z) - z$ is controlled for $|z| \leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}$, we need to introduce a localization procedure consisting in regularizing $1_{\left\{\mu^n([0,1]\times\left\{z\in\mathbb{R}:|z|>\epsilon_0\Delta_n^{-\frac{1}{\alpha}}\right\})=0\right\}}$. Let $\mathcal{I}$ be a smooth function defined on $\mathbb{R}$ and with values in $[0,1]$, such that $\mathcal{I}(x) = 1$ for $x \leq \frac{1}{2}$ and $\mathcal{I}(x) = 0$ for $x \geq 1$. Moreover, we denote by ζ a smooth function on $\mathbb{R}$, with values in $[0,1]$ such that $\zeta(z) = 0$ for $|z| \leq \frac{1}{2}$ and $\zeta(z) = 1$ for $|z| \geq 1$ and we set

$$V^n := \int_0^1 \int_{\mathbb{R}} \zeta(\frac{z\Delta_n^{\frac{1}{\alpha}}}{\epsilon_0}) \mu^n(ds,dz) = \int_0^1 \int_{\left\{\frac{1}{2}\epsilon_0\Delta_n^{-\frac{1}{\alpha}} \leq |z| \leq \epsilon_0\Delta_n^{-\frac{1}{\alpha}}\right\}} \zeta(\frac{z\Delta_n^{\frac{1}{\alpha}}}{\epsilon_0}) \mu^n(ds,dz) + \int_0^1 \int_{\left\{|z|\geq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}\right\}} \mu^n(ds,dz),$$

$$W^n := \mathcal{I}(V^n).$$

From the construction, W^n is a Malliavin differentiable random variable such that $W^n \neq 0$ implies $\mu^n([0,1] \times \left\{z \in \mathbb{R} : |z| > \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}\right\}) = 0$. It is possible to show, acting as we did in (110), that $\mathbb{P}(W^n \neq 1) \leq \mathbb{P}(\mu^n \text{ has a jump of size } > \frac{1}{2}\epsilon_0\Delta_n^{-\frac{1}{\alpha}}) = O(\Delta_n)$. From the latter, it is clear that the proof of the lemma reduces in proving the result on $|\mathbb{E}[h(L_1^n)W^n] - \mathbb{E}[h(L_1^{\alpha,n})W^n]|$. Considering a regularizing sequence (h_p) converging to h in L^1 norm, such that $\forall p$ h_p is $\mathcal{C}^1$ with bounded derivative and $\left\|h_p\right\|_{\infty} \leq \left\|h\right\|_{\infty}$, we may assume that h is $\mathcal{C}^1$ with bounded derivative too. Using the integration by part formula (115) and denoting by H any primitive function of h we can write $\mathbb{E}[h(L_1^n)W^n] = \mathbb{E}[H(L_1^n)\mathcal{H}_{L_1^n}(W^n)]$ where the Malliavin weight can be written, using (116) and the chain rule property of the operator Γ, as

$$\mathcal{H}_{L_1^n}(W^n) = W^n \mathcal{H}_{L_1^n}(1) - \frac{\Gamma(W^n, L_1^n)}{\Gamma(L_1^n, L_1^n)}. \tag{123}$$

Using the triangle inequality, we are now left to find upper bounds for the following two terms:

$$\tilde{T_1} := |\mathbb{E}[h(L_1^{\alpha,n})W^n] - \mathbb{E}[H(L_1^{\alpha,n})\mathcal{H}_{L_1^n}(W^n)]|,$$

$$\tilde{T_2} := |\mathbb{E}[H(L_1^{\alpha,n})\mathcal{H}_{L_1^n}(W^n)] - \mathbb{E}[H(L_1^n)\mathcal{H}_{L_1^n}(W^n)]|.$$

Let us start considering $\tilde{T_2}$. Using the Lipschitz property of the function H and (123) we have it is upper bounded by

$$\mathbb{E}[|h(\hat{L}_1)||L_1^{\alpha,n} - L_1^n||\mathcal{H}_{L_1^n}(W^n)|] \leq \mathbb{E}[|h(\hat{L}_1)||L_1^{\alpha,n} - L_1^n||W^n\mathcal{H}_{L_1^n}(1)|] + \mathbb{E}[|h(\hat{L}_1)||L_1^{\alpha,n} - L_1^n||\frac{\Gamma(W^n, L_1^n)}{\Gamma(L_1^n, L_1^n)}|] =$$

$$=: \tilde{T}_{2,1} + \tilde{T}_{2,2},$$

where $\hat{L}_1$ is between $L_1^{\alpha,n}$ and L_1^n. We focus on $\tilde{T}_{2,1}$. Using the definitions (117) and (120) of L_1^n and $L_1^{\alpha,n}$ it is

$$\tilde{T}_{2,1} \leq \mathbb{E}[|h(\hat{L}_1)|| \int_0^1 \int_{\mathbb{R}} (h_n(z) - z)\tilde{\mu}^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|] \leq \mathbb{E}[|h(\hat{L}_1)|| \int_0^1 \int_{|z|\leq 1} (h_n(z) - z)\tilde{\mu}^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|] +$$

$$+\mathbb{E}[|h(\hat{L}_1)|| \int_0^1 \int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}} (h_n(z)-z)\mu^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|], \tag{124}$$

where we have used that h_n is an odd function with the symmetry of the compensator $\bar{\mu}^n$ and the fact that on $W_n \neq 0$ we have $\mu^n([0,1]\times\left\{z\in\mathbb{R}:|z|>\epsilon_0\Delta_n^{-\frac{1}{\alpha}}\right\})=0$. For the sake of shortness, we only give the details of the proof in the case $\alpha\neq 1$. In the case $\alpha=1$, one needs to modify this control with an additional logarithmic term. For the small jumps term, from inequality 2.1.37 in [14] and the second point of Lemma 7 we deduce $\mathbb{E}[|\int_0^1\int_{|z|\leq 1}(h_n(z)-z)\tilde{\mu}^n(ds,dz)|^{q_1}]\leq C_{q_1}(\Delta_n+\Delta_n^{\frac{1}{\alpha}})^{q_1}$, $\forall q_1\geq 2$. Using it and Holder inequality with q_1 big and q_2 close to 1 we have

$$\mathbb{E}[|h(\hat{L}_1)|| \int_0^1 \int_{|z|\leq 1} (h_n(z)-z)\tilde{\mu}^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|] \leq C_{q_1}(\Delta_n+\Delta_n^{\frac{1}{\alpha}})\mathbb{E}[|h(\hat{L}_1)|^{q_2}|\mathcal{H}_{L_1^n}(1)|^{q_2}W^n]^{\frac{1}{q_2}} \leq$$

$$\leq C_{q_1}(\Delta_n+\Delta_n^{\frac{1}{\alpha}})\mathbb{E}[|h(\hat{L}_1)|^{p_1\,q_2}W^n]^{\frac{1}{q_2p_1}}\mathbb{E}[|\mathcal{H}_{L_1^n}(1)|^{q_2p_2}]^{\frac{1}{q_2p_2}}, \tag{125}$$

where in the last inequality we have used again Holder inequality, with p_2 big and p_1 close to 1. Using the first point of Lemma 6, we know that $\mathbb{E}[|\mathcal{H}_{L_1^n}(1)|^{q_2p_2}]^{\frac{1}{q_2p_2}}$ is bounded, hence (125) is upper bounded by

$$C_{q_1q_2p_2}\Delta_n\|h\|_\infty + C_{q_1q_2p_2}\Delta_n^{\frac{1}{\alpha}}\mathbb{E}[|h(\hat{L}_1)W^n|^{p_1\,q_2}]^{\frac{1}{q_2p_1}}, \tag{126}$$

where we have bounded $|h(\hat{L}_1)|$ with its infinity norm and used that $0\leq W^n\leq 1$. We remind that we are considering q_2 and p_1 next to 1, hence we can write q_2p_1 as $1+\epsilon$. We now introduce r in the following way:

$$\mathbb{E}[|h(\hat{L}_1)|^{1+\epsilon}W^n]^{\frac{1}{1+\epsilon}} = \mathbb{E}[|h(\hat{L}_1)|^{(1+\epsilon)r}|h(\hat{L}_1)|^{(1+\epsilon)(1-r)}W^n]^{\frac{1}{1+\epsilon}} \leq \|h\|_\infty^r\,\mathbb{E}[|h(\hat{L}_1)|^{(1+\epsilon)(1-r)}W^n]^{\frac{1}{1+\epsilon}} \leq$$

$$\|h\|_\infty^r\,\|h\|_{pol}^{1-r}\,\mathbb{E}[(1+|\hat{L}_1|^p)^{(1+\epsilon)(1-r)}W^n]^{\frac{1}{1+\epsilon}} \leq c\,\|h\|_\infty^r\,\|h\|_{pol}^{1-r} + c\,\|h\|_\infty^r\,\|h\|_{pol}^{1-r}\,\mathbb{E}[|\hat{L}_1|^{p(1+\epsilon)(1-r)}W^n]^{\frac{1}{1+\epsilon}}; \tag{127}$$

where we have estimated h with its norm ∞ and we have used the property (25) of h and that $0\leq W^n\leq 1$. We observe that $\hat{L}_1$ is between L_1^n and $L_1^{\alpha,n}$ hence $|\hat{L}_1|\leq|L_1^n|+|L_1^{\alpha,n}|$. Moreover we choose r such that $p(1+\epsilon)(1-r)=\alpha$; therefore $r=1-\frac{\alpha}{p(1+\epsilon)}$. In this way we have that (127) is upper bounded by

$$c\,\|h\|_\infty^{1-\frac{\alpha}{p(1+\epsilon)}}\,\|h\|_{pol}^{\frac{\alpha}{p(1+\epsilon)}}\,\frac{1}{1+\epsilon}log(\Delta_n^{-\frac{1}{\alpha}}). \tag{128}$$

Indeed, remarking that as a consequence of the second point of Lemma 7 there exists $c>0$ such that $|h_n(z)|\leq c|z|$, we can act on both L_1^n and $L_1^{\alpha,n}$ in the same way. Using also Lemma 2.1.5 in the appendix of [14] if $\alpha\in[1,2]$ and Jensen inequality if $\alpha\in[0,1)$, we have

$$\mathbb{E}[|\hat{L}_1|^\alpha W^n] \leq c\mathbb{E}[(|L_1^n|^\alpha+|L_1^{\alpha,n}|^\alpha)W^n] \leq c\mathbb{E}[|\int_0^1\int_{|z|\leq 1} z\tilde{\mu}^n(ds,dz)|] + c\mathbb{E}[|\int_0^1\int_{|z|\leq 1} h_n(z)\tilde{\mu}^n(ds,dz)|]+$$

$$+c\mathbb{E}[\int_0^1\int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|^\alpha\bar{\mu}^n(ds,dz)] + c\mathbb{E}[\int_0^1\int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|h_n(z)|^\alpha\bar{\mu}^n(ds,dz)].$$

We observe that, using Kunita inequality, the first term here above is bounded in L^p and, as a consequence of the second point of Lemma 7, the second term here above so does. Concerning the third term here above (and so, again, we act on the fourth in the same way), we have

$$c\mathbb{E}[\int_0^1\int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|^\alpha\bar{\mu}^n(ds,dz)] \leq c\int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|^{\alpha-1-\alpha}dz \leq c\,log(\Delta_n^{-\frac{1}{\alpha}}) \leq c|log(\Delta_n)|, \tag{129}$$

where we have also used definition (112) of $\bar{\mu}^n$. Replacing (128) in (126) we get

$$\mathbb{E}[|h(\hat{L}_1)|| \int_0^1 \int_{|z|\leq 1} (h_n(z)-z)\tilde{\mu}^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|] \leq C_{q_1q_2p_2}\Delta_n\|h\|_\infty + C_{q_1q_2p_2}\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1-\frac{\alpha}{p}+\epsilon}\|h\|_{pol}^{\frac{\alpha}{p}-\epsilon}log(\Delta_n^{-\frac{1}{\alpha}}), \tag{130}$$

where we have taken another ϵ, using its arbitrariness. The constants depend also on it.
Let us now consider the large jumps term in (124). Using the second point of Lemma 7 and the following basic inequality

$$\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|^\delta\mu^n(ds,dz) \leq \int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|^{\delta-1}\mu^n(ds,dz)\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}|z|\mu^n(ds,dz)$$

for $\delta \geq 1$, we get it is upper bounded by

$$\mathbb{E}[|h(\hat{L}_1)| \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} (\Delta_n^{\frac{1}{\alpha}}|z| + \Delta_n |z|^\alpha)\mu^n(ds,dz) \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|\mathcal{H}_{L_1^n}(1)|W^n]. \tag{131}$$

We now use Holder inequality with p_2 big and p_1 next to 1 and we observe that, from the second point of Lemma 6, it follows

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)\mathcal{H}_{L_1^n}(1)|^{p_2}]^{\frac{1}{p_2}} \leq C_{p_2}.$$

Hence (131) is upper bounded by

$$C_{p_2}\mathbb{E}[|h(\hat{L}_1)|^{p_1}| \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} (\Delta_n^{\frac{1}{\alpha}}|z| + \Delta_n |z|^\alpha)\mu^n(ds,dz)|^{p_1} W^n]^{\frac{1}{p_1}} \leq \tag{132}$$

$$\leq C_{p_2} \|h\|_\infty \Delta_n \mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|^\alpha \mu^n(ds,dz)|^{p_1}]^{\frac{1}{p_1}} + C_{p_2}\Delta_n^{\frac{1}{\alpha}}\mathbb{E}[|h(\hat{L}_1)|^{p_1}| \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{p_1} W^n]^{\frac{1}{p_1}}. \tag{133}$$

Concerning the first term of (133), we use Lemma 2.1.5 in the appendix of [14] with $p_1 = (1+\epsilon) \in [1,2]$ and the definition of F_n given in (113), getting

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|^\alpha \mu^n(ds,dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq \mathbb{E}[\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|^{\alpha(1+\epsilon)} \bar{\mu}^n(ds,dz)]^{\frac{1}{1+\epsilon}} \leq$$

$$\leq c(\int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|^{\alpha(1+\epsilon)-1-\alpha} dz)^{\frac{1}{1+\epsilon}} \leq c\Delta_n^{-\frac{\epsilon}{1+\epsilon}} = c\Delta_n^{-\epsilon}, \tag{134}$$

where we have used the arbitrariness of ϵ in the last equality.
On the second term of (133) we act differently depending on whether or not α is more than 1. If it does, we act as we did in (127), considering $p_1 = 1+\epsilon < \alpha$ and introducing r, this time we set it such that the following equality holds:

$$p(1+\epsilon)(1-r) + (1+\epsilon) = \alpha. \tag{135}$$

We also use the property (25) on h, hence it is upper bounded by

$$C_{p_2}\Delta_n^{\frac{1}{\alpha}} \|h\|_\infty^r \|h\|_{pol}^{1-r} \mathbb{E}[(1+|\hat{L}_1|^{p(1+\epsilon)(1-r)})| \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{1+\epsilon} W^n]^{\frac{1}{1+\epsilon}}. \tag{136}$$

Now on the first term here above we use that $0 \leq W^n \leq 1$ and Lemma 2.1.5 in the appendix of [14] as we did in (134) in order to get

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq c. \tag{137}$$

Moreover we observe, as we have already done, that $|\hat{L}_1| \leq |L_1^n| + |L_1^{\alpha,n}|$ and that, from the second point of Lemma 7, there exists $c > 0$ such that $|h_n(z)| \leq c|z|$; so we get

$$\mathbb{E}[|\hat{L}_1|^{p(1+\epsilon)(1-r)}| \int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{1+\epsilon} W^n]^{\frac{1}{1+\epsilon}} \leq c + \mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{p(1+\epsilon)(1-r)+(1+\epsilon)}]^{\frac{1}{1+\epsilon}} \leq$$

$$\leq c(\int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|^\alpha |z|^{-1-\alpha} dz)^{\frac{1}{1+\epsilon}} \leq c\frac{1}{1+\epsilon} log(\Delta_n^{-\frac{1}{\alpha}}) \leq c|log(\Delta_n)|, \tag{138}$$

having choosen a particular r just in order to have the exponent here above equal to α and so having found out the same computation of (129). We haven't considered the integral on $|z| \leq 1$ only because, as we have already seen above (129), the integral is bounded in L^p and so we simply get (137) again. From (135) we obtain $r = 1 + \frac{1}{p} - \frac{\alpha}{p(1+\epsilon)}$. Replacing it and using (137) and (138) we get (136) is upper bounded by

$$C_{p_2}\Delta_n^{\frac{1}{\alpha}} \|h\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}} \|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}} (c + |log(\Delta_n)|) = C_{p_2}\Delta_n^{\frac{1}{\alpha}} \|h\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}} \|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}} |log(\Delta_n)|. \tag{139}$$

If otherwise α is less than 1, then the second term of (133) is upper bounded by

$$C_{p_2}\Delta_n^{\frac{1}{\alpha}} \|h\|_\infty \mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{p_1} W^n]^{\frac{1}{p_1}} \leq C_{p_2}\Delta_n^{\frac{1}{\alpha}} \|h\|_\infty \Delta_n^{\frac{1}{1+\epsilon}-\frac{1}{\alpha}} = C_{p_2}\Delta_n^{\frac{1}{1+\epsilon}} \|h\|_\infty, \tag{140}$$

where we have taken $p_1 = 1+\epsilon$ and we have used the fact that $0 \leq W^n \leq 1$ and that, for $\alpha < 1$,

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 \Delta_n^{-\frac{1}{\alpha}}} |z|\mu^n(ds,dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq c\Delta_{n,i}^{\frac{1}{1+\epsilon}-\frac{1}{\alpha}}.$$

Using (133), (134), (139) and (140) it follows

$$\mathbb{E}[|h(\hat{L}_1)||\int_0^1 \int_{1\leq|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}} (h_n(z)-z)\mu^n(ds,dz)||\mathcal{H}_{L_1^n}(1)W^n|] \leq$$

$$\leq C_{p_2}\Delta_n^{1-\epsilon}\|h\|_\infty + C_{p_2}\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}}\|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}}|log(\Delta_n)|1_{\alpha>1}. \tag{141}$$

Now from (124), (130), and (141) it follows

$$\tilde{T}_{2,1} \leq C_{q_1q_2p_2}\Delta_n^{1-\epsilon}\|h\|_\infty + C_{q_1q_2p_2}\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1-\frac{\alpha}{p}+\epsilon}\|h\|_{pol}^{\frac{\alpha}{p}-\epsilon}|log(\Delta_n)| + C_{q_1q_2p_2}\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon}|log(\Delta_n)|1_{\alpha>1}. \tag{142}$$

Concerning $\tilde{T}_{2,2}$, it is already proved in Theorem 4.2 in [9] that

$$\tilde{T}_{2,2} \leq c\Delta_n\|h\|_\infty. \tag{143}$$

Let us now consider $\tilde{T}_1$. Using (114) and (116) we can write

$$\mathcal{H}_{L_1^n}(W^n) = \frac{-W^n L(L_1^n)}{\Gamma(L_1^n,L_1^n)} + L(\frac{W^n}{\Gamma(L_1^n,L_1^n)})L_1^n - L(\frac{L_1^n W^n}{\Gamma(L_1^n,L_1^n)}).$$

With computations using that L is a self-adjoint operator we get

$$\tilde{T}_1 = |\mathbb{E}[h(L_1^{\alpha,n})W^n] - \mathbb{E}[h(L_1^{\alpha,n})\frac{\Gamma(L_1^{\alpha,n},L_1^n)}{\Gamma(L_1^n,L_1^n)}W^n]| \leq \mathbb{E}[|h(\hat{L}_1)||\frac{\Gamma(L_1^n-L_1^{\alpha,n},L_1^n)}{\Gamma(L_1^n,L_1^n)}|W^n]. \tag{144}$$

Using equation (114), we have

$$\Gamma(L_1^n - L_1^{\alpha,n}, L_1^n) = \int_0^1 \int_{|z|<\frac{\eta}{2}\Delta_n^{-\frac{1}{\alpha}}} \rho(z)(1-h_n'(z))\mu^n(ds,dz).$$

Using the third point of Lemma 7 we deduce the following on the event $W^n \neq 0$:

$$|\Gamma(L_1^n - L_1^{\alpha,n}, L_1^n)| \leq c\int_0^1 \int_{|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}} \rho(z)(\Delta_n^{\frac{1}{\alpha}}|z| + \Delta_n|z|^\alpha)\mu^n(ds,dz) \leq c\int_0^1\int_{|z|\leq 1}\rho(z)(\Delta_n^{\frac{1}{\alpha}}|z|+\Delta_n|z|^\alpha)\mu^n(ds,dz) +$$

$$+c\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}\rho(z)\mu^n(ds,dz)\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}(\Delta_n^{\frac{1}{\alpha}}|z|+\Delta_n|z|^\alpha)\mu^n(ds,dz) \leq$$

$$\leq c\int_0^1\int_{\mathbb{R}}\rho(z)\mu^n(ds,dz)(\Delta_n^{\frac{1}{\alpha}}+\Delta_n) + c\int_0^1\int_{\mathbb{R}}\rho(z)\mu^n(ds,dz)\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}(\Delta_n^{\frac{1}{\alpha}}|z|+\Delta_n|z|^\alpha)\mu^n(ds,dz) =$$

$$= c(\Delta_n^{\frac{1}{\alpha}}+\Delta_n)\Gamma(L_1^n,L_1^n) + c\Gamma(L_1^n,L_1^n)(\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}(\Delta_n^{\frac{1}{\alpha}}|z|+\Delta_n|z|^\alpha)\mu^n(ds,dz)), \tag{145}$$

where we have used that z is always less than 1 in the first integral and that, since ρ is a positive function, we can upper bound the integrals considering whole set $\mathbb{R}$. Moreover, we have used the definition of $\Gamma(L_1^n, L_1^n)$. Replacing (145) in (144) we get

$$\tilde{T}_1 \leq c(\Delta_n^{\frac{1}{\alpha}}+\Delta_n)\mathbb{E}[|h(\hat{L}_1)|] + c\mathbb{E}[|h(\hat{L}_1)|\int_0^1\int_{1<|z|\leq\epsilon_0\Delta_n^{-\frac{1}{\alpha}}}(\Delta_n^{\frac{1}{\alpha}}|z|+\Delta_n|z|^\alpha)\mu^n(ds,dz))] =: \tilde{T}_{1,1}+\tilde{T}_{1,2}. \tag{146}$$

Concerning $\tilde{T}_{1,1}$, we have

$$\tilde{T}_{1,1} \leq c\Delta_n\|h\|_\infty + c\Delta_n^{\frac{1}{\alpha}}\mathbb{E}[|h(\hat{L}_1)|] \leq c\Delta_n\|h\|_\infty + c\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1-\frac{\alpha}{p}}\|h\|_{pol}^{\frac{\alpha}{p}}|log(\Delta_n)|), \tag{147}$$

where in the last inequality we have acted exactly like we did in (127) and (128) with the exponent on h that is exactly equal to 1 instead of $1+\epsilon$ and so we have choosen r such that $p(1-r) = \alpha$. Let us now consider $\tilde{T}_{1,2}$. We observe that it is exactly like (132) but with $p_1 = 1$ instead of $p_1 = 1+\epsilon$, with the only difference that computing (134) now we get $c\,log(\Delta_n^{-\frac{1}{\alpha}})$ instead of $c\Delta_n^{-\epsilon}$ and in the definition (135) we choose r such that $p(1-r)+1 = \alpha$. Acting exactly like we did above it follows

$$\tilde{T}_{1,2} \leq C_{p_2}\Delta_n|log(\Delta_n)|\|h\|_\infty + C_{p_2}\Delta_n^{\frac{1}{\alpha}}\|h\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}}\|h\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}}|log(\Delta_n)|1_{\alpha>1}. \tag{148}$$

Using (142), (143), (147) and (148), the lemma is proved. □

It follows Proposition 2, using also (118), (119) and (122). □

A Appendix

In this section we will prove the technical proposition and lemmas we have used.

A.1 Proof of Proposition 3

Proof. Proposition 3. In order to show (27), we reformulate $(\Delta X_i^J)^2\varphi_{\Delta_{n,i}^\beta}(\Delta X_i)$ as

$$(\Delta X_i^J)^2[\varphi_{\Delta_{n,i}^\beta}(\Delta X_i)-\varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)]+(\Delta X_i^J)^2[\varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)-\varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)]+(\Delta X_i^J-\Delta \tilde{X}_i^J)^2\varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)+ \tag{149}$$

$$+2\Delta \tilde{X}_i^J(\Delta X_i^J-\Delta \tilde{X}_i^J)\varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)+(\Delta \tilde{X}_i^J)^2\varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)=:\sum_{k=1}^5 I_k^n(i).$$

Comparing (27) with (149) it turns out that our goal is to show that $\sum_{k=1}^4 I_k^n(i)=o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1})$. In the sequel will prove that $\sum_{k=1}^4\mathbb{E}[|I_k^n(i)|]\leq c\Delta_{n,i}^{\beta(2-\alpha)+1}$; the same reasoning applies to the conditional version, that is $\sum_{k=1}^4\mathbb{E}_i[|I_k^n(i)|]\leq R(\Delta_{n,i}^{\beta(2-\alpha)+1},X_{t_i})$.
Let us start considering $I_1^n(i)$. We know that $\Delta X_i=\Delta X_i^c+\Delta X_i^J$, where we have denoted by ΔX_i^c the continuous part of the increments of the process X. We study

$$I_1^n(i)=I_{1,1}^n+I_{1,2}^n:=I_1^n(i)1_{\left\{|\Delta X_i|\geq 3\Delta_{n,i}^\beta\right\}}+I_1^n(i)1_{\left\{|\Delta X_i|<3\Delta_{n,i}^\beta\right\}}, \tag{150}$$

having omitted the dependence upon i in $I_{1,1}^n$ and $I_{1,2}^n$ in order to make the notation easier. Concerning $I_{1,1}^n$, we split again on the sets $\left\{|\Delta X_i^J|\geq 2\Delta_{n,i}^\beta\right\}$ and $\left\{|\Delta X_i^J|<2\Delta_{n,i}^\beta\right\}$. Recalling that $\varphi(\zeta)=0$ for $|\zeta|\geq 2\Delta_{n,i}^\beta$, we observe that if $|\Delta X_i^J|\geq 2\Delta_{n,i}^\beta$ then $I_{1,1}^n$ is just 0. Otherwise, if $|\Delta X_i^J|<2\Delta_{n,i}^\beta$, then it means that $|\Delta X_i^c|$ must be more than $\Delta_{n,i}^\beta$, so we can use (38). In the sequel the constant c may change value from line to line. Using the bound on $(\Delta X_i^J)^2$ and the boundedness of φ we get

$$\mathbb{E}[|I_{1,1}^n|]\leq c\Delta_{n,i}^{2\beta}\mathbb{E}[1_{\left\{|\Delta X_i|\geq 3\Delta_{n,i}^\beta,|\Delta X_i^J|<2\Delta_{n,i}^\beta\right\}}]\leq c\Delta_{n,i}^{2\beta}\mathbb{P}(|\Delta X_i^c|\geq\Delta_{n,i}^\beta)\leq c\Delta_{n,i}^{2\beta+(\frac{1}{2}-\beta)r}. \tag{151}$$

Hence

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{1,1}^n|]\leq c\Delta_{n,i}^{(\frac{1}{2}-\beta)r-1+\alpha\beta}, \tag{152}$$

that goes to 0 for $n\to\infty$ since for each choice of $\beta\in(0,\frac{1}{2})$ and $\alpha\in(0,2)$ we can always find r big enough such that the exponent on $\Delta_{n,i}$ is positive.
We now consider $I_{1,2}^n$ on the sets $\left\{|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}$ and $\left\{|\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}$. Using the boundedness of φ we have

$$\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}}]\leq c\mathbb{E}[(\Delta X_i^J)^2 1_{\left\{|\Delta X_i|<3\Delta_{n,i}^\beta,|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}}].$$

We observe that also in this case $|\Delta X_i|<3\Delta_{n,i}^\beta$ and $|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta$ involve $|\Delta X_i^c|\geq\Delta_{n,i}^\beta$. Moreover $(\Delta X_i^J)^2\leq c(\Delta X_i)^2+c(\Delta X_i^c)^2\leq c\Delta_{n,i}^{2\beta}+c(\Delta X_i^c)^2$, hence

$$\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}}]\leq c\Delta_{n,i}^{2\beta}\mathbb{P}(|\Delta X_i^c|\geq\Delta_{n,i}^\beta)+c\mathbb{E}[(\Delta X_i^c)^2 1_{\left\{|\Delta X_i^c|\geq\Delta_{n,i}^\beta\right\}}]\leq$$

$$\leq c\Delta_{n,i}^{2\beta+r(\frac{1}{2}-\beta)}+c\mathbb{E}[(\Delta X_i^c)^4]^{\frac{1}{2}}\mathbb{P}(|\Delta X_i^c|\geq\Delta_{n,i}^\beta)^{\frac{1}{2}}\leq c\Delta_{n,i}^{[2\beta+r(\frac{1}{2}-\beta)]\wedge[1+\frac{r}{2}(\frac{1}{2}-\beta)]}, \tag{153}$$

where we have used Cauchy Schwartz inequality, (38) and the fourth point of Lemma 1. Therefore we get

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}}]\leq c\Delta_{n,i}^{[r(\frac{1}{2}-\beta)-1+\alpha\beta]\wedge[\frac{r}{2}(\frac{1}{2}-\beta)-\beta(2-\alpha)]}, \tag{154}$$

that converges to 0 for $n\to\infty$ since we can always find $r\geq 1$ such that the exponent $\Delta_{n,i}$ is positive.
In order to conclude the study of $I_1^n(i)$, we study $I_{1,2}^n 1_{\left\{|\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}}$.

$$\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}}]\leq c\left\|\varphi'\right\|_\infty\Delta_{n,i}^{-\beta}\mathbb{E}[(\Delta X_i^J)^2|\Delta X_i-\Delta X_i^J|1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta,|\Delta X_i^J|\leq 4\Delta_{n,i}^\beta\right\}}], \tag{155}$$

where we have used the smoothness of φ. Using Holder inequality and the fourth point of Lemma 1 it is upper bounded by

$$c\Delta_{n,i}^{-\beta}\mathbb{E}[|\Delta X_i^c|^p]^{\frac{1}{p}}\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta,|\Delta X_i^J|\leq 4\Delta_{n,i}^\beta\right\}}]^{\frac{1}{q}}\leq c\Delta_{n,i}^{\frac{1}{2}-\beta}\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta,|\Delta X_i^J|\leq 4\Delta_{n,i}^\beta\right\}}]^{\frac{1}{q}}. \tag{156}$$

Now, since our indicator function $1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^{\beta},|\Delta X_i^J|\leq 4\Delta_{n,i}^{\beta}\right\}}$ is less then $1_{\left\{|\Delta X_i^J|\leq 4\Delta_{n,i}^{\beta}\right\}}$, we can use the first point of Lemma 4. Through the use of the conditional expectation we get

$$\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^{\beta},|\Delta X_i^J|\leq 4\Delta_{n,i}^{\beta}\right\}}]^{\frac{1}{q}}\leq c\Delta_{n,i}^{\frac{1+\beta(2q-\alpha)}{q}}\mathbb{E}[R(1,X_{t_i})]\leq c\Delta_{n,i}^{\frac{1+\beta(2q-\alpha)}{q}}, \tag{157}$$

where in the last inequality we have used the polynomial growth of R and the third point of Lemma 2. Replacing (157) in (156) and taking q small (next to 1), we obtain $\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_{n,i}^{\beta}\right\}}]\leq c\Delta_{n,i}^{\frac{1}{2}+\beta+1-\alpha\beta-\epsilon}$. It follows

$$\frac{\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_{n,i}^{\beta}\right\}}]}{\Delta_{n,i}^{\beta(2-\alpha)+1}}\leq c\Delta_{n,i}^{\frac{1}{2}-\beta-\epsilon}, \tag{158}$$

that goes to 0 for $n\to\infty$ since we can always find an ϵ as small as the exponent on $\Delta_{n,i}$ is positive, for $\beta\in(0,\frac{1}{2})$.
Let us now consider $I_2^n(i)$.

$$I_2^n(i)=I_2^n(i)\,1_{\left\{|\Delta X_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}+I_2^n(i)\,1_{\left\{|\Delta X_i^J|>2\Delta_{n,i}^{\beta}\right\}}=:I_{2,1}^n+I_{2,2}^n. \tag{159}$$

Concerning the first term of (159), we have

$$\mathbb{E}[|I_{2,1}^n|]\leq\Delta_{n,i}^{-\beta}\left\|\varphi'\right\|_{\infty}\mathbb{E}[(\Delta X_i^J)^2|\Delta X_i^J-\Delta\tilde{X}_i^J|1_{\left\{|\Delta X_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]\leq$$

$$\leq c\Delta_{n,i}^{-\beta}\mathbb{E}[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]^{\frac{1}{2}}\mathbb{E}[|\Delta X_i^J-\Delta\tilde{X}_i^J|^2]^{\frac{1}{2}}, \tag{160}$$

where we have used the smoothness of φ and Cauchy-Schwartz inequality. Using again the first point of Lemma 4, we have that

$$\mathbb{E}[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]^{\frac{1}{2}}=\mathbb{E}[\mathbb{E}_i[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]]^{\frac{1}{2}}\leq\Delta_{n,i}^{\frac{1+\beta(4-\alpha)}{2}}\mathbb{E}[R(1,X_{t_i})]\leq c\Delta_{n,i}^{\frac{1}{2}+2\beta-\frac{\alpha\beta}{2}}, \tag{161}$$

where we have also used the polynomial growth of R and the third point of Lemma 2.
We now introduce a lemma that will be proved later:

Lemma 9. *Suppose that Assumption 1 to 4 hold. Then*

1. *$\forall q\geq 2$ we have*
$$\mathbb{E}[|\Delta X_i^J-\Delta\tilde{X}_i^J|^q]\leq c\Delta_{n,i}^2, \tag{162}$$
2. *for $q\in[1,2]$ and $\alpha<1$, we have*
$$\mathbb{E}[|\Delta X_i^J-\Delta\tilde{X}_i^J|^q]^{\frac{1}{q}}\leq c\Delta_{n,i}^{\frac{1}{2}+\frac{1}{q}}. \tag{163}$$

Replacing (161) and (162) in (160) we get

$$\mathbb{E}[|I_{2,1}^n|]\leq c\Delta_{n,i}^{-\beta+\frac{1}{2}+2\beta-\frac{\alpha\beta}{2}+1}=c\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}. \tag{164}$$

Hence

$$\frac{\mathbb{E}[|I_{2,1}^n|]}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\leq c\Delta_{n,i}^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \tag{165}$$

that goes to 0 for $n\to\infty$ since the exponent on $\Delta_{n,i}$ is positive for $\beta<\frac{1}{2(1-\frac{\alpha}{2})}$, that is always true with α and β in the intervals choosen.
We now want to show that also $I_{2,2}^n$ is $o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1})$. We split $I_{2,2}^n$ on the sets $\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}$ and $\left\{|\Delta\tilde{X}_i^J|>2\Delta_{n,i}^{\beta}\right\}$. We observe that, by the definition of φ, $I_{2,2}^n$ is null on the second set. Adding and subtracting $\Delta\tilde{X}_i^J$ in $I_{2,2}^n 1_{\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}$ we have

$$\mathbb{E}[|I_{2,2}^n|1_{\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]\leq c\mathbb{E}[(\Delta X_i^J-\Delta\tilde{X}_i^J)^2|\varphi_{\Delta_{n,i}^{\beta}}(\Delta X_i^J)-\varphi_{\Delta_{n,i}^{\beta}}(\Delta\tilde{X}_i^J)|1_{\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta},|\Delta X_i^J|>2\Delta_{n,i}^{\beta}\right\}}]+$$

$$+c\mathbb{E}[(\Delta\tilde{X}_i^J)^2|\varphi_{\Delta_{n,i}^{\beta}}(\Delta X_i^J)-\varphi_{\Delta_{n,i}^{\beta}}(\Delta\tilde{X}_i^J)|1_{\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]. \tag{166}$$

On the second term of (166) we can act exactly as we have done in $I_{2,1}^n$, with $\Delta\tilde{X}_i^J$ instead of ΔX_i^J (and so using (32) instead of (31)). We get

$$\mathbb{E}[(\Delta\tilde{X}_i^J)^2|\varphi_{\Delta_{n,i}^{\beta}}(\Delta X_i^J)-\varphi_{\Delta_{n,i}^{\beta}}(\Delta\tilde{X}_i^J)|1_{\left\{|\Delta\tilde{X}_i^J|\leq 2\Delta_{n,i}^{\beta}\right\}}]\leq c\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}. \tag{167}$$

Concerning the first term of (166), by the definition of φ we know it is

$$\mathbb{E}[(\Delta X_i^J - \Delta \tilde{X}_i^J)^2| - \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)|1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_{n,i}^\beta, |\Delta X_i^J| > 2\Delta_{n,i}^\beta\right\}}] \leq c\mathbb{E}[(\Delta X_i^J - \Delta \tilde{X}_i^J)^2] \leq c\Delta_{n,i}^2, \quad (168)$$

where in the last inequality we have used (162). Using (166) - (168) it follows

$$\mathbb{E}[|I_{2,2}^n|] = \mathbb{E}[|I_{2,2}^n|1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_{n,i}^\beta\right\}}] \leq c\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}} + c\Delta_{n,i}^2 = c\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}, \quad (169)$$

considering that $\Delta_{n,i}^2$ is negligible compared to $\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}$ since $\beta < \frac{1}{2(1-\frac{\alpha}{2})}$. Hence

$$\frac{\mathbb{E}[|I_{2,2}^n|]}{\Delta_{n,i}^{1+\beta(2-\alpha)}} \leq c\Delta_{n,i}^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \quad (170)$$

that goes to 0 for $n \to \infty$.
Concerning $I_3^n(i)$, we have

$$\mathbb{E}[|I_3^n(i)|] \leq c\mathbb{E}[(\Delta X_i^J - \Delta \tilde{X}_i^J)^2] \leq c\Delta_{n,i}^2, \quad (171)$$

where the last inequality follows from (162). Hence $I_3^n(i) = o_{L^1}(\Delta_{n,i}^{\beta(2-\alpha)+1})$, indeed

$$\frac{\mathbb{E}[|I_3^n(i)|]}{\Delta_{n,i}^{1+\beta(2-\alpha)}} \leq c\Delta_{n,i}^{1-2\beta+\alpha\beta}, \quad (172)$$

that goes to 0 for $n \to \infty$ considering that the exponent on $\Delta_{n,i}$ is positive for $\beta < \frac{1}{2-\alpha}$, condition that is always satisfied for $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$.
Let us now consider $I_4^n(i)$. Using Cauchy-Schwartz inequality it is

$$\mathbb{E}[|I_4^n(i)|] \leq c\mathbb{E}[(\Delta X_i^J - \Delta \tilde{X}_i^J)^2]^{\frac{1}{2}}\mathbb{E}[(\Delta \tilde{X}_i^J)^2\varphi_{\Delta_{n,i}^\beta}^2(\Delta \tilde{X}_i^J)]^{\frac{1}{2}} \leq c\Delta_{n,i}\Delta_{n,i}^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)} = c\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}, \quad (173)$$

where we have used (162) and the first point of Lemma 4. It follows

$$\frac{\mathbb{E}[|I_4^n(i)|]}{\Delta_{n,i}^{1+\beta(2-\alpha)}} \leq c\Delta_{n,i}^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \quad (174)$$

that goes to 0 for $n \to \infty$ since the exponent on $\Delta_{n,i}$ is more than 0 if $\beta < \frac{1}{2(1-\frac{\alpha}{2})}$, that is always true. Using (149), (152), (154), (158), (165), (170), (172) and (174) we obtain (27).

In order to prove (28), we use again reformulation (149). Replacing it in the left hand side of (28) it turns out that our goal is to show that

$$\frac{1}{n}\sum_{i=0}^{n-1}(\sum_{k=1}^{4} I_k^n(i))\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \quad (175)$$

Using a conditional on $\mathcal{F}_{t_i}$ version of (159), (164) and (169) we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}_i[|I_2^n(i)f(X_{t_i})|]\frac{1}{\Delta_{n,i}} \leq \frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-1-\epsilon}, X_{t_i}) = \frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{\frac{1}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon}, X_{t_i}).$$

Since $\beta(1-\frac{\alpha}{2})$ is always more than zero and, $\forall\tilde{\epsilon} > 0$ we can always find ϵ smaller than it, we get

$$\frac{1}{n}\sum_{i=0}^{n-1} I_2^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \quad (176)$$

From a conditional version of (171) we get that $\frac{1}{n}\sum_{i=0}^{n-1} I_3^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}}$ is upper bounded in conditional norm 1 by $\frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{2-1-\epsilon}, X_{t_i}) = \frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{1-\epsilon}, X_{t_i})$ and so

$$\frac{1}{n}\sum_{i=0}^{n-1} I_3^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \quad (177)$$

Using a conditional version of (173) we get that $\frac{1}{n}\sum_{i=0}^{n-1} I_4^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}}$ is upper bounded in conditional norm 1 by $\frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-1-\epsilon}, X_{t_i}) = \frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{\frac{1}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon}, X_{t_i})$, hence

$$\frac{1}{n}\sum_{i=0}^{n-1} I_4^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \quad (178)$$

Concerning $I_1^n(i)$, we consider $I_{1,1}^n(i)$ and $I_{1,2}^n(i)$ as defined in (150). Using a conditional version of (151) on $I_{1,1}^n(i)$ it follows that $\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1} I_{1,1}^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}}$ is upper bounded in norm 1 by $\frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{(\frac{1}{2}-\beta)r+2\beta-1-\frac{1}{2}+\tilde{\epsilon}}, X_{t_i}) = \frac{1}{n}\sum_{i=0}^{n-1} R(\Delta_n^{(\frac{1}{2}-\beta)r+2\beta-\frac{3}{2}+\tilde{\epsilon}}, X_{t_i})$, that goes to zero because we can find r big enough such that the exponent on Δ_n is positive, hence

$$\frac{1}{n}\sum_{i=0}^{n-1} I_{1,1}^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{179}$$

Acting as we did in the proof of (27), we consider $I_{1,2}^n(i)$ on the sets $\left\{|\Delta X_i^J| \geq 4\Delta_{n,i}^\beta\right\}$ and $\left\{|\Delta X_i^J| < 4\Delta_{n,i}^\beta\right\}$. Again, from (153) and the arbitrariness of $r > 0$ it follows

$$\frac{1}{n}\sum_{i=0}^{n-1} I_{1,2}^n(i)1_{\left\{|\Delta X_i^J|\geq 4\Delta_{n,i}^\beta\right\}}\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{180}$$

When $|\Delta X_i^J| < 4\Delta_{n,i}^\beta$ we act in a different way, considering the development up to second order of $\varphi_{\Delta_{n,i}^\beta}$, centered in ΔX_i^J:

$$I_{1,2}^n(i)1_{\left\{|\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}} = [(\Delta X_i^J)^2\Delta X_i^c\varphi'_{\Delta_{n,i}^\beta}(\Delta X_i^J)\Delta_{n,i}^{-\beta} + (\Delta X_i^J)^2(\Delta X_i^c)^2\varphi''_{\Delta_{n,i}^\beta}(X_u)\Delta_{n,i}^{-2\beta}]1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}} =$$

$$=: \hat{I}_1^n(i)1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}} + \hat{I}_2^n(i)1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}},$$

where $X_u \in [\Delta X_i^J, \Delta X_i]$. Now, acting like we did in (155), (156) and (157), taking q next to 1 we get

$$\mathbb{E}_i[|\hat{I}_2^n(i)1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}}|] \leq R(\Delta_n^{1+\beta(2-\alpha)-\epsilon+1-2\beta}, X_{t_i}) = R(\Delta_n^{2-\alpha\beta-\epsilon}, X_{t_i}).$$

Since for each $\tilde{\epsilon} > 0$ we can find an ϵ such that $\tilde{\epsilon} - \epsilon > 0$ it follows, taking the conditional expectation

$$\frac{1}{n}\sum_{i=0}^{n-1} \hat{I}_2^n(i)1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}}\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{181}$$

Concerning $\hat{I}_1^n(i)1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}}$, we no longer consider the indicator function because it is

$$(\Delta X_i^J)^2\Delta X_i^c\varphi'_{\Delta_{n,i}^\beta}(\Delta X_i^J)\Delta_{n,i}^{-\beta} + (\Delta X_i^J)^2\Delta X_i^c\varphi'_{\Delta_{n,i}^\beta}(\Delta X_i^J)\Delta_{n,i}^{-\beta}(1_{\left\{|\Delta X_i|\leq 3\Delta_{n,i}^\beta, |\Delta X_i^J|<4\Delta_{n,i}^\beta\right\}} - 1)$$

and the second term here above is different from zero only on a set smaller that $\left\{|\Delta X_i| \geq 3\Delta_{n,i}^\beta\right\}$ or $\left\{|\Delta X_i^J| \geq 4\Delta_{n,i}^\beta\right\}$, on which we have already proved the result (see the study of $I_{1,1}^n(i)$ in (179) and $I_{1,2}^n(i)$ in (180)). We want to show that

$$\frac{1}{n}\sum_{i=0}^{n-1} \hat{I}_1^n(i)\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{182}$$

We start from the reformulation

$$\hat{I}_1^n(i) = \Delta X_i^c\Delta_{n,i}^{-\beta}[(\Delta X_i^J)^2(\varphi'_{\Delta_{n,i}^\beta}(\Delta X_i^J) - \varphi'_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)) + (\Delta X_i^J - \Delta\tilde{X}_i^J)^2\varphi'_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J) +$$

$$+2\Delta\tilde{X}_i^J(\Delta X_i^J - \Delta\tilde{X}_i^J)\varphi'_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J) + (\Delta\tilde{X}_i^J)^2\varphi'_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)] = \sum_{j=1}^4 \hat{I}_{1,j}^n(i).$$

and we observe that, after have used Holder inequality and have remarked that $\varphi'_{\Delta_{n,i}^\beta}$ acts like $\varphi_{\Delta_{n,i}^\beta}$, we can act on $\hat{I}_{1,1}^n$ as we did on I_2^n, on $\hat{I}_{1,2}^n$ as on I_3^n and on $\hat{I}_{1,3}^n$ as on I_4^n. So we get, using also Holder inequality and the fourth point of Lemma 1,

$$\mathbb{E}_i[|\hat{I}_{1,1}^n(i) + \hat{I}_{1,2}^n(i) + \hat{I}_{1,3}^n(i)|] \leq R(\Delta_{n,i}^{\frac{1}{2}-\beta}, X_{t_i})(\mathbb{E}_i[|I_2^n(i)|^q]^{\frac{1}{q}} + \mathbb{E}[|I_3^n(i)|^q]^{\frac{1}{q}} + \mathbb{E}[|I_4^n(i)|^q]^{\frac{1}{q}}). \tag{183}$$

Now, taking q next to 1, we need the following lemma that we will prove later:

Lemma 10. *Suppose that Assumption 1 to 4 hold. Then, $\forall \epsilon > 0$,*

$$\mathbb{E}_i[|I_2^n(i)|^{1+\epsilon} + |I_3^n(i)|^{1+\epsilon} + |I_4^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R(\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon}, X_{t_i}), \tag{184}$$

with $I_2^n(i)$, $I_3^n(i)$ and $I_4^n(i)$ as defined in (149).

From (183) and (184) it follows

$$\frac{1}{n}\sum_{i=0}^{n-1}[\hat{I}_{1,1}^n(i) + \hat{I}_{1,2}^n(i) + \hat{I}_{1,3}^n(i)]\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{185}$$

On $\frac{1}{n}\sum_{i=0}^{n-1}\hat{I}_{1,4}^n\frac{f(X_{t_i})}{\Delta_{n,i}} =: \sum_{i=0}^{n-1}\zeta_{n,i}$ we want to use Lemma 9 in [11]. By the independence between L and W we get

$$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}] = \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1}f(X_{t_i})\Delta_{n,i}^{-1-\beta}\mathbb{E}_i[(\Delta\tilde{X}_i^J)^2\varphi'_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)]\mathbb{E}_i[\Delta X_i^c] = 0 \tag{186}$$

and

$$\frac{\Delta_n^{1-2(\frac{1}{2}-\tilde{\epsilon})}}{n\Delta_n}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})\Delta_{n,i}^{-2-2\beta}\mathbb{E}_i[(\Delta\tilde{X}_i^J)^4\varphi'^2_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)]\mathbb{E}_i[(\Delta X_i^c)^2] \leq c\Delta_n^{1-1+2\tilde{\epsilon}-2-2\beta+1+1+\beta(4-\alpha)} = c\Delta_n^{2\tilde{\epsilon}+2\beta-\alpha\beta}, \tag{187}$$

where we have also used the fourth point of Lemma 1, the fact that $\frac{1}{n\Delta_n}$ is bounded and the first point of Lemma 4. Using (186) and (187) we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\hat{I}_{1,4}^n\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$$

that, joint with (185) and the fact that the convergence in norm 1 implies the convergence in probability, give us (182). Using also (176) - (181) we get (175) and so (28).

In order to prove (29), we reformulate $\Delta X_i^J\varphi_{\Delta_{n,i}^\beta}(\Delta X_i)$ as we have already done in (149) getting

$$\begin{aligned}(\int_{t_i}^{t_{i+1}}a(X_s)dW_s)\Delta X_i^J\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) &= (\int_{t_i}^{t_{i+1}}a(X_s)dW_s)(\Delta X_i^J)[\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)]+\\ +(\int_{t_i}^{t_{i+1}}a(X_s)dW_s)(\Delta X_i^J)[\varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)-\varphi_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)]+(\int_{t_i}^{t_{i+1}}a(X_s)dW_s)(\Delta X_i^J-\Delta\tilde{X}_i^J)\varphi_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J)+\\ +(\int_{t_i}^{t_{i+1}}a(X_s)dW_s)(\Delta\tilde{X}_i^J)\varphi_{\Delta_{n,i}^\beta}(\Delta\tilde{X}_i^J) &=: \sum_{j=1}^4\tilde{I}_j^n(i).\end{aligned} \tag{188}$$

Comparing (188) with (29) it turns out that our goal is to prove that $\frac{1}{\Delta_{n,i}^{\beta(2-\alpha)+1}}\sum_{j=1}^3\mathbb{E}[|\tilde{I}_j^n(i)|] \to 0$, for $n \to \infty$ (again, acting as we do in the sequel it is also possible to show that $\sum_{j=1}^3\mathbb{E}_i[|\tilde{I}_j^n(i)|] \leq R(\Delta_{n,i}^{\beta(2-\alpha)+1}, X_{t_i})$). Let us start considering $\tilde{I}_1^n(i)$. Using Holder inequality, its expected value is upper bounded by

$$\mathbb{E}[|\int_{t_i}^{t_{i+1}}a(X_s)dW_s|^{p_1}]^{\frac{1}{p_1}}\mathbb{E}[|\Delta X_i^J|^{p_2}|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)|^{p_2}]^{\frac{1}{p_2}}. \tag{189}$$

We now act on $\mathbb{E}[|\Delta X_i^J|^{p_2}|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)|^{p_2}]^{\frac{1}{p_2}}$ as we did in the study of $I_1^n(i)$:

$$\begin{aligned}|\Delta X_i^J|^{p_2}|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)|^{p_2} &= |\Delta X_i^J|^{p_2}|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)|^{p_2}1_{\{|\Delta X_i|\geq 3\Delta_{n,i}^\beta\}}+\\ +|\Delta X_i^J|^{p_2}|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - \varphi_{\Delta_{n,i}^\beta}(\Delta X_i^J)|^{p_2}1_{\{|\Delta X_i|<3\Delta_{n,i}^\beta\}} &=: \tilde{I}_{1,1}^n + \tilde{I}_{1,2}^n.\end{aligned}$$

Concerning $\tilde{I}_{1,1}^n$, if $|\Delta X_i^J| \geq 2\Delta_{n,i}^\beta$ it is just 0, otherwise we can act exactly as we have done on $I_{1,1}^n$, taking $p_2 = 2$. Hence, $\forall r \geq 1$,

$$\mathbb{E}[|\tilde{I}_{1,1}^n|]^{\frac{1}{2}} \leq (c\Delta_{n,i}^{2\beta+r(\frac{1}{2}-\beta)})^{\frac{1}{2}} = c\Delta_{n,i}^{\beta+\frac{r}{2}(\frac{1}{2}-\beta)}. \tag{190}$$

Let us now consider $\tilde{I}^n_{1,2}$. If $|\Delta X^J_i| \geq 4\Delta^\beta_{n,i}$, we act again like we did on $I^n_{1,2}$, taking $p_2 = 2$. It yields again

$$\mathbb{E}[|\tilde{I}^n_{1,2}|1_{\left\{|\Delta X^J_i|\geq 4\Delta^\beta_{n,i}\right\}}]^{\frac{1}{2}} \leq c\Delta^{\beta+\frac{r}{2}(\frac{1}{2}-\beta)}_{n,i}. \tag{191}$$

If $|\Delta X^J_i| < 4\Delta^\beta_{n,i}$ we use the smoothness of φ and Holder inequality getting

$$\mathbb{E}[|\tilde{I}^n_{1,2}|1_{\left\{|\Delta X^J_i|<4\Delta^\beta_{n,i}\right\}}] \leq \Delta^{-\beta}_{n,i}\mathbb{E}[|\Delta X^J_i|^{p_2}|\varphi'(\zeta)|^{p_2}|\Delta X^c_i|^{p_2}1_{\left\{|\Delta X_i|<3\Delta^\beta_{n,i},|\Delta X^J_i|<4\Delta^\beta_{n,i}\right\}}]^{\frac{1}{p_2}} \leq$$

$$\leq \Delta^{-\beta}_{n,i}\mathbb{E}[|\Delta X^c_i|^{p_2\,p}]^{\frac{1}{p_2\,p}}\mathbb{E}[|\varphi'(\zeta)|^{p_2\,q}|\Delta X^J_i|^{p_2\,q}1_{\left\{|\Delta X_i|<3\Delta^\beta_{n,i},|\Delta X^J_i|<4\Delta^\beta_{n,i}\right\}}]^{\frac{1}{p_2\,q}}, \tag{192}$$

with ζ a point between ΔX^J_i and ΔX_i.
Now we observe that, if $|\Delta X^c_i| \geq \frac{\Delta^\beta_{n,i}}{4}$, then taking $p_2\,q = 1+\epsilon$ we have

$$\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon}|\Delta X^J_i|^{1+\epsilon}1_{\left\{|\Delta X_i|<3\Delta^\beta_{n,i},|\Delta X^J_i|<4\Delta^\beta_{n,i},|\Delta X^c_i|\geq\frac{\Delta^\beta_{n,i}}{4}\right\}}]^{\frac{1}{1+\epsilon}} \leq c\Delta^{\beta+r(\frac{1}{2}-\beta)\frac{1}{1+\epsilon}}_{n,i}$$

where we have used the bound on $|\Delta X^J_i|$ given by the indicator function, the boundedness of φ' and (38). Otherwise, by the definition of φ, we know that $|\varphi'(\zeta)| \neq 0$ only if $|\zeta| \in (\Delta^\beta_{n,i}, 2\Delta^\beta_{n,i})$. Then $\Delta^\beta_{n,i} \leq |\zeta| \leq |\Delta X_i| + |\Delta X^J_i| \leq 2|\Delta X^J_i| + |\Delta X^c_i| \leq 2|\Delta X^J_i| + \frac{\Delta^\beta_{n,i}}{4}$, hence $|\Delta X^J_i| \geq \frac{3}{8}\Delta^\beta_{n,i} \geq \frac{\Delta^\beta_{n,i}}{4}$ and so we can say it is

$$\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon}|\Delta X^J_i|^{1+\epsilon}1_{\left\{|\Delta X_i|<3\Delta^\beta_{n,i},|\Delta X^J_i|<4\Delta^\beta_{n,i},|\Delta X^c_i|<\frac{\Delta^\beta_{n,i}}{4}\right\}}]^{\frac{1}{1+\epsilon}} \leq c\mathbb{E}[|\Delta X^J_i|^{1+\epsilon}1_{\left\{\frac{\Delta^\beta_{n,i}}{4}\leq|\Delta X^J_i|<4\Delta^\beta_{n,i},\right\}}].$$

Using the second point of Lemma 4, passing through the conditional expected value we get it is upper bounded by

$$\Delta^{1+\beta(1+\epsilon-\alpha)}_{n,i}\mathbb{E}[R(1, X_{t_i})] \leq c\Delta^{1+\beta(1+\epsilon-\alpha)}_{n,i},$$

where in the last inequality we have used the polynomial growth of R and the third point of Lemma 2. Hence

$$\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon}|\Delta X^J_i|^{1+\epsilon}1_{\left\{|\Delta X_i|<3\Delta^\beta_{n,i},|\Delta X^J_i|<4\Delta^\beta_{n,i}\right\}}]^{\frac{1}{1+\epsilon}} \leq c\Delta^{[\beta+r(\frac{1}{2}-\beta)-\epsilon]\wedge[1+\beta(1+\epsilon-\alpha)]\frac{1}{1+\epsilon}}_{n,i} = c\Delta^{[1+\beta(1+\epsilon-\alpha)]\frac{1}{1+\epsilon}}_{n,i}. \tag{193}$$

The last equality follows from the fact that, for each choice of $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$, we can always find $r \geq 1$ and $\epsilon > 0$ such that $\beta + r(\frac{1}{2}-\beta) - \epsilon > 1 + \beta(1+\epsilon-\alpha)$.
Replacing (193) in (192) and using the fourth point of Lemma 1 we have that

$$\mathbb{E}[|\tilde{I}^n_{1,2}|1_{\left\{|\Delta X^J_i|<4\Delta^\beta_{n,i}\right\}}]^{\frac{1}{p_2}} \leq c\Delta^{[\frac{1}{2}-\beta+1+\beta(1+\epsilon-\alpha)]\frac{1}{p_2}}_{n,i} = c\Delta^{(\frac{3}{2}-\alpha\beta-\epsilon)\frac{1}{p_2}}_{n,i} = c\Delta^{\frac{3}{2}-\alpha\beta-\epsilon}_{n,i}, \tag{194}$$

the last equality follows from the choice of both p_2 and q next to 1. Using (190), (191) and (194) we get

$$\mathbb{E}[|\Delta X^J_i|^{p_2}|\varphi_{\Delta^\beta_{n,i}}(\Delta X_i) - \varphi_{\Delta^\beta_{n,i}}(\Delta X^J_i)|^{p_2}]^{\frac{1}{p_2}} \leq c\Delta^{[\beta+\frac{r}{2}(\frac{1}{2}-\beta)]\wedge[\frac{3}{2}-\alpha\beta-\epsilon]}_{n,i} = c\Delta^{\frac{3}{2}-\alpha\beta-\epsilon}_{n,i}. \tag{195}$$

Replacing (37) and (195) in (189) it follows

$$\mathbb{E}[|\tilde{I}^n_1(i)|] \leq c\Delta^{2-\alpha\beta-\epsilon}_{n,i}, \tag{196}$$

hence

$$\frac{\mathbb{E}[|\tilde{I}^n_1(i)|]}{\Delta^{1+\beta(2-\alpha)}_{n,i}} \leq c\Delta^{1-2\beta-\epsilon}_{n,i}. \tag{197}$$

Since we can always find an $\epsilon > 0$ such that $1-2\beta-\epsilon > 0$, the expected value above goes to 0 for $n \to \infty$. Concerning $\tilde{I}^n_2(i)$, we split again on $\tilde{I}^n_{2,1} := \tilde{I}^n_2(i)1_{\left\{|\Delta X^J_i|\leq 2\Delta^\beta_{n,i}\right\}}$ and $\tilde{I}^n_{2,2} := \tilde{I}^n_2(i)1_{\left\{|\Delta X^J_i|>2\Delta^\beta_{n,i}\right\}}$.

$$\mathbb{E}[|\tilde{I}^n_{2,1}|] = \mathbb{E}[|\tilde{I}^n_2(i)|1_{\left\{|\Delta X^J_i|\leq 2\Delta^\beta_{n,i}\right\}}] \leq c\Delta^{-\beta}_{n,i}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s||\Delta X^J_i||\Delta X^J_i - \Delta\tilde{X}^J_i|1_{\left\{|\Delta X^J_i|\leq 2\Delta^\beta_{n,i}\right\}}] \leq$$

$$\leq c\Delta^{-\beta}_{n,i}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s|^2|\Delta X^J_i|^2 1_{\left\{|\Delta X^J_i|\leq 2\Delta^\beta_{n,i}\right\}}]^{\frac{1}{2}}\mathbb{E}[|\Delta X^J_i - \Delta\tilde{X}^J_i|^2]^{\frac{1}{2}} \leq$$

$$\leq c\Delta^{1-\beta}_{n,i}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s|^{2p}]^{\frac{1}{2p}}\mathbb{E}[|\Delta X^J_i|^{2q}1_{\left\{|\Delta X^J_i|\leq 2\Delta^\beta_{n,i}\right\}}]^{\frac{1}{2q}},$$

where we have used Cauchy-Schwartz inequality, (162) and Holder inequality. Now we take p big and q next to 1, using (37) and the first point of Lemma 4 we get

$$\mathbb{E}[|\tilde{I}^n_{2,1}|] \leq c\Delta_{n,i}^{1-\beta+\frac{1}{2}+\frac{1}{2}+\frac{\beta}{2}(2-\alpha)-\epsilon} \tag{198}$$

and so

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}^n_{2,1}|] \leq \Delta_{n,i}^{1-2\beta+\frac{\alpha\beta}{2}-\epsilon}. \tag{199}$$

It goes to 0 for $n \to \infty$ because we can always find an $\epsilon > 0$ such that the exponent in Δ_{n_i} is positive. Let us now consider $\tilde{I}^n_{2,2} = \tilde{I}^n_{2,2}1_{\{|\Delta\tilde{X}^J_i|\leq 2\Delta^\beta_{n,i}\}} + \tilde{I}^n_{2,2}1_{\{|\Delta\tilde{X}^J_i|>2\Delta^\beta_{n,i}\}}$. From the definition of φ, $\tilde{I}^n_{2,2}1_{\{|\Delta\tilde{X}^J_i|>2\Delta^\beta_{n,i}\}} = 0$.

$$\mathbb{E}[|\tilde{I}^n_{2,2}|1_{\{|\Delta\tilde{X}^J_i|\leq 2\Delta^\beta_{n,i}\}}] = \mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s||\Delta\tilde{X}^J_i||\varphi_{\Delta^\beta_{n,i}}(\Delta X^J_i)-\varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}^J_i)|1_{\{|\Delta\tilde{X}^J_i|\leq 2\Delta^\beta_{n,i},|\Delta X^J_i|>2\Delta^\beta_{n,i}\}}]+$$

$$+\mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s||\Delta X^J_i - \Delta\tilde{X}^J_i||\varphi_{\Delta^\beta_{n,i}}(\Delta X^J_i) - \varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}^J_i)|1_{\{|\Delta\tilde{X}^J_i|\leq 2\Delta^\beta_{n,i},|\Delta X^J_i|>2\Delta^\beta_{n,i}\}}] \leq$$

$$\leq c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon} + \mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s||\Delta X^J_i - \Delta\tilde{X}^J_i|| - \varphi_{\Delta^\beta_{n,i}}(\Delta\tilde{X}^J_i)|],$$

where we have acted exactly like we did in $\tilde{I}^n_{2,1}$, using that $\Delta\tilde{X}^J_i$ is less then $2\Delta^\beta_{n,i}$. We have also used that, by the definition of φ, evaluated in ΔX^J_i it is zero. Now we use Holder inequality, (37) and the boundedness of φ to get

$$\mathbb{E}[|\tilde{I}^n_{2,2}|] \leq c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon}+\mathbb{E}[|\int_{t_i}^{t_{i+1}} a(X_s)dW_s|^p]^{\frac{1}{p}}\mathbb{E}[|\Delta X^J_i-\Delta\tilde{X}^J_i|^q]^{\frac{1}{q}} \leq c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon}+c\Delta_{n,i}^{\frac{1}{2}}\mathbb{E}[|\Delta X^J_i-\Delta\tilde{X}^J_i|^q]^{\frac{1}{q}}.$$

Now, if $\alpha < 1$ we use (163), with $q = 1+\epsilon$, getting

$$\mathbb{E}[|\tilde{I}^n_{2,2}|] \leq c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon} + c\Delta_{n,i}^{\frac{1}{2}+\frac{1}{2}+\frac{1}{1+\epsilon}} = c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon}. \tag{200}$$

Therefore, for $\alpha < 1$, we have

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|I^n_{2,2}|] \leq c\Delta_{n,i}^{1-2\beta+\frac{\alpha\beta}{2}-\epsilon}. \tag{201}$$

We can find an $\epsilon > 0$ such that the exponent on $\Delta_{n,i}$ is positive hence, if $\alpha < 1$, then $I^n_{2,2} = o_{L^1}(\Delta_{n,i}^{1+\beta(2-\alpha)})$. Otherwise, if $\alpha \geq 1$, we use (162) having taken $q = 2$. We get

$$\mathbb{E}[|\tilde{I}^n_{2,2}|] \leq c\Delta_{n,i}^{2-\frac{\alpha\beta}{2}-\epsilon} + c\Delta_{n,i}^{\frac{1}{2}+1} = c\Delta_{n,i}^{\frac{3}{2}}.$$

It follows that, for $\alpha \geq 1$, it is

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|I^n_{2,2}|] \leq c\Delta_{n,i}^{\frac{1}{2}-\beta(2-\alpha)}. \tag{202}$$

We observe that the exponent on $\Delta_{n,i}$ is more than 0 if $\beta < \frac{1}{2}\frac{1}{(2-\alpha)}$, that is always true for $\beta \in (0, \frac{1}{2})$ and $\alpha \in [1, 2)$.
To conclude, we use on $\tilde{I}_3(i)$ Holder inequality, (37), the boundedness of φ and then we act as we did on $\tilde{I}^n_{2,2}$, using (163) or (162), depending on whether or not α is less than 1. In the case $\alpha < 1$ we get

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}^n_3(i)|] \leq \frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}c\Delta_{n,i}^{\frac{1}{2}+\frac{1}{2}+\frac{1}{1+\epsilon}} = c\Delta_{n,i}^{1-\beta(2-\alpha)-\epsilon}, \tag{203}$$

that goes to 0 for $n \to \infty$ since we can always find $\epsilon > 0$ such that the exponent on $\Delta_{n,i}$ is positive. Otherwise it follows

$$\frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}^n_3(i)|] \leq \frac{1}{\Delta_{n,i}^{1+\beta(2-\alpha)}}c\Delta_{n,i}^{\frac{3}{2}} = c\Delta_{n,i}^{\frac{1}{2}-\beta(2-\alpha)}. \tag{204}$$

The exponent on $\Delta_{n,i}$ is positive if $\beta < \frac{1}{2}\frac{1}{(2-\alpha)}$, that is always true since we are in the case $\alpha \geq 1$. Hence $\tilde{I}^n_3(i) = o_{L^1}(\Delta_{n,i}^{1+\beta(2-\alpha)})$.
From (197) - (204) and the reformulation (188), it follows (29).

Replacing reformulation (188) in the left hand side of (30), it turns out that the theorem is proved if

$$\frac{1}{n}\sum_{i=0}^{n-1}(\sum_{k=1}^{3}\tilde{I}_k^n(i))\frac{f(X_{t_i})}{\Delta_{n,i}} = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{205}$$

Using a conditional version of equations (196), (198), (200), (203) and (204) (adding in the last two $\beta(2-\alpha)$ in the exponent of $\Delta_{n,i}$) we easily get (205) and so (30). □

A.2 Proof of Lemma 3

Proof. In this proof, we emphasize that the sampling scheme $(t_i)_{i=0,\dots,n}$depends on n, by noting $t_i = T_{n,i}$, and we have $T_{n,j} = \sum_{i=0}^{j-1}\Delta_{n,i}$. We define $X_{n,j} := \frac{1}{T_{n,j}}\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} h(X_s)ds$ and we observe that

$$T_{n,j+1}X_{n,j+1} - T_{n,j}X_{n,j} = \int_0^{T_{n,j+1}} h(X_s)ds - \int_0^{T_{n,j}} h(X_s)ds = \int_{T_{n,j}}^{T_{n,j+1}} h(X_s)ds =$$

$$= \int_{T_{n,j}}^{T_{n,j+1}} [h(X_s) - h(X_{T_{n,j}})]ds + \Delta_{n,j}h(X_{T_{n,j}}).$$

Hence

$$S_n := \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}h(X_{T_{n,i}}) = \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}\Delta_{n,i}h(X_{T_{n,i}}) =$$

$$= \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}[T_{n,i+1}X_{n,i+1} - T_{n,i}X_{n,i}] + \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}(\int_{T_{n,i}}^{T_{n,i+1}}[h(X_{T_{n,i}}) - h(X_s)]ds). \tag{206}$$

Now, concerning the second term of (206), we have that its norm 1 is upper bounded by

$$\frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}(\int_{T_{n,i}}^{T_{n,i+1}}\|h'\|_\infty\,\mathbb{E}[|X_{T_{n,i}} - X_s|]ds) \le \frac{c}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}\int_{T_{n,i}}^{T_{n,i+1}}\|h'\|_\infty|s-T_{n,i}|ds \le$$

$$\le \frac{c}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta+\frac{1}{2}} \le c\Delta_n^{\frac{1}{2}},$$

where we have used the regularity of h, Cauchy-Schwartz inequality, the first point of Lemma 1 and the fact that $\Delta_{n,i} \le \Delta_n$. Therefore the second term of (206) converges to zero in norm 1, that implies the convergence to zero in probability.
Concerning the first term of (206), it is

$$\tilde{S}_n := \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta-1}[T_{n,i+1}X_{n,i+1} - T_{n,i}X_{n,i}] = \frac{\sum_{i=1}^{n-1}T_{n,i}X_{n,i}(\Delta_{n,i-1}^{\delta-1} - \Delta_{n,i}^{\delta-1})}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}} + \frac{\Delta_{n,n-1}^{\delta-1}T_{n,n}X_{n,n}}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}.$$

We define

$$a_{n,j} := \frac{1}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}T_{n,j}(\Delta_{n,j-1}^{\delta-1} - \Delta_{n,j}^{\delta-1}) \qquad \text{for } j \le n-2,$$

$$a_{n,n-1} = \frac{\Delta_{n,n-1}^{\delta-1}T_{n,n}}{\sum_{i=0}^{n-1}\Delta_{n,i}^{\delta}}.$$

Let us start assuming the following two conditions, that we will prove later:

$$\exists c > 0 : \sup_{n\ge 1}\sum_{i=0}^{n-1}|a_{n,i}| < c < \infty, \tag{207}$$

$$\sum_{i=0}^{n-1}a_{n,i} = 1. \tag{208}$$

We now write

$$\tilde{S}_n = \sum_{i=0}^{n-1}a_{n,i}X_{n,i} = X + \sum_{i=0}^{n-1}a_{n,i}(X_{n,i} - X),$$

with $X := \int_{\mathbb{R}} h(x)\pi(dx)$. In the last equality here above we have used (208). In order to show that

$$\sum_{i=0}^{n-1} a_{n,i}(X_{n,i} - X) \xrightarrow{\mathbb{P}} 0, \tag{209}$$

we first prove that, $\forall i \in \{0, ..., n-1\}$, $T_{n,i} = O(\frac{i}{n} T_{n,n})$. Indeed, we clearly have

$$n \min_k \Delta_{n,k} \leq T_{n,n} = \sum_{j=0}^{n-1} \Delta_{n,j} \leq n \max_k \Delta_{n,k}.$$

Using the first point of Assumption S2 it follows

$$\frac{T_{n,n}}{nc_2} \leq \frac{1}{c_2} \max_k \Delta_{n,k} \leq \min_k \Delta_{n,k} \leq \frac{T_{n,n}}{n} \qquad \text{and so}$$

$$\frac{T_{n,n}}{nc_2} \leq \min_k \Delta_{n,k} \leq \frac{T_{n,n}}{n}, \quad \frac{T_{n,n}}{n} \leq \max_k \Delta_{n,k} \leq \frac{c_2 T_{n,n}}{n}.$$

Hence

$$\frac{i\, T_{n,n}}{nc_2} \leq T_{n,i} = \sum_{j=0}^{i-1} \Delta_{n,j} \leq \frac{i\, c_2\, T_{n,n}}{n}. \tag{210}$$

Now, using ergodic theorem, we know that $\forall \epsilon > 0$ $\exists T_\epsilon > 0$ such that, $\forall T \geq T_\epsilon$,

$$|\frac{1}{T} \int_0^T h(X_s)ds - \int_{\mathbb{R}} h(x)\pi(dx)| < \epsilon. \tag{211}$$

By the equation (210), we choose $\eta > 0$, $\eta < 1$ such that, $\forall i \geq \eta n$, $T_{n,i} \geq T_\epsilon$.
We can see $\sum_{i=0}^{n-1} a_{n,i}(X_{n,i} - X)$ as $\sum_{i=0}^{\lfloor \eta n \rfloor} a_{n,i}(X_{n,i} - X) + \sum_{i=\lfloor \eta n \rfloor + 1}^{n-1} a_{n,i}(X_{n,i} - X)$. Using (207) and (211) we get

$$\sum_{i=\lfloor \eta n \rfloor + 1}^{n-1} |a_{n,i}||X_{n,i} - X| \leq \epsilon c. \tag{212}$$

Concerning $\sum_{i=0}^{\lfloor \eta n \rfloor} a_{n,i}(X_{n,i} - X)$, we use that $|X_{n,i} - X|$ is bounded and that, by its definition, $T_{n,i}$ is upper bounded by $T_{n,\eta n}$. Therefore, using also that $\delta > 0$ and $\delta - 1 < 0$, we have

$$\sum_{i=0}^{\lfloor \eta n \rfloor} |a_{n,i}||X_{n,i} - X| \leq c \sum_{i=0}^{\lfloor \eta n \rfloor} \frac{T_{n,\eta n} |\Delta_{n,j-1}^{\delta-1} - \Delta_{n,j}^{\delta-1}|}{\sum_{j=0}^{n-1} \Delta_{n,j}^{\delta}} \leq c \frac{n\eta \max_k \Delta_{n,k}}{n(\min_k \Delta_{n,k})^\delta} \sum_{i=0}^{\lfloor \eta n \rfloor} |\Delta_{n,j-1}^{\delta-1} - \Delta_{n,j}^{\delta-1}|. \tag{213}$$

We use the first and the second point of Assumption S2, getting

$$\sum_{i=0}^{\lfloor \eta n \rfloor} |a_{n,i}||X_{n,i} - X| \leq c\eta. \tag{214}$$

From (212) and (214) and the arbitrariness of both ϵ and η it follows that (209) holds almost surely and so in probability. If we show (207) and (208), the lemma is therefore proved. Concerning (207), we observe it is enough to study the behavior of $\frac{1}{\sum_{i=0}^{n-1} \Delta_{n,i}^{\delta}} \sum_{i=1}^{n-2} T_{n,i}(\Delta_{n,i-1}^{\delta-1} - \Delta_{n,i}^{\delta-1})$. Indeed if it converges then

$$\sup_{n \geq 1} \sum_{i=1}^{n-1} |a_{n,i}| \leq \sup_{n \geq 1} \sum_{i=1}^{n-2} |a_{n,i}| + \sup_{n \geq 1} |a_{n,n-1}| \leq \sup_{n \geq 1} \sum_{i=1}^{n-2} |a_{n,i}| + c_2 < \infty.$$

We focus on $\sum_{i=1}^{n-2} |a_{n,i}|$ and we act like we did in (213), using this time that $T_{n,i} \leq T_{n,n}$. We get

$$\sup_{n \geq 1} \sum_{i=1}^{n-2} |a_{n,i}| \leq \sup_{n \geq 1} c \frac{(n-1) \max_k \Delta_{n,k}}{n(\min_k \Delta_{n,k})^\delta} \sum_{i=0}^{n-2} |\Delta_{n,i-1}^{\delta-1} - \Delta_{n,i}^{\delta-1}|.$$

Again, using the first and the second point of Assumption S2, we get it is bounded by a constant.
To conclude, we observe that $T_{n,i} = T_{n,i-1} + \Delta_{n,i-1}$ and so it is enough to compute $\sum_{i=1}^{n-1} a_{n,i}$ to get it is equal to 1. □

A.3 Proof of Lemma 5

Proof. By the definition of $d(\zeta_n)$, as in law we have that $S_1^\alpha = -S_1^\alpha$, we get $d(\zeta_n) = d(|\zeta_n|)$ and thus we can assume that $\zeta_n > 0$. Using a change of variable we obtain

$$d(\zeta_n) = \mathbb{E}[(S_1^\alpha)^2\varphi(S_1^\alpha\zeta_n)] = \int_{\mathbb{R}} z^2\varphi(z\zeta_n)f_\alpha(z)dz = (\zeta_n)^{-3}\int_{\mathbb{R}} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du. \tag{215}$$

We want to use an asymptotic expansion of the density (see Theorem 7.22 in [18], with $d = 1$ and $\sigma = 1$) which states that, if z is big enough, then a development up to order N of $f_\alpha(z)$ is

$$\frac{c_\alpha}{|z|^{1+\alpha}} + \frac{1}{\pi}\frac{1}{|z|}\sum_{k=2}^{N}\frac{a_k}{k!}(|z|^{-\alpha})^k + o_{\mathbb{P}}(|z|^{-\alpha N}), \tag{216}$$

for some coefficients a_k. We therefore take $M > 0$ big enough such that, for $\frac{u}{\zeta_n} > M$, we can use (216). Hence the right hand side of (215) can be seen as

$$(\zeta_n)^{-3}\int_{|u|\le\zeta_n M} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du + (\zeta_n)^{-3}\int_{|u|>\zeta_n M} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du =: I_1^n + I_2^n. \tag{217}$$

We have that, $\forall\hat{\epsilon} > 0$, $I_1^n = o_{\mathbb{P}}(\zeta_n^{-\hat{\epsilon}})$. Indeed, using that φ and f_α are both bounded, we get

$$\frac{I_1^n}{\zeta_n^{-\hat{\epsilon}}} \le \zeta_n^{-3+\hat{\epsilon}}\int_{|u|\le\zeta_n M} u^2du \le c\zeta_n^{\hat{\epsilon}}, \tag{218}$$

that goes to zero because we have assumed that $\zeta_n \to 0$. I_2^n is

$$(\zeta_n)^{-3}\int_{|u|>\zeta_n M} u^2\varphi(u)c_\alpha(\zeta_n)^{1+\alpha}|u|^{-1-\alpha}du + (\zeta_n)^{-3}\int_{|u|>\zeta_n M} u^2\varphi(u)[f_\alpha(\frac{u}{\zeta_n}) - \frac{c_\alpha}{|u|^{1+\alpha}}|\zeta_n|^{1+\alpha}]du. \tag{219}$$

The first term here above can be seen as

$$(\zeta_n)^{\alpha-2}c_\alpha\int_{\mathbb{R}}|u|^{1-\alpha}\varphi(u)du - (\zeta_n)^{\alpha-2}c_\alpha\int_{|u|\le\zeta_n M}|u|^{1-\alpha}\varphi(u)du = (\zeta_n)^{\alpha-2}c_\alpha\int_{\mathbb{R}}|u|^{1-\alpha}\varphi(u)du + o_{\mathbb{P}}((\zeta_n)^{-\hat{\epsilon}}).$$

Indeed, using that φ is bounded, we have

$$\frac{1}{(\zeta_n)^{-\hat{\epsilon}}}|(\zeta_n)^{\alpha-2}c_\alpha\int_{|u|\le\zeta_n M}|u|^{1-\alpha}\varphi(u)du| \le c(\zeta_n)^{\hat{\epsilon}+\alpha-2}\int_{|u|\le\zeta_n M}|u|^{1-\alpha}du \le c(\zeta_n)^{\hat{\epsilon}}, \tag{220}$$

that goes to zero for $n \to \infty$.
Replacing (218), (219) and (220) in (217) and comparing it with (97), it turns out that our goal is to show that the second term of (219) is $o_{\mathbb{P}}(\zeta_n^{(-\hat{\epsilon})\wedge(2\alpha-2-\hat{\epsilon})})$. Using on it (216) with $N = 2$, which implies $|f_\alpha(z) - \frac{c_\alpha}{|z|^{1+\alpha}}| \le \frac{c}{|z|^{1+2\alpha}}$ for $|z| > M$ and some $c > 0$, we can upper bound it with $c(\zeta_n)^{2\alpha-2}\int_{|u|\le\zeta_n M}|u|^{1-2\alpha}du$.
By the definition of φ we have

$$\int_{|u|>\zeta_n M}|u|^{1-2\alpha}\varphi(u)du = \int_{-2}^{-\zeta_n M}(-u)^{1-2\alpha}\varphi(u)du + \int_2^{\zeta_n M}u^{1-2\alpha}\varphi(u)du \le c + c(\zeta_n)^{2-2\alpha}. \tag{221}$$

Therefore we get that the second term of (219) is upper bounded by

$$c\zeta_n^{2\alpha-2} + c.$$

The first term here above is clearly $o_{\mathbb{P}}(\zeta_n^{2\alpha-2-\hat{\epsilon}})$ while the second is $o_{\mathbb{P}}(\zeta_n^{-\hat{\epsilon}})$, hence the sum is $o_{\mathbb{P}}(\zeta_n^{(-\hat{\epsilon})\wedge(2\alpha-2-\hat{\epsilon})})$.
The lemma is therefore proved. □

A.4 Proof of Lemma 9

Proof. We observe that, $\forall\alpha \in [0, 2]$, we have

$$\mathbb{E}[|\Delta X_i^J - \Delta\tilde{X}_i^J|^2] = \mathbb{E}[(\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}}[\gamma(X_{s-}) - \gamma(X_{t_i})]z\tilde{\mu}(ds, dz))^2] = \mathbb{E}[\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}}[\gamma(X_{s-}) - \gamma(X_{t_i})]^2|z|^2\bar{\mu}(ds, dz)] \le$$

$$\le c\int_{t_i}^{t_{i+1}}\mathbb{E}[|X_s - X_{t_i}|^2]ds\int_{\mathbb{R}}|z|^2F(z)dz \le c\int_{t_i}^{t_{i+1}}\Delta_{n,i}ds \le c\Delta_{n,i}^2, \tag{222}$$

where we have used Ito isometry, the regularity of γ and the first point of Lemma 1.
We have in this way proved (162) and showed that (163) holds with $q = 2$. For $q > 2$, using Kunita inequality and acting like we did here above we get

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q] \leq \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})]^q |z|^q \bar{\mu}(ds, dz)] + \mathbb{E}[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})]^2 |z|^2 \bar{\mu}(ds, dz))^{\frac{q}{2}}] \leq$$

$$\leq c \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_s - X_{t_i}|^q] ds + \mathbb{E}[(\int_{t_i}^{t_{i+1}} |X_s - X_{t_i}|^2 ds)^{\frac{q}{2}}] \leq c\Delta_{n,i}^2 + c\mathbb{E}[\Delta_{n,i}^{\frac{q}{2}-1} \int_{t_i}^{t_{i+1}} |X_s - X_{t_i}|^q ds] = c\Delta_{n,i}^2 + c\Delta_{n,i}^{\frac{q}{2}-1} \leq c\Delta_{n,i}^2,$$

where we have also used Jensen inequality.
In order to prove (163) we observe that, if $\alpha < 1$, then we have

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|] \leq \mathbb{E}[|\int_{t_i}^{t_{i+1}} \int_{|z| \geq 2\Delta_{n,i}^\beta} [\gamma(X_{s-}) - \gamma(X_{t_i})] z \tilde{\mu}(ds, dz)|] + \mathbb{E}[|\int_{t_i}^{t_{i+1}} \int_{|z| \leq 2\Delta_{n,i}^\beta} [\gamma(X_{s-}) - \gamma(X_{t_i})] z \tilde{\mu}(ds, dz)|]. \tag{223}$$

The first term in the right hand side of (223) is upper bounded by

$$\|\gamma'\|_\infty \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{|z| \geq 2\Delta_{n,i}^\beta} |X_{s-} - X_{t_i}| |z| F(z) dz ds] \leq c \int_{t_i}^{t_{i+1}} \int_{|z| \geq 2\Delta_{n,i}^\beta} \mathbb{E}[|X_{s-} - X_{t_i}|^2]^{\frac{1}{2}} ds |z| F(z) dz \leq$$

$$\leq c \int_{t_i}^{t_{i+1}} \Delta_{n,i}^{\frac{1}{2}} (\int_{|z| \geq 2\Delta_{n,i}^\beta} |z| F(z) dz) ds \leq c\Delta_{n,i}^{\frac{3}{2}}, \tag{224}$$

where we have used the compensation formula, the regularity of γ, Cauchy-Schwartz inequality in order to use the first point of Lemma 1 and the boundedness of the integral for $|z| \geq 2\Delta_{n,i}^\beta$. Moreover, acting in the same way, the second term in the right hand side of (223) is upper bounded by

$$\|\gamma'\|_\infty \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{|z| \leq 2\Delta_{n,i}^\beta} |X_{s-} - X_{t_i}| |z| F(z) dz ds] \leq c \int_{t_i}^{t_{i+1}} \Delta_{n,i}^{\frac{1}{2}} (\int_{|z| \geq 2\Delta_{n,i}^\beta} |z|^{-\alpha} dz) ds \leq c\Delta_{n,i}^{\frac{3}{2}+\beta(1-\alpha)}, \tag{225}$$

using again compensation formula, the regularity of γ and Cauchy-Schwartz inequality in order to use the first point of Lemma 1. We have also used the third point of Assumption 4 and computed the integral on z. Using (223) - (225) we get

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|] \leq c\Delta_{n,i}^{\frac{3}{2} \wedge [\frac{3}{2}+\beta(1-\alpha)]} = c\Delta_{n,i}^{\frac{3}{2}}, \tag{226}$$

since $\alpha < 1$ and so $(1 - \alpha) > 0$. We now use interpolation theorem (see below theorem 1.7 in Chapter 4 of [7]) getting

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}} \leq \mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|]^\theta (\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^2]^{\frac{1}{2}})^{1-\theta},$$

with $\frac{1}{q} = \theta + \frac{1-\theta}{2}$, hence $\theta = \frac{2}{q} - 1$. Using (222) and (226) it follows

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}} \leq c\Delta_{n,i}^{\frac{3}{2}\theta} \Delta_{n,i}^{1-\theta} = c\Delta_{n,i}^{\frac{1}{2}\theta+1} = c\Delta_{n,i}^{\frac{1}{q}+\frac{1}{2}},$$

where we have also replaced θ. □

A.5 Proof of Lemma 10

Proof. We want to use a conditional version of the interpolation theorem, therefore we have to estimate the norm 2 of $I_2^n(i)$, $I_3^n(i)$ and $I_4^n(i)$. Observing that φ is a bounded function and using Kunita inequality we get

$$\mathbb{E}_i[|I_2^n(i)|^2] \leq \mathbb{E}_i[|\Delta X_i^J|^4] \leq c\mathbb{E}_i[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\gamma(X_{s-})|^4 |z|^4 \bar{\mu}(ds, dz)] + c\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\gamma(X_{s-})|^2 |z|^2 \bar{\mu}(ds, dz))^2] \leq$$

$$\leq c(\int_{\mathbb{R}} |z|^4 F(z) dz) \mathbb{E}_i[\int_{t_i}^{t_{i+1}} |\gamma(X_{s-})|^4 ds] + c\mathbb{E}_i[(\int_{\mathbb{R}} |z|^2 F(z) dz)^2 (\int_{t_i}^{t_{i+1}} |\gamma(X_{s-})|^2 ds)^2] \leq$$

$$\leq R(\Delta_{n,i}, X_{t_i}) + R(\Delta_{n,i}^2, X_{t_i}) = R(\Delta_{n,i}, X_{t_i}), \tag{227}$$

where in the last inequality we have also used the polynomial growth of γ and the third point of Lemma 1.

Concerning the norm 2 of $I_3^n(i)$, we use the conditional version of the first point of Lemma 9 for $q = 2$ to get

$$\mathbb{E}_i[|I_3^n(i)|^2] \leq \mathbb{E}_i[|\Delta X_i^J - \Delta \tilde{X}_i^J|^4] \leq R(\Delta_{n,i}^2, X_{t_i}). \tag{228}$$

We now consider $I_4^n(i)$. Using Cauchy-Schwartz inequality and a conditional version of both the first point of Lemma 9 for $q = 2$ and (32) in Lemma 4, where φ acts like the indicator function, we have

$$\mathbb{E}_i[|I_4^n(i)|^2]^{\frac{1}{2}} \leq c\mathbb{E}_i[|\Delta X_i^J - \Delta \tilde{X}_i^J|^4]^{\frac{1}{2}} \mathbb{E}_i[|\Delta \tilde{X}_i^J \varphi_{\Delta_{n,i}^\beta}(\Delta \tilde{X}_i^J)|^4]^{\frac{1}{2}} \leq R(\Delta_{n,i}^{\frac{3}{2}+\frac{\beta}{2}(4-\alpha)}, X_{t_i}). \tag{229}$$

Using interpolation theorem it follows, $\forall j \in \{2, 3, 4\}$,

$$\mathbb{E}_i[|I_j^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq \mathbb{E}_i[|I_j^n(i)|]^\theta (\mathbb{E}_i[|I_j^n(i)|^2]^{\frac{1}{2}})^{1-\theta}, \tag{230}$$

with θ such that $\frac{1}{1+\epsilon} = \theta + \frac{1-\theta}{2}$, hence $\theta = \frac{2}{1+\epsilon} - 1 = 1 - \frac{2\epsilon}{1+\epsilon}$.
From a conditional version of (159), (164), (169) and equations (227) and (230) it follows

$$\mathbb{E}_i[|I_2^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R(\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}, X_{t_i})^\theta R(\Delta_{n,i}^{\frac{1}{2}}, X_{t_i})^{1-\theta} =$$

$$= R(\Delta_{n,i}^{(\frac{3}{2}+\beta-\frac{\alpha\beta}{2})(1-\frac{2\epsilon}{1+\epsilon})+\frac{\epsilon}{1+\epsilon}}, X_{t_i}) = R(\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-\frac{\epsilon}{1+\epsilon}(2+2\beta-\alpha\beta)}, X_{t_i}). \tag{231}$$

Since $2 + 2\beta - \alpha\beta$ is always more than zero we can just see the exponent on $\Delta_{n,i}$ as $\frac{3}{2} + \beta - \frac{\alpha\beta}{2} - \epsilon$.
From a conditional version of (171), (228) and (230) it follows

$$\mathbb{E}_i[|I_3^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R(\Delta_{n,i}^2, X_{t_i})^\theta R(\Delta_{n,i}, X_{t_i})^{1-\theta} = R(\Delta_{n,i}^{1+\theta}, X_{t_i}) = R(\Delta_{n,i}^{2-\frac{2\epsilon}{1+\epsilon}}, X_{t_i}). \tag{232}$$

In the same way, using a conditional version of (173), (229) and (230) it follows

$$\mathbb{E}_i[|I_4^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R(\Delta_{n,i}^{(\frac{3}{2}+\beta-\frac{\alpha\beta}{2})(1-\frac{2\epsilon}{1+\epsilon})+\frac{2\epsilon}{1+\epsilon}(\frac{3}{2}+2\beta-\frac{\alpha\beta}{2})}, X_{t_i}) = R(\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}+\frac{2\beta\epsilon}{1+\epsilon}}, X_{t_i}). \tag{233}$$

The result (184) is a consequence of (231), (232), (233) and that 2 is always more than $\frac{3}{2} + \beta - \frac{\alpha\beta}{2}$. □

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