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THE LANE-EMDEN EQUATION IN STRIPS

LOUIS DUPAIGNE AND ABDELLAZIZ HARRABI

ABSTRACT. We study the Lane-Emden equation in strips.

1. INTRODUCTION

In this work, we investigate the celebrated Lane-Emden equation, see (1.1) below. We restrict to the case where the domain is a strip: given integers $n \geq 2$ and $k \geq 1$, we set

$$\Omega = \mathbb{R}^{n-k} \times \omega, \quad \text{where } \omega \subset \mathbb{R}^k \text{ is a smoothly bounded domain.}$$

For $p > 1$, we consider the equation

$$(1.1) \quad \begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The case of homogeneous Neumann boundary condition

$$(1.2) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

will also be considered. Strips provide an interesting example of unbounded domains where, as we shall see, rather sharp classification results can be obtained. Let us begin by observing that the symmetries of the domain allow solutions of the form¹

$$(1.3) \quad u(x', x'') = \bar{u}(x''), \quad \text{for } x = (x', x'') \in \Omega$$

and, more generally,

$$(1.4) \quad u(x', x'') = u(\rho, x''), \quad \text{for } x = (x', x'') \in \Omega, \rho = |x'|.$$

Solutions of the form (1.3) are simply solutions in the bounded domain $\omega \subset \mathbb{R}^k$. As can be read in any introductory book on PDEs, in the simplest situation e.g. when ω is strictly star-shaped, such solutions with Dirichlet boundary condition exist if and only if $1 < p < p_S(k)$, where $p_S(k)$ is the classical Sobolev exponent in dimension k . Similarly, if $n - k \geq 2$, solutions of the form (1.4) can be constructed when $1 < p < p_S(n)$, by applying the mountain pass lemma in the space H of functions $u \in H_0^1(\Omega)$ having cylindrical symmetry, i.e. such that (1.4) holds a.e., see Esteban [5]. The case $p \geq p_S(n)$ seems rather unexplored for strips. To gain further insight into the problem, we observe that the aforementioned solutions have

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¹with a standard abuse of notation, we have used the same letter u to denote distinct functions

very distinct stability properties. Recall that a solution of (1.1) is said to be stable in an open subset $\Omega' \subset \Omega$ if

$$(1.5) \quad \int_{\Omega'} |\nabla \varphi|^2 dx - p \int_{\Omega'} |u|^{p-1} \varphi^2 dx \geq 0, \quad \text{for every } \varphi \in C_c^1(\Omega').$$

We prove

Proposition 1.1. *Every solution of the form (1.3) is unstable outside every compact set, while if $n-k \geq 2$, the mountain-pass solution in the space H of H^1 functions with symmetry (1.4) is stable outside a compact set.*

Remark 1.2. The first part of the proposition is essentially due to Farina (see Example 1 in [6]).

Remark 1.3. For quite general nonlinearities, in the case $n-k \leq 2$, Dancer [2] proved that every bounded stable solution is after rotation in the x' variable, a function of x_1, x'' only, is monotone in the x_1 direction, and the limits of $u(x_1, x'')$ as $x_1 \rightarrow \pm\infty$ are stable solutions of the problem on ω with the same energy.

The above results suggest that in the supercritical case, the class of stable solutions might also be relevant. We obtain the following result, also valid for nodal solutions.

Theorem 1.4. *Assume $p > 1$. The Dirichlet problem (1.1) has no nontrivial stable solution $u \in C^2(\overline{\Omega})$. More generally, for $p \geq p_S(n)$, there is no nontrivial solution which is stable outside a compact set.*

Remark 1.5. Farina [6, Theorem 7] proved the above result in the special case where u is stable and $1 < p < p_c(n-k)$, where $p_c(n-k)$ is the Joseph-Lundgren stability exponent in dimension $n-k$, see e.g. [4] for the definition. He also observed that the exponent $p_c(n)$ is sharp when $\Omega = \mathbb{R}^n$. Our result shows that this exponent is irrelevant when Ω is a strip. This also corroborates a recent result of Chen, Lin and Zou [1], who proved, in the case of the half-space $\Omega = \mathbb{R}_+^n$, that given any $p > 1$, there is no positive solution to (1.1). Note that in this case, every positive solution is known to be stable.

Remark 1.6. Recall that for $1 < p < p_S(n)$, there exists a positive solution of the form (1.4) and so, according to Proposition 1.1, Theorem 1.4 is sharp.

We turn now to the case of homogeneous Neumann boundary conditions. Here, stability in Ω means that (1.5) holds for every $\varphi \in H^1(\Omega)$.

Theorem 1.7. *Assume that $1 < p < p_c(n-k)$. Then, the Neumann problem has no nontrivial stable solution $u \in C^2(\overline{\Omega})$. More generally, if $p_S(n-k) < p < p_c(n-k)$, there is no nontrivial solution which is stable outside a compact set. Finally, if $p = p_S(n-k)$, any solution which is stable outside a compact set is a function of x' only and it has finite energy*

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{p+1} < \infty$$

Remark 1.8. We do not know whether the second statement of Theorem 1.7 remains true in the case $1 < p < p_S(n-k)$. But the theorem is sharp in the case $p \geq p_S(n-k)$. Indeed, there exists a (radial) stable solution of the Lane-Emden equation

in \mathbb{R}^{n-k} whenever $p \geq p_c(n-k)$ which is also clearly² a stable solution of the Neumann problem of the form (1.4). Similarly, when $p = p_S(n-k)$, the standard bubble in \mathbb{R}^{n-k} is a solution of Morse index 1 in \mathbb{R}^{n-k} . As such, it is stable outside a compact set of \mathbb{R}^{n-k} and so it is also stable outside a compact set of Ω .

Notation. Without further notice, we shall use the following notation. A point $x \in \mathbb{R}^n$ is written $x = (x', x'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. The same applies to the operators $\nabla = (\nabla', \nabla'')$ and $\Delta = \Delta' + \Delta''$. Polar coordinates in the x' variable are written $x' = \rho\theta$, where $\rho = |x'|$ and $\theta = x'/|x'|$ whenever $x' \neq 0$. In particular, $\partial_\rho = \nabla \cdot x'/\rho$ differentiates functions in the ρ -variable and $\Delta' = \rho^{1+k-n}\partial_\rho(\rho^{n-k-1}\partial_\rho) + \rho^{-2}\Delta_\theta$, where Δ_θ is the Laplace-Beltrami operator on the unit sphere S^{n-k-1} of \mathbb{R}^{n-k} .

2. Proof of Proposition 1.1

As mentioned in the introduction, the fact that solutions of the form (1.3) are unstable outside any compact set is an obvious generalization of Example 1 in [6], so we skip it.

Turning to mountain-pass solutions in the space H of functions in $H_0^1(\Omega)$ having the cylindrical symmetry (1.4), we recall that, thanks to Solimini's work [9], they have Morse index 1 in H . In particular, there exists a function $\phi \in H$ such that $Q_u(\phi) < 0$, where the second variation of the energy is defined as usual by

$$Q_u(\phi) = \int_{\Omega} |\nabla \phi|^2 dx - p \int_{\Omega} |u|^{p-1} \phi^2 dx.$$

Without loss of generality, we may assume in addition that $\phi \in C_c^\infty(\Omega)$. Fix $R > 0$ large enough so that ϕ is supported in $B'(0, R) \times \omega$. We claim that u is stable outside $B'(0, R) \times \omega$. To see this, assume by contradiction that a function $\varphi \in C_c^\infty(\Omega \setminus \overline{B'(0, R) \times \omega})$ satisfies $Q_u(\varphi) < 0$. Let

$$\chi(\rho, x'') = \left[\frac{1}{|S^{n-k-1}|} \int_{S^{n-k-1}} \varphi^2(\rho\theta, x'') d\sigma(\theta) \right]^{1/2}$$

denote the quadratic average on the unit sphere of \mathbb{R}^{n-k} of φ and $\phi_0(x', x'') = \chi(|x'|, x'')$. Clearly, $\phi_0 \in H$ has disjoint support from ϕ . We are going to prove that $Q_u(\phi_0) < 0$, so that u would have Morse index in H at least equal to 2, a contradiction. Note that

$$(2.1) \quad \int_{\Omega} |u|^{p-1} \varphi^2 = \int_{\Omega} |u|^{p-1} \phi_0^2.$$

For $\epsilon > 0$, let

$$\chi_\epsilon(\rho, x'') = [\chi^2(\rho, x'') + \epsilon]^{1/2} \quad \text{and} \quad \phi_\epsilon = \chi_\epsilon(|x'|, x'').$$

Clearly, χ_ϵ is differentiable in $[0; +\infty[\times \omega$, and

$$\frac{\partial \chi_\epsilon}{\partial \rho}(\rho; x'') = \frac{\frac{1}{|S^{n-k-1}|} \int_{S^{n-k-1}} \varphi \frac{\partial \varphi}{\partial \rho}(\rho\theta, x'') d\sigma(\theta)}{\chi_\epsilon},$$

and

$$\frac{\partial \chi_\epsilon}{\partial x_j}(\rho, x'') = \frac{\frac{1}{|S^{n-k-1}|} \int_{S^{n-k-1}} \varphi \frac{\partial \varphi}{\partial x_j}(\rho\theta, x'') d\sigma(\theta)}{\chi_\epsilon}, \quad n-k+1 \leq j \leq n.$$

²the stability in Ω is a direct consequence of the stability in \mathbb{R}^{n-k} and Fubini's theorem

Therefore, $\phi_\epsilon \in C^1(\overline{\Omega})$ and by Hölder's inequality, we have

$$|\nabla\phi_\epsilon(x', x'')|^2 \leq \frac{1}{|S^{n-k-1}|} \int_{S^{n-k-1}} |\nabla\varphi(\rho\theta, x'')|^2 d\sigma(\theta),$$

which also implies that

$$\int_{\Omega} |\nabla\phi_\epsilon|^2 \leq \int_{\Omega} |\nabla\varphi|^2.$$

So, (ϕ_ϵ) is bounded in H^1 converges weakly in H^1 and a.e. to ϕ_0 as ϵ converges to 0. In particular, $\phi_0 \in H^1(\Omega)$ and

$$\int_{\Omega} |\nabla\phi_0|^2 \leq \int_{\Omega} |\nabla\varphi|^2.$$

So, $Q_u(\phi_0) < 0$, and we obtain the desired contradiction.

3. Proof of Theorem 1.4

Step 1. Capacitary estimate. The following estimate is due to Farina (see [6, Proposition 6]). Let u denote a solution which is stable outside a compact set. For every $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1]$, there exist constants $C_1, C_2 > 0$ such that every $R > 0$,

$$(3.1) \quad \int_{B'_R \times \omega} \left(|\nabla|u|^{\frac{\gamma-1}{2}} u|^2 + |u|^{p+\gamma} \right) dx \leq C_1 + C_2 R^{(n-k)-2\frac{p+\gamma}{p-1}}.$$

Step 2. In fact, if u is stable, one can choose $C_1 = 0$. As in Farina's work we readily deduce, by letting $R \rightarrow +\infty$, that there is no nontrivial stable solution of (1.1), in the special case $1 < p < p_c(n-k)$.

Step 3. If $1 < p \leq p_s(n-k)$, and u a solution of (1.1), which is stable outside a compact set, then in view of (3.1), we obtain

$$(3.2) \quad \int_{\Omega} |\nabla u|^2 + |u|^{p+1} < \infty.$$

By dominated convergence, it follows from (3.2) that if $A_R = \{R < |x'| < 2R\}$, then

$$(3.3) \quad \int_{A_R \times \omega} |\nabla u|^2 + |u|^{p+1} = o(1), \quad \text{as } R \rightarrow \infty.$$

Consider a test function $\phi_R(x) = \phi(|x'|/R)$, where $\phi \in C_c^2([0; \infty])$, is a standard cut-off function satisfying $\phi(t) = 1$, if $0 \leq t \leq 1$ and $\phi(t) = 0$, if $t \geq 2$. Multiplying equation (1.1) by $u\phi_R$ and integrating by parts over $B'_{2R} \times \omega$, we get:

$$\int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R - \int_{B'_{2R} \times \omega} |u|^{p+1} \phi_R = \frac{1}{2} \int_{B'_{2R} \times \omega} u^2 \Delta \phi_R.$$

Using Hölder's inequality and (3.3), we conclude that

$$\int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R - \int_{B'_{2R} \times \omega} |u|^{p+1} \phi_R = o(1), \quad \text{as } R \rightarrow \infty.$$

Hence, letting $R \rightarrow \infty$, we get

$$(3.4) \quad \int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{p+1}$$

To complete the proof of Theorem 1.4 for $p_s(n) \leq p \leq p_s(n-k)$, we need the following Pohozaev identity:

Proposition 3.1. *Let u be a solution of (1.1) which is stable outside a compact set and $z = (0, z) \in \mathbb{R}^{n-k} \times \omega$. Then, if $1 < p \leq p_s(n-k)$, we have*

$$(3.5) \quad \left(1 - \frac{2n}{(n-2)(p+1)}\right) \int_{\Omega} |u|^{p+1} = -\frac{1}{n-2} \int_{\mathbb{R}^{n-k} \times \partial\omega} |\nabla'' u|^2 (x-z)'' \cdot \nu''.$$

where ν denotes the outward unit vector normal on $B'_{2R} \times \partial\omega$.

Proof of Proposition 3.1. Multiplying equation (1.1) by $\nabla u \cdot (x-z)\phi_R(x')$ and integrating over $B'_{2R} \times \omega$, we find

$$(3.6) \quad \int_{B'_{2R} \times \omega} -\Delta u \nabla u \cdot (x-z)\phi_R = \int_{B'_{2R} \times \omega} |u|^{p-1} u \nabla u \cdot (x-z)\phi_R.$$

For the left hand side of (3.6), integrating by parts there holds

$$\begin{aligned} I_1(R) &:= \int_{B'_{2R} \times \omega} -\Delta u \nabla u \cdot (x-z)\phi_R \\ &= \frac{1}{2} \int_{B'_{2R} \times \omega} \nabla(|\nabla u|^2) \cdot (x-z)\phi_R + \int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R \\ &\quad + \int_{B'_{2R} \times \omega} (\nabla u \cdot \nabla \phi_R)(\nabla u \cdot (x-z)) - \int_{B'_{2R} \times \partial\omega} (\nabla u \cdot \nu)(\nabla u \cdot (x-z))\phi_R, \end{aligned}$$

Integrating again by parts the first term of the last equality, we get

$$(3.7) \quad \begin{aligned} I_1(R) &= \frac{2-n}{2} \int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R + \frac{1}{2} \int_{B'_{2R} \times \partial\omega} |\nabla u|^2 (x-z) \cdot \nu \phi_R \\ &\quad - \int_{B'_{2R} \times \partial\omega} (\nabla u \cdot \nu)(\nabla u \cdot (x-z))\phi_R + o(1). \end{aligned}$$

where in view of (3.3) and the definition of ϕ_R ,

$$o(1) = \int_{B'_{2R} \times \omega} (\nabla u \cdot \nabla \phi_R)(\nabla u \cdot (x-z)) + \frac{1}{2} \int_{B'_{2R} \times \omega} |\nabla u|^2 \nabla \phi_R \cdot (x-z), \text{ as } R \rightarrow \infty$$

Taking in account that $\nu = (0, \nu'')$, and $u = 0$ on $\mathbb{R}^{n-k} \times \partial\omega$, at any point $x \in \mathbb{R}^{n-k} \times \partial\omega$ where $\nabla'' u \neq 0$, we have $\nu'' = \epsilon \frac{\nabla'' u}{|\nabla'' u|}$ where $\epsilon \in \{-1, 1\}$. Therefore, (3.7) becomes

$$I_1(R) = \frac{2-n}{2} \int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R - \frac{1}{2} \int_{B'_{2R} \times \partial\omega} |\nabla u|^2 (x-z) \cdot \nu \phi_R + o(1).$$

Now, integrate by part the right hand side of (3.6) to obtain

$$(3.8) \quad \begin{aligned} I_2(R) &:= \int_{B'_{2R} \times \omega} |u|^{p-1} u \nabla u \cdot (x-z)\phi_R = -\frac{n}{p+1} \int_{B'_{2R} \times \omega} |u|^{p+1} \phi_R \\ &\quad - \frac{1}{p+1} \int_{B'_{2R} \times \omega} |u|^{p+1} \nabla \phi_R \cdot (x-z) \\ &= -\frac{n}{p+1} \int_{B'_{2R} \times \omega} |u|^{p+1} \phi_R + o(1). \end{aligned}$$

Since $I_1(R) = I_2(R)$ combining (3.7), (3.8), and letting $R \rightarrow \infty$, then from (3.6), we get

$$\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 - \frac{n}{p+1} \int_{\Omega} |u|^{p+1} = -\frac{1}{2} \int_{\mathbb{R}^{n-k} \times \partial\omega} |\nabla'' u|^2 (x-z)'' \cdot \nu''.$$

From (3.4), we derive the desired result. \square

As a consequence of Proposition 3.1, if ω is strictly star-shaped with respect to z , there holds $(x-z)'' \cdot \nu'' > 0$ on $\mathbb{R}^{n-k} \times \partial\omega$ and so $u \equiv 0$ when $p_s(n) < p \leq p_s(n-k)$. The case $p = p_s(n)$, requires more analysis. In fact, from (3.5), one has $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. In addition, applying (3.1) with $\gamma = p$ and recalling that $n-k-2\frac{2p}{p-1} < n-k-2\frac{p+1}{p-1} < 0$ for $p = p_s(n)$, we find

$$\int_{\Omega} |u|^{2p} < \infty \quad \text{i.e.} \quad |u|^{p-1}u \in L^2(\Omega).$$

Recall that Poincaré's inequality holds in $H_0^1(\Omega)$. By L^p -elliptic theory, we deduce that $u \in H^2 \subset L^{\frac{2N}{N-4}}$ and by a standard boot-strap argument, one has $u \in W^{2,r}$ with $r > \frac{n}{2}$ and then $u \in L^\infty(\Omega)$. Therefore (1.1) becomes $\Delta u + qu = 0$ where $q = |u|^{p-1} \in L^\infty(\Omega)$ with $u \in H^2(\Omega)$ satisfying $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. By the unique continuation principle for Cauchy data (see [7]), it follows that $u \equiv 0$.

Step 4. We derive a variant of a monotonicity formula due to Pacard [8], see also K. Wang [10]. For $\lambda > 0$, define u^λ by

$$u^\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x', x''), \quad \text{for all } x = (x', x'') \in \mathbb{R}^{n-k} \times \omega$$

and

$$E(u; \lambda) = \int_{B_1' \times \omega} \left(\frac{1}{2} [|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2] - \frac{1}{p+1} |u^\lambda|^{p+1} \right) dx + \frac{1}{p-1} \int_{\partial B_1' \times \omega} |u^\lambda|^2 d\sigma.$$

We claim that E is a nondecreasing function of λ . Furthermore, E is differentiable and

$$\frac{dE}{d\lambda} = \lambda \left[\int_{\partial B_1' \times \omega} |\partial_\lambda u^\lambda|^2 d\sigma + \int_{B_1' \times \omega} |\nabla'' u^\lambda|^2 dx \right]$$

To prove our claim, we note that u^λ solves

$$(3.9) \quad -\Delta' u^\lambda - \lambda^2 \Delta'' u^\lambda = |u^\lambda|^{p-1} u^\lambda \quad \text{in } \mathbb{R}^{n-k} \times \omega,$$

that

$$(3.10) \quad E(u; \lambda) = E(u^\lambda; 1),$$

and that

$$(3.11) \quad \lambda \partial_\lambda u^\lambda = \frac{2}{p-1} u^\lambda + \rho \partial_\rho u^\lambda \quad \text{for } x \in \Omega, \lambda > 0,$$

where we recall that $\rho = |x'|$ and $\partial_\rho = \nabla \cdot \frac{x'}{\rho}$. So, if

$$(3.12) \quad E_1 = \int_{B_1' \times \omega} \left(\frac{1}{2} |\nabla' u^\lambda|^2 - \frac{1}{p+1} |u^\lambda|^{p+1} \right) dx,$$

then

$$\frac{dE_1}{d\lambda} = \int_{B_1' \times \omega} (\nabla' u^\lambda \cdot \nabla' \partial_\lambda u^\lambda - |u^\lambda|^{p-1} u^\lambda \partial_\lambda u^\lambda) dx.$$

Integrating by parts and using (3.9),

$$\frac{dE_1}{d\lambda} = \lambda^2 \int_{B_1' \times \omega} \Delta'' u^\lambda \partial_\lambda u^\lambda + \int_{\partial B_1' \times \omega} \partial_\rho u^\lambda \partial_\lambda u^\lambda$$

Integrating by parts again and using the boundary condition, the first addend is equal to

$$\begin{aligned} & \lambda^2 \int_{B'_1 \times \omega} \Delta'' u^\lambda \partial_\lambda u^\lambda = -\lambda^2 \int_{B'_1 \times \omega} \nabla'' u^\lambda \cdot \nabla'' \partial_\lambda u^\lambda \\ & = -\frac{\lambda^2}{2} \frac{d}{d\lambda} \int_{B'_1 \times \omega} |\nabla'' u^\lambda|^2 = -\frac{d}{d\lambda} \left[\frac{\lambda^2}{2} \int_{B'_1 \times \omega} |\nabla'' u^\lambda|^2 \right] + \lambda \int_{B'_1 \times \omega} |\nabla'' u^\lambda|^2. \end{aligned}$$

Thanks to (3.11), the second addend is equal to

$$\begin{aligned} & \int_{\partial B'_1 \times \omega} \partial_\rho u^\lambda \partial_\lambda u^\lambda = \int_{\partial B'_1 \times \omega} \left(\lambda \partial_\lambda u^\lambda - \frac{2}{p-1} u^\lambda \right) \partial_\lambda u^\lambda \\ & = \lambda \int_{\partial B'_1 \times \omega} |\partial_\lambda u^\lambda|^2 - \frac{1}{p-1} \frac{d}{d\lambda} \int_{\partial B'_1 \times \omega} |u^\lambda|^2 \end{aligned}$$

and the result follows.

Step 5. Blow-down analysis for stable solutions.

From Step 1 (applied to u on a ball of radius λR), we know that given $R > 0$,

$$(3.13) \quad \int_{B'_R \times \omega} (|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2 + |u^\lambda|^{p+1}) \, dx \leq CR^{(n-k)-2\frac{p+1}{p-1}}.$$

So, $(u^\lambda)_{\lambda \geq 1}$ is uniformly bounded in $H^1 \cap L^{p+1}(B'_R \times \omega)$, for any $R > 0$. In particular, a sequence (u^{λ_n}) converges weakly to some function u^∞ in $H^1 \cap L^{p+1}(B'_R \times \omega)$ for every $R > 0$, as $\lambda_n \rightarrow +\infty$. Note also that u^λ satisfies the following PDE

$$(3.14) \quad -\Delta'' u^\lambda = \lambda^{-2} (\Delta' u^\lambda + |u^\lambda|^{p-1} u^\lambda).$$

Taking limits in the sense of distributions, it follows that

$$-\Delta'' u^\infty = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^{n-k} \times \omega)$$

The maximum principle, applied for a.e. $x' \in \mathbb{R}^{n-k}$ to the function $u^\infty(x', \cdot)$ implies that $u^\infty \equiv 0$.

Actually, the full family (u^λ) converges *strongly* to $u^\infty = 0$ in $L^{p+1}(B'_R \times \omega)$. Indeed, by Rellich's theorem, (u^λ) is compact in $L^2(B'_R \times \omega)$, while it remains bounded in $L^{p+\gamma}(B'_R \times \omega)$ for some $\gamma > 1$, thanks to Step 1. By Hölder's inequality, (u^λ) is compact in $L^{p+1}(B'_R \times \omega)$. $u^\infty = 0$ being its only cluster point, the claim follows.

Now, multiply equation (1.1) by $pu\zeta^2$, where $\zeta \in C_c^1(\overline{\Omega})$ is a cut-off function to be specified soon³. We find

$$p \int_{\Omega} \nabla u \cdot \nabla (u\zeta^2) = p \int_{\Omega} |u|^{p+1} \zeta^2$$

The left-hand side is equal to

$$p \int_{\Omega} |\nabla(u\zeta)|^2 - u^2 |\nabla \zeta|^2$$

while the right-hand side is bounded above by $\int_{\Omega} |\nabla(u\zeta)|^2$, since u is stable. It follows that

$$(3.15) \quad (p-1) \int_{\Omega} |\nabla(u\zeta)|^2 \leq p \int_{\Omega} u^2 |\nabla \zeta|^2.$$

³Such test functions can indeed be used in the stability inequality, thanks to the Dirichlet boundary condition, see [6, Remark 5]

Choose now $\zeta(x) = \zeta_0\left(\frac{x'}{\lambda}\right)$, where $\zeta_0 \equiv 1$ in B'_1 and $\zeta_0 \equiv 0$ outside B'_2 . Then,

$$\int_{B'_\lambda \times \omega} |\nabla u|^2 \leq C\lambda^{-2} \int_{B'_{2\lambda} \times \omega} u^2$$

Scaling back, we arrive at

$$\int_{B'_1 \times \omega} |\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2 \leq C \int_{B'_2 \times \omega} |u^\lambda|^2$$

Recalling that (u^λ) converges to zero in $L^{p+1}(B'_R \times \omega)$, thus also in $L^2(B'_2 \times \omega)$, we conclude that

$$\lim_{\lambda \rightarrow +\infty} E_2(u; \lambda) = \lim_{\lambda \rightarrow +\infty} E_2(u^\lambda; 1) = 0,$$

where E_2 is given by

$$E_2(u^\lambda; 1) = \int_{B_1 \times \omega} \left[\frac{1}{2} (|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2) - \frac{1}{p+1} |u^\lambda|^{p+1} \right] dx$$

We claim that the same holds true for E . To see this, simply observe that since E is nondecreasing,

$$\begin{aligned} (3.16) \quad E(u^\lambda, 1) &= E(u, \lambda) \leq \frac{1}{\lambda} \int_\lambda^{2\lambda} E(u, t) dt = \\ &= \frac{1}{\lambda} \int_\lambda^{2\lambda} E_2(u, t) dt + \frac{1}{p-1} \lambda^{-1} \int_\lambda^{2\lambda} t^{n-k-1-\frac{4}{p-1}} \int_{\partial B'_t \times \omega} |u|^2 \\ &\leq \sup_{t \geq \lambda} E_2(u, t) + C \int_{B'_2 \times \omega} |u^\lambda|^2 \end{aligned}$$

Thanks to this, we deduce that

$$\lim_{\lambda \rightarrow +\infty} E(u, \lambda) = \lim_{\lambda \rightarrow +\infty} E(u^\lambda, 1) = 0.$$

In addition, since u is C^2 , one easily verifies that

$$E(u, 0) = 0.$$

And so, $E(u, \lambda) \equiv 0$, since E is nondecreasing. So, $\frac{dE}{d\lambda} = 0$, which means that u is homogeneous and independent of x'' . Thanks to the boundary condition, we readily deduce that $u \equiv 0$.

Step 6. Blow-down analysis for solutions which are stable outside a compact set.

We assume that $p > p_S(n-k)$. As before, by Step 1, (u^λ) is uniformly bounded in $L^{p+\gamma}(B'_R \times \omega)$ for some $\gamma > 1$. In addition, $|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2$ is bounded in $L^1(B'_R \times \omega)$, for any $R > 0$. As in Step 5, this is enough to conclude that (u^λ) converges strongly to $u^\infty \equiv 0$ in $L^{p+1}(B'_R \times \omega)$. In addition, $E_2(u^\lambda; 1)$ remains bounded. This time however, (3.15) remains valid only for cut-off functions $\zeta \in C_c^1(\bar{\Omega} \setminus B_{R_0} \times \omega)$, for R_0 sufficiently large. So, choose $\zeta(x) = \zeta_0(x'/\lambda)$, where $\zeta_0 \equiv 0$ in $B'_{\varepsilon/2}$, $\zeta_0 \equiv 1$ in $B'_1 \setminus B'_\varepsilon$ and $\zeta_0 \equiv 0$ outside B'_2 . Then, for $\lambda > R_0/\varepsilon$,

$$\int_{B'_\lambda \setminus B'_{\varepsilon\lambda} \times \omega} |\nabla u|^2 \leq C\lambda^{-2} \int_{B'_{2\lambda} \times \omega} u^2$$

Scaling back yields

$$\int_{B'_1 \setminus B'_\varepsilon \times \omega} |\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2 \leq C \int_{B'_2 \times \omega} |u^\lambda|^2.$$

And so,

$$\begin{aligned}
E_2(u^\lambda; 1) &= \int_{B'_1 \times \omega} \left(\frac{1}{2} [|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2] - \frac{1}{p+1} |u^\lambda|^{p+1} \right) dx \\
&= \int_{B'_\varepsilon \times \omega} + \int_{B'_1 \setminus B'_\varepsilon \times \omega} \\
&= \varepsilon^{n-k-2\frac{p+1}{p-1}} E_2(u; \lambda\varepsilon) + \int_{B'_1 \setminus B'_\varepsilon \times \omega} \\
&\leq C \left(\varepsilon^{n-k-2\frac{p+1}{p-1}} + \int_{B'_2 \times \omega} |u^\lambda|^2 \right)
\end{aligned}$$

Letting $\lambda \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$, we deduce that $\lim_{\lambda \rightarrow +\infty} E_2(u; \lambda) = 0$. The remaining part of the proof of Step 5 can be used unchanged. \square

4. PROOF OF THEOREM 1.7

We indicate here how to adapt the proof of Theorem 1.4 in this case.

The case $p > p_S(n-k)$

Here the only difference comes from the classification of the blow-down limit u^∞ . In fact, multiplying (3.9) by $u_\infty \phi_R(x')$, we see easily that $\int_{B'_{2R} \times \omega} (\nabla'' u_{\lambda_j} \cdot \nabla'' u_\infty) \phi_R$ converges to 0 as $j \rightarrow \infty$. Since (u_{λ_j}) converges weakly to u_∞ in $H^1(B'_{2R} \times \omega)$, we deduce that $\int_{B'_{2R} \times \omega} |\nabla'' u_\infty|^2 \phi_R = 0$, $\forall R > 0$. In other words, u^∞ is a function of x' only. But then, integrating (3.9) in the x'' variable and passing again to the weak limit implies that $u^\infty = \frac{1}{|\omega|} \int_\omega u^\infty dx''$ is an energy solution of

$$-\Delta' u^\infty = |u^\infty|^{p-1} u^\infty \quad \text{in } \mathbb{R}^{n-k}$$

In addition, since u is stable outside a compact set, u^∞ is stable outside the origin. If $n-k \geq 2$, points have zero Newtonian capacity and so u^∞ is stable in all of \mathbb{R}^{n-k} . By Farina's [6, Theorem 1] (which still holds for energy solutions), $u^\infty \equiv 0$. If $n-k = 1$, then u^∞ is stable only outside the compact set $\{0\}$. But $p_S(n-k) = p_c(n-k) = +\infty$, so we can apply e.g. [6, Theorem 2] to arrive at the same conclusion.

The case $p = p_S(n-k)$

First we need the following version of Pohozaev identity:

Proposition 4.1. *Let u be a solution of (1.1) which is stable outside a compact set, then we have*

$$\left(1 - \frac{2(n-k)}{(p+1)(n-k-2)} \right) \int_\Omega |u|^{p+1} = -\frac{2}{n-k-2} \int_\Omega |\nabla'' u|^2.$$

Proof of Proposition 4.1. Note that (3.2), (3.3) and (3.4) hold for (1.2). As in the proof of Proposition 3.1, multiplying equation (1.2) by $\nabla' u \cdot (x-z)' \phi_R(|x'|)$ and integrating over $B'_{2R} \times \omega$, we get

$$(4.1) \quad \int_{B'_{2R} \times \omega} -\Delta u \nabla' u \cdot (x-z) \phi_R = \int_{B'_{2R} \times \omega} |u|^{p-1} u \nabla' u \cdot (x-z) \phi_R.$$

For the left hand side of (4.1), integrating by parts there holds

$$\begin{aligned} J_1(R) &:= \int_{B'_{2R} \times \omega} -\Delta u \nabla' u \cdot (x-z)' \phi_R \\ &= \frac{1}{2} \int_{B'_{2R} \times \omega} \nabla' (|\nabla u|^2) \cdot (x-z)' \phi_R + \int_{B'_{2R} \times \omega} |\nabla' u|^2 \phi_R \\ &\quad + o(1) \end{aligned}$$

Integrating again by parts the first term of the last equality, we get

$$\begin{aligned} J_1(R) &= \frac{k-n}{2} \int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R \\ &\quad + \int_{B'_{2R} \times \omega} |\nabla' u|^2 \phi_R + o(1) \\ (4.2) \quad &= -\frac{n-k-2}{2} \int_{B'_{2R} \times \omega} |\nabla u|^2 \phi_R - \int_{B'_{2R} \times \omega} |\nabla'' u|^2 \phi_R + o(1) \end{aligned}$$

Now, integrating by part the right hand side of (4.1) to obtain

$$J_2(R) := \int_{B'_{2R} \times \omega} |u|^{p-1} u \nabla' u \cdot (x-z)' \phi_R = -\frac{n-k}{p+1} \int_{B'_{2R} \times \omega} |u|^{p+1} \phi_R + o(1).$$

Since $J_1(R) = J_2(R)$ combining (4.2), (4.3), and letting $R \rightarrow \infty$, then from (4.1), we get

$$(4.3) \quad \frac{n-k-2}{2} \int_{\Omega} |\nabla u|^2 - \frac{n-k}{p+1} \int_{\Omega} |u|^{p+1} = - \int_{\Omega} |\nabla'' u|^2.$$

Therefore, (4.3) becomes

$$\int_{\Omega} |\nabla u|^2 - \frac{2(n-k)}{(p+1)(n-k-2)} \int_{\Omega} |u|^{p+1} = -\frac{2}{n-k-2} \int_{\Omega} |\nabla'' u|^2$$

From (3.4), we derive

$$\left(1 - \frac{2(n-k)}{(p+1)(n-k-2)}\right) \int_{\Omega} |u|^{p+1} = -\frac{2}{n-k-2} \int_{\Omega} |\nabla'' u|^2$$

If $p = p_s(n-k)$, from (4.4), one has

$$\int_{\Omega} |\nabla'' u|^2 = 0.$$

This gives the following classification: $u(x', x'') = u(x')$ in Ω and u satisfies

$$-\Delta' u = |u|^{p-1} u, \text{ in } \mathbb{R}^{n-k}, \text{ with } p = p_s(n-k).$$

□

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