



HAL
open science

THE GAMMA KUMARASWAMY-G FAMILY OF DISTRIBUTIONS: THEORY, INFERENCE AND APPLICATIONS

Rana Muhammad Imran Arshad, Christophe Chesneau, Farrukh Jamal,
Muhammad Mansoor, Muhamad Zubair, M Arslan Nasir

► **To cite this version:**

Rana Muhammad Imran Arshad, Christophe Chesneau, Farrukh Jamal, Muhammad Mansoor, Muhamad Zubair, et al.. THE GAMMA KUMARASWAMY-G FAMILY OF DISTRIBUTIONS: THEORY, INFERENCE AND APPLICATIONS. 2019. hal-02103938

HAL Id: hal-02103938

<https://hal.science/hal-02103938v1>

Preprint submitted on 19 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE GAMMA KUMARASWAMY-G FAMILY OF DISTRIBUTIONS: THEORY, INFERENCE AND APPLICATIONS

RANA MUHAMMAD IMRAN ARSHAD ¹, CHRISTOPHE CHESNEAU ², FARRUKH JAMAL ³,
MUHAMMAD MANSOOR ⁴, MUHAMAD ZUBAIR ⁴ AND M. ARSLAN NASIR ⁴

ABSTRACT. In this paper, we introduce a new family of univariate continuous distributions called the Gamma Kumaraswamy generalized family of distributions. We study some general mathematical properties of this family, including analysis of the shapes of the probability density and hazard rate functions, quantile function, skewness, kurtosis, linear representations of the cumulative distribution and probability density functions, moments and derived quantities, stochastic ordering, reliability parameter and order statistics. Then, we give a special attention to a particular member of the family with four parameters called the Gamma Kumaraswamy exponential distribution. Among its advantages, the corresponding probability density function can have symmetrical, left-skewed, right-skewed and reversed-J shapes, and the corresponding hazard rate function can have (near) constant, increasing, decreasing, upside-down bathtub, and bathtub shapes. Then, the inference on the Gamma Kumaraswamy exponential model is performed. The method of maximum likelihood is applied to estimate the model parameters. We illustrate the interest of the model by the analyses of two practical data sets, with favorable results in comparison to other competitive models in the field. It is hoped that the Gamma Kumaraswamy model will be attractive for the practitioner in many applied areas.

Keywords: Kumaraswamy distribution; Gamma distribution; generalized family; maximum likelihood method; data analysis.

AMS Subject: 9A60; 62E15; 62H10.

1. INTRODUCTION

In order to satisfy a legitimate scientific exigence, most of the modern experiments require a high degree of precision in the analysis of data. Unfortunately, this exigence can not be reached by the use of standard statistical models. For this reason, the creation of new flexible models, well adapted to the context, remains a passionating challenge for the statisticians. From a probabilistic point of view, attractive models can be derived from families of distributions enjoying remarkable mathematical and practical properties. Such families of distributions can be defined by the use of effective techniques introducing new tuning parameters to well-established distributions. Among others, a popular technique consists in compounding well-known distributions depending on their own parameters. The resulting families of distributions are often characterized by sophisticated but flexible functions, which can be easily handle thanks to the computational and analytical facilities available in modern programming softwares (as R, Maple, Mathematica. . .). In particular,

the use of these softwares can easily tackle the problems involved in computing eventual special functions. Among the families of distributions having a high impact in statistical modelling, there are the beta-G family by [16] and [21], the Kumaraswamy-G (Kw-G) family by [11], the McDonald-G (Mc-G) family by [2], the gamma-G type 1 family by [36] and [7], the gamma-G type 2 family by [31] and [7], the odd-gamma-G type 3 family by [34], the logistic-G family by [35], the odd exponentiated generalized (odd exp-G) family by [12], the transformed-transformer (T-X) (Weibull-X and gamma-X) family by [5], the exponentiated T-X family by [6], the odd Weibull-G family by [9], the exponentiated half-logistic by [13], the T-X{Y}-quantile based approach family by [3], the odd Burr-III-G family by [20], the Kumaraswamy odd Burr-G family by [27], the T-R{Y} family by [4], the generalized odd gamma-G family by [19].

In this study, we introduce a new family of distributions derived to the Kumaraswamy-G family of distributions introduced by [11] and the odd Gamma-G family of distributions established by [34], offering new perspectives of models. Before going further, let us briefly describe these two well-recognized families, beginning with the Kumaraswamy-G family of distributions. Let $a > 0$, $b > 0$, $G(x)$ be the cumulative distribution function (cdf) of an univariate continuous distribution and $g(x)$ be the corresponding probability distribution function (pdf). Then, the Kumaraswamy-G family of distributions is characterized by the cdf given by

$$H(x) = 1 - \{1 - G(x)^a\}^b, \quad x \in \mathbb{R}. \quad (1)$$

Also, the corresponding pdf is given by

$$h(x) = abg(x)G(x)^{a-1} \{1 - G(x)^a\}^{b-1}, \quad x \in \mathbb{R}. \quad (2)$$

The feature of Kumaraswamy-G family of distributions is to add two shape parameters a and b to the former distribution characterized by the cdf $G(x)$, increasing mechanically its flexible properties. Among others, the presence of a and b allow the construction of more flexible model to analyze a wide variety of data sets, as developed in [11] for the normal, Weibull, gamma, Gumbel and inverse Gaussian distributions. The Kumaraswamy-G family of distributions is also known to be a simple alternative to the beta-G family of distribution established by [16], since it deals with more tractable cdf and pdf. Interesting facts about the standard Kumaraswamy distribution are developed in [22]. Current developments and extensions of the Kumaraswamy-G family of distributions can be found in, e.g., [30], [15], [17] and [32].

On the other side, Torabi and Montazari [34] introduced the odd Gamma-G family of distributions, briefly described below. Let $\alpha > 0$, $H(x)$ be the cdf of an univariate continuous distribution, $\bar{H}(x) = 1 - H(x)$ and $h(x)$ be the corresponding pdf. Let $\gamma_1(\alpha, z)$ be the regularized lower incomplete gamma function defined by $\gamma_1(\alpha, z) = \gamma(\alpha, z)/\Gamma(\alpha)$, where $\gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt$ and $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$. Then, the odd Gamma-G family of distributions "with $G = H$ " is characterized by the cdf given by

$$F(x) = \gamma_1 \left(\alpha, \frac{H(x)}{\bar{H}(x)} \right), \quad x \in \mathbb{R}. \quad (3)$$

Also, the corresponding pdf is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{h(x)H(x)^{\alpha-1}}{\bar{H}(x)^{\alpha+1}} \exp\left(-\frac{H(x)}{\bar{H}(x)}\right), \quad x \in \mathbb{R}. \quad (4)$$

The odd-gamma-G family of distributions gives an alternative to the useful gamma-G type 1 family of distributions introduced by [36] in the sense that the following stochastic ordering holds: $F(x) \geq K(x)$, where $K(x) = \gamma_1(\alpha, -\log[\bar{H}(x)])$ is the cdf corresponding to the gamma-G type 1 family of distributions. Also, the merits of the odd-gamma-G family of distributions have been highlighted by [34], [19] and [29] via the exploration of various theoretical and practical aspects. In particular, it is shown that the former distribution characterized by the cdf $G(x)$ can take the benefits of the considered polynomial-exponential transformation with α as tuning parameter, allowing the construction of new flexible statistical models. In particular, for appropriated $G(x)$, the analyses of a wide broad of real life data sets are favorable to the odd-gamma-G models in comparison to well-recognized competitors.

Thus, by combining the Kumaraswamy-G and odd Gamma-H families of distributions, we aim to create a new generalized family of distributions benefiting of the respective qualities of these two families, enlarging the horizon of fields of applications. The corresponding family of distributions is called the Gamma Kumaraswamy-G (GKw-G) family of distributions. This study explores in both theoretical and practical terms the properties of the GKw-G family of distributions. A special member defined with the exponential distribution as baseline, called the GKw-E distribution, will serve as statistical model. The complete analyses of two practical data sets are proposed, showing that the GKw-E model presents better fit to eight notorious models in the field.

The rest of the article is organized as follows. In Section 2, we present the main functions of the GKw-G family of distributions, including the cdf, pdf and hrf. The shapes of the pdf and hrf are then studied analytically. Some general mathematical properties are presented in Section 3, as quantile function, skewness, kurtosis, linear representation of the cumulative distribution and probability density functions, moments and consorts, stochastic ordering, reliability parameter and order statistics. In Section 4, the GKw-E distribution is introduced, as well as some of its structural properties. In Section 5, the GKw-E model parameters are estimated by the maximum likelihood method and a simulation study is performed to verify the convergence properties. In Section 6, the usefulness of the GKw-E model is illustrated by means of two practical data sets. Finally, Section 7 offers some concluding remarks.

2. THE GAMMA KUMARASWAMY-G FAMILY OF DISTRIBUTIONS

2.1. Cumulative and probability density functions. We characterize the GKw-G family of distributions by the cdf of the odd Gamma-H family of distributions given by (3), defined with the cdf $H(x)$ of Kumaraswamy-G family of distributions given by (1). Hence, by noticing that $H(x)/\bar{H}(x) = \{1 - G(x)^a\}^{-b} - 1$, the corresponding cdf is given

by

$$F(x) = \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right), \quad x \in \mathbb{R}. \quad (5)$$

One can remark that, if $b = 1$, we rediscover the cdf of the generalized odd Gamma-G family introduced by [19], i.e., $F(x) = \gamma_1(\alpha, G(x)^a/[1 - G(x)^a])$, $x \in \mathbb{R}$. In this sense, the GKw-G family of distributions can be viewed as generalization of this family. The parameter b is of importance however, as we shall see in the coming mathematical properties and applications.

The corresponding pdf can be obtained by putting (1) and (2) into (4). More directly by differentiation of $F(x)$, it is given by

$$f(x) = \frac{ab}{\Gamma(\alpha)} g(x) G(x)^{a-1} \{1 - G(x)^a\}^{-b-1} \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha-1} \\ \times \exp \left[1 - \{1 - G(x)^a\}^{-b} \right]. \quad x \in \mathbb{R}. \quad (6)$$

Some special members of the GKw-G family of distributions characterized by their cdfs are presented in Table 1.

TABLE 1. Some members of the GKw-G family of distributions characterized by their cdfs.

cdf $G(x)$	Support	GKw-G cdf $F(x)$	Parameters
Uniform	$(0, \theta)$	$\gamma_1 \left(\alpha, \{1 - (x/\theta)^a\}^{-b} - 1 \right)$	(α, a, b, θ)
Exponential	$(0, +\infty)$	$\gamma_1 \left(\alpha, \{1 - [1 - e^{-\lambda x}]^a\}^{-b} - 1 \right)$	(α, a, b, λ)
Weibull	$(0, +\infty)$	$\gamma_1 \left(\alpha, \{1 - [1 - e^{-(\lambda x)^\beta}]^a\}^{-b} - 1 \right)$	(α, a, b, λ)
Inverse Weibull	$(0, +\infty)$	$\gamma_1 \left(\alpha, \{1 - e^{-a(\lambda/x)^\beta}\}^{-b} - 1 \right)$	$(\alpha, a, b, \lambda, \beta)$
Burr XII	$(0, +\infty)$	$\gamma_1 \left(\alpha, \{1 - \{1 - [1 + (x/s)^c]^{-k}\}^a\}^{-b} - 1 \right)$	(α, a, b, c, k, s)
Logistic	\mathbb{R}	$\gamma_1 \left(\alpha, \{1 - [1 + e^{-(x-\mu)/s}]^{-a}\}^{-b} - 1 \right)$	(α, a, b, μ, s)
Gumbel	\mathbb{R}	$\gamma_1 \left(\alpha, \{1 - \exp(-ae^{-(x-\mu)/\sigma})\}^{-b} - 1 \right)$	$(\alpha, a, b, \mu, \sigma)$
Normal	\mathbb{R}	$\gamma_1 \left(\alpha, \{1 - \Phi((x-\mu)/\sigma)\}^{-b} - 1 \right)$	$(\alpha, a, b, \mu, \sigma)$
Cauchy	\mathbb{R}	$\gamma_1 \left(\alpha, \{1 - [(1/\pi) \arctan((x-x_0)/\theta) + 1/2]^a\}^{-b} - 1 \right)$	$(\alpha, a, b, x_0, \theta)$

Thanks to its simplicity in the definition, the special member of the GKw-G family of distributions using the exponential distribution as baseline will be the object of all the attentions in Sections 4, 5 and 6.

2.2. Survival, hazard rate and cumulative hazard rate functions. Other basics functions related to the GKw-G family of distributions, playing a central role in the reliability theory, are given below.

The survival function (sf) of the GKw-G family of distributions is given by

$$S(x) = 1 - F(x) = 1 - \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right), \quad x \in \mathbb{R},$$

the corresponding hazard rate function (hrf) is given by

$$\begin{aligned} \pi(x) &= \frac{f(x)}{S(x)} \\ &= \frac{ab}{\Gamma(\alpha)} \frac{g(x)G(x)^{a-1} \{1 - G(x)^a\}^{-b-1} \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha-1} \exp \left[1 - \{1 - G(x)^a\}^{-b} \right]}{1 - \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right)}, \\ & \quad x \in \mathbb{R} \end{aligned}$$

and the corresponding cumulative hazard rate function (chrf) is given by

$$\Omega(x) = -\log [1 - F(x)] = -\log \left[1 - \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right) \right], \quad x \in \mathbb{R}.$$

2.3. Asymptotic properties. The two following propositions investigate the asymptotic properties of the cdf, sf, pdf and hrf of the GKw-G family of distributions.

Proposition 2.1. *The asymptotic properties of the cdf, pdf and hrf of the GKw-G family of distributions when $G(x) \rightarrow 0$ are, respectively, given by*

$$F(x) \sim \frac{b^\alpha}{\alpha\Gamma(\alpha)} G(x)^{\alpha\alpha}, \quad f(x) \sim \frac{ab^\alpha}{\Gamma(\alpha)} g(x)G(x)^{\alpha\alpha-1}, \quad h(x) \sim \frac{ab^\alpha}{\Gamma(\alpha)} g(x)G(x)^{\alpha\alpha-1}.$$

Proof. The proof follows from the following equivalences: when $y \rightarrow 0$, we have $(1-y^a)^{-b} \sim 1 + by^a$ and $\gamma_1(\alpha, y) \sim y^\alpha / (\alpha\Gamma(\alpha))$. \square

Proposition 2.2. *The asymptotic properties of the sf, pdf and hrf of the GKw-G family of distributions when $G(x) \rightarrow 1$ are, respectively, given by*

$$\begin{aligned} S(x) &\sim \frac{a^{-b(\alpha-1)}}{\Gamma(\alpha)} \{1 - G(x)\}^{-b(\alpha-1)} e^{1-a^{-b}\{1-G(x)\}^{-b}}, \\ f(x) &\sim \frac{ba^{-\alpha b}}{\Gamma(\alpha)} g(x) \{1 - G(x)\}^{-\alpha b-1} e^{1-a^{-b}\{1-G(x)\}^{-b}} \end{aligned}$$

and

$$h(x) \sim ba^{-b} g(x) \{1 - G(x)\}^{-b-1}.$$

Proof. The proof follows from the following equivalences: when $y \rightarrow +\infty$, we have $\gamma_1(\alpha, y) \sim 1 - y^{\alpha-1}e^{-y}/\Gamma(\alpha)$ and, when $y \rightarrow 1$, we have $y^a \sim 1 - a(1-y)$. \square

Propositions 2.1 and 2.2 show the roles of $G(x)$, $g(x)$, α , a and b on the asymptotic properties of the cdf, sf, pdf and hrf of the GKw-G family of distributions. In particular, we see that the parameter b has strong impact, mainly when $G(x) \rightarrow 1$.

2.4. Critical points. The study of the critical points of the pdf and hrf of the GKw-G family of distributions are crucial to understand the complexity of their shapes. They can be determined by solving the nonlinear equations $\partial \log[f(x)]/\partial x = 0$ and $\partial \log[h(x)]/\partial x = 0$, respectively, both given by

$$\begin{aligned} & \frac{\partial g(x)/\partial x}{g(x)} + (a-1)\frac{g(x)}{G(x)} + a(b+1)\frac{g(x)G(x)^{a-1}}{1-G(x)^a} \\ & + ab(\alpha-1)\frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}}{\{1-G(x)^a\}^{-b}-1} - abg(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} = 0 \end{aligned} \quad (7)$$

and

$$\begin{aligned} & \frac{\partial g(x)/\partial x}{g(x)} + (a-1)\frac{g(x)}{G(x)} + a(b+1)\frac{g(x)G(x)^{a-1}}{1-G(x)^a} \\ & + ab(\alpha-1)\frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}}{\{1-G(x)^a\}^{-b}-1} - abg(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} \\ & + \frac{ab}{\Gamma(\alpha)} \frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} \left\{ \{1-G(x)^a\}^{-b}-1 \right\}^{\alpha-1} \exp \left[1 - \{1-G(x)^a\}^{-b} \right]}{1 - \gamma_1 \left(\alpha, \{1-G(x)^a\}^{-b}-1 \right)} \\ & = 0. \end{aligned} \quad (8)$$

As usual, the nature of these critical points can be determined by investigating the signs of $\partial^2 \log[f(x)]/\partial x^2$ and $\partial^2 \log[h(x)]/\partial x^2$ taken at these points, respectively.

3. MAIN FEATURES

3.1. Quantile function. Let $Q_G(x)$ be the quantile function corresponding to $G(x)$, i.e., satisfying $G(Q_G(p)) = Q_G(G(p)) = p$ for any $p \in (0, 1)$. Then, the quantile function of the GKw-G family of distributions is given by

$$Q(p) = Q_G \left(\left[1 - \{1 + \gamma_1^{-1}(\alpha, p)\}^{-1/b} \right]^{1/a} \right), \quad p \in (0, 1), \quad (9)$$

where $\gamma_1^{-1}(\alpha, p)$ denotes the inverse function of $\gamma_1(\alpha, p)$, i.e., satisfying $\gamma_1(\alpha, \gamma_1^{-1}(\alpha, p)) = \gamma_1^{-1}(\alpha, \gamma_1(\alpha, p)) = p$ for any $p \in (0, 1)$. Further details on $\gamma_1^{-1}(\alpha, p)$ can be found in [1, Section 6.5]. In particular, the median of the GKw-G family of distributions is given by

$$M = Q(1/2) = Q_G \left(\left[1 - \{1 + \gamma_1^{-1}(\alpha, 0.5)\}^{-1/b} \right]^{1/a} \right).$$

Also, the three quartiles are given by $Q_1 = Q(1/4)$, $Q_2 = M$ and $Q_3 = Q(3/4)$ and the seven octiles are given by $O_1 = Q(1/8)$, $O_2 = Q(2/8)$, $O_3 = Q(3/8)$, $O_4 = Q(4/8)$, $O_5 = Q(5/8)$, $O_6 = Q(6/8)$ and $O_7 = Q(7/8)$.

The quantile function and its related values are useful to evaluate some properties of GKw-G family of distributions, as the skewness, kurtosis and central probabilistic results. Some of them are presented in the next two subsections.

3.2. Skewness and kurtosis. A measure of the skewness of the GKw-G family of distributions is given by

$$S = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}. \quad (10)$$

In full generality, for given $G(x)$, α , a and b , when the corresponding GKw-G distribution is symmetric, we have $S = 0$, when it is right skewed, we have $S > 0$ and when it is left skewed, we have $S < 0$. See [23].

Also, a measure of the kurtosis of the GKw-G family of distributions is given by

$$K = \frac{O_3 - O_1 + O_7 - O_5}{O_6 - O_2}. \quad (11)$$

For given $G(x)$, α , a and b , as K increases, the tail of the corresponding GKw-G distribution becomes heavier. We refer to [26].

The advantages of these measures are to be robust in presence of outliers and they always exist (even if the distribution does not admit moments).

3.3. Some results in distribution. As usual, for any random variable U following the uniform distribution over $(0, 1)$, the random variable $X = Q(U)$ has the cdf $F(x)$. For given $G(x)$, α , a and b , this characterization is useful to generate random data distributed according to the related GKw-G distribution.

We say that a random variable follows the Gamma distribution $\mathcal{G}_{am}(1, \alpha)$ if it has the cdf given by $K(x) = \gamma_1(\alpha, x)$ with $x > 0$. If X is a random variable having the cdf of the GKw-G family of distributions, then the random variable Y defined by $Y = \{1 - G(X)^a\}^{-b} - 1$ follows the Gamma distribution $\mathcal{G}_{am}(1, \alpha)$.

Also, if Y is a random variable following the Gamma distribution $\mathcal{G}_{am}(1, \alpha)$, then the random variable X defined by $X = Q_G \left(\left[1 - \{1 + Y\}^{-1/b} \right]^{1/a} \right)$ has the cdf of the GKw-G family of distributions.

3.4. Linear representations. This subsection is devoted to linear representations for the pdf and cdf of the GKw-G family of distributions.

Proposition 3.1. *We have the following linear representations for the cdf and pdf of the GKw-G family of distributions:*

$$F(x) = \sum_{i=0}^{+\infty} w_i G(x)^{ai}, \quad f(x) = \sum_{i=0}^{+\infty} w_i [aig(x)G(x)^{ai-1}], \quad (12)$$

where

$$w_i = \sum_{j,k=0}^{+\infty} \frac{(-1)^{i+j+k}}{\Gamma(\alpha)k!(\alpha+k)} \binom{\alpha+k}{j} \binom{b(j-\alpha-k)}{i} \quad (13)$$

and $\binom{b}{a}$ denotes the generalized binomial coefficient, i.e., $\binom{b}{a} = b(b-1)\dots(b-a+1)/a!$.

Proof. By using the regularized lower incomplete gamma function series expansion, i.e.,

$$\gamma_1(\alpha, y) = \sum_{k=0}^{+\infty} (-1)^k \frac{y^{\alpha+k}}{\Gamma(\alpha)k!(\alpha+k)}, \quad y \geq 0,$$

and after some simplifications, we can express $F(x)$ as

$$\begin{aligned} F(x) &= \gamma_1 \left(\alpha, \frac{1 - \{1 - G(x)^a\}^b}{\{1 - G(x)^a\}^b} \right) \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha)k!(\alpha+k)} \{1 - G(x)^a\}^{-b(\alpha+k)} \underbrace{\left[1 - \{1 - G(x)^a\}^b \right]^{\alpha+k}}_A. \end{aligned}$$

By virtue of the generalized binomial series expansion, the quantity A can expressed as

$$A = \sum_{j=0}^{+\infty} (-1)^j \binom{\alpha+k}{j} \{1 - G(x)^a\}^{bj}.$$

By putting the previous equalities together, we get

$$F(x) = \sum_{j,k=0}^{+\infty} \frac{(-1)^{j+k}}{\Gamma(\alpha)k!(\alpha+k)} \binom{\alpha+k}{j} \underbrace{\{1 - G(x)^a\}^{b(j-\alpha-k)}}_B.$$

By using again generalized binomial series expansion, the quantity B becomes

$$B = \sum_{i=0}^{+\infty} (-1)^i \binom{b(j-\alpha-k)}{i} G(x)^{ai}.$$

The desired linear representation of $F(x)$ follows from the combination of all the equalities above. By differentiation, we derive the linear representation of $f(x)$. This completes the proof of Proposition 3.1. \square

Since it depends on the well-known exp-G family of distributions (with parameter ai for any integer i), the linear representations presented in Proposition 3.1 are useful to derive related analytical and numerical properties. Some of them are explored in the subsections below.

3.5. Moments and derivations. In this subsection, we assume that all the presented integrals and sum exist (which is not necessarily the case, depending on the definition of $G(x)$, among others). Let r be an integer. Then, the r -th moment of GKw-G family of

distributions is given by

$$\begin{aligned}\mu'_r &= \int_{-\infty}^{+\infty} x^r f(x) dx \\ &= \int_{-\infty}^{+\infty} x^r \frac{ab}{\Gamma(\alpha)} g(x) G(x)^{a-1} \{1 - G(x)^a\}^{-b-1} \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha-1} \\ &\quad \times \exp \left[1 - \{1 - G(x)^a\}^{-b} \right] dx.\end{aligned}\quad (14)$$

By using the quantile function given by (9), with the change of variable $x = Q(p)$, we can express μ'_r as

$$\mu'_r = \int_0^1 [Q(p)]^r dp = \int_0^1 \left[Q_G \left(\left[1 - \{1 + \gamma_1^{-1}(\alpha, p)\}^{-1/b} \right]^{1/a} \right) \right]^r dp. \quad (15)$$

For given $G(x)$, α , a and b , this integral can be computed numerically via any mathematical softwares (R, Maple, Matlab, Mathematica. . .).

A linear expression of μ'_r can be deduced from (12). Indeed, we have

$$\mu'_r = \sum_{i=0}^{+\infty} w_i \int_{-\infty}^{+\infty} x^r [aig(x)G(x)^{ai-1}] dx = \sum_{i=0}^{+\infty} w_i ai \int_0^1 p^{ai-1} [Q_G(p)]^r dp.$$

Among others, one can deduce the mean given by $\mu = \mu'_1$, the variance given by $\sigma^2 = \mu'_2 - (\mu'_1)^2$, the r -th central moment given by

$$\mu_r = \int_{-\infty}^{+\infty} (x - \mu'_1)^r f(x) dx = \sum_{k=0}^r \binom{r}{k} (-1)^k (\mu'_1)^k \mu'_{r-k}, \quad (16)$$

the coefficient of skewness given by $CS = \mu_3/\mu_2^{3/2}$, the coefficient of kurtosis given by $CK = \mu_4/\mu_2^2$ and the moment generating function given by

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx = \sum_{r=0}^{+\infty} \frac{t^r}{r!} \mu'_r.$$

Alternatively, we can use (12) to have a linear representation for $M(t)$ without using moments. Indeed, we have

$$M(t) = \sum_{i=0}^{+\infty} w_i \int_{-\infty}^{+\infty} e^{tx} [aig(x)G(x)^{ai-1}] dx = \sum_{i=0}^{+\infty} w_i ai \int_0^1 p^{ai-1} e^{tQ_G(p)} dp.$$

Finally, let us mention that the incomplete moments can be expressed in a similar way, giving expressions for the Bonferroni and Lorenz curves, mean residual-life, mean waiting-time, mean deviation about the mean and mean deviation about the median. For instance, we refer to the methodology of [19].

3.6. Stochastic ordering. We now prove a result on the stochastic ordering involving the GKw-G family of distributions with a and b as common parameters. Further details on stochastic ordering can be found in [33].

Proposition 3.2. *Let X be a random variable having the pdf $f_1(x)$ given by (6) with parameters α_1 , a and b and Y be a random variable having the pdf $f_2(x)$ given by (6) with parameters α_2 , a and b . Then, if $\alpha_1 \leq \alpha_2$, we have $X \leq_{lr} Y$, i.e., $f_1(x)/f_2(x)$ is decreasing.*

Proof. We have

$$\frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)} \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha_1 - \alpha_2}.$$

By differentiating with respect to x , since $\alpha_1 \leq \alpha_2$, we have

$$\begin{aligned} \frac{\partial}{\partial x} \frac{f_1(x)}{f_2(x)} &= \\ \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)} (\alpha_1 - \alpha_2) \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha_1 - \alpha_2 - 1} & abg(x)G(x)^{a-1} \{1 - G(x)^a\}^{-b-1} \leq 0. \end{aligned}$$

Hence we have $X \leq_{lr} Y$. This ends the proof of Proposition 3.2. \square

3.7. Reliability parameter. The reliability parameter plays an important role in the area of engineering. It is a measure of component reliability. Further details can be found in [24]. Here, we present a result on this parameter defined with the GKw-G family of distributions. Let X be a random variable having the pdf $f_1(x)$ given by (6) with parameters α_1 , a_1 and b_1 and Y be a random variable having the cdf $F_2(x)$ given by (5) with parameters α_2 , a_2 and b_2 . We suppose that X and Y are independent. Then the reliability parameter is defined by

$$R = \mathbb{P}(Y < X) = \int_{-\infty}^{+\infty} f_1(x)F_2(x)dx.$$

By using the expressions of $f_1(x)$ and $F_2(x)$, we have

$$\begin{aligned} R &= \int_{-\infty}^{+\infty} \frac{a_1 b_1}{\Gamma(\alpha_1)} g(x)G(x)^{a_1-1} \{1 - G(x)^{a_1}\}^{-b_1-1} \left\{ \{1 - G(x)^{a_1}\}^{-b_1} - 1 \right\}^{\alpha_1-1} \\ &\quad \times \exp \left[1 - \{1 - G(x)^{a_1}\}^{-b_1} \right] \gamma_1 \left(\alpha_2, \{1 - G(x)^{a_2}\}^{-b_2} - 1 \right) dx. \end{aligned}$$

Alternatively, with the change of variable $x = Q_1(p)$, where $Q_1(p)$ denotes the quantile function given by (9) corresponding to $f_1(x)$, we have

$$\begin{aligned} R &= \int_0^1 F_2(Q_1(p))dp \\ &= \int_0^1 \gamma_1 \left(\alpha_2, \left\{ 1 - \left[1 - \{1 + \gamma_1^{-1}(\alpha_1, p)\}^{-1/b_1} \right]^{a_2/a_1} \right\}^{-b_2} - 1 \right) dp. \end{aligned}$$

In particular, from this expression, we see that R does not depend on the baseline distribution characterized by the cdf $G(x)$.

To the best of our knowledge, there is close form for the previous integrals. A linear expression can be given by using (12). Indeed, let us consider the expansions:

$$F_2(x) = \sum_{i=0}^{+\infty} w_i[\alpha_2, a_2, b_2]G(x)^{a_2i}, \quad f_1(x) = \sum_{j=0}^{+\infty} w_j[\alpha_1, a_1, b_1] [a_1jg(x)G(x)^{a_1j-1}],$$

where $w_i[\alpha_2, a_2, b_2]$ and $w_j[\alpha_1, a_1, b_1]$ are given by (13) with the parameters α_2, a_2, b_2 and α_1, a_1, b_1 , respectively. Then,

$$R = \sum_{i,j=0}^{+\infty} w_i[\alpha_2, a_2, b_2]w_j[\alpha_1, a_1, b_1]a_1j \int_{-\infty}^{+\infty} g(x)G(x)^{a_1j+a_2i-1}dx \\ \sum_{i,j=0}^{+\infty} w_i[\alpha_2, a_2, b_2]w_j[\alpha_1, a_1, b_1] \frac{a_1j}{a_1j + a_2i}.$$

As usual, when $\alpha_1 = \alpha_2, a_1 = a_2$ and $b_1 = b_2$ (corresponding to the identically distributed case), we have $R = 1/2$.

3.8. Order statistics. The order statistics naturally arise in many applications involving data relating to survival testing studies. All the details can be found in the book of [14]. This subsection is devoted to the order statistics of the GKw-G family of distributions. Let X_1, \dots, X_n be the random sample from the GKw-G family of distributions and $X_{i:n}$ be the i -th order statistic. Then the pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1} [1 - F(x)]^{n-i}, \quad x \in \mathbb{R}. \quad (17)$$

Hence, by using (5) and (6), we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \frac{ab}{\Gamma(\alpha)} g(x)G(x)^{a-1} \{1 - G(x)^a\}^{-b-1} \left\{ \{1 - G(x)^a\}^{-b} - 1 \right\}^{\alpha-1} \\ \exp \left[1 - \{1 - G(x)^a\}^{-b} \right] \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right)^{i-1} \left[1 - \gamma_1 \left(\alpha, \{1 - G(x)^a\}^{-b} - 1 \right) \right]^{n-i}.$$

In particular, the pdfs of $X_{1:n} = \inf(X_1, \dots, X_n)$ and $X_{n:n} = \sup(X_1, \dots, X_n)$ are respectively given by $f_{1:n}(x)$ and $f_{n:n}(x)$.

The proposition below presents a result characterizing this pdf.

Proposition 3.3. *The pdf of $X_{i:n}$ can be expressed as a linear combination of pdfs of the exp-G family of distributions.*

Proof. Let us consider the expression of $f_{i:n}(x)$ given by (17). It follows from the binomial formula and (12) that

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) [F(x)]^{j+i-1} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left\{ \sum_{\ell=0}^{+\infty} w_{\ell} \left[a \ell g(x) G(x)^{a\ell-1} \right] \right\} \left[\sum_{k=0}^{+\infty} w_k G(x)^{ak} \right]^{j+i-1}. \end{aligned}$$

By virtue of a result established by [18, Section 0.314], we have

$$\left[\sum_{k=0}^{+\infty} w_k G(x)^{ak} \right]^{j+i-1} = \sum_{m=0}^{+\infty} d_{j+i-1,m} G(x)^{am},$$

where $d_{j+i-1,0} = w_0^{j+i-1}$ and, for any integer $m \geq 1$,

$$d_{j+i-1,m} = \frac{1}{mw_0} \sum_{k=1}^m (k(j+i) - m) w_k d_{j+i-1,m-k}.$$

By putting the equalities above together, we obtain

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{\ell,m=0}^{+\infty} \binom{n-i}{j} (-1)^j w_{\ell} d_{j+i-1,m} \frac{\ell}{\ell+m} q_{\ell,m}(x), \quad (18)$$

where $q_{\ell,m}(x) = a(\ell+m)g(x)G(x)^{a(\ell+m)-1}$. Since $q_{\ell,m}(x)$ is a pdf of the exp-G family of distributions with parameter $a(\ell+m)$, the proof of Proposition 3.3 is complete. \square

By using the existing results on the exp-G family of distributions, we can use Proposition 3.3 to derive mathematical properties of the distribution of the i -th order statistics, as moments and all the related quantities.

4. GKw-EXPONENTIAL DISTRIBUTION

4.1. Definition. In this section, we focus our attention on the special member of the GKw-G family of distributions using the exponential distribution as baseline. Hence, by substituting the cdf $G(x) = 1 - e^{-\lambda x}$, $x > 0$, into (5), the cdf of this special distribution is given by

$$F_{GKw-E}(x) = \gamma_1 \left(\alpha, \left\{ 1 - \left(1 - e^{-\lambda x} \right)^a \right\}^{-b} - 1 \right), \quad x > 0. \quad (19)$$

The related distribution is called the GKw-Exponential (GKw-E) distribution. The corresponding pdf is given by

$$\begin{aligned} f_{GKw-E}(x) = & \\ & \frac{ab\lambda}{\Gamma(\alpha)} e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b-1} \left\{\left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b} - 1\right\}^{\alpha-1} \\ & \times \exp\left[1 - \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b}\right]. \quad x > 0. \end{aligned} \quad (20)$$

Other basics functions related to the GKw-E distribution are given below. The corresponding sf is given by

$$S_{GKw-E}(x) = 1 - \gamma_1\left(\alpha, \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b} - 1\right), \quad x > 0, \quad (21)$$

the corresponding hrf is given by

$$\begin{aligned} \pi_{GKw-E}(x) = & \\ & \frac{ab\lambda}{\Gamma(\alpha)} \frac{e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b-1} \left\{\left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b} - 1\right\}^{\alpha-1}}{1 - \gamma_1\left(\alpha, \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b} - 1\right)}, \\ & \times \exp\left[1 - \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b}\right], \quad x > 0 \end{aligned} \quad (22)$$

and the corresponding chrh is given by

$$\Omega_{GKw-E}(x) = -\log\left[1 - \gamma_1\left(\alpha, \left\{1 - (1 - e^{-\lambda x})^a\right\}^{-b} - 1\right)\right], \quad x > 0.$$

Let us now investigate some asymptotic properties of $F_{GKw-E}(x)$, $S_{GKw-E}(x)$, $f_{GKw-E}(x)$ and $h_{GKw-E}(x)$. When $x \rightarrow 0$, by using Proposition 2.1 with $G(x) \sim \lambda x$ and $g(x) \sim \lambda$, we have

$$F_{GKw-E}(x) \sim \frac{b^\alpha \lambda^{\alpha\alpha}}{\alpha \Gamma(\alpha)} x^{\alpha\alpha}, \quad f_{GKw-E}(x) \sim \frac{ab^\alpha \lambda^{\alpha\alpha}}{\Gamma(\alpha)} x^{\alpha\alpha-1}, \quad h_{GKw-E}(x) \sim \frac{ab^\alpha \lambda^{\alpha\alpha}}{\Gamma(\alpha)} x^{\alpha\alpha-1}.$$

Hence, when $x \rightarrow 0$, if $a\alpha < 1$, we have $f_{GKw-E}(x) \rightarrow +\infty$, if $a\alpha = 1$, we have $f_{GKw-E}(x) \rightarrow ab^{1/a}\lambda/\Gamma(\alpha)$, and if $a\alpha > 1$, we have $f_{GKw-E}(x) \rightarrow 0$. Similarly, if $a\alpha < 1$, we have $h_{GKw-E}(x) \rightarrow +\infty$, if $a\alpha = 1$, we have $h_{GKw-E}(x) \rightarrow ab^{1/a}\lambda/\Gamma(\alpha)$, and if $a\alpha > 1$, we have $h_{GKw-E}(x) \rightarrow 0$.

When $x \rightarrow +\infty$, by using Proposition 2.2, we have

$$S_{GKw-E}(x) \sim \frac{a^{-b(\alpha-1)}}{\Gamma(\alpha)} e^{\lambda b(\alpha-1)x} e^{1-a^{-b}e^{\lambda bx}}, \quad f_{GKw-E}(x) \sim \frac{\lambda b a^{-\alpha b}}{\Gamma(\alpha)} e^{\lambda b \alpha x} e^{1-a^{-b}e^{\lambda bx}}$$

and

$$h_{GKw-E}(x) \sim \lambda b a^{-b} e^{\lambda bx}.$$

Hence, when $x \rightarrow +\infty$, we $f_{GKw-E}(x) \rightarrow 0$ and $h_{GKw-E}(x) \rightarrow +\infty$.

The critical points of the GKw-E pdf and hrf can be determined by using the non-linear equations given by (7) and (8).

In order to give more concrete illustrations on their shapes, Figures 1 and 2 display some plots of the GKw-E pdf and hrf for specified parameters values. Figure 1 indicates that the GKw-E distribution is right-skewed, left skewed and reversed-J shaped. Also, Figure 2 shows that GKw-E hrf can produce various shapes such as increasing, decreasing, bathtub and upside-down bathtub.

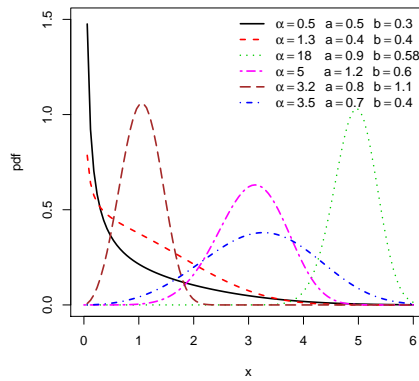


FIGURE 1. Plots of GKw-E pdfs for some parametric values with fixed $\lambda = 1$.

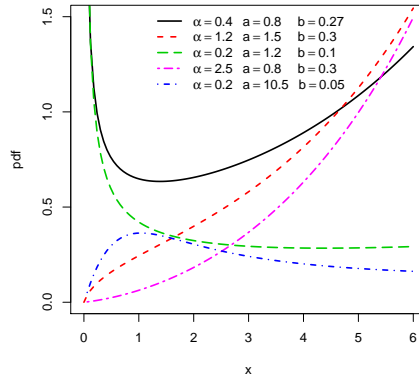


FIGURE 2. Plots of GKw-E hrfs for some parametric values with fixed $\lambda = 1$.

4.2. Other properties. All the general properties determined in Section 3 can be transposed to the GKw-E distribution. The most significant of them are described below.

Since $Q_G(p) = -(1/\lambda) \log(1-p)$, based on (9), the GKw-E quantile function is given by

$$Q_{GKw-E}(p) = -\frac{1}{\lambda} \log \left[1 - \left[1 - \{1 + \gamma_1^{-1}(\alpha, p)\}^{-1/b} \right]^{1/a} \right], \quad p \in (0, 1).$$

Form this definition, the quartiles and octiles can be determined, as well as skewness and kurtosis as given by (10) and (11), respectively, and some results on distributions, as the useful one: for a random variable U following the uniform distribution on $(0, 1)$, $Q_{GKw-E}(U)$ follows the GKw-E distribution.

A result on linear representations of $F(x)$ and $f(x)$ in terms of exponential functions is presented below.

Proposition 4.1. *We have the following linear representations for the cdf and pdf of the GKw-E distribution:*

$$F_{GKw-E}(x) = \sum_{m=0}^{+\infty} w_m^* e^{-\lambda m x}, \quad f_{GKw-E}(x) = \sum_{m=0}^{+\infty} w_m^{**} e^{-\lambda m x}, \quad x > 0, \quad (23)$$

where

$$w_m^* = \sum_{i,j,k=0}^{+\infty} \frac{(-1)^{i+j+k+m}}{\Gamma(\alpha)k!(\alpha+k)} \binom{\alpha+k}{j} \binom{b(j-\alpha-k)}{i} \binom{\alpha i}{m}, \quad w_m^{**} = -\lambda m w_m^*. \quad (24)$$

Proof. For any positive integer i , by virtue of the generalized binomial formula, we have

$$G(x)^{\alpha i} = (1 - e^{-\lambda x})^{\alpha i} = \sum_{m=0}^{+\infty} \binom{\alpha i}{m} (-1)^m e^{-\lambda m x}.$$

It follows from Proposition 3.1 that

$$F_{GKw-E}(x) = \sum_{i=0}^{+\infty} w_i G(x)^{\alpha i} = \sum_{m=0}^{+\infty} w_m^* e^{-\lambda m x},$$

where $w_m^* = \sum_{i=0}^{+\infty} \binom{\alpha i}{m} (-1)^m w_i$. The corresponding pdf is obtained by differentiation of $F_{GKw-E}(x)$. This ends the proof of Proposition 4.1. \square

Thanks to Proposition 4.1, several of structural properties of the GKw-E distribution can be derived. Some of them are described below.

The r -th moment of the GKw-E distribution is given by

$$\mu'_r = \sum_{m=0}^{+\infty} w_m^{**} \int_0^{+\infty} x^r e^{-\lambda m x} dx = \frac{1}{\lambda^{r+1}} \Gamma(r+1) \sum_{m=0}^{+\infty} w_m^{**} \frac{1}{m^{r+1}}.$$

We thus deduce the mean, the variance, the r -th central moment by using the formula (16), the coefficient of skewness and the coefficient of kurtosis.

In a similar manner, the moment generating function can be expressed as, for $t \leq 0$,

$$M(t) = \sum_{m=0}^{+\infty} w_m^{**} \int_0^{+\infty} e^{tx} e^{-\lambda mx} dx = \sum_{m=0}^{+\infty} w_m^{**} \frac{1}{\lambda m - t}.$$

The r -th incomplete moment is given by, for $t \geq 0$,

$$\begin{aligned} I_r(t) &= \int_{-\infty}^t x^r f_{GKw-E}(x) dx = \sum_{m=0}^{+\infty} w_m^{**} \int_0^t x^r e^{-\lambda mx} dx \\ &= \frac{1}{\lambda^{r+1}} \sum_{m=0}^{+\infty} w_m^{**} \frac{1}{m^{r+1}} \gamma(r+1, \lambda mt). \end{aligned}$$

The incomplete moments are useful to determine other important mathematical quantities as the Bonferroni and Lorenz curves, mean residual-life, mean waiting-time, mean deviation about the mean and mean deviation about the median.

We end this subsection by a result about the i -th order statistics related to the GKw-E distribution, mainly based on Proposition 3.3.

Proposition 4.2. *Let X_1, \dots, X_n be the random sample from the GKw-E distribution and $X_{i:n}$ be the i -th order statistic. Then the pdf of $X_{i:n}$ can be expressed as a linear combination of simple exponential functions.*

Proof. By applying Proposition 3.3, and more precisely, the equality (18) in the proof, we can write

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{\ell, m=0}^{+\infty} \binom{n-i}{j} (-1)^j w_\ell d_{j+i-1, m} \frac{\ell}{\ell+m} q_{\ell, m}(x),$$

where $q_{\ell, m}(x) = a(\ell+m)g(x)G(x)^{a(\ell+m)-1}$. It follows from the generalized binomial theorem that

$$q_{\ell, m}(x) = a(\ell+m)\lambda \sum_{k=0}^{+\infty} \binom{a(\ell+m)-1}{k} (-1)^k e^{-\lambda(k+1)x}.$$

Hence we can write

$$f_{i:n}(x) = \sum_{k=0}^{+\infty} v_k e^{-\lambda(k+1)x},$$

with

$$v_k = \frac{n!}{(i-1)!(n-i)!} a\lambda \sum_{j=0}^{n-i} \sum_{\ell, m=0}^{+\infty} \binom{n-i}{j} \binom{a(\ell+m)-1}{k} (-1)^{j+k} \ell w_\ell d_{j+i-1, m}.$$

The proof of Proposition 4.2 is completed. \square

Proposition 4.2 allows the determination of structural properties for $X_{i:n}$, as moments, moment generating function, incomplete moments. . .

5. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we adopt the GKw-E distribution as model and consider the estimation of the unknown parameters by the maximum likelihood method.

5.1. Characterization. The usefulness of the maximum likelihood estimates (MLEs) in statistical inference is due to their theoretical and practical merits. Indeed, they have a limiting normal distribution which are easily to handle either analytically or numerically. The log-likelihood function for the vector of parameters $\Omega = (a, b, \alpha, \lambda)^\top$ is given by

$$\begin{aligned} \ell(\Omega) &= \sum_{i=1}^n \log[f_{GKw-E}(x_i)] \\ &= n \log(a) + n \log(b) - n \log[\Gamma(\alpha)] + n \log(\lambda) - \lambda \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \log[1 - e^{-\lambda x_i}] \\ &\quad - (b+1) \sum_{i=1}^n \log\left[1 - \left(1 - e^{-\lambda x_i}\right)^a\right] + (\alpha-1) \sum_{i=1}^n \log\left[\left\{1 - \left(1 - e^{-\lambda x_i}\right)^a\right\}^{-b} - 1\right] + n \\ &\quad - \sum_{i=1}^n \left\{1 - \left(1 - e^{-\lambda x_i}\right)^a\right\}^{-b}. \end{aligned}$$

The first partial derivatives of $\ell(\Omega)$ with respect to a , b , α and λ are given by

$$\begin{aligned} \frac{\partial}{\partial a} \ell(\Omega) &= \frac{n}{a} + \sum_{i=1}^n \log[1 - e^{-\lambda x_i}] + (b+1) \sum_{i=1}^n \frac{(1 - e^{-\lambda x_i})^a \log[1 - e^{-\lambda x_i}]}{1 - (1 - e^{-\lambda x_i})^a} \\ &\quad + b(\alpha-1) \sum_{i=1}^n \frac{\{1 - (1 - e^{-\lambda x_i})^a\}^{-b-1} (1 - e^{-\lambda x_i})^a \log[1 - e^{-\lambda x_i}]}{\{1 - (1 - e^{-\lambda x_i})^a\}^{-b} - 1} \\ &\quad - b \sum_{i=1}^n \left\{1 - \left(1 - e^{-\lambda x_i}\right)^a\right\}^{-b-1} \left(1 - e^{-\lambda x_i}\right)^a \log[1 - e^{-\lambda x_i}], \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial b} \ell(\Omega) &= \frac{n}{b} - \sum_{i=1}^n \log\left[1 - \left(1 - e^{-\lambda x_i}\right)^a\right] \\ &\quad - (\alpha-1) \sum_{i=1}^n \frac{\{1 - (1 - e^{-\lambda x_i})^a\}^{-b} \log[1 - (1 - e^{-\lambda x_i})^a]}{\{1 - (1 - e^{-\lambda x_i})^a\}^{-b} - 1} \\ &\quad + \sum_{i=1}^n \left\{1 - \left(1 - e^{-\lambda x_i}\right)^a\right\}^{-b} \log\left[1 - \left(1 - e^{-\lambda x_i}\right)^a\right], \end{aligned}$$

$$\frac{\partial}{\partial \alpha} \ell(\Omega) = -n \frac{\Gamma(\alpha)'}{\Gamma(\alpha)} + \sum_{i=1}^n \log\left[\left\{1 - \left(1 - e^{-\lambda x_i}\right)^a\right\}^{-b} - 1\right],$$

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \ell(\Omega) &= \frac{n}{\lambda} - \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} + a(b+1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1}}{1 - (1 - e^{-\lambda x_i})^a} \\
&+ ba(\alpha-1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1} \{1 - (1 - e^{-\lambda x_i})^a\}^{-b-1}}{\{1 - (1 - e^{-\lambda x_i})^a\}^{-b} - 1} \\
&- ab \sum_{i=1}^n x_i e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{a-1} \{1 - (1 - e^{-\lambda x_i})^a\}^{-b-1}.
\end{aligned}$$

Setting these equations to zero and solving them simultaneously yields the MLEs of the GKw-E parameters. Since there are no close form for these MLEs, one can use a standard statistical software or numerical techniques to solve them. Also, let us mention that the observed Fisher information for the MLEs can be computed, allowing the construction of confidence intervals for the parameters based on the limiting normal distribution. In particular, this is useful to examine the probability coverage of these interval through simulation, which is done the next subsection.

5.2. A numerical study. Now we assess the performance of the maximum likelihood method for estimating the GKw-E parameters using Monte Carlo simulations. The simulation study is repeated 5000 times each with sample sizes $n = 50, 100, 200$ and the following parameter scenarios: I: $a = 0.5, b = 0.5, \alpha = 0.5$, and $\lambda = 1$, II: $a = 0.3, b = 1.5, \alpha = 0.7$, and $\lambda = 2.5$ and III: $a = 1.7, b = 0.7, \alpha = 0.2$, and $\lambda = 0.3$, IV: $a = 0.1, b = 2.5, \alpha = 1.1$, and $\lambda = 1.5$, V: $a = 2.5, b = 1.7, \alpha = 2.5$, and $\lambda = 1$, VI: $a = 1.8, b = 1.7, \alpha = 2.1$, and $\lambda = 0.1$. Under this setting, Table 2 gives the average biases (Bias) of the MLEs, mean square errors (MSEs) and model-based coverage probabilities (CPs) for the parameters a, b, α and λ . Based on these results, we conclude that the MLEs perform well in estimating the parameters of the GKw-E distribution. The CPs of the confidence intervals are quite close to the 95% nominal levels. Therefore, the MLEs and their asymptotic results can be adopted for efficiently estimating and constructing confidence intervals for the model parameters.

TABLE 2. Monte Carlo simulation results for the GKw-E distribution: Biases, MSEs and CPs.

Parameter	n	I			II			III		
		Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
a	50	-0.015	0.051	0.98	-0.008	0.044	0.94	0.810	14.386	0.85
	100	0.007	0.047	0.97	0.023	0.049	0.95	0.616	4.488	0.90
	200	0.039	0.045	0.96	0.004	0.037	0.95	0.576	2.908	0.95
b	50	-0.140	0.162	0.97	-0.404	1.318	0.90	0.244	3.047	0.97
	100	-0.125	0.127	0.97	-0.217	0.918	0.96	0.307	2.484	0.98
	200	-0.113	0.104	0.95	-0.072	0.477	0.99	0.287	0.977	0.99
α	50	0.153	0.257	0.91	0.465	1.300	0.92	0.452	1.404	0.83
	100	0.084	0.116	0.91	0.307	0.710	0.93	0.225	0.989	0.89
	200	0.046	0.082	0.89	0.306	0.628	0.96	0.139	0.958	0.96
λ	50	1.807	6.527	0.95	2.601	2.726	0.92	0.752	1.324	1.00
	100	1.461	4.742	0.94	1.136	1.129	0.93	0.555	1.002	1.00
	200	1.180	3.136	0.95	0.202	0.847	0.97	0.364	0.743	0.97
Parameter	n	IV			V			VI		
		Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
a	50	-0.904	1.154	0.65	0.146	0.535	0.94	0.441	1.253	0.95
	100	-0.665	0.461	0.92	0.164	0.309	0.95	0.194	0.579	0.96
	200	-0.002	0.019	0.97	0.195	0.228	0.97	0.015	0.263	0.99
b	50	-0.032	0.349	0.98	0.172	0.241	1.00	0.018	0.893	0.95
	100	0.014	0.333	0.98	0.053	0.065	0.96	0.072	0.633	0.96
	200	-0.051	0.052	0.96	0.001	0.031	0.97	0.136	0.438	0.98
α	50	0.477	0.480	0.89	0.311	0.163	0.99	-0.158	0.112	0.97
	100	0.270	0.163	0.96	0.271	0.132	0.95	-0.145	0.106	0.96
	200	-0.051	0.052	0.98	0.222	0.100	0.96	-0.148	0.110	0.97
λ	50	0.337	0.601	0.99	-0.062	0.022	0.95	0.179	0.298	0.95
	100	0.214	0.284	0.96	-0.059	0.017	0.96	0.204	0.323	0.96
	200	0.243	0.814	0.98	-0.051	0.011	0.98	0.253	0.392	0.97

6. APPLICATIONS

In this section, we compare the proposed GKw-E model with well-known models using two real data sets.

Application 1. The first data set is reported in [31]. The data represent the annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado from 1900 through 1999. The data are as follows: 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241.

In the statistical literature, several models are appropriate to the analysis of such kinds of data. The most commonly used are the lognormal, generalized logistic (GL), Gumbel, gamma, Weibull and generalized binomial exponential 2 (GBE2) models. Several extensions have also been introduced by this purpose. Here, to highlight the potentiality of the GKw-E model, the comparison is made between the GKw-E model and eight notorious models: the Kumaraswamy Weibull (Kw-W) model studied by [10], the Beta Weibull (BW) model due to [25], the exponentiated generalized Weibull (EGW) model by [28], the generalized binomial exponential 2 (GBE2) model introduced by [8], the generalized logistic (GL) model, the Gumbel model, the gamma model and the Weibull model. We estimate the unknown models parameters by the maximum likelihood method (as described in Section 5 for the GKw-E model). The log-likelihood function is evaluated at the MLEs ($\hat{\ell}$). For model comparison, we consider three well-known statistics: Akaike information criterion (AIC), Anderson-Darling (A^*) Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) measures, where lower values of these statistics indicate good fits.

Table 3 lists the MLEs and standard errors for the considered models. Table 4 lists the AIC, A^* , W^* , K-S and p-values for the considered models. The values of the statistics in Table 4 indicate that the GKw-E model shows small values of the statistics and thus provides the best fit compared to the other models. Figures 3 and 4 show the plots of the estimated pdfs and cdfs over the histogram of the data, respectively.

TABLE 3. MLEs and their standard errors (in parentheses) for Precipitation data.

Distribution	α	β	a	b	μ	σ	θ	λ
GKw-E	0.2975 (0.1594)	-	67.1975 (24.7418)	0.1802 (0.0599)	-	-	-	0.0261 (0.0072)
Kw-W	0.0228 (0.0053)	1.3122 (0.2462)	13.4486 (7.6120)	0.2461 (0.1229)	-	-	-	-
BW	0.0243 (0.0033)	1.4375 (0.0193)	12.6298 (5.5638)	0.1734 (0.0446)	-	-	-	-
EGW	0.3105 (0.0148)	0.7061 (0.0117)	0.2357 (0.0276)	27.1942 (7.6257)	-	-	-	-
GBE2	9.0774 (1.9764)	-	-	-	-	-	0.0222 (0.3265)	0.0165 (0.0029)
GL	13.5845 (6.8592)	0.0174 (0.0015)	-	-	-8.5348 (35.1221)	-	-	-
Gumbel	-	-	-	-	139.8754 (6.0596)	57.8420 (4.7356)	-	-
Gamma	33.2955 (4.7925)	5.2761 (0.7239)	-	-	-	-	-	-
Weibull	0.0051 (0.0002)	2.2608 (0.1628)	-	-	-	-	-	-

TABLE 4. The statistics AIC, A^* , W^* and K-S for Precipitation data.

Distribution	AIC	A^*	W^*	K-S
GKw-E	1137.2320	0.1664	0.0187	0.0421
Kw-W	1138.0280	0.1831	0.0212	0.0430
BW	1137.7220	0.1844	0.0210	0.0429
EGW	1138.7100	0.2045	0.0259	0.0481
GBE2	1138.9210	0.3655	0.0482	0.0573
GL	1143.1390	0.6335	0.0872	0.0565
Gumbel	1139.2900	0.4990	0.0675	0.0640
Gamma	1141.9400	0.7732	0.1088	0.0600
Weibull	1156.2860	1.8272	0.2927	0.0950

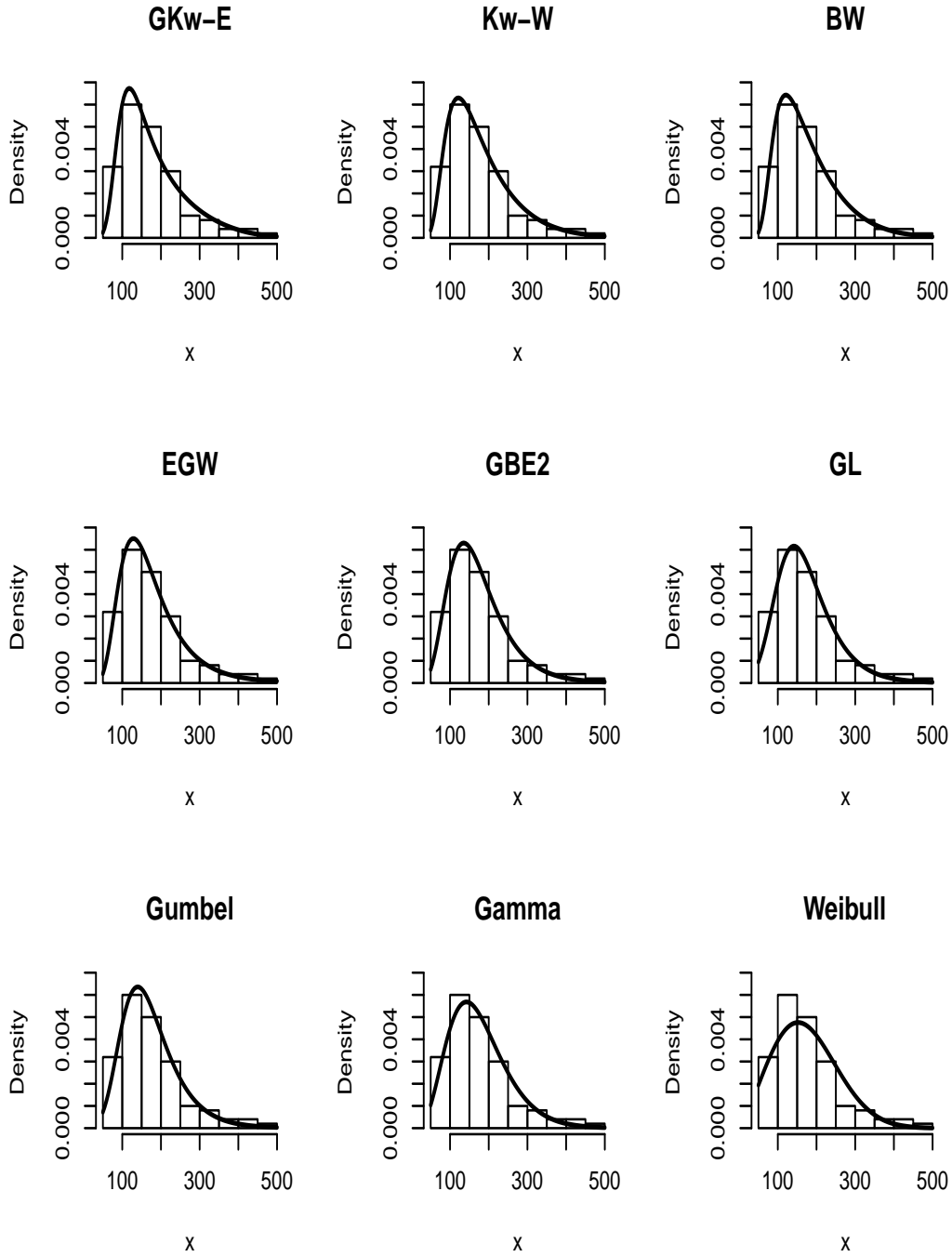


FIGURE 3. Estimated pdfs of the considered models for Precipitation data.

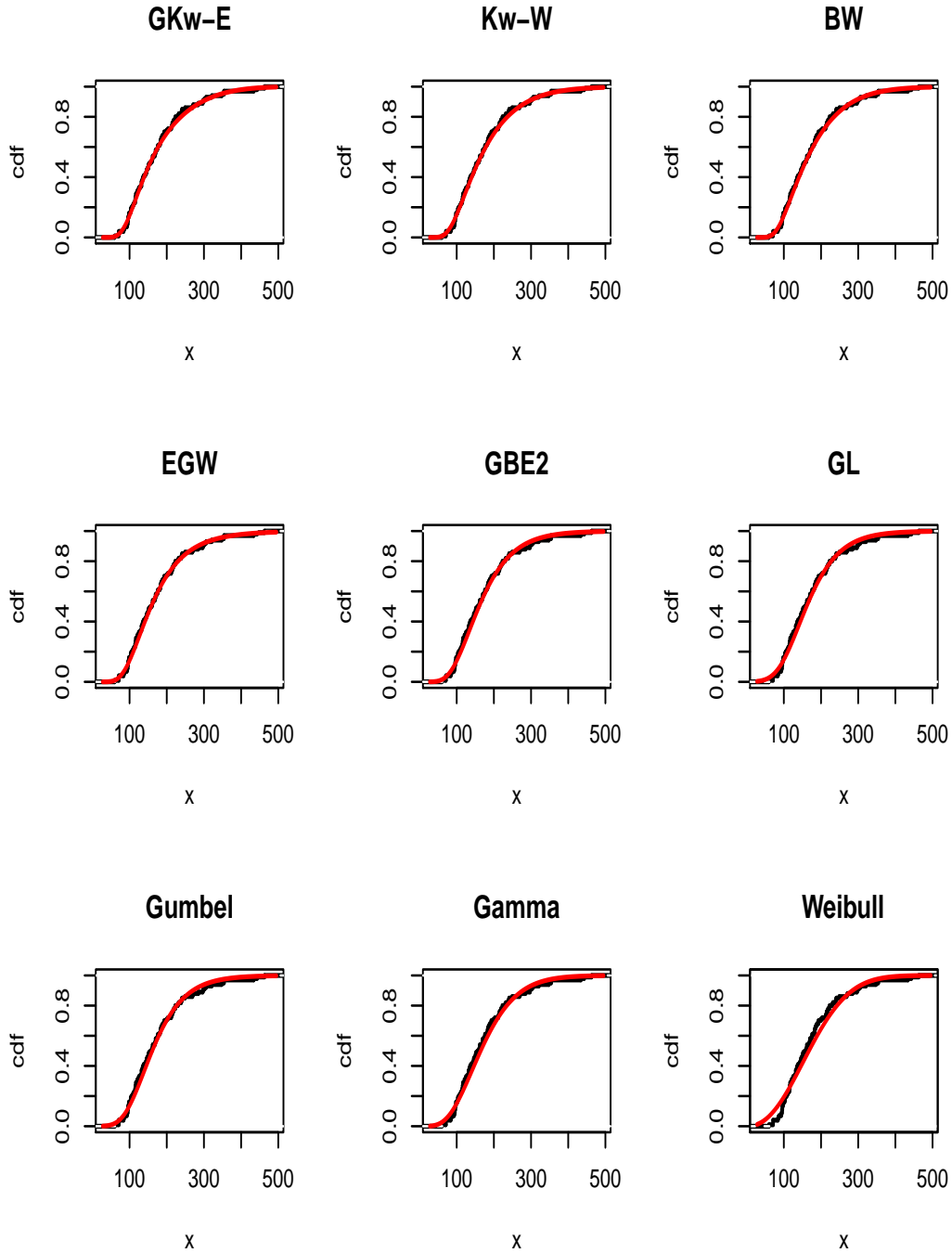


FIGURE 4. Estimated cdfs of the considered models for Precipitation data.

Application 2. The second data set were reported by professor Jim Irish and can be obtained at <http://www.statsci.org/data/oz/kiama.html>. It is about the Kiama Blowhole eruptions. The data are as follows: 83, 51, 87, 60, 28,95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35,47, 77, 36, 17, 21, 36, 18, 40,10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 5 lists the MLEs and standard errors for the considered models. Table 6 lists the AIC, A^* , W^* , K-S and p-values for the considered models. It is clear that, the GKw-E model provides a better fit than the other tested models, because it has the smallest value among AIC, A^* , W^* and K-S. Figures 5 and 6 show the plots of the estimated pdfs and cdfs over the histogram of the data, respectively.

TABLE 5. MLEs and their standard errors (in parentheses) for the Kiama Blowhole eruptions data.

Distribution	α	β	a	b	μ	σ	θ	λ
GKw-E	0.4154 (0.0545)	-	17.7076 (0.2513)	0.0481 (0.0072)	-	-	-	0.2063 (0.0046)
Kw-W	0.3410 (0.0026)	0.8685 (0.0022)	10.4397 (0.0083)	0.1396 (0.0168)	-	-	-	-
BW	0.5484 (0.0025)	0.7937 (0.0025)	13.5819 (4.8229)	0.1336 (0.0177)	-	-	-	-
EGW	2.5406 (9.4366)	0.3714 (0.2260)	0.7506 (3.0932)	26.1285 (0.8858)	-	-	-	-
GBE2	1.7325 (0.3190)	-	-	-	-	-	0.0048 (0.5680)	0.0350 (0.0111)
GL	21.5045 (6.5526)	0.0473 (0.0048)	-	-	-38.5692 (7.8114)	-	-	-
Gumbel	-	-	-	-	25.6833 (2.8506)	21.8407 (2.3260)	-	-
Gamma	24.5722 (4.6509)	1.6207 (0.2623)	-	-	-	-	-	-
Weibull	0.0230 (0.0023)	1.2701 (0.1199)	-	-	-	-	-	-

TABLE 6. The statistics AIC, A^* , W^* and K-S for the Kiama Blowhole eruptions data.

Distribution	AIC	A^*	W^*	K-S
GKw-E	589.2545	0.4614	0.0530	0.0708
Kw-W	591.0460	0.6231	0.0819	0.0954
BW	591.6412	0.6366	0.0840	0.1023
EGW	595.9134	0.8324	0.1134	0.0946
GBE2	597.3321	0.9009	0.1287	0.1227
GL	612.7799	1.5554	0.2440	0.1517
Gumbel	609.6039	1.5124	0.2361	0.1493
Gamma	595.7988	0.9220	0.1324	0.1215
Weibull	597.8029	1.0058	0.1467	0.1111

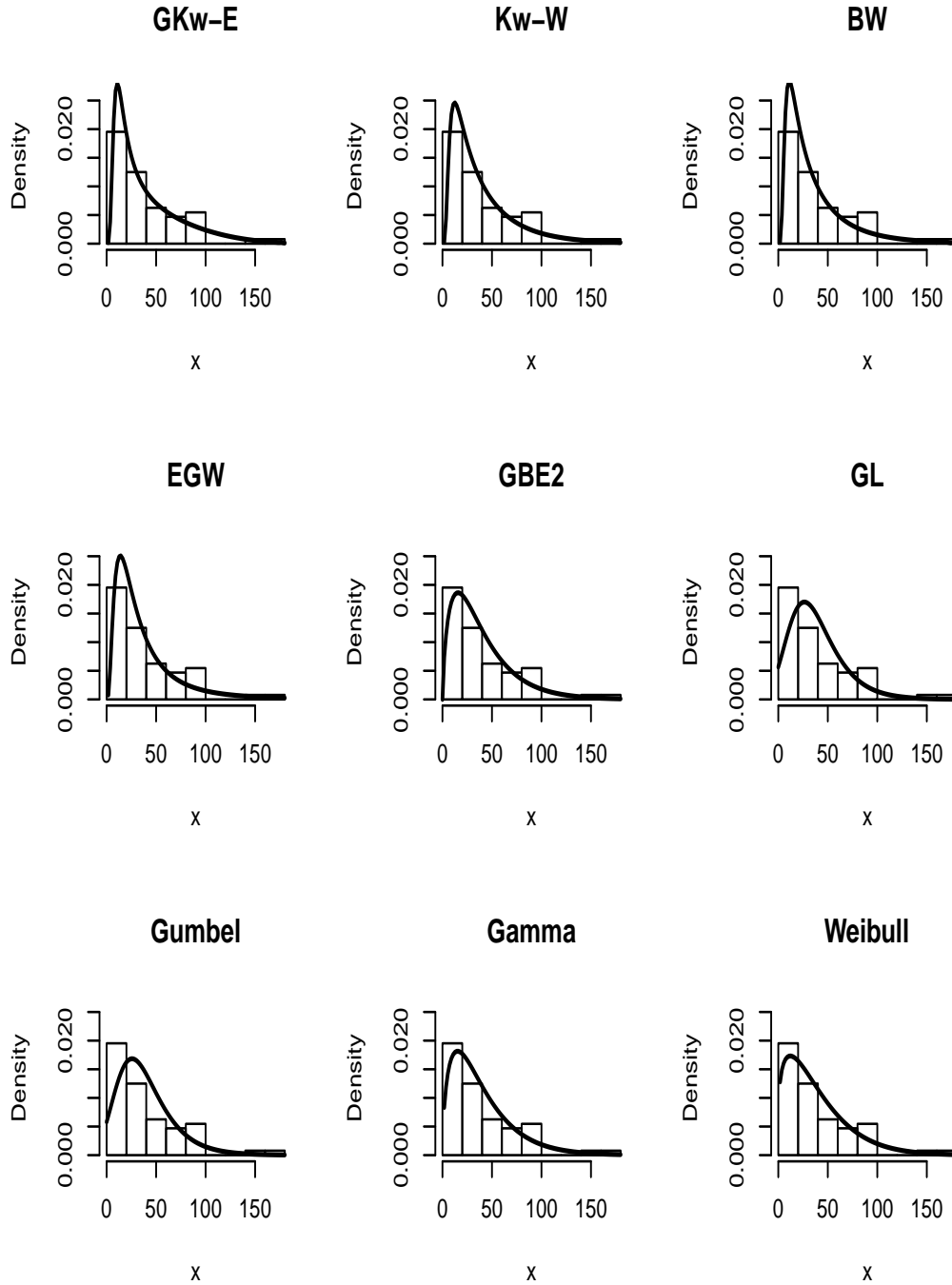


FIGURE 5. Estimated pdfs of competitive models for Kiama Blowhole eruptions data.

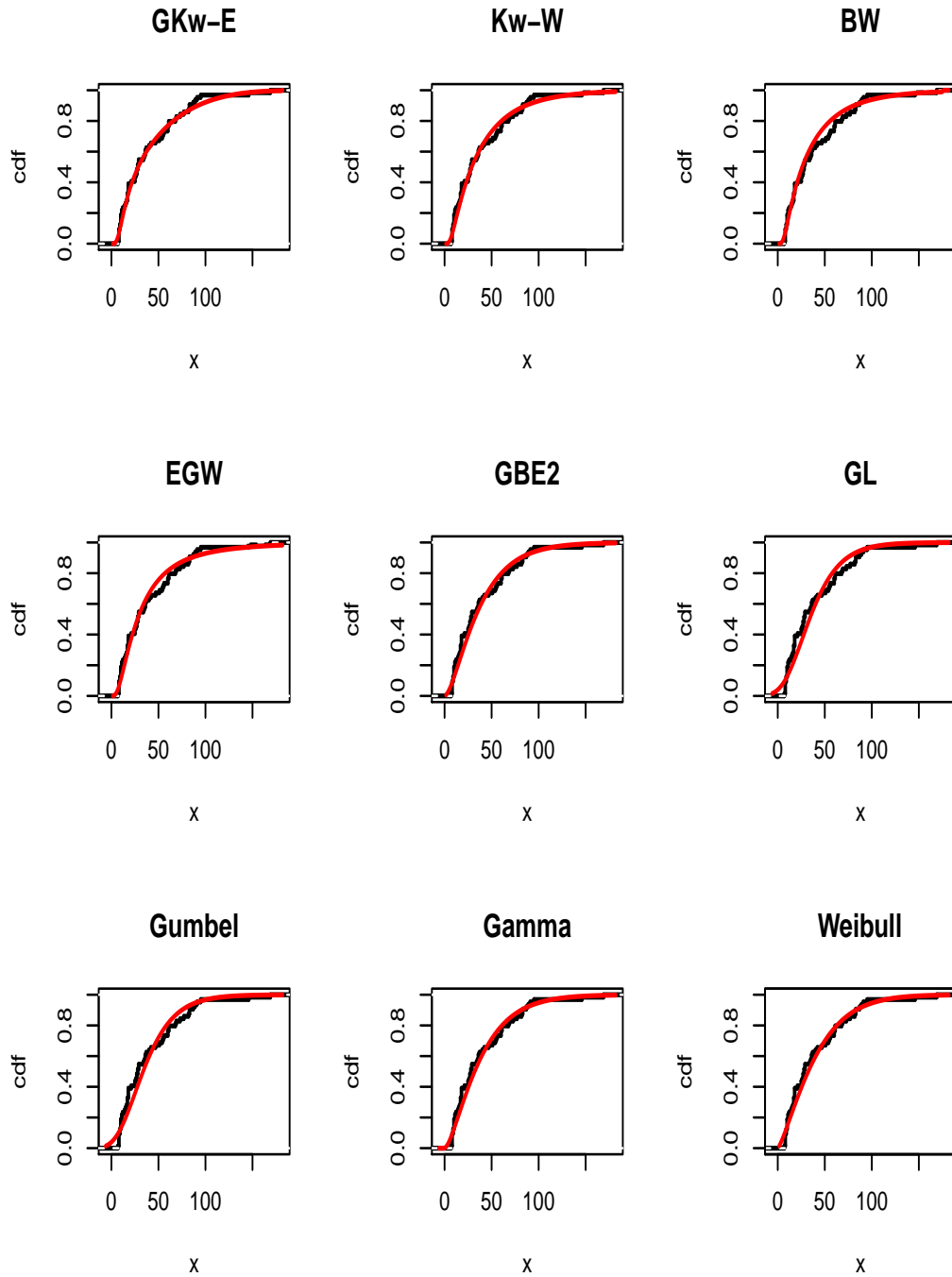


FIGURE 6. Estimated cdfs of competitive models for Kiama Blowhole eruptions data.

7. CONCLUDING REMARKS

In this paper, we propose and study the new GKw-G family of distributions. We investigate some of its structural properties including skewness, kurtosis, linear representations of the cumulative distribution and probability density functions, moments and derived quantities, stochastic ordering, reliability parameter and order statistics. Then a special model is considered, the GKw-E model, using the exponential distribution as baseline. The maximum likelihood method is employed for estimating the model parameters. We analyze two practical data sets, with fair comparison to other models, to demonstrate the usefulness of the new family. The results are strictly favorable to the GKw-E model. We hope that the proposed family and its generated models will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics, among others.

REFERENCES

- [1] Abramowitz, M. and Stegun, I. A. (1965). *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Math. Series 55, Dover Publications.
- [2] Alexander, C., Cordeiro, G. M., Ortega, E. M. M. and Sarabia, J. M. (2012). Generalized beta-generated distributions. *Comput. Stat. Data Anal.* **56** 1880–1897.
- [3] Aljarrah, M. A., Lee, C. and Famoye, F. (2014). A method of generating T-X family of distributions using quantile functions. *J. Stat. Dist. Appl.* **1** Article No. 2.
- [4] Alzaatreh, A., Famoye, F. and Lee, C. (2014). T-normal family of distributions: A new approach to generalize the normal distribution. *J. Stat. Dist. Appl.* **1** Article No. 16.
- [5] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of distributions. *Metron* **71** 63–79.
- [6] Alzaghal, A., Lee, C. and Famoye, F. (2013). Exponentiated T-X family of distributions with some applications. *Int. J. Probab. Statist.* **2** 31–49.
- [7] Amini, M., MirMostafaei, S. M. T. K. and Ahmadi, J. (2014). Log-gamma-generated families of distributions. *Statistics* **48** 913–932.
- [8] Asgharzadeh, A., Bakouch, H. S. and Habibi, M. (2016). A generalized binomial exponential 2 distribution: modeling and applications to hydrologic events. *Journal of Applied Statistics* **44** 2368–2387.
- [9] Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. *J. Data Sci.* **12** 53–68.
- [10] Cordeiro, G. M., Ortega, E. M. M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute* **347** 1399–1429.
- [11] Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *J. Stat. Comput. Simul.* **81** 883–893.
- [12] Cordeiro, G. M., Ortega, E. M. M. and da Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *J. Data Sci.* **11** 1–27.
- [13] Cordeiro, G. M., Alizadeh, M. and Ortega, E. M. M. (2014). The exponentiated half-logistic family of distributions: Properties and applications. *J. Probab. Statist.* Article ID 864396, 21 pages.
- [14] David, H. A. and Nagaraja, H. N. (2003). *Order Statistics*. John Wiley and Sons, New Jersey.
- [15] de Pascoa, M. A. R., Ortega, E. M. M. and Cordeiro, G. M. (2011). The Kumaraswamy Weibull distribution with application to failure data. *Journal of Franklin Institute* **347** 1399–1429.
- [16] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* **31** 497–512.
- [17] Gomes, A. E., da Silva, C. Q., Cordeiro, G. M. and Ortega, E. M. M. (2014). A new lifetime model: The Kumaraswamy generalized Rayleigh distribution. *Journal of Statistical Computation and Simulation* **84** 290–309.

- [18] Gradshteyn, I. S. and Ryzhik I. M. (2000). *Table of Integrals, Series and Products*. Academic Press, New York.
- [19] Hosseini, B., Afshari, M. and Alizadeh, M. (2018). The Generalized Odd Gamma-G Family of Distributions: Properties and Applications. *Austrian Journal of Statistics* **47** 69–89.
- [20] Jamal, F., Nasir, A., Tahir, M. H. and Montazeri, N. H. (2017). The odd Burr-III family of distributions. *Journal of Statistics Applications and Probability* **6** 105–122.
- [21] Jones, M. C. (2004). Families of distributions arising from the distributions of order statistics. *Test* **13** 1–43.
- [22] Jones, M. C. (2008). Kumaraswamy’s distribution: A beta-type distribution with some tractability advantages. *Statistical Methodology* **6** 70–81.
- [23] Kenney, J. and Keeping, E. (1962). *Mathematics of Statistics*. Vol. 1, 3rd edition, Princeton: NJ, Van Nostrand.
- [24] Kotz, S., Lumelskii, Y. and Penskey, M. (2003). *The stress-strength model and its generalizations and applications*. World Scientific, Singapore.
- [25] Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull Distribution: Some Properties and Applications to Censored Data. *Journal of Modern Applied Statistical Methods* **6** 173–186.
- [26] Moors, J. J. A. (1998). A quantile alternative for kurtosis. *Statistician* **37** 25–32.
- [27] Nasir, A., Bakouch, H. S. and Jamal, F. (2018). Kumaraswamy Odd Burr G Family of Distributions with Applications to Reliability Data. *Studia Scientiarum Mathematicarum Hungarica* **55** 1–21.
- [28] Oguntunde, P. E., Odetunmibi, O. A. and Adejumo, A. O. (2015). On the Exponentiated Generalized Weibull Distribution: A Generalization of the Weibull Distribution. *Indian Journal of Science and Technology* **8** 1–7.
- [29] Oluyede, B. O., Pu, S., Makubate, B., and Qiu, Y. (2018). The Gamma-Weibull-G Family of Distributions with Applications. *Austrian Journal of Statistics* **47** 45–76.
- [30] Paranaíba, P. F., Ortega, E. M. M., Cordeiro, G. M. and de Pascoa, M. A. D. (2012). The Kumaraswamy Burr XII distribution: Theory and practice. *Journal of Statistical Computation and Simulation* **82** 1–27.
- [31] Ristić, M., and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation* **82** 1191–1206.
- [32] Rodrigues, J. A. and Silva, A. P. C. (2015). The exponentiated Kumaraswamy-exponential distribution. *British Journal of Applied Science and Technology* **10** 1–12.
- [33] Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic orders and their applications*. Academic Press, New York.
- [34] Torabi, H. and Montazari, N. H. (2012). The gamma-uniform distribution and its application. *Kybernetika* **48** 16–30.
- [35] Torabi, H. and Montazari, N. H. (2014). The logistic-uniform distribution and its application, *Commun. Stat. Simul. Comput.* **43** 2551–2569.
- [36] Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Stat. Methodol.* **6** 344–362.

¹ DEPARTMENT OF STATISTICS, THE ISLAMIA UNIVERSITY OF BAHAWALPUR, PUNJAB 63100, PAKISTAN; IMRANARSHAD.STAT@GMAIL.COM, ² DEPARTMENT OF MATHEMATICS, LMNO, UNIVERSITY OF CAEN, CAEN 14032, FRANCE; CHRISTOPHE.CHESNEAU@GMAIL.COM, ³ DEPARTMENT OF STATISTICS, GOVT. S.A POSTGRADUATE COLLEGE DERA NAWAB SAHIB, BAHAWALPUR, PUNJAB 63360, PAKISTAN; DRFARRUKH1982@GMAIL.COM ⁴ GOVT S.E COLLEGE BAHAWALPUR, PAKISTAN: MANSOOR.ABBASI143@GMAIL.COM (M. M.), ZUBAIR.STAT@YAHOO.COM (M. Z.) AND ARSLANNASIR147@GMAIL.COM (M. A. N.)