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Riemann's Conjecture, a "One Page Proof (new)"

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Abstract

We provide a very short and simple proof of the RC (short for "Riemann Conjecture"), using the series expansion about $z=1/2$ of the Xi function $\xi(z)$ and the Hilbert spaces Theory. The proof is so simple that the author wonders why a great mathematician like Riemann did not see it; therefore F. Galetto thinks that somewhere in the purported proof there should be an error.

1. Introduction

This paper is a preprint of a paper submitted to American Mathematical Monthly.

On 5th of April 2019 the author found in a Wolfram MathWorld page [4] that the entire function $\xi(z)$ can be expressed as $\xi(z) = \sum_{n=0}^{\infty} a_n (z - 0.5)^{2n}$ a "series expansion about $z=1/2$ " with suitable coefficients a_n , shown later.

Working on this fact the author shows a very simple proof of the Riemann Conjecture.

It is well known that for over a century mathematicians have been trying to prove the Riemann Conjecture (also known as Hypothesis), RC for short, a conjecture claimed by Riemann [professor at University of Gottingen in Germany], near 1859 in a 8-page paper "On the number of primes less than a given magnitude" shown at Berlin Academy, and dated/published in 1859; at that time, Bernhard Riemann conjectured that all the zeros of the zeta function should have their real part σ equal to 0.5; unfortunately he left the problem aside by writing "... it is very probable that all roots are real. Without doubt it would be desirable to have a rigorous proof of this proposition; however I have left this research aside for the time being after some quick unsuccessful attempts, because it appears to be unnecessary for the immediate goal of my study...". The comment was related to the real function $\Xi(t)$ [named $\xi(t)$ by B. Riemann] obtained from the zeta function $\zeta(1/2+it)$ with $\sigma=0.5$, (the line $\sigma=\text{Re}(z)=1/2$ is named *Critical Line*).

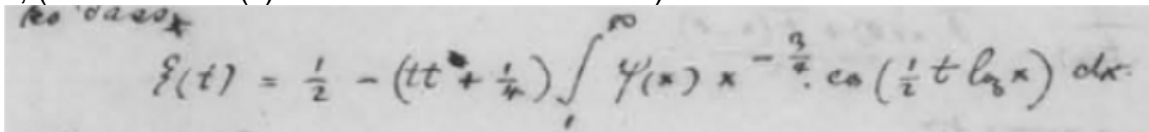

$$\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^{\infty} \psi(x) x^{-\frac{3}{2}} \cos(\frac{1}{2}t \log x) dx$$

Figure 1 From the original B. Riemann manuscript

2. The proof of RC

To get our result we use the entire function $\xi(z)$ which has the functional equation $\xi(z) = \xi(1-z)$ and is analytic, defined by $\xi(z) = \frac{1}{2} z(z-1) \Gamma\left(\frac{z}{2}\right) \pi^{-\frac{z}{2}} \zeta(z)$ [no poles].

The nontrivial zeros of the Riemann Zeta Function $\zeta(z)$ exactly correspond to those of $\xi(z)$; putting $z=1/2+it$ (i.e. for real t , the z points are on the Critical Line) the roots of $\xi(1/2+it)$ are the same as those of $\zeta(1/2+it)$; moreover $\xi(1/2+it)=\Xi(t)$ is a purely real function, with (see fig. 1, with different notation) $\Xi(t) = -\frac{1}{2}(t^2 + \frac{1}{4}) \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \pi^{-\frac{1}{4} + \frac{it}{2}} \zeta\left(\frac{1}{2} - it\right)$.

The coefficients a_n , in Wolfram MathWorld [4], are given by the formula, which depends only on $2n$, $a_{2n} = 4 \int_1^{\infty} \frac{d[x^{3/2} \psi'(x)]}{dx} \frac{[0.5 \ln(x)]^{2n}}{(2n)!} x^{-0.25} dx$; once computed a_{2n} we have only to compute $\xi(z)$.

To get our proof we modify the expansion from $\xi(z) = \sum_{n=0}^{\infty} a_n(z-0.5)^{2n}$ to $\xi(z) = \sum_{n=0}^{\infty} b_n(z-0.5)^{2n}/(2n)!$; it is easily seen that $(z-0.5)^{2n}/(2n)!$ are the entries of the expansion of the analytic function $\cosh(z-0.5)$, so that we derive the relationship $\xi(z) = 4 \int_1^{\infty} \frac{d[x^{3/2}\psi'(x)]}{dx} \cosh[(z-0.5)0.5 \ln(x)] x^{-0.25} dx$; this allows us to find the roots of $\xi(z)=0$.

Let's consider the two infinite dimensional vectors (the 1st real and the 2nd complex) $b=[b_0, b_2, \dots, b_{2n}, \dots, b_{\infty}]$ and $c(z)=[1, (z-0.5)^2/2!, (z-0.5)^4/4!, \dots, (z-0.5)^{2n}/(2n)!, \dots]$. They are such that $\xi(z)$ is the inner product [or scalar product] $c(z)b'=\xi(z)$; the norm $\|b\|<\infty$ and $\|c(z)\|<\infty$, because $c(z)c'(\bar{z})<\cosh[(x-0.5)^2+y^2]<\infty$.

Now let $z_1=x_1+y$ and $z_2=x_2+y$ be two zeros symmetric to the Critical Line. We have $\xi(z_1)=\xi(z_2)=0$, that is $c(z_1)b' = c(1-\bar{z}_1)b' = 0$, which means orthogonality in an ℓ^2 Hilbert space (on the field \mathbb{C} of complex numbers).

The two vectors $c(z_1)$ and $c(1-\bar{z}_1)$ are **both orthogonal to b** and then they are

- either coincident
- or they intersect

Using the following two theorems

Theorem 1: IF y and $x \neq 0$ are two vectors and $\lambda \in \mathbb{C}$, THEN one can find a unique vector $y-\lambda_0 x$ orthogonal to x . (See [3] and [5] for the proof.)

Theorem 2: the vector $y-\lambda_0 x$ is, among all the vectors $y-\lambda x$, the unique with the minimum norm. (See [3] and [5] for the proof.)

we find that $c(z_1)$, $c(1-\bar{z}_1)$ are the same vector $c(z_1) = c(1-\bar{z}_1)$, that is $z_1 = 1-\bar{z}_1$, which means $x_1=x_2=0.5$: the two assumed zeros are actually "one" zero on the *Critical Line*: $x=1/2$, which contradicts "our previous hypothesis that RH was false".

Since it was proved (Titchmarsh 1986) that there are infinite zeros of the *Riemann zeta function* $\zeta(z)$ in the Critical Strip, there are infinite values $z_k=\sigma_k+i\tau_k$ such that $\zeta(z_k)=0=\xi(z_k)$, $[0 < \sigma_k < 1]$. For any τ_k such that $\zeta(z_k)=\xi(z_k)=0$, either there is only one zero with $\sigma_k=1/2$ (on the Critical Line) or two zeros, symmetric to the critical line, with different real parts σ_k and $1-\sigma_k$, that we proved, above, impossible for a particular value τ_k such that $\zeta(z_k)=0=\xi(z_k)$. Since we can repeat the same argument for any couple of zeros assumed symmetric [for any τ_k such that $\xi(z_k)=0$, for any nontrivial zero z_k] to the *Critical Line*: any zero has $\sigma=1/2$ and therefore **RH is true.**

References

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