



Stochastic transport to quantify errors in geophysical fluid dynamic simulations

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Motivations

- More rigorously identified subgrid dynamics effects
- Quantification of modeling errors (UQ)



Ensemble forecasts and data assimilation

Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrization

Part I

Stochastic transport

Idea of SALT & LU :
Adding random velocity

$$\mathbf{v} = \mathbf{w} + \sigma \dot{\mathbf{B}}$$

Idea of SALT & LU :

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

Idea of SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Idea of SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$v = w + \sigma \dot{B}$$



Large scales:

\boldsymbol{w}

Small scales:

$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance
tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Idea of SALT & LU :

Adding random velocity

$$d\boldsymbol{X}_t = \boldsymbol{w}^*(\boldsymbol{X}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ d\boldsymbol{B}_t$$

$$= \boldsymbol{w}(\boldsymbol{X}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)d\boldsymbol{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Idea of SALT & LU : Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Idea of SALT & LU : Adding random velocity

$$d\mathbf{X}_t = \overset{\text{Stratonovich}}{w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t}$$

$$= \underset{\text{Ito}}{w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t}$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Idea of SALT & LU : Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References :

Mikulevicius &
Rozovskii, 2004
Flandoli, 2011

Memin, 2014

Resseguier et al. 2017 a, b, c
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019

Holm, 2015

Holm and
Tyranowski, 2016
Arnaudon et al. 2017

Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al. 2018 a, b
Cotter and al. 2019

Cotter and al. 2017 Resseguier et al. 2019 a, b



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Idea of SALT & LU : Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

LU

Memin, 2014
Resseguier et al. 2017 a, b, c
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019

SALT

Holm, 2015
Holm and
Tyranowski, 2016
Arnaudon et al. 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al. 2018 a, b
Cotter and al. 2019

Cotter and al. 2017 Resseguier et al. 2019 a, b



References :

Mikulevicius &
Rozovskii, 2004
Flandoli, 2011

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Dynamics:

e.g. advection of tracer Θ

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Dynamics:

e.g. advection of tracer Θ

Large scales:

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Variance
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Dynamics:

e.g. advection of tracer Θ

Stratonovich notations:

Large scales:

w

Small scales:

$\sigma \dot{B}$

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tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Dynamics:

e.g. advection of tracer Θ

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Dynamics:

e.g. advection of tracer Θ

Stratonovich notations:

Conservative
multiplicative
random
forcing

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Large scales:

w

Small scales:

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Variance tensor:

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$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Part II

Stochastic Navier-Stokes

SALT vs LU

Comparison

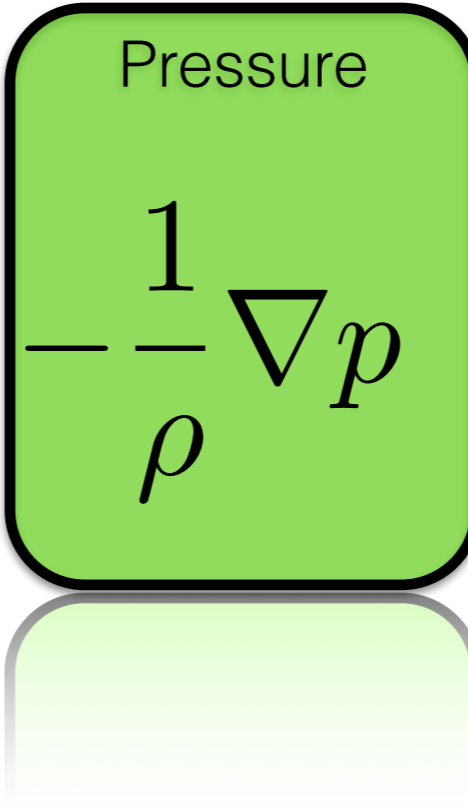
	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓



Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

$$\frac{Dw}{Dt} = \text{Pressure} \left[-\frac{1}{\rho} \nabla p \right]$$


Navier-Stokes LU

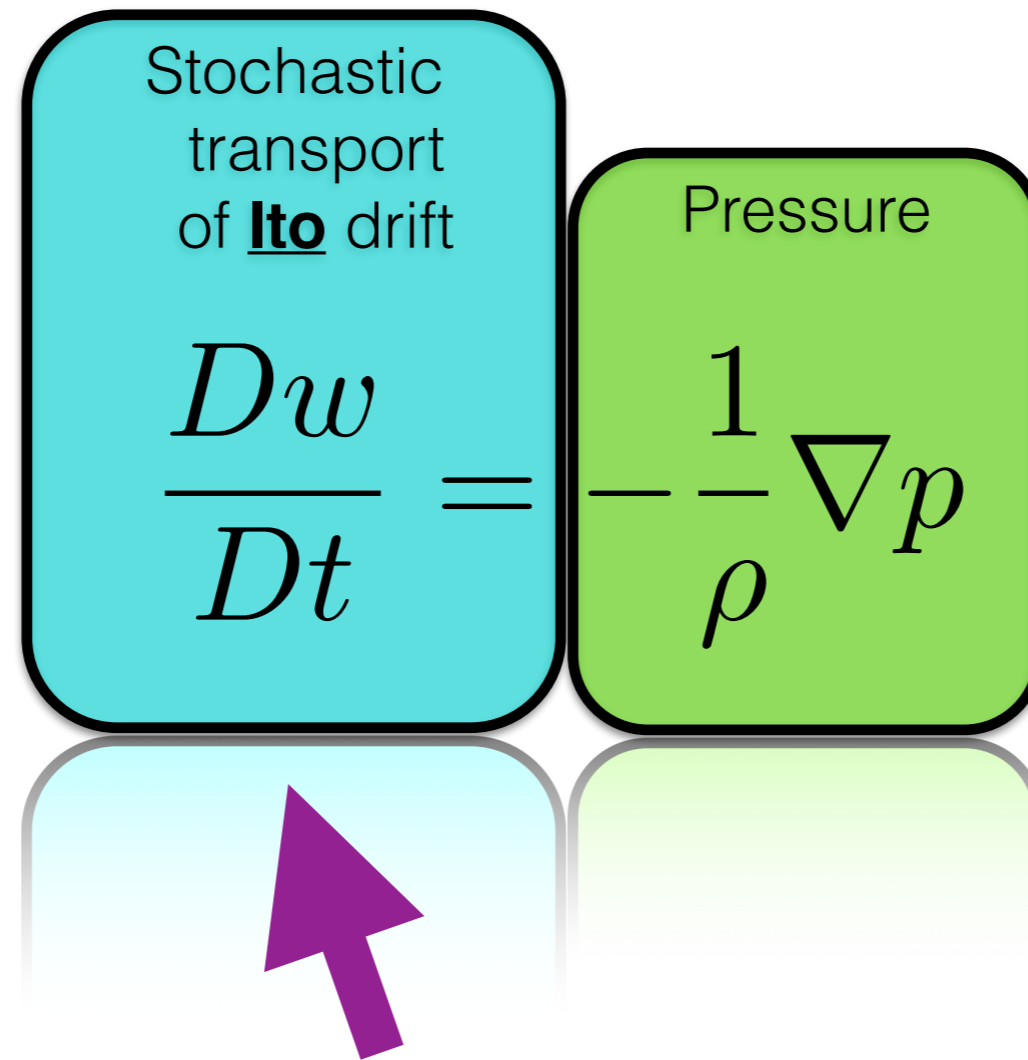
Stochastic
transport
of Ito drift

$$\frac{Dw}{Dt} =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

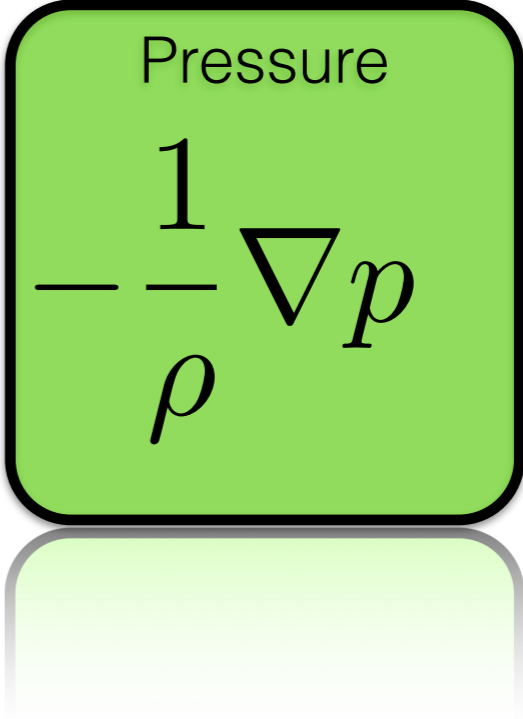
$$\begin{array}{|c|c|} \hline \text{Stochastic transport of Ito drift} & \text{Pressure} \\ \hline \frac{Dw}{Dt} = & -\frac{1}{\rho} \nabla p \\ \hline \end{array}$$


Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \text{Pressure} \frac{1}{\rho} \nabla p$$


Navier-Stokes SALT

Stochastic
transport of
Stratonovich drift

$$\frac{Dw^*}{Dt}$$

$$+ \nabla (\sigma \circ \dot{B})^T w^* =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

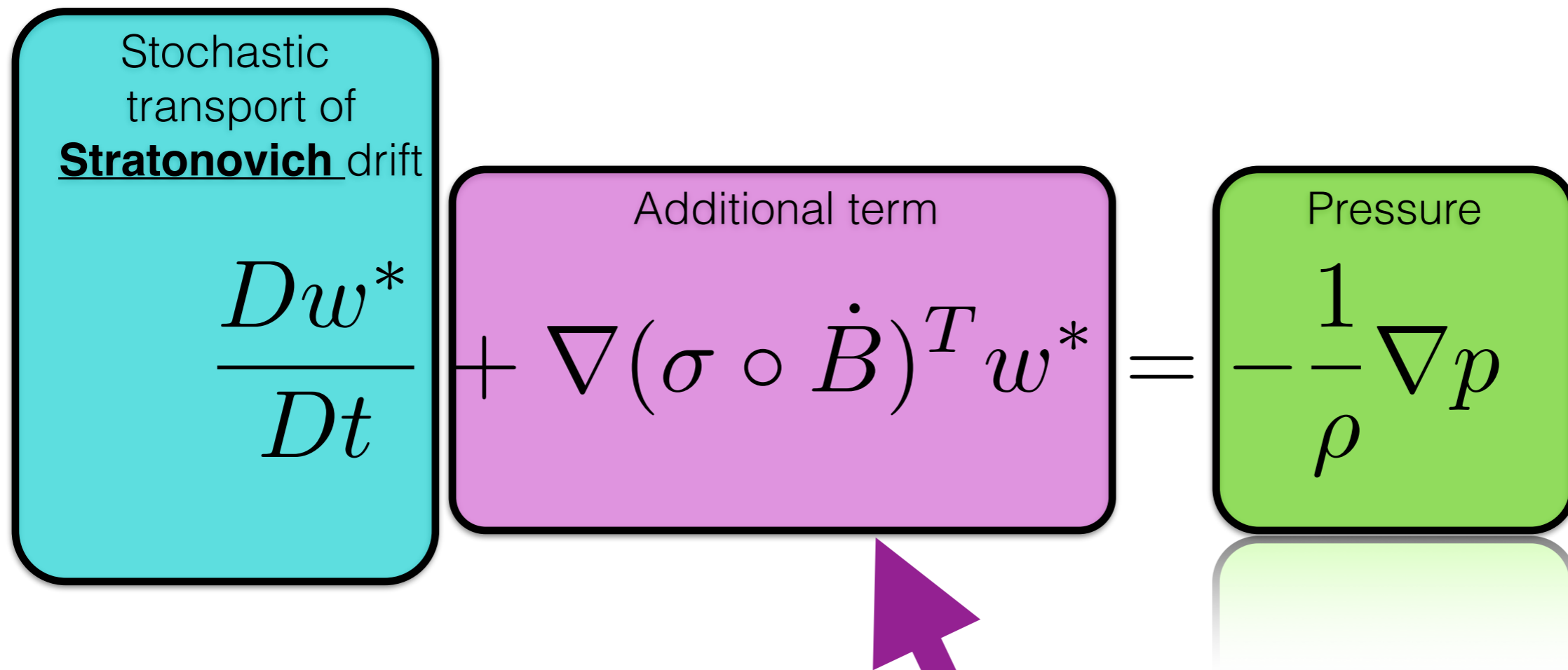
Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Additional term

Pressure



Conserve (Stratonovich) helicity

Part III

Unresolved velocity
parametrization
for *both* SALT & LU

Part III

Unresolved velocity
parametrization
for *both* SALT & LU

$\sigma = ?$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

EOF decomposition

(stationary but heterogeneous $\sigma \dot{B}$)

EOF

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

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$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

Large scales:

w

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EOF

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

weighted
EOFs learned
on data

iid
Brownian
motion

EOF decomposition

(stationary but heterogeneous $\sigma \dot{B}$)

Large scales:
 w

Small scales:
 $\sigma \dot{B}$

Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
 Brownian
 motion

weighted
 EOFs learned
 on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Reference:
 Cotter, Crisan, Holm, Pan, Shevchenko 2018



EOF decomposition

(stationary but heterogeneous $\sigma \dot{B}$)

Large scales:
 w

Small scales:
 $\sigma \dot{B}$

Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

$$\sigma d\mathbf{B}_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

weighted
EOFs learned
on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Flow from **high-resolution** velocity

Flow from **low-resolution** velocity

Reference:
Cotter, Crisan, Holm, Pan, Shevchenko 2018



Absolute Diffusivity Spectral Density

MU ADSD

Large scales:

w

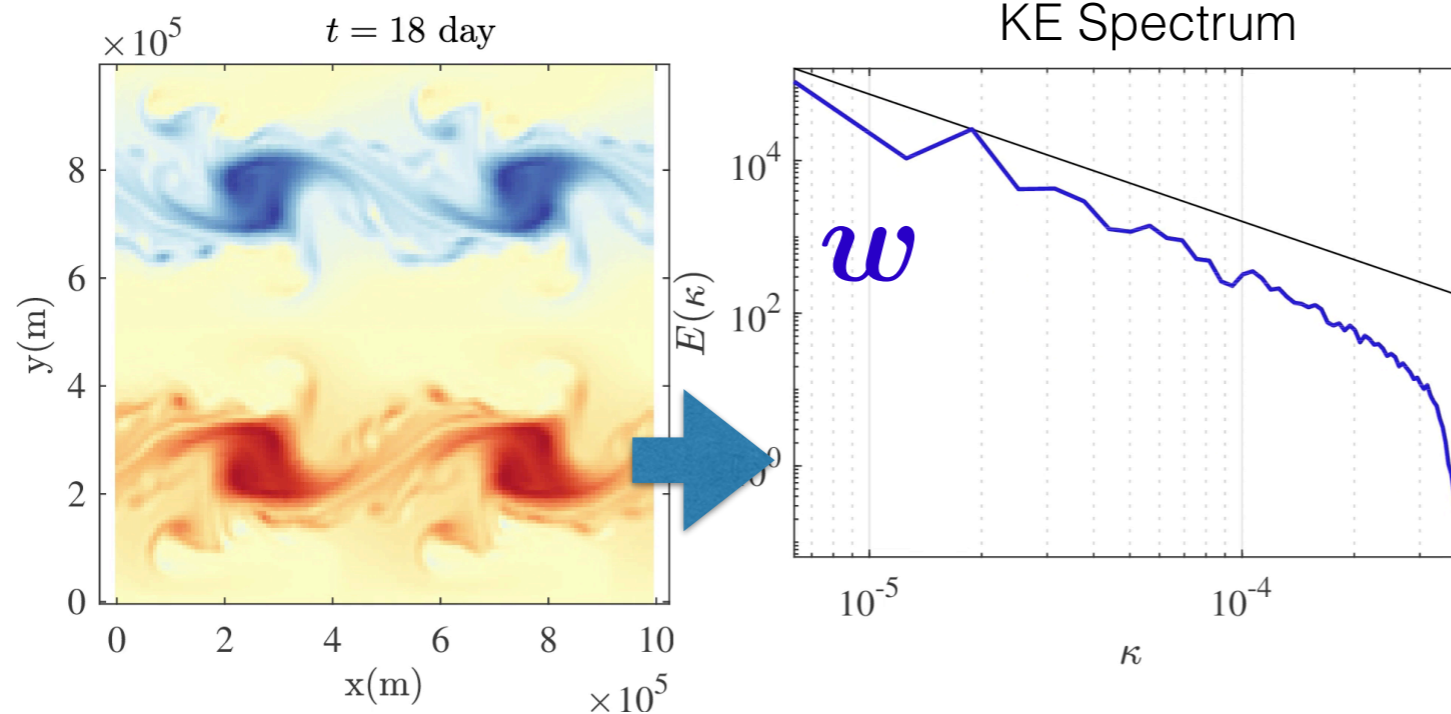
Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)



Absolute Diffusivity Spectral Density

MU ADSD

Large scales:

w

Small scales:

$\sigma \dot{B}$

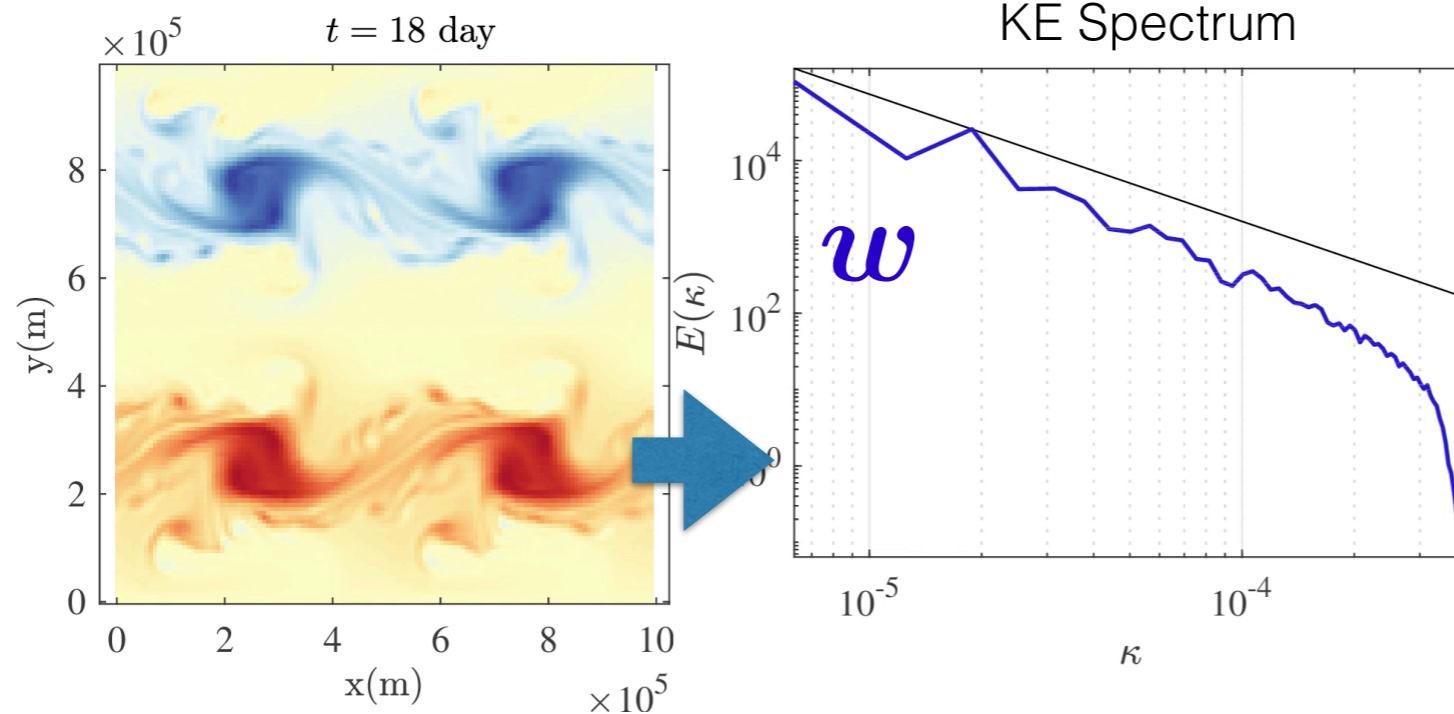
Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Absolute Diffusivity
Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa)$$

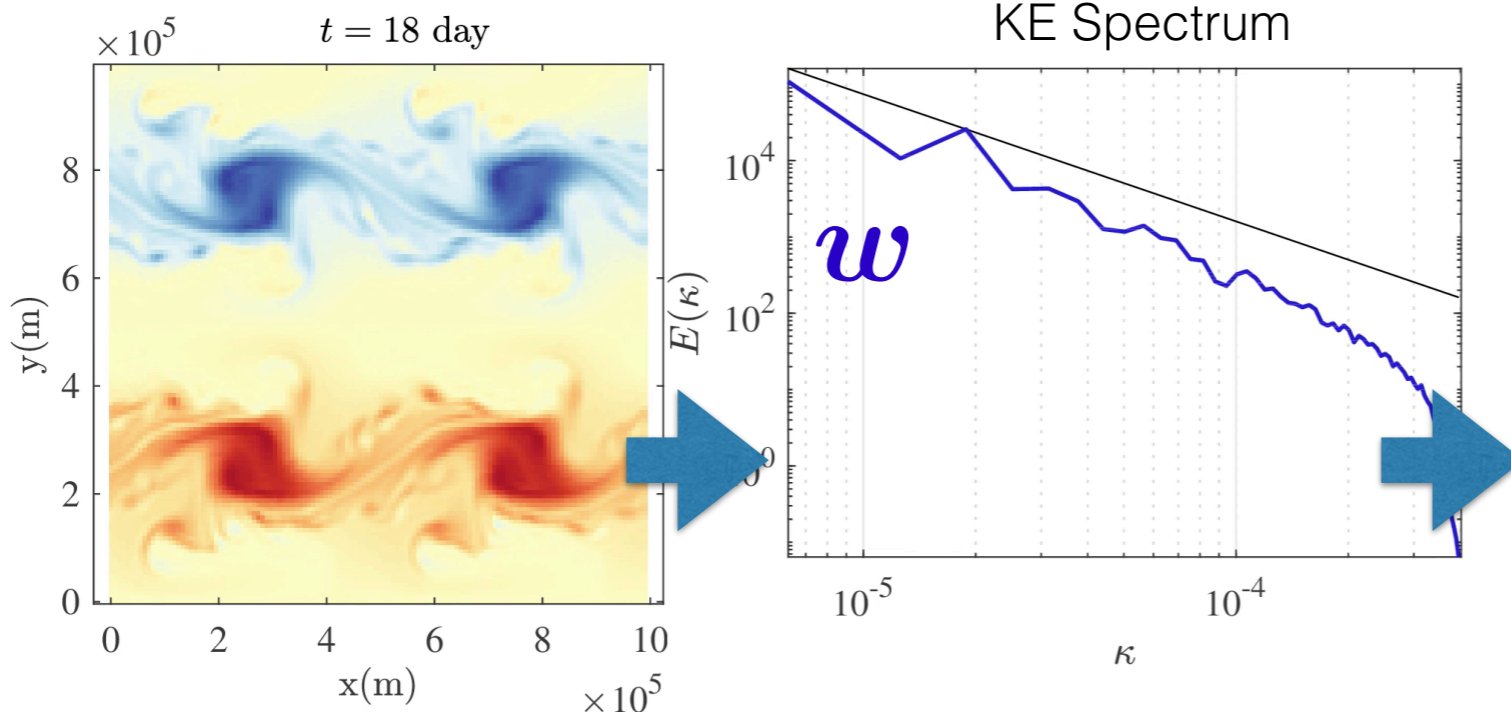


Absolute Diffusivity Spectral Density

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Large scales:
 w
 Small scales:
 $\sigma \dot{B}$
 Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Absolute Diffusivity Spectral Density $A(\kappa) = E(\kappa)\tau(\kappa)$



Reference:
Resseguier, Pan & Fox-Kemper 2019

Absolute Diffusivity Spectral Density

MU ADSD

Large scales:

w

Small scales:

$\sigma \dot{B}$

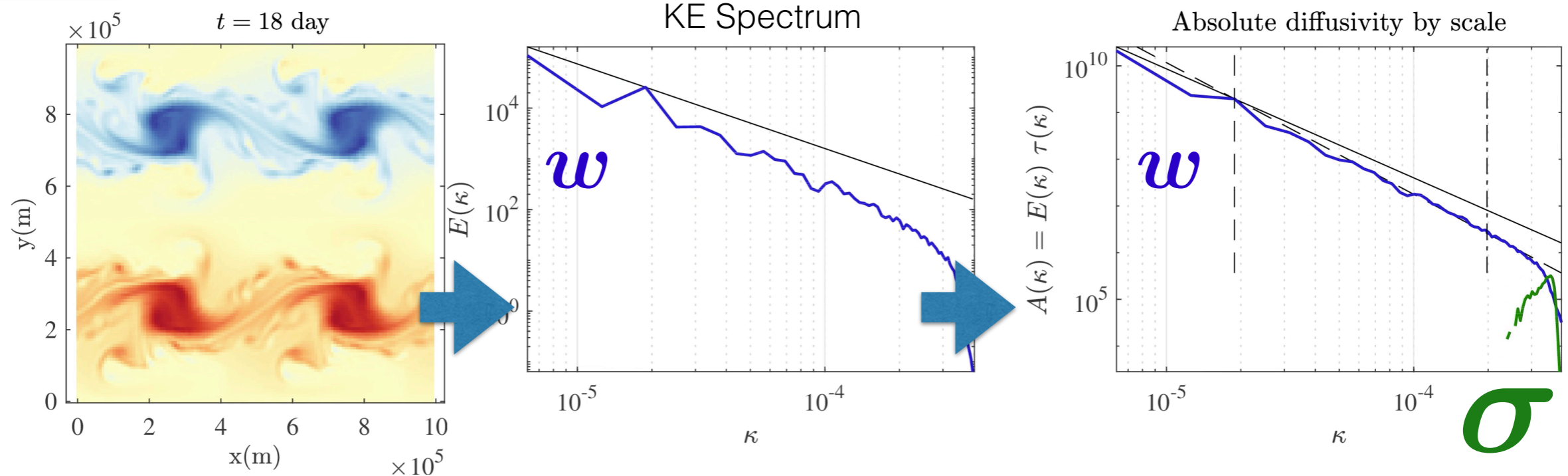
Variance
tensor:

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Absolute Diffusivity
Spectral Density

$$A(\kappa) = E(\kappa) \tau(\kappa)$$



Absolute Diffusivity Spectral Density

MU ADSD

Large scales:

w

Small scales:

$\sigma \dot{B}$

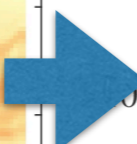
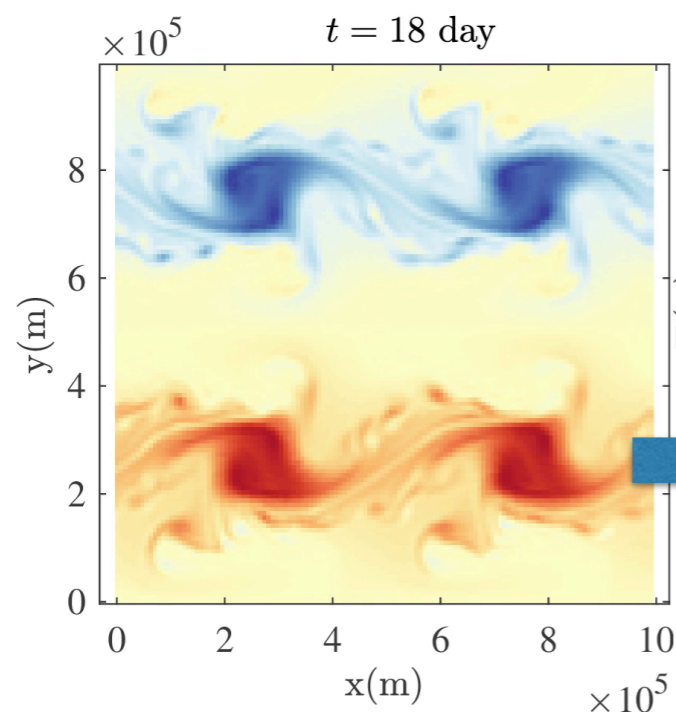
Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

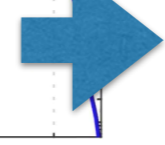
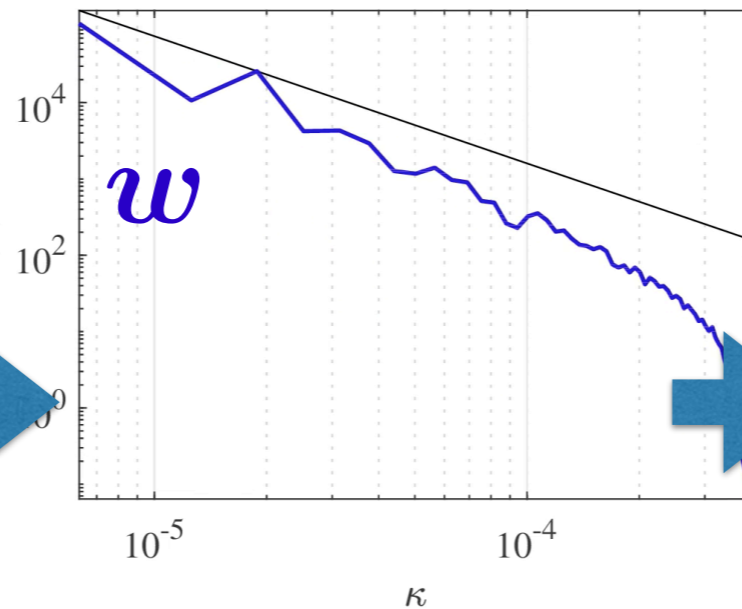
(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Absolute Diffusivity
Spectral Density

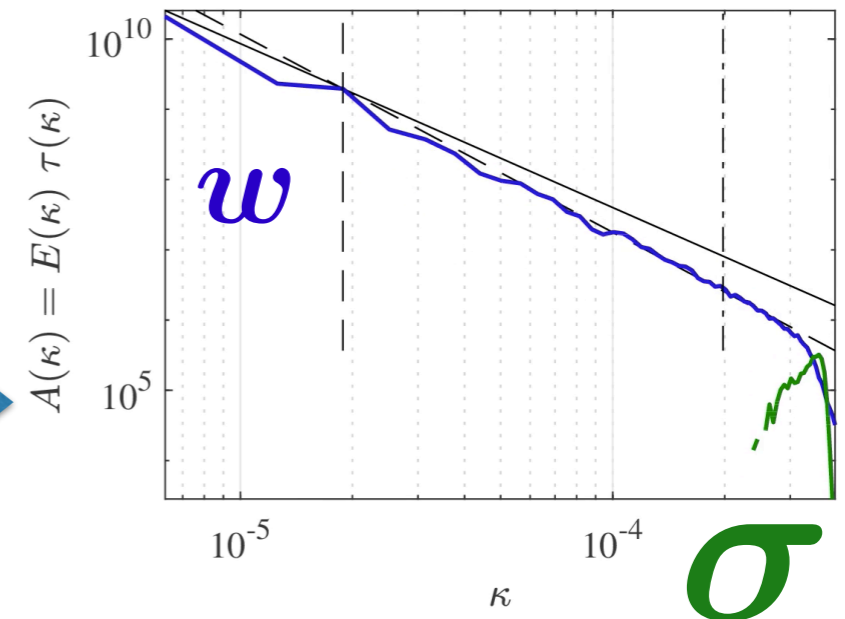
$$A(\kappa) = E(\kappa)\tau(\kappa)$$



KE Spectrum



Absolute diffusivity by scale



Absolute Diffusivity Spectral Density

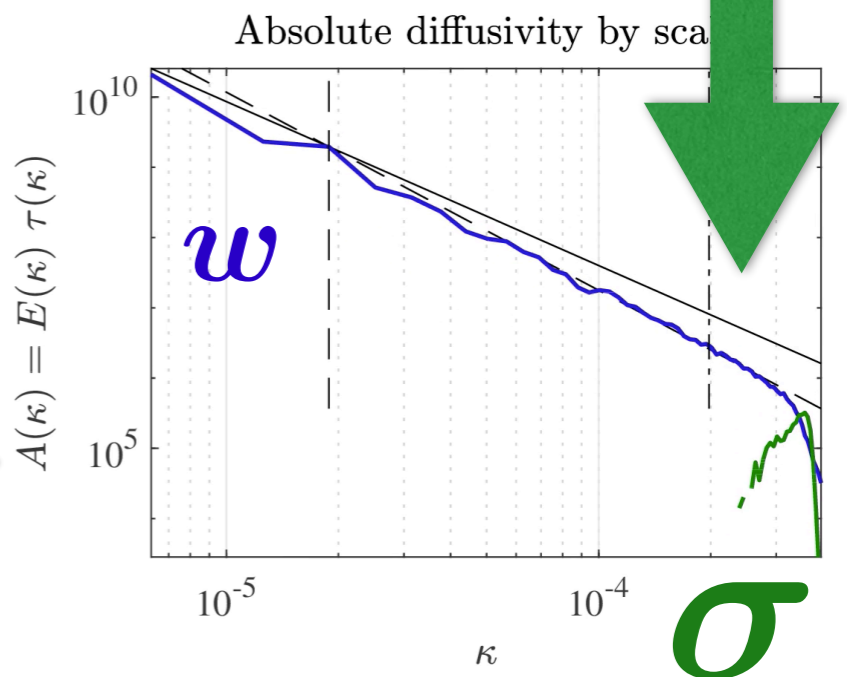
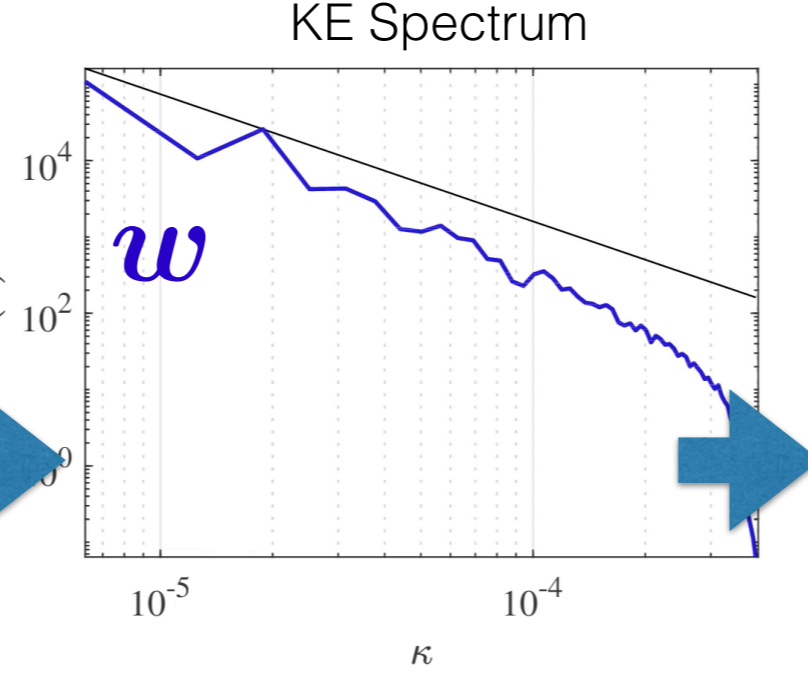
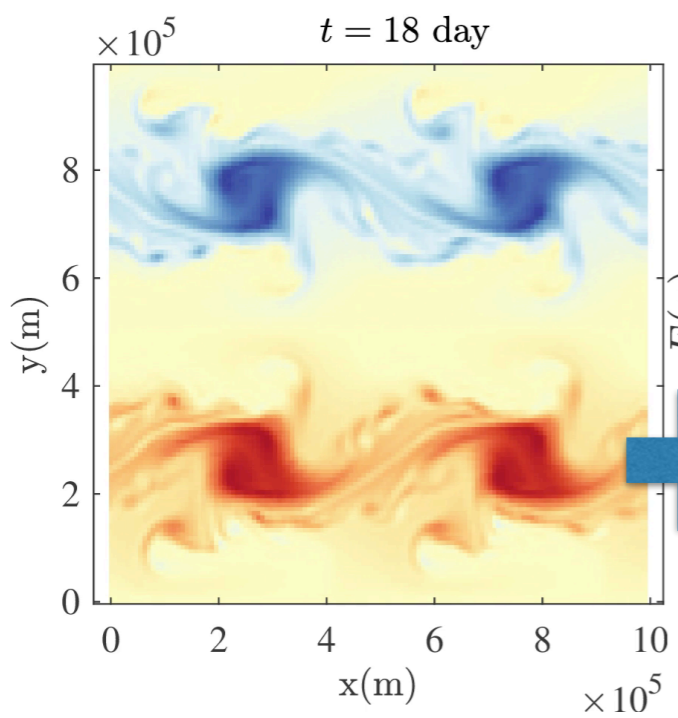
MU ADSD

Large scales:
 w
 Small scales:
 $\sigma \dot{B}$
 Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Absolute Diffusivity Spectral Density $A(\kappa) = E(\kappa)\tau(\kappa)$

Residual non-stationary ADSD



Reference:
Resseguier, Pan & Fox-Kemper 2019

Absolute Diffusivity Spectral Density

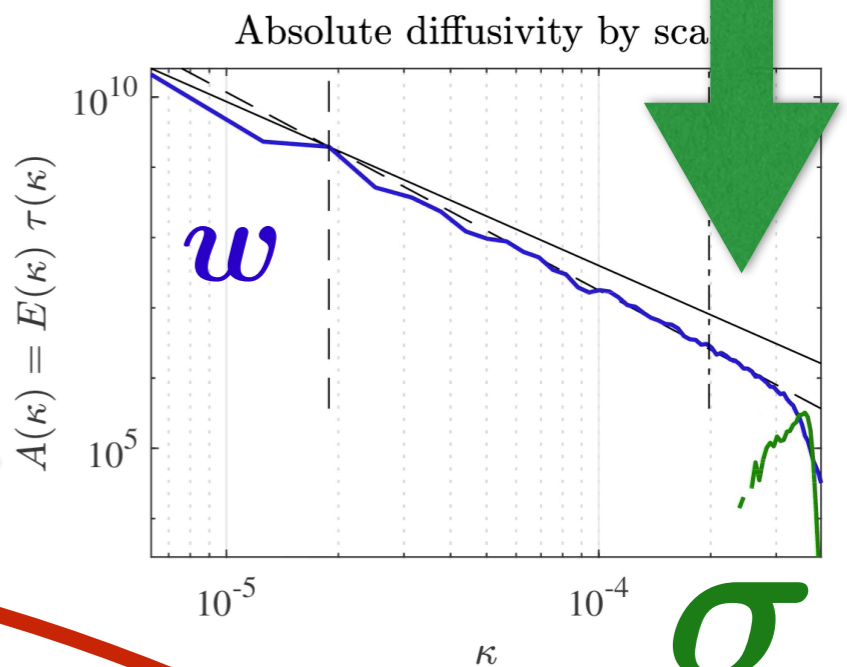
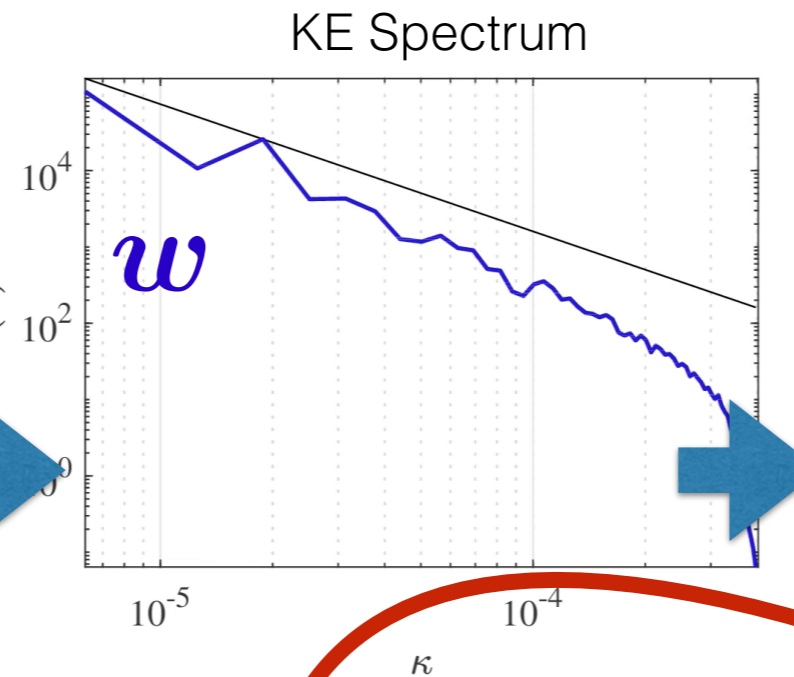
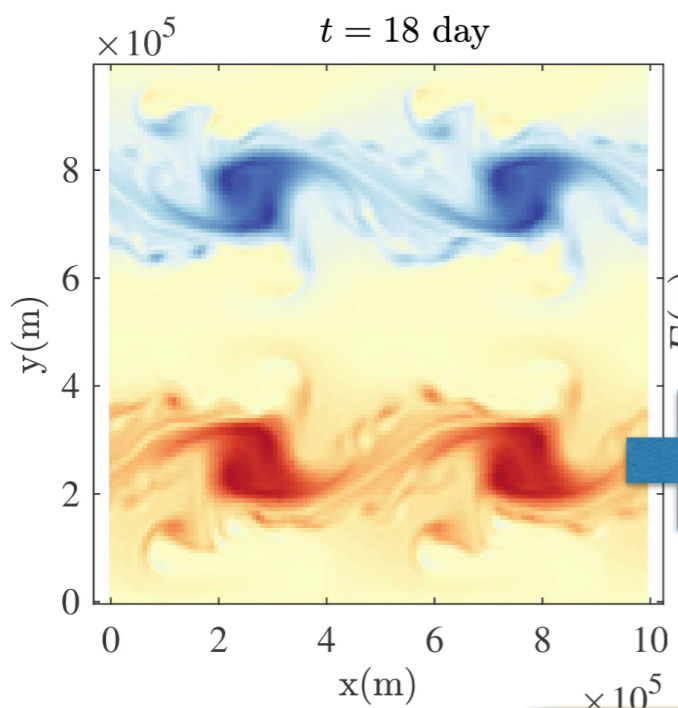
MU ADSD

Large scales: w
 Small scales: $\sigma \dot{B}$
 Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

(homogeneous but non-stationary and tuning-free $\sigma \dot{B}$)

Absolute Diffusivity Spectral Density $A(\kappa) = E(\kappa)\tau(\kappa)$

Residual non-stationary ADSD



$\sigma \dot{B} = (\text{filter}) * (\text{white noise})$

Reference:
 Resseguier, Pan & Fox-Kemper 2019



$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

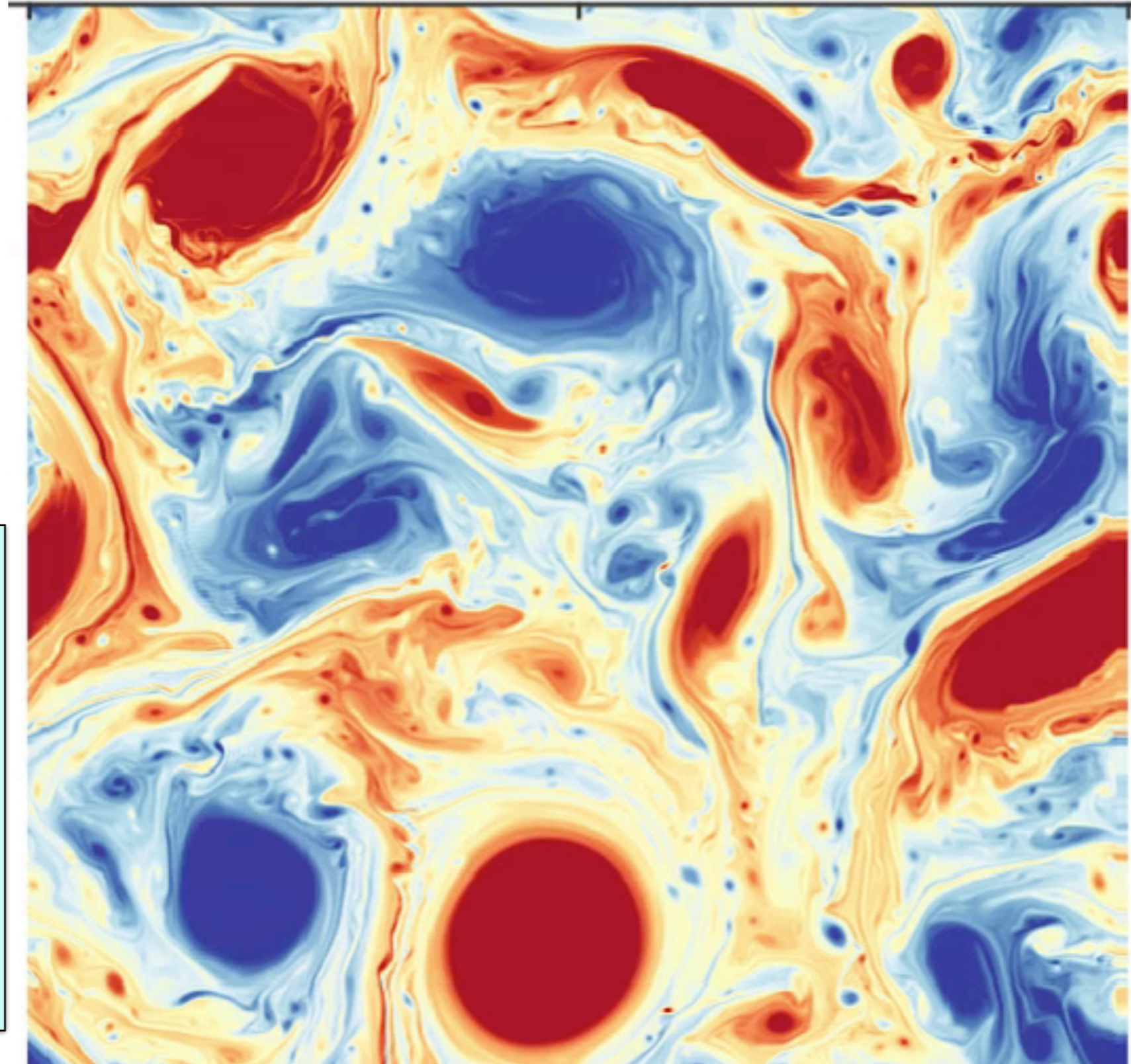
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

deterministic

SQG

512 x 512



$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

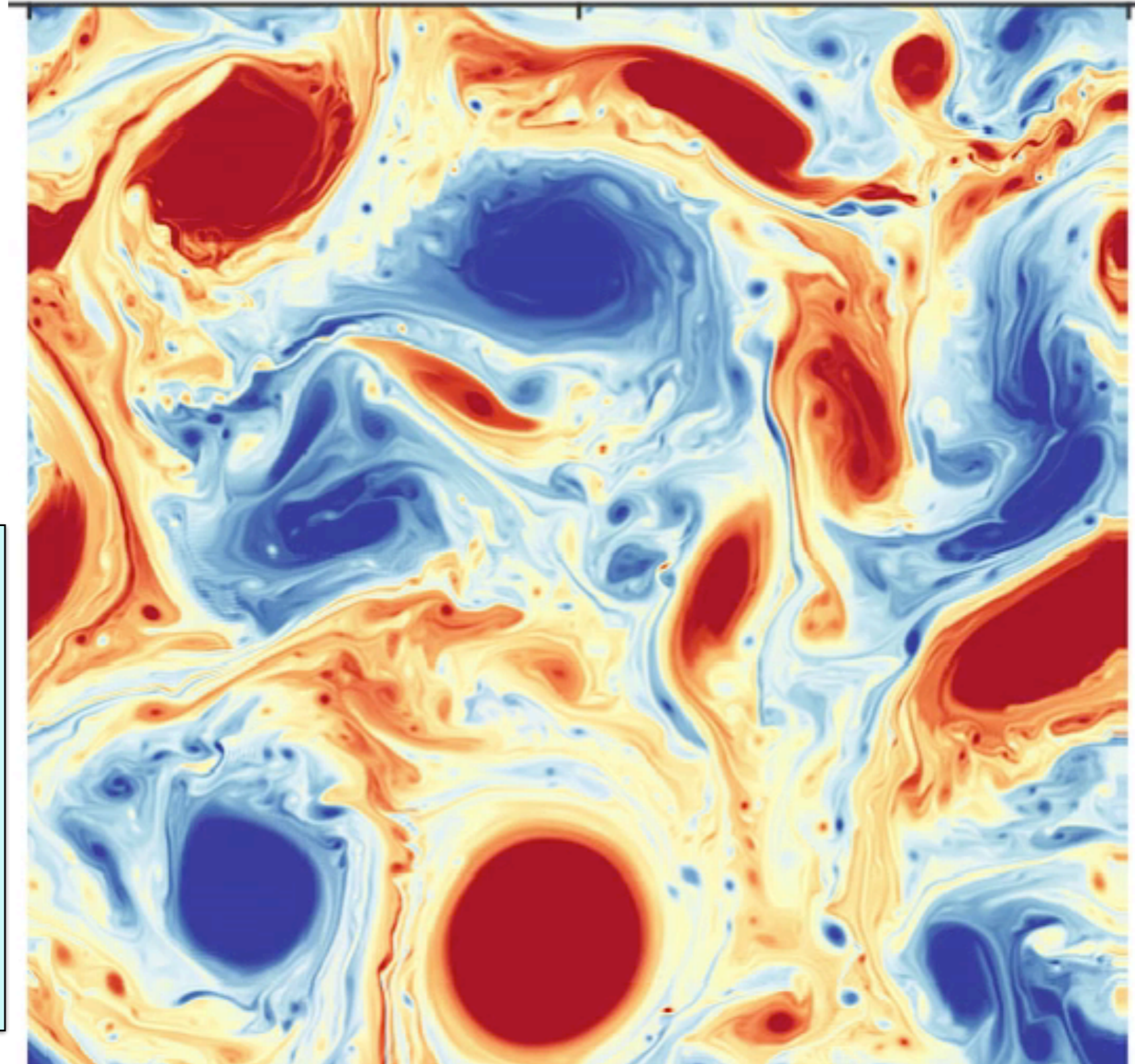
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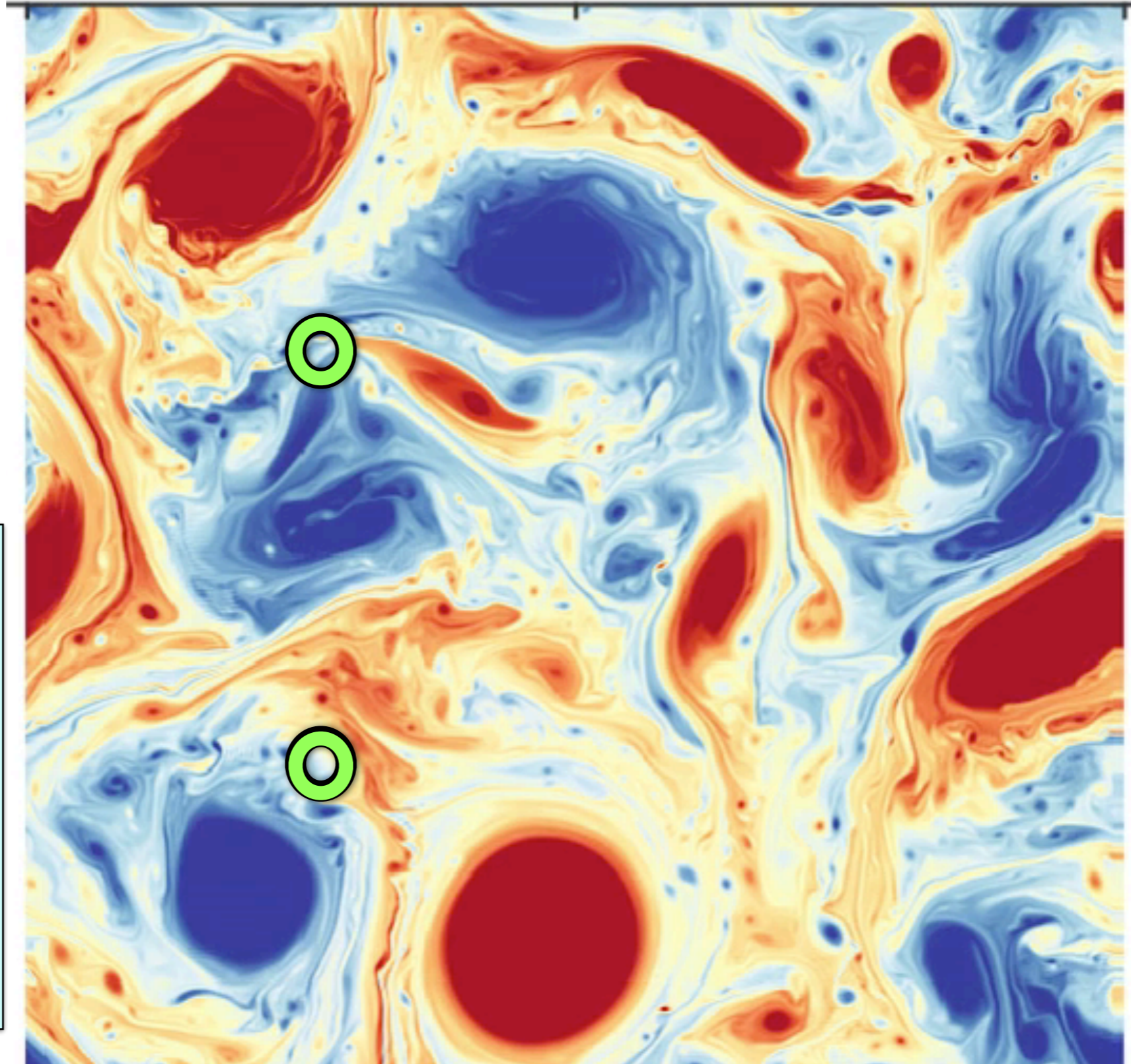
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

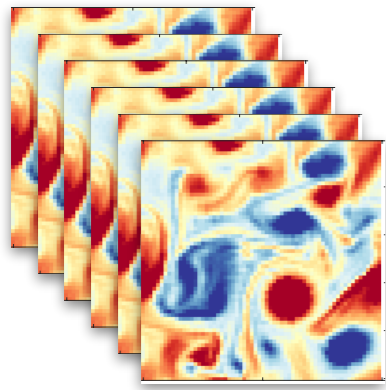
Reference flow:

deterministic

SQG

512 x 512





**20 realizations
at 64 x 64**

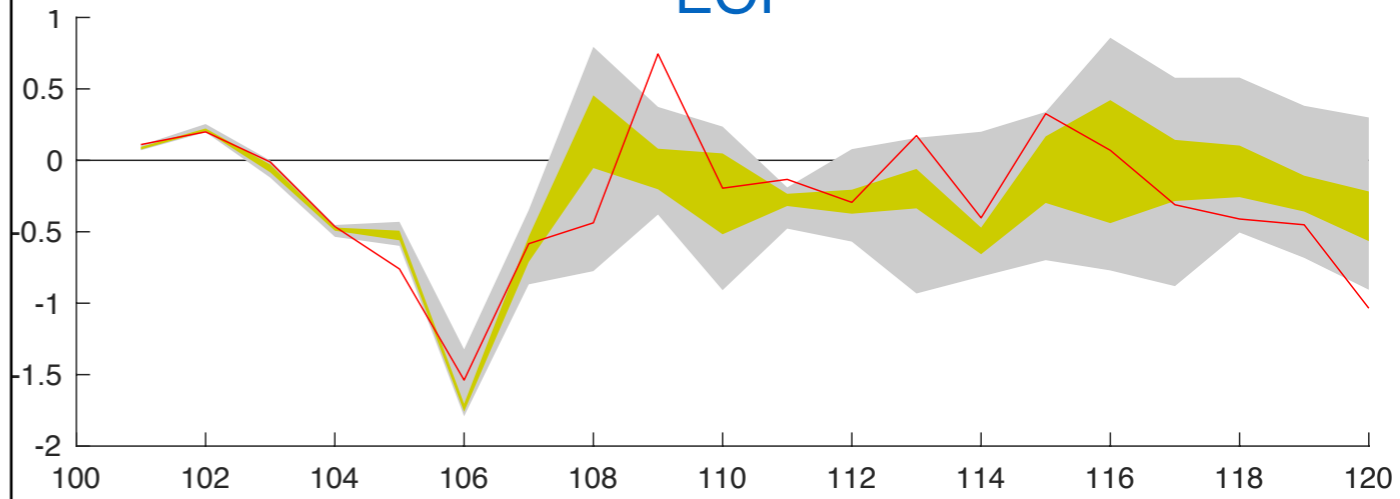
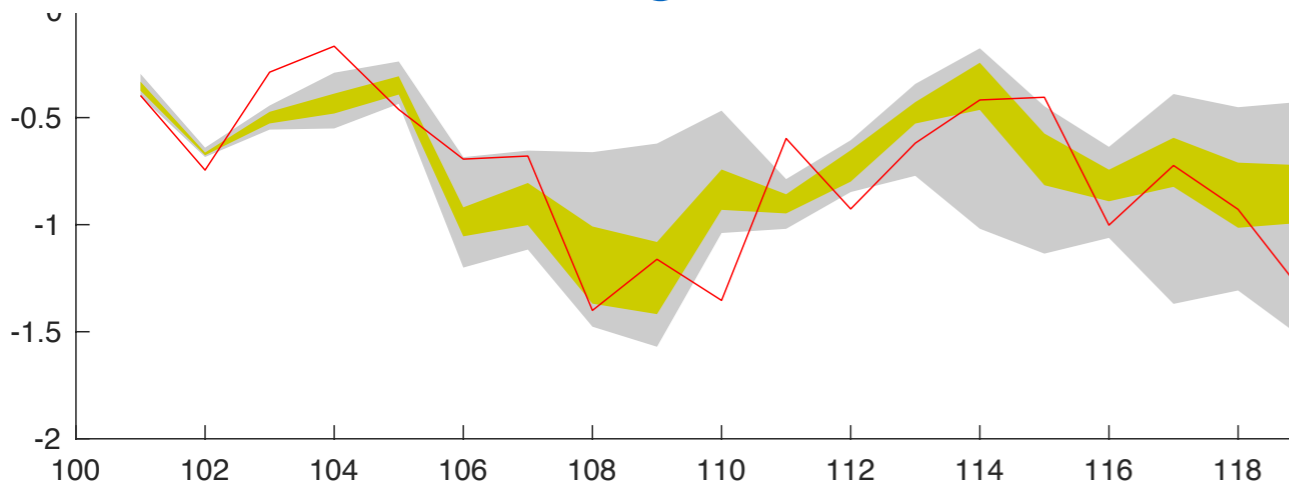
Ensemble : uncertainty quantification

At point (250km,500km) 

At point (500km,500km) 

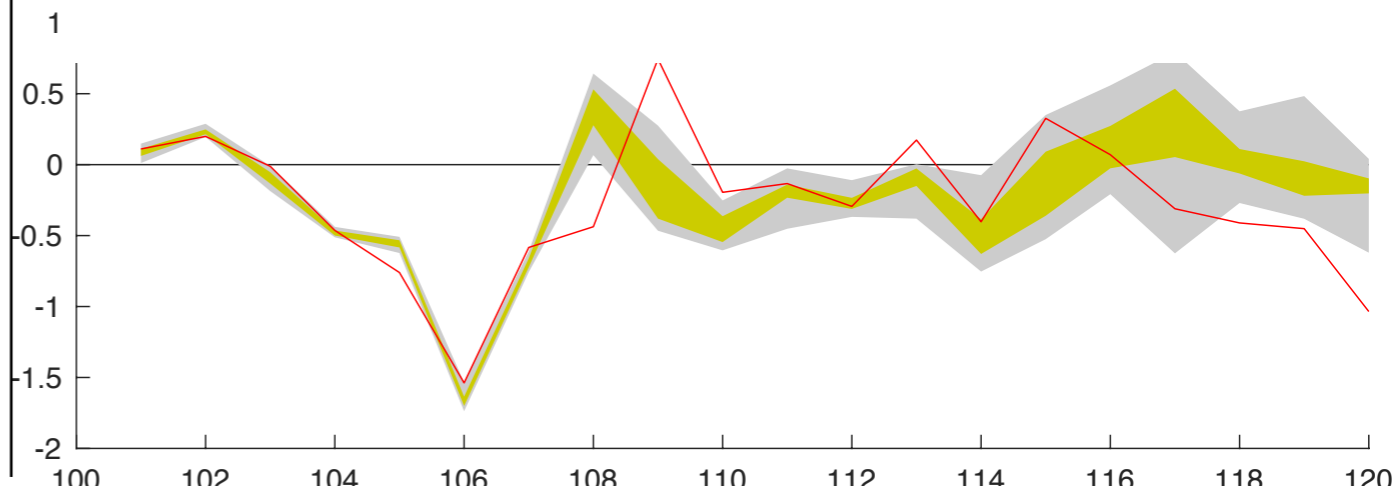
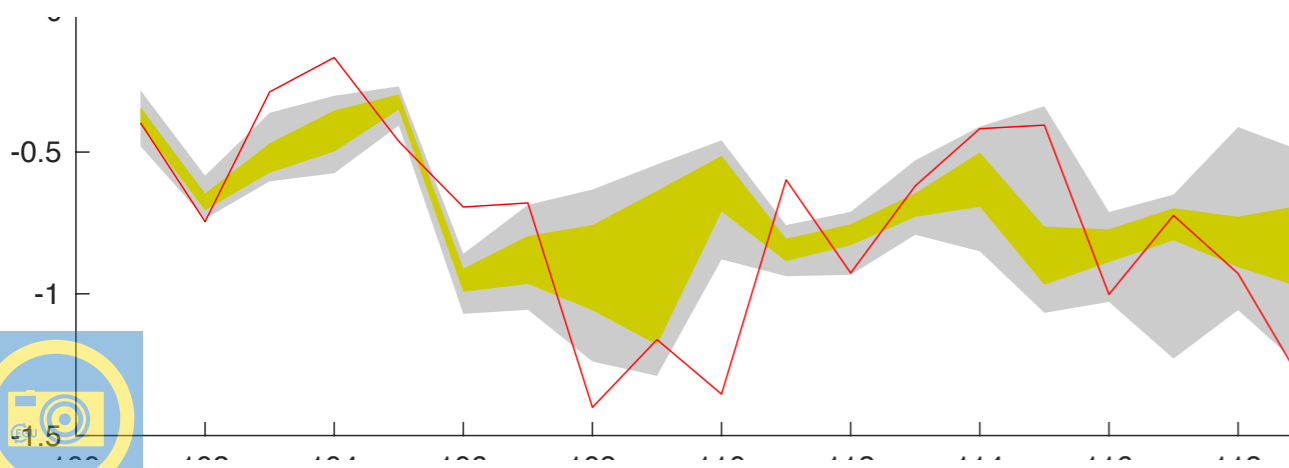
EOF

EOF



ADSD

ADSD



Conclusion

Conclusion

- LU conserves kinetic energy / SALT conserves helicity
- Both EOF and ADSD small-scale velocity parametrization lead to accurate uncertainty quantification
(to address filter divergence)