




Wave - current interactions through stochastic physics

Valentin Resseguier,
Bertrand Chapron, Fabrice Collard



Motivations

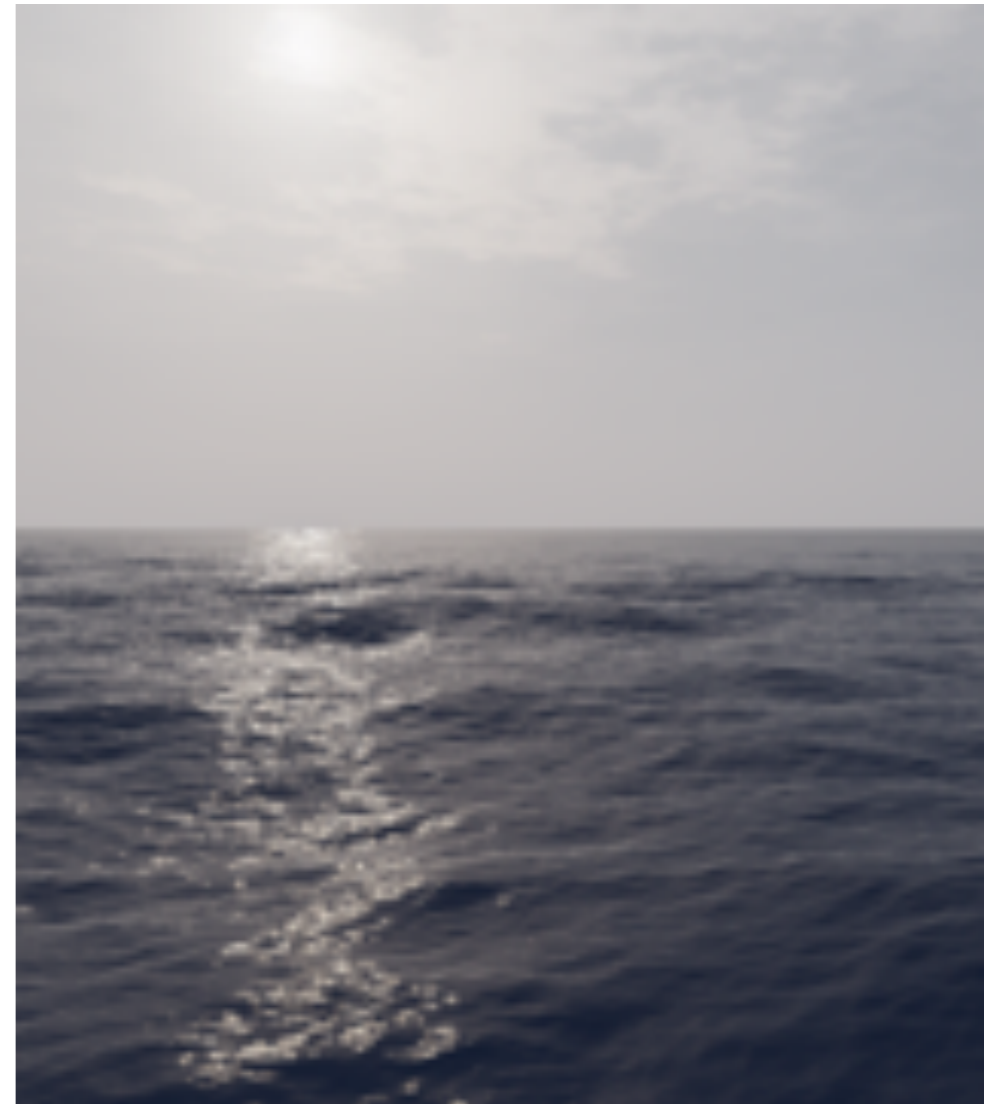
- Mimic interaction wave / (small-scale) current
 - Small-scale currents are hardly observed or simulated
 - Waves see them during very short time
- Quantification of associated modelling errors
 -  Ensemble forecasts and data assimilation
- Improve the Scalian's wave simulator
- Study influence of swell / current interactions onto satellite altimetry

Part I

Background

SeaMotion-radar simulator

- Simple and fast simulation of surface gravity waves over very large domain
- Coupled with radar simulation
- Lot of physical phenomena taking into account
- Used to study satellite altimetry measurements



Large scale
group velocity:

$$\boldsymbol{w} + \nabla_{\boldsymbol{k}} \omega_0$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|\boldsymbol{k}\|}$$

Wave-current interaction

Random ray

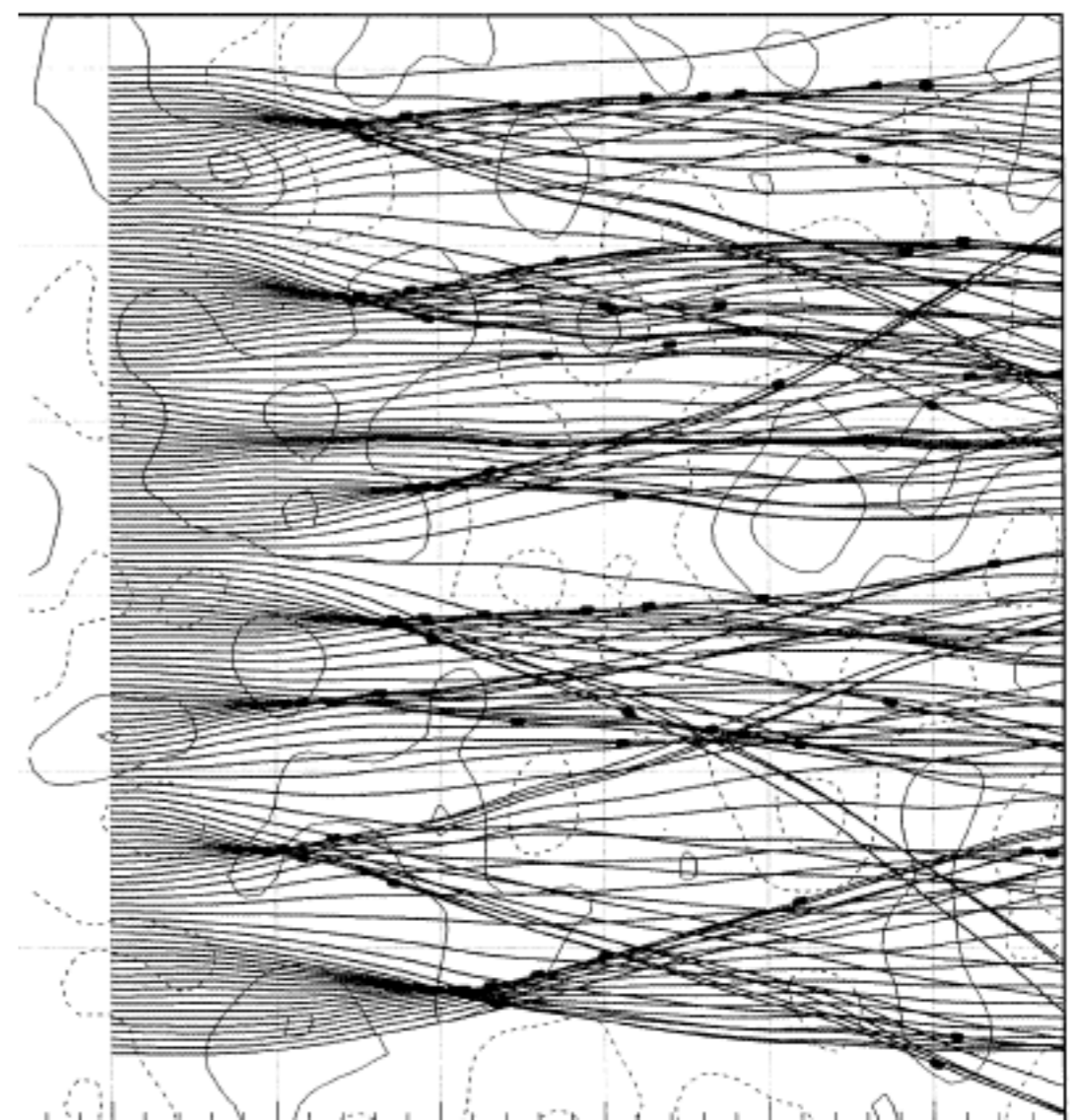
$$\dot{\boldsymbol{X}}_R = \boldsymbol{u}_g$$

Stretching of phase

$$\dot{\boldsymbol{k}} = -\nabla \omega^T \boldsymbol{k}$$

Conservation of
action

$$\text{cst.} = A = \frac{a^2}{\omega_0}$$



White & Fornberg 1997
(stationary velocity)

Large scale group velocity:

$$\mathbf{w} + \nabla_{\mathbf{k}} \omega_0$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|\mathbf{k}\|}$$

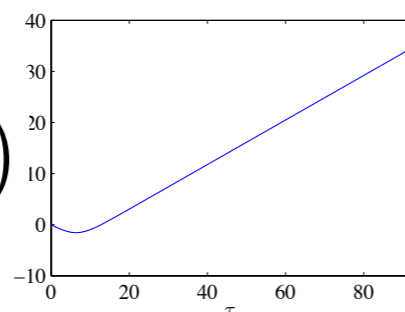
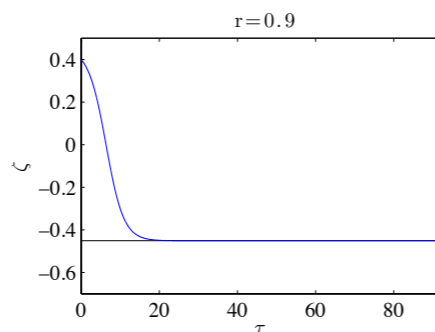
Simple linear case 1:
stationary deterministic,
divergence-free and
linear in x large-scale velocity

Stretching of phase

$$\dot{\mathbf{k}} = -\nabla \mathbf{w}^T \mathbf{k}$$

$$|r| < 1$$

Wave shortening



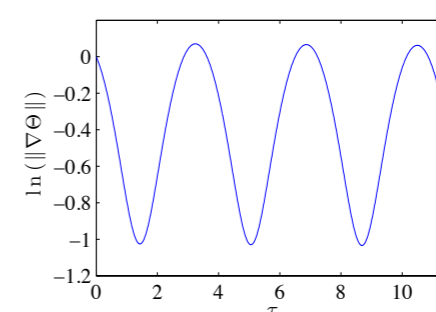
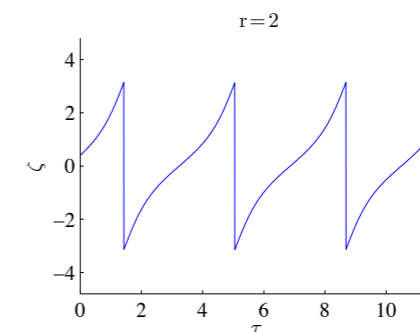
$\ln(\|\mathbf{k}\|)$

Ratio
vorticity / strain rate

$$r = \frac{\omega}{\sigma_w}$$

$$|r| > 1$$

Wave rotation



$\ln(\|\mathbf{k}\|)$

$$\frac{D\zeta}{D\tau} = r - \cos(\zeta)$$

$$\frac{D}{D\tau} \ln(\|\mathbf{k}\|) = -\sin(\zeta)$$

Part II

Waves under location uncertainty

w
σḂ

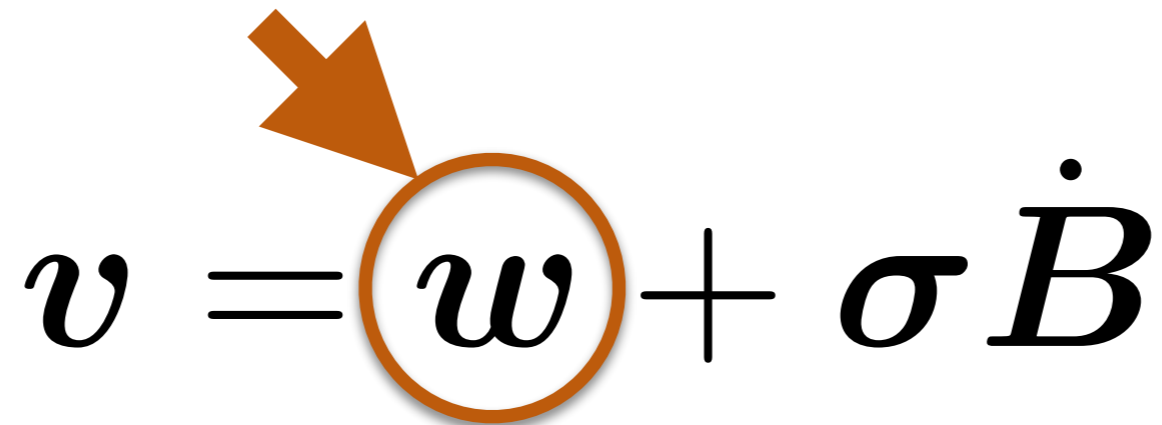
Models under
location uncertainty (LU) :
Adding random velocity

$$\mathbf{v} = \mathbf{w} + \sigma \dot{\mathbf{B}}$$

w
σḂ

Models under
location uncertainty (LU) :
Adding random velocity

Resolved
large scales



The diagram shows the equation $v = w + \sigma \dot{B}$. The variable w is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" above to this circle.

$$v = w + \sigma \dot{B}$$

w
 $\sigma \dot{B}$

Models under
location uncertainty (LU) :
Adding random velocity

Resolved
large scales

White-in-time
small scales

$v = w + \sigma \dot{B}$

The diagram shows the equation $v = w + \sigma \dot{B}$. The term w is enclosed in a brown circle, and the term $\sigma \dot{B}$ is enclosed in a purple circle. A brown arrow points from the text 'Resolved large scales' to the w circle, and a purple arrow points from the text 'White-in-time small scales' to the $\sigma \dot{B}$ circle.

Large scales:

w

Small scales:

$\sigma \dot{B}$

Models under
location uncertainty (LU) :
Adding random velocity

Resolved
large scales

White-in-time
small scales

The diagram shows the equation $v = w + \sigma \dot{B}$. The term w is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" to it. The term $\sigma \dot{B}$ is enclosed in a purple circle, and a purple arrow points from the text "White-in-time small scales" to it.

$$v = w + \sigma \dot{B}$$

Large scales:
 w
 Small scales:
 $\sigma \dot{B}$

Models under location uncertainty (LU) : Adding random velocity

Resolved large scales

White-in-time small scales

$$v = w + \sigma \dot{B}$$

LU

Memin, 2014
 Resseguier et al. 2017 a, b, c
 Cai et al. 2017
 Chapron et al. 2018
 Yang & Memin 2019

SALT

Holm, 2015
 Holm and Tyranowski, 2016
 Arnaudon et al. 2017
 Crisan et al., 2017
 Gay-Balmaz & Holm 2017
 Cotter and al. 2018 a, b
 Cotter and al. 2019

Cotter and al. 2017 Resseguier et al. 2019 a, b

References : Mikulevicius & Rozovskii, 2004
 Flandoli, 2011

Large scales:
 w
 Small scales:
 $\sigma \dot{B}$

Models under location uncertainty (LU) : Adding random velocity

See talks of
 • W. Pan (NP1.1)
 • I. Shevchenko (NP1.1)
 • V. Resseguier (NP2.3)

Initially for uncertainty quantification & data assimilation

Resolved large scales

White-in-time small scales

$$v = w + \sigma \dot{B}$$

	LU	SALT
References :	Mikulevicius & Rozovskii, 2004 Flandoli, 2011 Memin, 2014 Resseguier et al. 2017 a, b, c Cai et al. 2017 Chapron et al. 2018 Yang & Memin 2019	Holm, 2015 Holm and Tyranowski, 2016 Arnaudon et al. 2017 Crisan et al., 2017 Gay-Balmaz & Holm 2017 Cotter and al. 2018 a, b Cotter and al. 2019
	Cotter and al. 2017 Resseguier et al. 2019 a, b	

Large scale
group velocity:

$$\boldsymbol{w} + \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Dispersion ratio

At the first order
in steepness :

Large scale
group velocity:

$$\boldsymbol{w} + \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Dispersion ratio

At the first order
in steepness :

$$\frac{D\phi}{Dt} = \omega_0(k) = \sqrt{g \|k\|}$$

Large scale
group velocity:

$$\boldsymbol{w} + \nabla_{\boldsymbol{k}} \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Dispersion ratio

At the first order
in steepness :

Stochastic
transport

$$\frac{D\phi}{Dt}$$

$$= \omega_0(k) = \sqrt{g \|k\|}$$

Large scale group velocity:

$$\boldsymbol{w} + \nabla_{\boldsymbol{k}} \omega_0$$

Small scale group velocity:

$$\sigma \dot{\boldsymbol{B}}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|\boldsymbol{k}\|}$$

Dispersion ratio

At the first order
in steepness :

Stochastic transport

$$\frac{D\phi}{Dt}$$

Frequency without currents

$$\omega_0(\boldsymbol{k}) = \sqrt{g \|\boldsymbol{k}\|}$$

Large scale
group velocity:

$$+ \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Simple linear case 2:

no large-scale current
+ homogeneous and divergence-free
small-scale velocity

$$\beta(x, t) e^{i(\omega_0 t + k \cdot \sigma B_t)} \quad \text{Wave with Brownian phase}$$

Stochastic stretching of phase

$$\dot{k} = -\nabla(\sigma \dot{B})^T k \quad \text{Log-normal wave-number}$$

Conservation of action

$$\text{cst.} = A = \frac{a^2}{\omega_0} \quad \text{Log-normal amplitude}$$

Large scale
group velocity:

$$w + \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Simple linear case 3:

divergence-free and linear-in-x large-scale current
+ homogeneous and divergence-free
small-scale velocity

Stochastic stretching
of phase

$$\dot{k} = -\nabla (w + \sigma \dot{B})^T k$$

Action

$$\text{cst.} = \frac{a^2}{\omega_0}$$

Large scale group velocity:

$$w + \nabla_k \omega_0$$

Small scale group velocity:

$$\sigma \dot{B}$$

Wave:

$$ae^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Simple linear case 3:

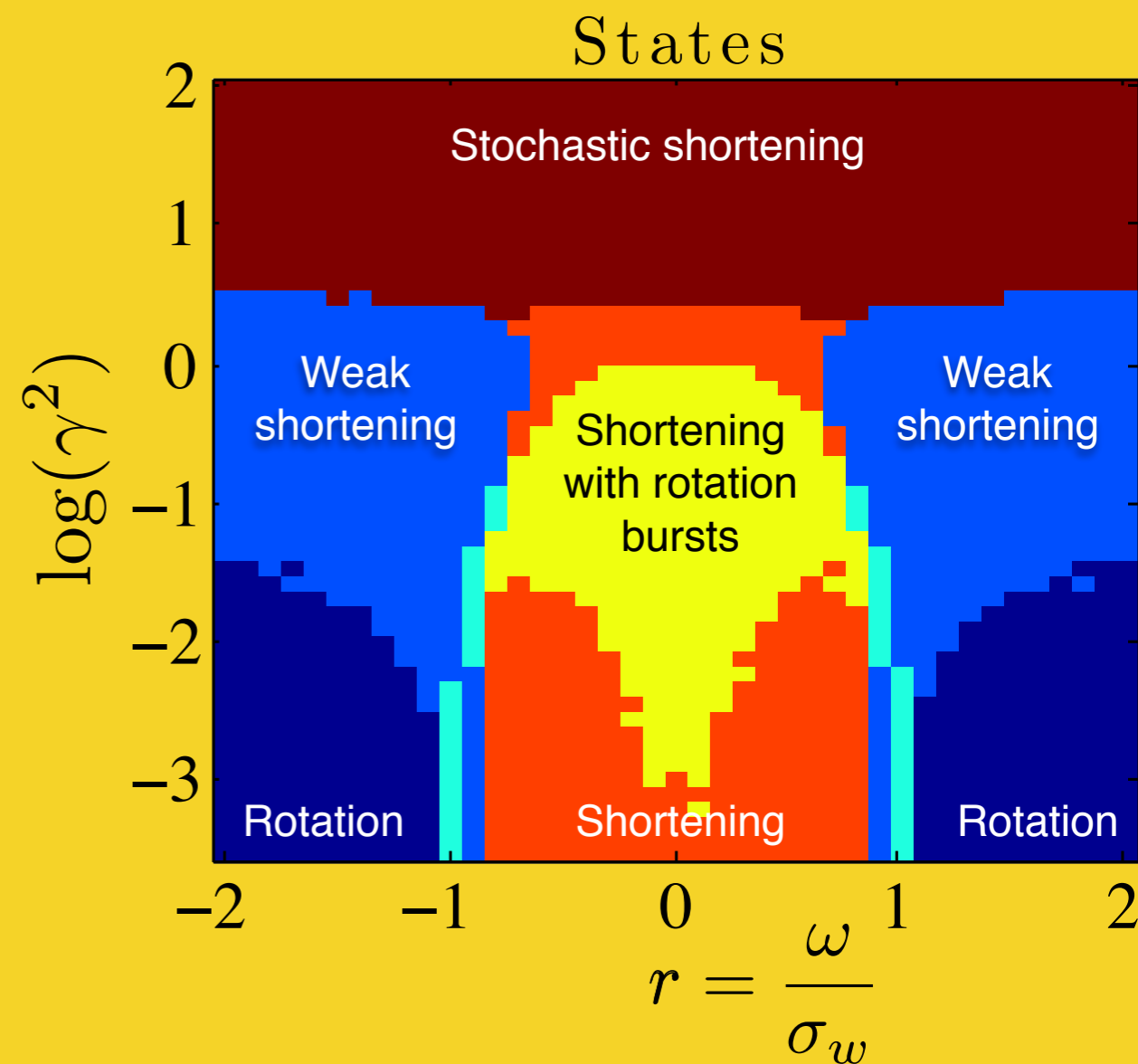
divergence-free and linear-in-x large-scale current
+ homogeneous and divergence-free small-scale velocity

Stochastic stretching of phase

$$\dot{k} = -\nabla (w + \sigma \dot{B})^T k$$

Action

$$\text{cst.} = \frac{a^2}{\omega_0}$$



Large scale group velocity:

$$w + \nabla_k \omega_0$$

Small scale group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Simple linear case 3:

divergence-free and linear-in-x large-scale current
+ homogeneous and divergence-free
small-scale velocity

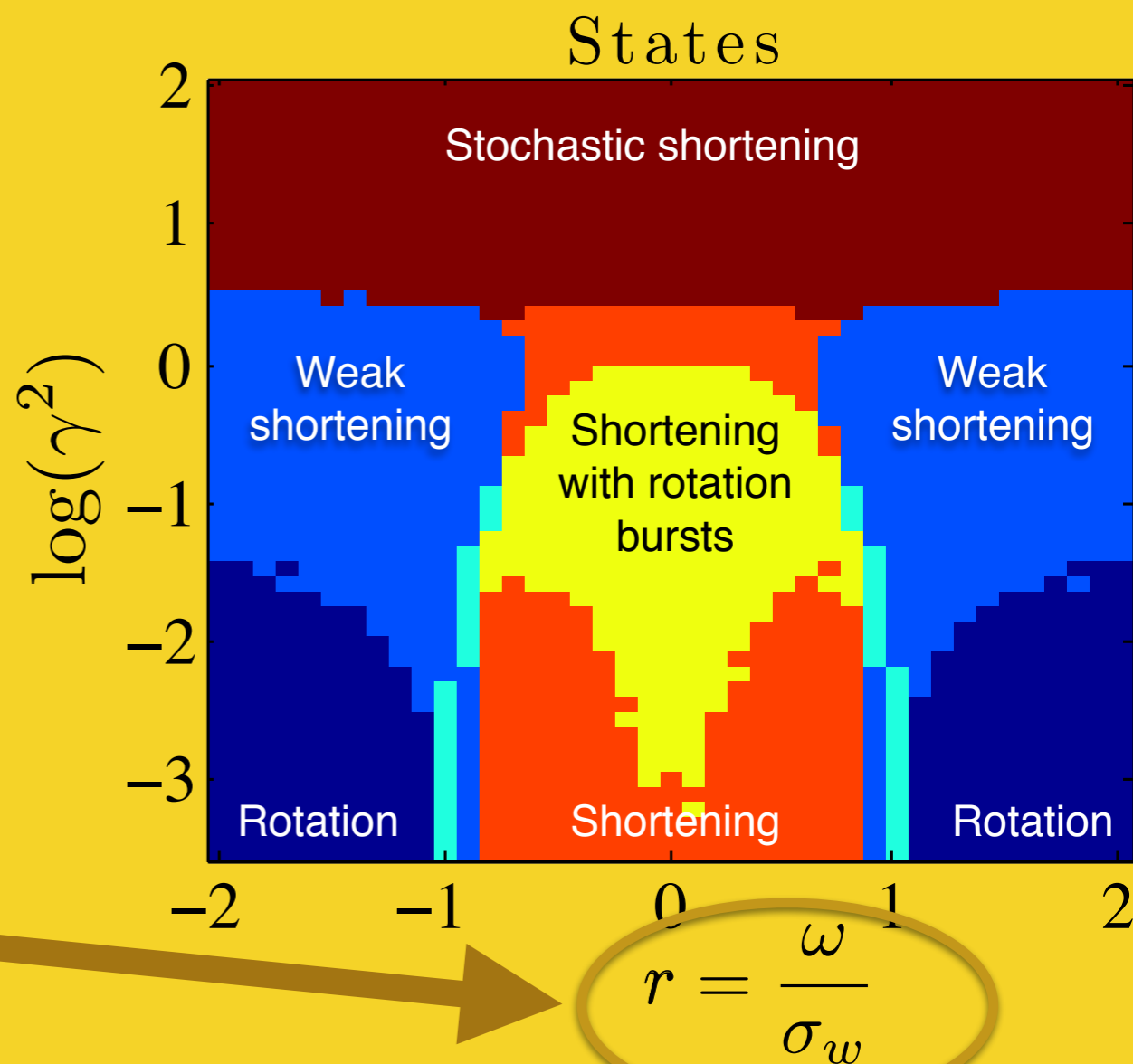
Stochastic stretching
of phase

$$\dot{k} = -\nabla (w + \sigma \dot{B})^T k$$

Action

$$\text{cst.} = \frac{a^2}{\omega_0}$$

Ratio
vorticity / strain rate



Large scale group velocity:

$$w + \nabla_k \omega_0$$

Small scale group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

Simple linear case 3:

divergence-free and linear-in-x large-scale current
+ homogeneous and divergence-free small-scale velocity

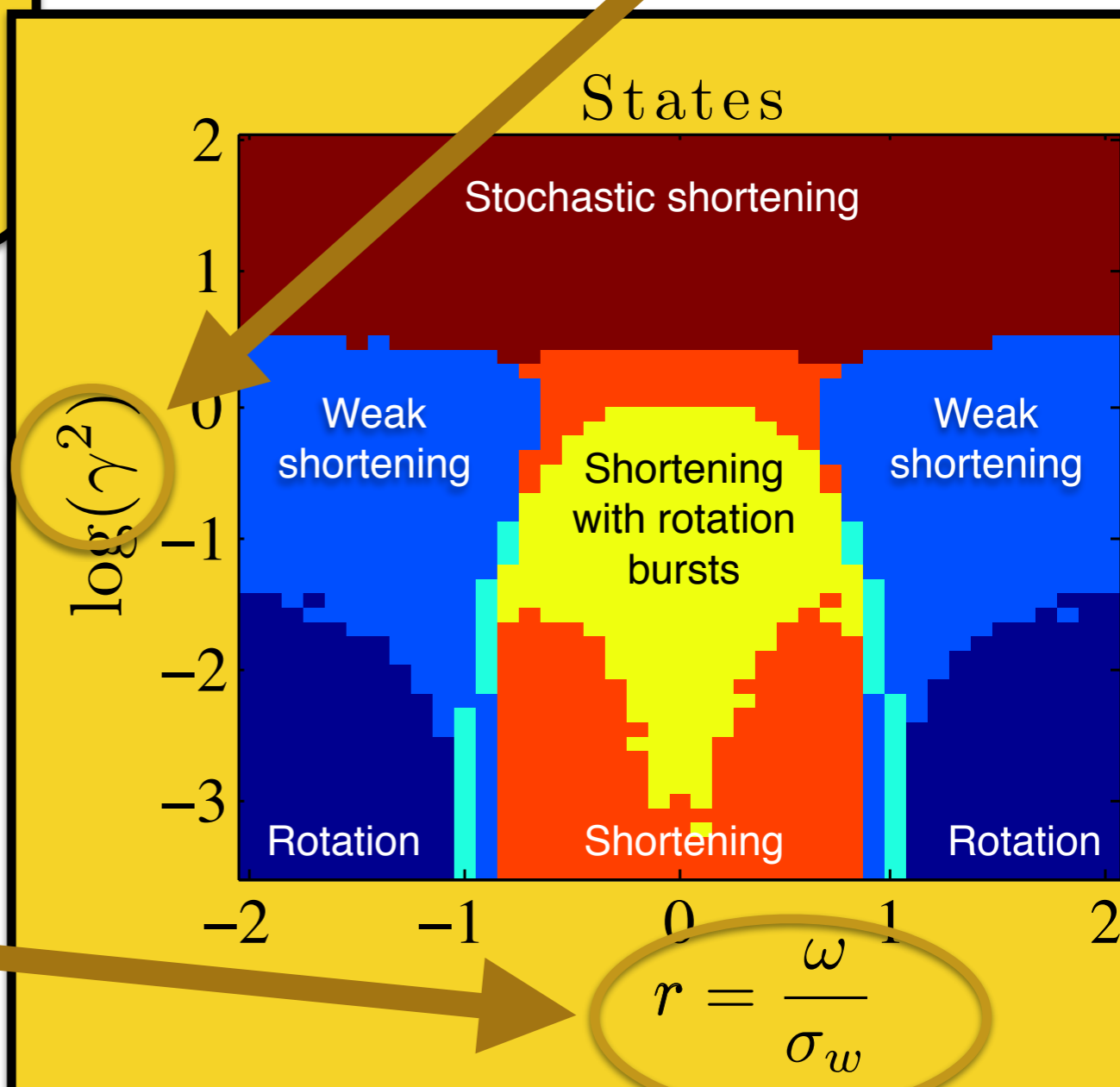
Stochastic stretching of phase

$$\dot{k} = -\nabla (w + \sigma \dot{B})^T k$$

Action

$$\text{cst.} = \frac{a^2}{\omega_0}$$

Depends on small scale's statistics



Ratio vorticity / strain rate

Large scale
group velocity:

$$w + \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

General case

$$\dot{X}_R = u_g \quad \text{Random ray}$$

Stochastic stretching of phase

$$\dot{k} = \left(-\nabla (w + \sigma \dot{B})^T + (\nabla \sigma^T)^2 \right) k \quad \left| \quad \text{Wave-vector} \right.$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad \left| \quad A = \frac{a^2}{\omega_0} \quad \text{Amplitude} \right.$$

Large scale
group velocity:

$$w + \nabla_k \omega_0$$

Small scale
group velocity:

$$\sigma \dot{B}$$

Wave:

$$ae^{\frac{i}{\epsilon} \phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

General case

$$\dot{X}_R = \mathcal{U}_g \quad \text{Random ray}$$

Stochastic stretching of phase

$$\dot{k} = \left(-\nabla(w + \sigma \dot{B})^T + (\nabla \sigma^T)^2 \right) k \quad \text{Wave-vector}$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad \left| \quad A = \frac{a^2}{\omega_0} \quad \text{Amplitude}$$

Large scale group velocity:

$$w + \nabla_k \omega_0$$

Small scale group velocity:

$$\sigma \dot{B}$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

General case

$$\dot{X}_R = \mathcal{U}_g \quad \text{Random ray}$$

Stochastic stretching of phase

$$\dot{k} = \left(-\nabla (w + \sigma \dot{B})^T + (\nabla \sigma^T)^2 \right) k \quad \text{Wave-vector}$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad \left| \quad A = \frac{a^2}{\omega_0} \quad \text{Amplitude}$$

Large scale group velocity:

$$w + \nabla_k \omega_0$$

Small scale group velocity:

$$\sigma \dot{B}$$

Wave:

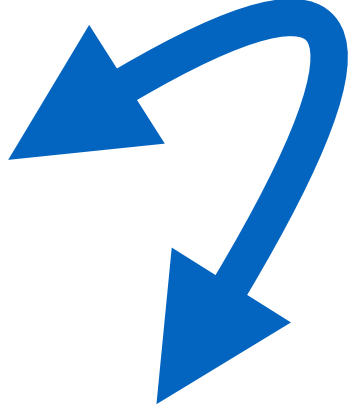
$$ae^{\frac{i}{\epsilon} \phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g \|k\|}$$

General case

Non-linear coupling

$$\dot{X}_R = \underbrace{u_g}_{\text{Function of } k} \quad \text{Random ray}$$


Stochastic stretching of phase

$$\dot{k} = \underbrace{\left(-\nabla (w + \sigma \dot{B})^T + (\nabla \sigma^T)^2 \right)}_{\text{Function of } x} k \quad \left| \quad \text{Wave-vector} \right.$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g \quad A \quad \left| \quad A = \frac{a^2}{\omega_0} \quad \text{Amplitude} \right.$$

Conclusion

Conclusion and outlooks

- Method of characteristics : low computational costs
We can simulate a lot of realisations, and initial conditions
- Understanding the dynamics with increasing complexity
- With satellite measurements,
try to predict wave spectra variations due to Agulhas courant
- Integration into the SeaMotion Radar
- Which are the swell / current effects on satellite measurements?
- Outlook : stochastic Wave Watch III