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SCALING LIMIT OF SUB-BALLISTIC 1D RANDOM WALK AMONG BIASED CONDUCTANCES: A STORY OF WELLS AND WALLS

QUENTIN BERGER AND MICHELE SALVI

ABSTRACT. We consider a one-dimensional random walk among biased i.i.d. conductances, in the case where the random walk is transient but sub-ballistic: this occurs when the conductances have a heavy-tail at $+\infty$ or at 0. We prove that the scaling limit of the process is the inverse of an α -stable subordinator, which indicates an aging phenomenon, expressed in terms of the generalized arcsine law. In analogy with the case of an i.i.d. random environment studied in details in [ESZ09a, ESTZ13], some “traps” are responsible for the slowdown of the random walk. However, the phenomenology is somehow different (and richer) here. In particular, three types of traps may occur, depending on the fine properties of the tails of the conductances: (i) a very large conductance (a *well* in the potential); (ii) a very small conductance (a *wall* in the potential); (iii) the combination of a large conductance followed shortly after by a small conductance (a *well-and-wall* in the potential).

1. INTRODUCTION

Random walks in random environment have been studied extensively over the past forty years, both from a physical and a mathematical perspective. In dimension one, they were introduced in the mathematical literature by Solomon in [Sol75], who gave a criterion for transience/recurrence, and identified three possible regimes: recurrent, transient with positive speed, transient with zero speed.

The behavior of recurrent random walks in (genuine) random environment has been further studied by Sinai in [Sin82], who showed a strong slowdown of the walk, with an unusual scaling $(\log n)^2$. The limiting law has then been identified in [Gol86, Kes86], and the scaling limit of the process is an interesting singular diffusion in random environment, see [FIN02]. The case of transient sub-ballistic random walks in i.i.d. random environment has been considered in [KKS75], and then in [ESZ09a, ESZ09b, ESTZ13]. In particular, the scaling limit of the walk is the inverse of an α -stable subordinator, and the random walk is shown to exhibit an aging phenomenon, *i.e.* the two-time correlation function converges when both times tend to infinity with a fixed ratio, the limit being a non-trivial function of the ratio (aging has been extensively studied in the physics literature, in the context of out-of-equilibrium glassy disordered systems, see [BCKM98] for an overview).

In the present paper we consider the case of a one-dimensional random walk among i.i.d. random conductances: in this case, the random walk is recurrent. One central question in Physics (see e.g. [NJB⁺10]) is to understand the effect of an external field of intensity λ (producing a bias in the conductances) on the behavior of the walk. In particular, as soon as $\lambda > 0$, the walk is transient to the right. Under some integrability condition

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of the conductances, the walk is ballistic; this regime has been studied for instance in [LD16, FS18] (see also references therein). Here we investigate the regime where the random walk among biased random conductances is sub-ballistic: we show that its scaling limit is the inverse of an α -stable subordinator, confirming the universality of the aging phenomenon, cf. [BAČ08, Zin09].

As mentioned above, a similar behavior is observed for sub-ballistic transient random walks in i.i.d. environment. We show however that as soon as the independence hypothesis on the environment is dropped (conductances introduce automatically a strong dependence between neighbours), the trapping mechanism for the walk changes dramatically and might become more irregular. We also point out that convergence to the inverse of an α -stable subordinator has also been shown for walks among biased random conductances in dimension $d \geq 2$ in [FK18]. The one-dimensional case, though, presents once more a wilder zoology of possible trapping mechanisms. In particular, we might have three different kind of traps, given either by a very large conductance, either by a very small conductance or by a combination of the two. The kind of traps that contributes the most to the slowdown of the walk is given by the fine properties of the tails of the conductances at $+\infty$ and at 0. The distribution of the depth of such traps (this is roughly the amount of time that the walk will spend on the trap) has to be studied in terms of the product of random variables with regularly varying tails, which might exhibit unexpected behaviors.

1.1. General setting of the paper. Let $(X_n)_{n \in \mathbb{N}}$ be a discrete-time random walk on an environment ω given by a sequence $\{c_x\}_{x \in \mathbb{Z}}$ of random conductances that are i.i.d. under measure \mathbb{P} . For a fixed realization of ω , we call P^ω the law of the random walk $(X_n)_{n \in \mathbb{N}}$ which starts at the origin and has transition probabilities

$$P^\omega(X_{n+1} = x+1 \mid X_n = x) = \frac{c_x}{c_{x-1} + c_x}, \quad P^\omega(X_{n+1} = x-1 \mid X_n = x) = \frac{c_{x-1}}{c_{x-1} + c_x}.$$

In the usual language of random walks in random environment, where the probability of jumping from x to $x+1$ is called ω_x , we have $\omega_x := \frac{c_x}{c_{x-1} + c_x}$. For $x \in \mathbb{Z}$ define $\rho_x := \frac{1-\omega_x}{\omega_x} = \frac{c_{x-1}}{c_x}$: since the $\{c_x\}_{x \in \mathbb{Z}}$ are independent, we get that $\mathbb{E}[\log \rho_0] = 0$ (provided that $\log c_0$ is integrable). As a consequence, Theorem 2.1.2 in [TZ04] tells us that $(X_n)_{n \geq 0}$ is P^ω -a.s. recurrent for \mathbb{P} -a.e. ω .

We stress that while in the seminal paper by Solomon [Sol75] (and in many other works) the $\{\omega_x\}_{x \in \mathbb{Z}}$ are taken i.i.d., this is clearly not true anymore in the case of conductances. On the other hand, the conductance model boasts the important feature of being reversible with respect to the measure $\pi(x) = c_{x-1} + c_x$. We refer to [Bis11] for an extensive account on the random conductance model, which has been widely studied in the literature.

Now, we introduce an external field of intensity $\lambda > 0$, which corresponds to a tilt of the conductances:

$$c_x^\lambda = e^{\lambda x} c_x. \quad (1.1)$$

The process $(X_n)_{n \in \mathbb{N}}$ on these tilted conductances is called random walk among Biased Random Conductances (which we abbreviate as BiRC). In the BiRC setting, for $x \in \mathbb{Z}$ we have $\rho_x = \rho_x(\lambda) := e^{-\lambda} c_{x-1}/c_x$ (here and in the rest of the paper, we drop the dependence of ρ_x on λ). We get that $\mathbb{E}[\log \rho_0] = -\lambda < 0$ (provided that $\log c_0$ is integrable), so that $(X_n)_{n \in \mathbb{N}}$ is P^ω -a.s. transient to the right for \mathbb{P} -a.e. ω , see [TZ04, Thm 2.1.2]. Additionally, the asymptotic velocity $\mathbf{v}(\lambda) = \lim_{n \rightarrow +\infty} X_n/n$ exists and is $\mathbb{P} \otimes P^\omega$ -a.s. equal to

$$\mathbf{v}(\lambda) = \frac{1}{\mathbb{E}[\bar{S}]} \quad \text{with} \quad \mathbb{E}[\bar{S}] = 1 + 2\mathbb{E}[c_0]\mathbb{E}[1/c_0] \frac{e^{-\lambda}}{1 - e^{-\lambda}},$$

cf. [TZ04, Thm. 2.1.9]. Hence, for $\lambda > 0$, $\mathbf{v}(\lambda) > 0$ if and only if $\mathbb{E}[c_0] < +\infty$ and $\mathbb{E}[1/c_0] < +\infty$. The zero velocity regime can therefore occur for two different reasons: the conductances have some heavy tail at $+\infty$ or they have some heavy tail at 0.

In our previous work [BS19], we consider the sub-ballistic regime, where $\mathbb{E}[c_0] = +\infty$ or $\mathbb{E}[1/c_0] = +\infty$, and we find the correct order for the scaling of X_n . More precisely, we assume that there are some $\alpha_\infty, \alpha_0 \in [0, +\infty]$ with $\alpha := \min(\alpha_0, \alpha_\infty) \leq 1$, such that

$$\lim_{t \rightarrow +\infty} \frac{\log \mathbb{P}(c_0 > t)}{\log t} = -\alpha_\infty, \quad \lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{P}(c_0 < \varepsilon)}{\log \varepsilon} = \alpha_0. \quad (1.2)$$

In Theorem 1.1 of [BS19] we prove that for each $\lambda > 0$

$$\lim_{n \rightarrow \infty} \frac{\log X_n}{\log n} = \alpha \quad \mathbb{P} \otimes P^\omega - a.s.$$

The main goal of the present paper is to make this result much more precise, and prove the convergence of X_n rescaled by n^α (properly corrected by a slowly varying function) to the inverse of an α -stable subordinator. In [ESZ09b, ESTZ13] the authors prove this type of result for one-dimensional random walks in random environment, where the transition probabilities $\{\omega_x\}_{x \in \mathbb{Z}}$ are i.i.d. Their study is based on the analysis of the so-called potential of the environment: the walk is slowed-down by large regions with an atypical value of the potential. As we shall see, this is in sharp contrast with our setting, where the trapping parts of the environment are determined by one or at most two abnormal values of the conductances, see the discussion in Section 1.3.

1.2. Main assumption and main result. Having only (1.2) is not sufficient for describing a functional limit theorem for the BiRC. In the present paper we will make the following stronger assumption.

For two functions f and g , we write that $f(t) \sim g(t)$ if $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$.

Assumption “Traps”. *There exist some slowly varying functions $L_\infty(\cdot), L_0(\cdot)$ and some $\alpha_\infty, \alpha_0 > 0$ such that, for $t > 1$ and $\varepsilon < 1$,*

$$\mathbb{P}(c_0 > t) = L_\infty(t) t^{-\alpha_\infty} \quad \text{and} \quad \mathbb{P}(c_0 < \varepsilon) = L_0(1/\varepsilon) \varepsilon^{\alpha_0}, \quad (1.3)$$

with $\alpha := \min\{\alpha_0, \alpha_\infty\} < 1$. If $\alpha_0 = \alpha_\infty$, we additionally assume that $v \mapsto L_\infty(e^v)$, $v \mapsto L_0(e^v)$ are regularly varying with respective indices γ_∞, γ_0 , i.e. there are slowly varying functions $\varphi_0, \varphi_\infty$ such that $L_\infty(x) = \varphi_\infty(\log x)(\log x)^{\gamma_\infty}$, $L_0(x) = \varphi_0(\log x)(\log x)^{\gamma_0}$. We suppose that $\gamma_\infty \neq -1, \gamma_0 \neq -1$.

As it will be clear below, large values of ρ_x will be associated to “traps” responsible for the slowdown of the random walk $(X_n)_{n \in \mathbb{N}}$: it is therefore crucial to obtain the tail behavior of ρ_x , in order to be able to quantify the depth of the traps. Assumption “Traps” serves that purpose, and is the condition that enables one to obtain the behavior of $\mathbb{P}(\rho_0 > t)$ as $t \rightarrow +\infty$. Indeed, Corollary 5 in [Cli86] gives the the following proposition:

Proposition 1.1. *Assume that Assumption “Traps” holds. Then, there is a slowly varying function $\psi(\cdot)$ such that*

$$\mathbb{P}(\rho_0 > t) = \mathbb{P}(e^{-\lambda} c_{-1}/c_0 > t) \sim \psi(t) t^{-\alpha} \quad \text{as } t \rightarrow +\infty. \quad (1.4)$$

The function $\psi(t)$ is asymptotic to $e^{\lambda\alpha}$ times:

$$\begin{aligned} & \mathbb{E}[c_0^\alpha] L_0(t) \mathbf{1}_{\{\mathbb{E}[c_0^\alpha] < +\infty\}} + \mathbb{E}[1/c_0^\alpha] L_\infty(t) \mathbf{1}_{\{\mathbb{E}[1/c_0^\alpha] < +\infty\}} \quad \text{if } \mathbb{E}[c_0^\alpha] < +\infty \text{ or } \mathbb{E}[1/c_0^\alpha] < +\infty; \\ & \alpha \frac{\Gamma(1+\gamma_0)\Gamma(1+\gamma_\infty)}{\Gamma(2+\gamma_0+\gamma_\infty)} (\log t) L_0(t) L_\infty(t) \quad \text{if } \mathbb{E}[c_0^\alpha] = +\infty \text{ and } \mathbb{E}[1/c_0^\alpha] = +\infty. \end{aligned}$$

In view of Assumption “*Traps*”, we have that $v \mapsto \psi(e^v)$ is regularly varying if $\alpha_0 = \alpha_\infty$.

We stress that in the case $\alpha_0 = \alpha_\infty$, the fact that $L_\infty(e^t), L_0(e^t)$ are regularly varying is crucial to obtain Proposition 1.1—however, the case $L_{\infty/0}(t) = \exp(\pm(\log t)^a)$ with $a \in (0, 1)$ has to be excluded. Some more comments on the tail of ρ_0 are made in Appendix B.

We are now ready to state our main theorem:

Theorem 1.2. *Suppose that Assumption “*Traps*” holds, and recall (1.4). Then, on the space $\mathbf{D}([0, 1])$ of càdlàg functions $[0, 1] \rightarrow \mathbb{R}$ with the uniform topology, we have the following convergence in distribution, as $n \rightarrow +\infty$: under $\mathbb{P} \otimes P^\omega$*

$$\left(\frac{X_{\lfloor un \rfloor}}{n^\alpha / \psi(n)} \right)_{u \in [0, 1]} \Longrightarrow \frac{\sin(\pi\alpha)}{\pi\alpha \mathbb{E}[\zeta^\alpha]} ((\mathcal{S}_\alpha)^{-1}(u))_{u \in [0, 1]}.$$

Here $(\mathcal{S}_\alpha(u))_{u \geq 0}$ is an α -stable subordinator satisfying $\mathbf{E}[e^{-t\mathcal{S}_\alpha(u)}] = e^{-ut^\alpha}$, and $\zeta = \zeta_\lambda(\omega)$ is the random variable defined in (7.1) (if $\mathbb{E}[c_0^\alpha], \mathbb{E}[1/c_0^\alpha] = +\infty$, we get that $\zeta = 2$).

This scaling limit is similar to that found in [ESTZ13] (and [Zin09]), but we stress that the trapping mechanism is different. Note also that the inverse of an α -stable subordinator is easily shown to exhibit an aging phenomenon expressed in terms of generalized arcsine laws: for any fixed $h > 1$, we have

$$\lim_{t \rightarrow +\infty} \mathbb{P}\left((S_\alpha)^{-1}(th) = (S_\alpha)^{-1}(t)\right) = \frac{\sin(\pi\alpha)}{\pi} \int_0^{1/h} y^{-(1-\alpha)}(1-y)^{-\alpha} dy, \quad (1.5)$$

see e.g. [Ber99] (arcsine laws are shown for last-passage times, but the proofs easily adapt to the above statement). An analogous aging result for the random walk $(X_n)_{n \in \mathbb{N}}$ should hold: using the same techniques as in [ESZ09a], one could prove that, for any sequence $j_n \rightarrow +\infty$ and any $h > 1$, $\mathbb{P}(|X_{\lfloor hn \rfloor} - X_n| \leq j_n)$ converges as $n \rightarrow +\infty$ to the right-hand side of (1.5). Our proof actually shows some *localization* property of the walk, which is typically “stuck” at a trap, with a value of ρ_x of order d_n (see (2.2)) which is regularly varying with index $1/\alpha$; this is the key idea to prove aging result.

Remark 1.3. In [BS19], we also considered the biased range-one Mott random walk in the sub-diffusive regime. This process is the simplified version of the Variable-Range Hopping model used in Physics (cfr. [FGS18]), and essentially corresponds to a random walk among random conductances with a different type of bias. In [BS19] we identify the correct scaling exponent for the walk as a specific function of the field intensity λ (in contrast with the BiRC where the scaling α does not depend on λ). The techniques developed in the present work would allow us to push this result further and obtain the scaling limit of the range-one Mott walk, but the result would be less interesting since in that model the conductances are bounded from above and hence only one kind of trap is present.

In dimension $d \geq 2$, the rescaled sub-ballistic BiRC also converges to the inverse of a stable subordinator, as proven in [FK18]. In that case, the sub-ballistic behavior is due to very large conductances, the walk being trapped between the two endpoints of the edge. On the other hand, small conductances do not represent a problem for the walk, which will typically go around them. In dimension $d = 1$ however, the lattice geometry forces the walk to pass through each edge of the positive half line, so the walk can be slowed down both by large conductances (as in the higher dimensional case) and by very small, hard-to-overjump, conductances. Furthermore, in a particular range of the parameters, a new kind of trap arises, as the combination of a small conductance followed shortly after by a large conductance. The fine properties of the tails of c_0 at $+\infty$ and at 0 dictate which kind of traps represent the biggest contribution to the slow-down.

1.3. The different types of traps. In his seminal work, Sinai [Sin82] described the notion of traps in the environment via “valleys” in the potential function V , defined by $V(x) = \sum_{i=1}^x \log \rho_i$ for $x \in \mathbb{Z}_+$ ($V(0) = 0$ and a minus sign is added for $x \in \mathbb{Z}_-$). In the case of i.i.d. ω_i , the potential V is itself a random walk, and valleys in the potential V are due to large regions where the sum of $\log \rho_i$ ’s is abnormally large, cf. [KKS75] (see also [ESTZ13]). In that setting, if H denotes the height of a valley in the potential (for a precise definition, see [ESTZ13, Sec. 3]), then a key result by Iglehart [Igl72] gives that $\mathbb{P}(H > v) \sim C_I e^{-\kappa v}$, for some specific κ and some explicit constant C_I .

This is in sharp contrast with what happens in the BiRC setting: as already noticed in [BS19, § 1.3], we have here $V(x) = \log c_0 - \lambda x - \log c_x$ (for $x \geq 1$), and valleys are caused by isolated large values of $V(x) - V(x-1) = \log \rho_x$. Hence, Proposition 1.1 is crucial in the understanding of the depth of traps, and the tail behavior of c_0 and $1/c_0$ plays a key role in the deviations for $\log \rho_x$, and it gives rise to a much richer phenomenology than in the case of i.i.d. ω_i ’s. In particular, the probability of observing a large valley in the potential behaves as $\mathbb{P}(\log \rho_0 > v) \sim \psi(e^v) e^{-\alpha v}$. The extra slowly varying function $\psi(\cdot)$ depends on the fine asymptotics of the tails (1.3) of c_0 and $1/c_0$, cf. Proposition 1.1, and may be ruled by different types of mechanisms that need to be treated separately. Propositions 1.4-1.5 below consider the distribution of c_{-1}, c_0 conditionally on having $\rho_0 = e^{-\lambda c_{-1}/c_0} > t$, as $t \rightarrow +\infty$: this enables us to understand whether large values of ρ_x are typically due to large values of c_{x-1} (wells), small values of c_x (walls), or a combination of a large value of c_{x-1} and of a small value of c_x (well-and-walls), see Figure 1 for an illustration.

We will distinguish two cases.

Assumption “Simple Traps”. *The distribution of the conductances satisfy Assumption “Traps”, and $\mathbb{E}[c_0^\alpha] < \infty$ or $\mathbb{E}[1/c_0^\alpha] < \infty$.*

Assumption “Well-and-walls”. *The distribution of the conductances satisfy Assumption “Traps”, and $\mathbb{E}[c_0^\alpha] = \infty$ and $\mathbb{E}[1/c_0^\alpha] = \infty$.*

We state here some results that will be proven in Appendix B. Let us introduce some notation. If $\mathbb{E}[c_0^\alpha] < +\infty$, then \bar{c}_{-1} is a r.v. with c.d.f. $F_{\bar{c}_{-1}}(u) = \frac{1}{\mathbb{E}[c_{-1}^\alpha]} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} \leq u\}}]$. If $\mathbb{E}[1/c_0^\alpha] < +\infty$, then $1/\bar{c}_0$ is a r.v. with c.d.f. $F_{1/\bar{c}_0}(u) = \frac{1}{\mathbb{E}[1/c_0^\alpha]} \mathbb{E}[1/c_0^\alpha \mathbf{1}_{\{1/c_0 \leq u\}}]$.

Proposition 1.4. *Suppose that Assumption “Simple Traps” holds. Then the distribution of $(c_{-1}, 1/c_0)$, conditionally on $\rho_0 > t$, converges as $t \rightarrow +\infty$ to the r.v.*

$$(1 - B) \cdot (\bar{c}_{-1}, +\infty) + B \cdot (+\infty, 1/\bar{c}_0), \quad (1.6)$$

where B is a Bernoulli r.v. independent of $\bar{c}_{-1}, 1/\bar{c}_0$, with parameter $q \in [0, 1]$. If $\mathbb{E}[1/c_0^\alpha] = +\infty$, then $q = 0$; if $\mathbb{E}[c_0^\alpha] = +\infty$ then $q = 1$; if $\mathbb{E}[c_0^\alpha], \mathbb{E}[1/c_0^\alpha] < +\infty$ ($\alpha = \alpha_\infty = \alpha_0$) then $q = \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[c_0^\alpha] L_0(t)}{\mathbb{E}[c_0^\alpha] L_0(t) + \mathbb{E}[1/c_0^\alpha] L_\infty(t)}$ (we assume that this limit exists, to avoid working with subsequences).

Proposition 1.4 tells that under Assumption “Simple Traps”, only two types of traps can occur: conditionally on having a large trap (we postpone the precise definition of traps to Definitions 3.1 and 8.1, but one can just think of having ρ_x large), then either

- (i) $B = 0$, $c_{x-1} \asymp 1$, $c_x \ll 1$, corresponding to a *wall* in the potential V ;
- (ii) $B = 1$, $c_{x-1} \gg 1$, $c_x \asymp 1$, corresponding to a *well* in the potential V .

If $q = 0$ (e.g. if $\mathbb{E}[1/c_0^\alpha] = +\infty$), then $B = 0$ and only walls can occur. If $q = 1$ (e.g. if $\mathbb{E}[c_0^\alpha] = +\infty$) then only wells can occur. If $q \in (0, 1)$ (e.g. if $\alpha = \alpha_\infty = \alpha_0$ with

$L_0(t) \sim cL_\infty(t)$ for some constant $c > 0$), then both walls or wells may occur, with respective probability $1 - q$ and q , but not simultaneously.

Proposition 1.5. *Suppose that Assumption “Well-and-walls” holds. Then the distribution of $(c_{-1}, 1/c_0)$, conditionally on $\rho_0 > t$, converges as $t \rightarrow +\infty$ to the r.v. $(+\infty, +\infty)$.*

Proposition 1.5 tells that under Assumption “Well-and-walls”, only one type of traps can occur: conditionally on having a large trap (say ρ_x large), then we necessarily have

(iii) $c_{x-1} \gg 1$, $c_x \ll 1$, corresponding to a *well-and-wall* in the potential V .

It turns out that under Assumption “Well-and-walls”, also “ k -distant” well-and-walls traps may occur, with $k \geq 1$: they consist of the combination of a large conductance c_{x-1} followed shortly after by a small conductance c_{x+k} (this makes $\rho_x^{(k)} := \rho_x \cdots \rho_{x+k} = e^{-\lambda(k+1)} c_{x-1}/c_{x+k}$ large).

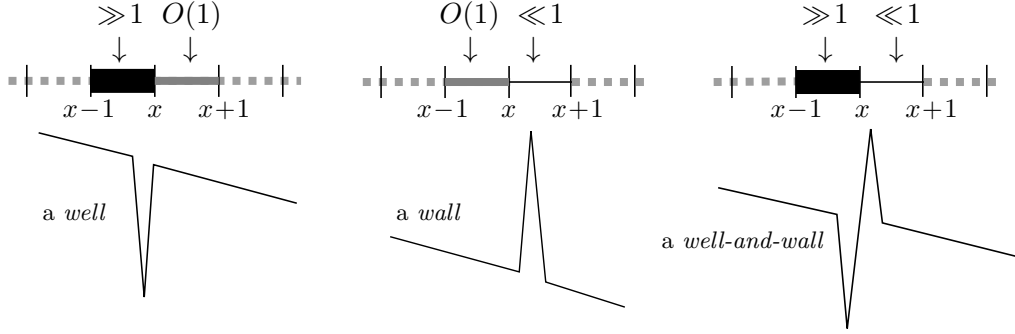


FIGURE 1. The three types of traps are represented above, together with the shape of the potential $x \mapsto V(x)$ associated to each type of trap (recall $V(x) = \log c_0 - \lambda x - \log c_x$). A trap, which roughly speaking is a large value of $\log \rho_x$, can occur in three different ways (from left to right): under Assumption “Simple Traps”, we have either (i) $c_{x-1} \gg 1$, $c_x \asymp 1$ well-shaped trap, or (ii) $c_{x-1} \asymp 1$, $c_x \ll 1$ wall-shaped trap; under Assumption “Well-and-walls”, we have only (iii) $c_{x-1} \gg 1$, $c_x \ll 1$ well-and-wall-shaped trap.

1.4. Organization of the paper and overview of the proof. Let us present briefly how the rest of the paper is organized. In Section 2, we give some preliminary results: we define first-passage times T_n , and then state the convergence of the passage-time process towards an α -stable subordinator, see Theorem 2.1. We prove then that the random walk cannot backtrack more than $C \log n$ and conclude the section by showing how Theorem 1.2 follows from Theorem 2.1.

The rest of the paper is devoted to the proof of Theorem 2.1: in Section 3 we give the precise definition of simple traps and list some important properties. In Section 4 to 7, we prove Theorem 2.1 under Assumption “Simple Traps” (we detail the scheme of the proof below). In Section 8, we define well-and-wall traps, list their properties and adapt the proof of Theorem 2.1 to Assumption “Well-and-walls”: on the one hand we need to additionally deal with k -distant traps, but on the other hand some simplifications occur (due to the fact that $\zeta = 2$ in Theorems 1.2-2.1).

Some technical results are collected in the Appendix: in Appendix A we recall useful formulas for resistor networks; in Appendix B we deal with the tail of ρ_0 and prove some consequences, such as Propositions 1.4-1.5; finally in Appendix C we collect the proofs of the properties of the traps presented in Sections 3 and 8.

Let us now sketch how the proof of Theorem 2.1 works in the case of Assumption “Simple Traps”, i.e. we give a summary of Sections 4–7.

Part 1: Section 4. First, we divide the interval $[0, n]$ into blocks of size $C \log n$ and show that the main contribution to the first-passage time T_n comes from so-called *trapping* blocks, which contain a (unique) large value of ρ_x (see Section 3 for a definition). In Proposition 4.1, we show that with high $\mathbb{P} \otimes P^\omega$ -probability, $T_n \simeq \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)$ where \mathcal{J}_n is the set of indices of trapping blocks, and $T(\mathcal{B}_j)$ is the crossing time of such a block.

Part 2: Section 5. Our second step consists in identifying the trapping mechanism on trapping blocks. We show that, if \mathcal{B} is a (good) trapping block, then the time to cross the block \mathcal{B} is dominated by the time to overcome the deep trap where ρ_x is very large: we prove in Proposition 5.1 that $T(\mathcal{B})/\rho_x \simeq \tau_{\mathcal{B}}$ with high probability. Here, $\tau_{\mathcal{B}}$ is a random variable associated with the environment in the block \mathcal{B} around x . In particular, to overcome a trap, one needs to account for

- the number of attempts to cross the edge $(x, x+1)$;
- the time between two attempts at crossing the edge $(x, x+1)$;
- the number of times the random walk falls anew in the trap.

This is why we write $\tau_{\mathcal{B}}$ as $\theta_{\mathcal{B}} \sum_{j=1}^G E_j$ (cfr. (5.1)), where $\theta_{\mathcal{B}} = \theta_{\mathcal{B}}(\omega)$ is the average time between two attempts to cross $(x, x+1)$; G is the geometric random variable of parameter $p_{\mathcal{B}} = p_{\mathcal{B}}(\omega)$ counting the number of times we fall anew in the trap ($p_{\mathcal{B}}$ is the probability of never returning to x , starting from $x+1$); $(E_j)_{j \geq 1}$ are i.i.d. exponential random variables of parameter 1, which approximate the geometric number of attempts to cross $(x, x+1)$, renormalized by ρ_x . Let us stress here that if $c_{x-1} \gg 1$ (well), then $\theta_{\mathcal{B}} \approx 2$, while if $c_x \ll 1$ (wall), then $p_{\mathcal{B}} \approx 1$. In the case “Well-and-walls” of Section 8, we will therefore show that $T(\mathcal{B})/\rho_x \simeq 2e_{\mathcal{B}}$, where $e_{\mathcal{B}}$ is just one exponential random variable of parameter 1.

Part 3: Section 6. We prove that the crossing time of all good blocks is dominated by the crossing times of the blocks with values of $\rho_x \geq \varepsilon d_n$, cf. Proposition 6.1, up to an error η (that can be made arbitrarily small by taking ε small). Here d_n is roughly the order of the deepest trap between 0 and n , cf. (2.2). The main technical difficulty is that one needs to exclude the possibility that many smaller traps with $\rho_x < \varepsilon d_n$ would slowdown a lot the random walk: in particular, we need to have a (uniform) control on the tail of $\tau_{\mathcal{B}}$ conditionally on having ρ_x large (cf. Lemma 6.3), in order to be sure that the sum of $\tau_{\mathcal{B}}$ on blocks with $\rho_x < \varepsilon d_n$ cannot be large.

Part 4: Section 7. Finally, we prove the convergence for the first-passage time reduced to blocks with $\rho_x \geq \varepsilon d_n$. We show that the positions, depths and crossing times $\tau_{\mathcal{B}}$ of blocks with $\rho_x \geq \varepsilon d_n$ converge to a Poisson Point Process, with an explicit intensity, cf. Proposition 7.2: this allows us to show that T_n/d_n converges to a stable distribution. We then conclude proof of Theorem 2.1 by proving the convergence of the whole process to an α -stable subordinator.

2. PRELIMINARIES: RELATION BETWEEN $(X_n)_{n \in \mathbb{N}}$ AND FIRST-PASSAGE TIMES

A classical way of analyzing the properties of the random walk $(X_n)_{n \in \mathbb{N}}$ is to study its first-passage times. Let

$$T_n := \inf\{j \geq 1, X_j = n\} \quad (2.1)$$

be the first-passage time to n of the walk. Recall Proposition 1.1 and define the sequence d_n , up to asymptotic equivalence, by

$$\mathbb{P}(\rho_0 > d_n) \sim \psi(d_n) d_n^{-\alpha} \sim 1/n \quad \text{as } n \rightarrow \infty, \quad (2.2)$$

so that d_n corresponds to the order of $\max_{1 \leq x \leq n} \rho_x$. We see that d_n is a regularly varying function with index $\gamma = 1/\alpha$. The core of the paper is the proof of the following result.

Theorem 2.1. *Suppose that Assumption “Traps” holds. Then, on the space $\mathbf{D}([0, 1])$ of càdlàg functions $[0, 1] \rightarrow \mathbb{R}$ with Skorohod M_1 -topology, we have the following convergence in distribution, as $n \rightarrow +\infty$: under $\mathbb{P} \otimes P^\omega$ we have*

$$\left(\frac{T_{\lfloor un \rfloor}}{d_n} \right)_{u \in [0, 1]} \Longrightarrow \left(\frac{\pi \alpha \mathbb{E}[\zeta^\alpha]}{\sin(\pi \alpha)} \right)^{1/\alpha} (\mathcal{S}_\alpha(u))_{u \in [0, 1]}.$$

where $(\mathcal{S}_\alpha(u))_{u \geq 0}$ is an α -stable subordinator which satisfies $\mathbf{E}[e^{-t\mathcal{S}_\alpha(u)}] = e^{-ut^\alpha}$, and ζ is a r.v. defined in (7.1).

We use here Skorohod M_1 -topology, which is weaker than the J_1 -topology (roughly speaking, it allows for *intermediate jumps*), since it will be sufficient for our purpose. We refer to [Whi02] for a detailed account on the M_1 - and J_1 -topologies on \mathbf{D} .

Before showing how to derive Theorem 1.2 from Theorem 2.1, we need the following proposition, which slightly improves Proposition 2.1 in [BS19]: we prove that the walk $(X_j)_{j \geq 0}$ cannot backtrack more than $C \log n$ before reaching distance n . We will use this property several times in the proof of the main theorem, but it already tells us that the map $n \mapsto T_n$ is the inverse of the map $j \mapsto X_j$, up to an error of at most a constant times $\log n$.

Proposition 2.2. *There exists $C > 0$ such that the following holds: define the events $\bar{A}_x^n := \bigcap_{j \geq 0} \{X_{T_x+j} \geq x - C \log n\}$ that the random walk does not backtrack more than $C \log n$ after having reached x . Then $\mathbb{P} \otimes P^\omega$ -almost surely there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, \bar{A}_x^n holds for all $0 \leq x \leq n$.*

Proof We have that $(\bar{A}_x^n)^c := \{X_m < x - C \log n, \text{ for some } m \geq T_x\}$, and then

$$P^\omega((\bar{A}_x^n)^c) \leq P_x^\omega(T_{x-C \log n} < +\infty) = \lim_{M \rightarrow \infty} \left(\sum_{j=x}^M \frac{1}{c_j^\lambda} \right) \left(\sum_{j=x-C \log n}^M \frac{1}{c_j^\lambda} \right)^{-1} \quad (2.3)$$

where P_x^ω is the law of the random walk in random environment ω , starting from x . For the equality we have used (A.5). Keeping only the conductance at position $x - C \log n$ in the second sum and recalling (1.1), (2.3) can be bounded by

$$P^\omega((\bar{A}_x^n)^c) \leq e^{-\lambda C \log n} c_{x-C \log n} \sum_{j=x}^{\infty} e^{-\lambda(j-x)} \frac{1}{c_j} =: e^{-\lambda C \log n} K_x^n(\omega).$$

Then, we may use that for any x, n , $\mathbb{P}(K_x^n > t) = t^{-\alpha+o(1)}$ (see [BS19, Lem. 2.5]), to get that \mathbb{P} -a.s., for n large enough, $\max_{0 \leq x \leq n} K_x^n \leq n^{3/\alpha}$. As a consequence, for n large enough we have that $P^\omega(\bigcup_{0 \leq x \leq n} (\bar{A}_x^n)^c) \leq n^{1+3/\alpha} e^{-\lambda C \log n}$. For C large enough this is summable, so Borel-Cantelli gives that $\mathbb{P} \otimes P^\omega$ -a.s., for n large enough, \bar{A}_x^n holds for all $0 \leq x \leq n$. \square

Proof of Theorem 1.2 From Theorem 2.1 and Proposition 2.2, we are able to prove Theorem 1.2. Let us denote $Y_n := \inf\{j: T_j > n\}$, i.e. the unique integer such that $T_{Y_n-1} \leq n < T_{Y_n}$. By construction, $X_n \leq Y_n$ for all $n \in \mathbb{N}$, and Proposition 2.2 tells that a.s., for n large enough, $X_j \geq Y_j - 1 - C \log n$ for all $0 \leq j \leq n$. This shows that we only have to prove the convergence of $(\frac{\psi(n)}{n^\alpha} Y_{\lfloor un \rfloor})_{u \in [0, 1]}$ as $n \rightarrow +\infty$.

Now, define for any $u \geq 0$

$$\mathcal{T}_n(u) := \frac{T_{\lfloor un \rfloor}}{d_n} \quad \text{and} \quad \mathcal{T}_n^{-1}(u) := \inf\{s \geq 0 : \mathcal{T}_n(s) > u\} \quad (2.4)$$

the inverse map of \mathcal{T}_n . Let f_n be an inverse of d_n , i.e. a sequence of real numbers such that $d_{f_n} = n$ (we may assume that $n \mapsto d_n$ is defined on \mathbb{R}_+): note that by (2.2), we have that $n^\alpha \psi(n)^{-1} \sim f_n$ as $n \rightarrow +\infty$. Then, we can write $Y_{\lfloor un \rfloor}$ in terms of $\mathcal{T}_{f_n}^{-1}$:

$$\frac{Y_{\lfloor un \rfloor}}{f_n} = \inf\{j/f_n : T_j > un\} = \inf\{s \geq 0 : \frac{1}{n}T_{\lfloor sf_n \rfloor} > u\} = \mathcal{T}_{f_n}^{-1}(u)$$

Now, we may apply Theorem 2.1 with f_n in place of n , together with Corollary 13.6.4 of [Whi02] which says that the inverse map from non-decreasing functions of (\mathbf{D}, M_1) to non-decreasing functions of $(\mathbf{D}, \|\cdot\|_\infty)$ is continuous at strictly increasing functions. Since $(\mathcal{S}_\alpha(u))_{u \geq 0}$ is a.s. strictly increasing, see [Ber99], we obtain the following convergence in distribution as $n \rightarrow +\infty$, under $\mathbb{P} \otimes P^\omega$

$$(\mathcal{T}_{f_n}^{-1}(u))_{u \in [0,1]} \implies \frac{\sin(\pi\alpha)}{\pi\alpha \mathbb{E}[\zeta^\alpha]} ((\mathcal{S}_\alpha)^{-1}(u))_{u \in [0,1]}.$$

Note that we also used the scaling relation $(\mathbf{c}\mathcal{S}_\alpha(u))_{u \geq 0} \stackrel{(d)}{=} (\mathcal{S}_\alpha(\mathbf{c}^\alpha u))_{u \geq 0}$ to express in a simpler way the inverse of $(\mathbf{c}\mathcal{S}_\alpha(u))_{u \geq 0}$ with $\mathbf{c} = (\frac{\pi\alpha}{\sin(\pi\alpha)} \mathbb{E}[\zeta^\alpha])^{1/\alpha}$.

This concludes the proof since $\mathcal{T}_{f_n}^{-1}(u) = Y_{\lfloor un \rfloor}/f_n$, with $f_n \sim n^\alpha/\psi(n)$ as $n \rightarrow +\infty$. \square

3. DEFINITION OF SIMPLE TRAPS AND THEIR PROPERTIES

In this section (and up to Section 7), we suppose that Assumption “Simple Traps” holds, that is $\mathbb{E}[c_0^\alpha] < +\infty$ or $\mathbb{E}[(1/c_{-1})^\alpha] < +\infty$. We recall that under these hypothesis there will be only well-traps or wall-traps, but never well-and-wall traps, cf. Proposition 1.4.

3.1. Definition of simple traps. By definition (2.2) of d_n , the maximal depth of a trap between 0 and n will be of order d_n . We give here the definition of simple n -traps, that is traps with near-maximal depth that are either well-shaped or wall-shaped. We will drop the “ n ” in the name whenever possible.

Definition 3.1. Let $q_n := (\log n)^{1/4}$. A site x is a simple n -trap if $\rho_x > d_n e^{-q_n}$. It is

- ◆ of well type if $c_{x-1} > d_n e^{-q_n^2}$,
- ◆ of wall type if $\frac{1}{c_x} > d_n e^{-q_n^2}$.

We call $\mathcal{W}_x := \{\rho_x > d_n e^{-q_n}\}$ the event that there is a trap in x .

For $n \in \mathbb{N}$ we let $C_n := \lceil C \log n \rceil$, with $C > 0$ the constant in Proposition 2.2. We divide \mathbb{Z} in disjoint n -blocks, where an n -block is a box of the form $\{jC_n, jC_n+1, \dots, (j+1)C_n-1\}$ for some $j \in \mathbb{Z}$. A n -triblock is a sequence of three consecutive n -blocks: we denote $\mathcal{B}_j := \{(j-1)C_n, \dots, (j+2)C_n-1\}$ the j -th triblock, and we let $k_n := \lceil n/C_n \rceil$ be the number of n -triblocks between 0 and n .

Definition 3.2. For a given $n \in \mathbb{N}$, we define a trapping environment as $\Gamma_n := \bigcup_{x=0}^{C_n-1} \mathcal{W}_x$, that is, an environment with a simple n -trap in the first n -block. A n -triblock \mathcal{B}_j is called a trapping n -triblock if $\theta^{jC_n} \omega \in \Gamma_n$ (with $\theta^a \omega := (\omega_{x+a})_{x \in \mathbb{Z}}$). We define $\mathcal{J}_n := \{j \in \{0, \dots, k_n\} : \theta^{jC_n} \omega \in \Gamma_n\}$ the set of indices of the trapping triblocks.

Definition 3.3. A trapping n -triblock with a n -trap at site x is said to be good if

- (i) $c_x \leq e^{4q_n}$, $c_{x-1} \geq e^{-4q_n}$;

(ii) for all $y \neq x-1, x$ in the n -triblock, $c_y \leq e^{4q_n}$ and $c_y \geq e^{-4q_n}$.

We denote

$$\bar{\Gamma}_n := \bigcup_{x=0}^{C_n-1} \left(\mathcal{W}_x \cap \left\{ c_x, \frac{1}{c_{x-1}} \leq e^{4q_n} \right\} \bigcap_{\substack{-C_n \leq y < 2C_n \\ y \neq x-1, x}} \{c_y \in [e^{-4q_n}, e^{4q_n}]\} \right)$$

3.2. Properties of simple traps. In this section we collect some important properties of simple traps and of trapping triblocks:

1. Traps are isolated, in fact distant by at least ne^{-5q_n} (Lemma 3.4);
2. Traps are not too deep: they cannot be deeper than $d_ne^{q_n}$ (Lemma 3.5);
3. All trapping triblocks are good (Lemma 3.6);
4. Traps are the only annoying parts of the environment (Lemma 3.7).

We now state the lemmas corresponding to these four properties, and postpone their proofs to Appendix C—they will be treated together with the analogous statements in the case “Well-and-walls”.

Lemma 3.4. *Let \mathcal{D}_n be the event*

$$\mathcal{D}_n := \bigcup_{\substack{-n \leq x < y \leq n \\ |x-y| \leq ne^{-5q_n}}} \mathcal{W}_x \cap \mathcal{W}_y.$$

Then \mathbb{P} -a.s. \mathcal{D}_n occurs for only finitely many n .

The proof of Lemma 3.4 can be found in Section C.1 of the Appendix. As a consequence of Lemma 3.4, we have a.s. that $|\mathcal{J}_n| \leq e^{5q_n}$ for n large enough.

Lemma 3.5. *\mathbb{P} -a.s., there is some n_0 such that for all $n \geq n_0$*

$$M_n := \sup_{-n \leq x \leq n} \rho_x \leq d_ne^{q_n}.$$

The proof of Lemma 3.5 can be found in Section C.2 of the Appendix.

Lemma 3.6. *Let \mathcal{G}_n be the event*

$$\mathcal{G}_n := \bigcup_{-n \leq x \leq n} \mathcal{W}_x \cap \left(\bigcup_{\substack{y=x-2C_n \\ y \neq x-1, x}}^{x+2C_n} \left\{ c_x, \frac{1}{c_{x-1}} > e^{4q_n} \right\} \cup \{c_y \notin [e^{-4q_n}, e^{4q_n}]\} \right).$$

Then \mathbb{P} -a.s. \mathcal{G}_n occurs only for finitely many n .

The proof of Lemma 3.6 can be found in Section C.3 of the Appendix.

The last property mentioned above may be more mysterious. Let us define a generalization of ρ_x : for all $x \in \mathbb{Z}$ and $k \in \mathbb{N}$, set

$$\rho_x^{(k)} := e^{-\lambda(k+1)} \frac{c_{x-1}}{c_{x+k}} = \rho_x \cdots \rho_{x+k}. \quad (3.1)$$

As it will be clear in the next sections, high values of $\rho_x^{(k)}$ are also responsible for the trapping of the walk. What we mean by property 4 is that, under Assumption “Simple Traps”, high values of $\rho_x^{(k)}$ are only observed in the presence of a well or of a wall (that is, they cannot come from the combination of an almost-well and an almost-wall). This is to ensure that triblocks that are not trapping triblocks (according to Definition 3.2) will not slowdown the walk for a long time.

Lemma 3.7. *Let $\varepsilon_n = q_n^{-\delta}$ with some $\delta > 0$, and let*

$$\mathcal{H}_n := \bigcup_{-n \leq x \leq n} \bigcup_{k \geq 1} \{\rho_x^{(k)} > \varepsilon_n e^{-\lambda k/2} d_n\} \cap \mathcal{W}_x^c \cap \mathcal{W}_{x+k}^c. \quad (3.2)$$

Then, if δ is sufficiently small, we have that $\mathbb{P}(\mathcal{H}_n) \rightarrow 0$ as $n \rightarrow +\infty$.

The proof of Lemma 3.7 can be found in Section C.5 of the Appendix.

4. REDUCTION TO LARGE TRAPS

For a triblock \mathcal{B}_j , we define its crossing time:

$$T(\mathcal{B}_j) := T_{(j+2)C_n} - T_{jC_n}, \quad (4.1)$$

which is the time it takes for the random walk, started at the beginning of the middle n -block of \mathcal{B}_j , to exit \mathcal{B}_j from the right. In view of Proposition 2.2, provided that n is sufficiently large, the walk X_i remains in \mathcal{B}_j for all i between times T_{jC_n} and $T_{(j+2)C_n}$. In this section we prove that the time T_n to reach distance n is close to the crossing time of all trapping n -triblocks, so that the other blocks can be neglected.

Proposition 4.1. *Let $\varepsilon_n := q_n^{-\delta}$ with δ small enough, as defined in Lemma 3.7: we have that*

$$P^\omega\left(\frac{1}{d_n} \left|T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)\right| > \varepsilon_n^{(1-\alpha)/2}\right) \leq \varepsilon_n^{(1-\alpha)/2}.$$

with \mathbb{P} -probability going to 1.

Before we start the proof, recall the definition (3.1) of $\rho_x^{(k)} := e^{-\lambda(k+1)\frac{c_x-1}{c_{x+k}}}$. Recall also Proposition 1.1: using Potter's bound (cf. [BGT89, Thm. 1.5.6]), we easily have that there is a constant $c > 0$ such that for any $k \geq 0$

$$\mathbb{P}(\rho_x^{(k)} > t) \leq c e^{-\lambda \alpha k/2} \psi(t) t^{-\alpha} \quad \text{for all } t > 1. \quad (4.2)$$

Proof of Proposition 4.1 We write that a.s., for n large enough

$$T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j) = \sum_{j=0}^{k_n} (T_{(j+1)C_n} - T_{jC_n}) \mathbf{1}_{\{j, j-1 \notin \mathcal{J}_n\}} \mathbf{1}_{A_j^n}. \quad (4.3)$$

Here, we let $A_j^n := \{X_{T_{jC_n}+i} > (j-1)C_n \text{ for all } i \geq 0\}$ be the event that the random walk does not backtrack from the j -th n -block to position $(j-1)C_n$: Proposition 2.2 tells that $\mathbb{P} \otimes P^\omega$ -a.s., for n large enough, $\mathbf{1}_{A_j^n} = 1$ for all $0 \leq j \leq k_n$. We also used that there cannot be two consecutive good blocks, see Lemma 3.4. Let us now remark that thanks to formula (A.9)

$$\begin{aligned} E^\omega[(T_{(j+1)C_n} - T_{jC_n}) \mathbf{1}_{A_j^n}] &= E_{jC_n}^\omega[T_{(j+1)C_n} \mathbf{1}_{A_j^n}] \\ &\leq C_n + \sum_{z=(j-1)C_n}^{jC_n-1} \sum_{k=jC_n-z}^{(j+1)C_n-z} \rho_z^{(k)} + \sum_{z=jC_n}^{(j+1)C_n-1} \sum_{k=0}^{(j+1)C_n-z} \rho_z^{(k)}. \end{aligned}$$

Here we have first set all conductances to the left of $(j-1)C_n$ equal to 0 thanks to the indicator of A_j^n (so the $\rho_z^{(k)}$ are equal to 0 for $z < (j-1)C_n$), and then we have dropped the indicator. Thanks to Lemma 3.7, we have that with high \mathbb{P} -probability all the $\rho_z^{(k)}$

involved in the sum (*i.e.* for $j, j-1 \notin \mathcal{J}_n$) are smaller than $\varepsilon_n e^{-\lambda k/2} d_n$. Going back to (4.3), we therefore get that on the event \mathcal{H}_n^c (recall \mathcal{H}_n is defined in (3.2)),

$$\begin{aligned} E^\omega \left[T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j) \right] &\leq n + 2 \sum_{z=-C_n}^n \sum_{k=0}^{2C_n} \rho_z^{(k)} \mathbf{1}_{\{\rho_z^{(k)} \leq \varepsilon_n e^{-\lambda k/2} d_n\}} \\ &\leq n + 2 \sum_{z=-C_n}^n \sum_{k=0}^{\ell_n} \rho_z^{(k)} \mathbf{1}_{\{\rho_z^{(k)} \leq \varepsilon_n e^{-\lambda k/2} d_n\}} + 2 \sum_{z=-C_n}^n \sum_{k=\ell_n}^{C_n} \rho_z^{(k)} \mathbf{1}_{\{\rho_z^{(k)} \leq e^{-\lambda \ell_n/2} d_n\}}, \end{aligned} \quad (4.4)$$

where we set $\ell_n := (\log \log n)^2$ and take the integer part when dealing with non-integer indices.

We first deal with the first sum in (4.4). Let us set $R_z^n := \sum_{k=0}^{\ell_n} \rho_z^{(k)}$. We have that

$$\sum_{k=0}^{\ell_n} \rho_z^{(k)} \mathbf{1}_{\{\rho_z^{(k)} \leq \varepsilon_n e^{-\lambda k/2} d_n\}} \leq R_z^n \mathbf{1}_{\{R_z^n \leq \ell_n \varepsilon_n d_n\}} =: \bar{R}_z^n.$$

For any fixed $1 - \alpha < \delta' < 1$, we can bound

$$\mathbb{P} \left(\sum_{z=-C_n}^n \bar{R}_z^n > (\varepsilon_n)^{\delta'} d_n \right) \leq 3\ell_n \mathbb{P} \left(\sum_{j=-C_n/3\ell_n}^{n/3\ell_n} \bar{R}_{3j\ell_n}^n > \frac{(\varepsilon_n)^{\delta'}}{3\ell_n} d_n \right) \quad (4.5)$$

where we used a union bound for the last inequality, splitting the sum $\sum_{z=-C_n}^n \bar{R}_z^n$ as $\sum_{x=0}^{3\ell_n-1} \sum_{j=-C_n/3\ell_n}^{n/3\ell_n-1} \bar{R}_{3j\ell_n+x}^n$. Now, we notice that the $\bar{R}_{3j\ell_n}^n$ for $j \in \{-C_n/3\ell_n, \dots, n/3\ell_n\}$ are independent. Also, thanks to (4.2) and a union bound, we have that $\mathbb{P}(R_0^n > t) \leq c \psi(t) t^{-\alpha}$. Then, using a Fuk-Nagaev inequality (see [Nag79, Thm. 1.1]), we have that there is a constant c such that for any $m \geq 1$ and any $y \leq x$

$$\mathbb{P} \left(\sum_{j=1}^m R_{3j\ell_n}^n > x, \max_{1 \leq j \leq m} R_{3j\ell_n}^n \leq y \right) \leq \left(c m \frac{y}{x} \psi(y) y^{-\alpha} \right)^{x/y}.$$

We therefore get that there are constants $c, c', c'' > 0$ such that

$$\begin{aligned} \mathbb{P} \left(\sum_{j=-C_n/3\ell_n}^{n/3\ell_n} \bar{R}_{3j\ell_n}^n > \frac{(\varepsilon_n)^{\delta'}}{3\ell_n} d_n \right) &\leq \left(c \frac{n}{\ell_n} \frac{\ell_n \varepsilon_n}{(\varepsilon_n)^{\delta'}/\ell_n} \psi(\ell_n \varepsilon_n d_n) (\ell_n \varepsilon_n d_n)^{-\alpha} \right)^{(\varepsilon_n)^{\delta'-1}/(3\ell_n^2)} \\ &\leq \left(c' n \ell_n (\varepsilon_n)^{(1-\alpha-\delta')/2} \psi(d_n) d_n^{-\alpha} \right)^{(\varepsilon_n)^{-(1-\delta')/2}} \leq c'' e^{-(\varepsilon_n)^{-(1-\delta')/2}}. \end{aligned}$$

For the second inequality, we used Potter's bound to get that $\psi(\ell_n \varepsilon_n d_n)$ is bounded by a constant times $\ell_n^\alpha (\varepsilon_n)^{(\delta'+\alpha-1)/2} \psi(d_n)$. We also used that $(3\ell_n)^{-2} \leq (\varepsilon_n)^{-(1-\delta')/2}$ for n large enough (recall $\varepsilon_n = q_n^{-\delta}$ with $q_n = (\log n)^{1/4}$). For the last inequality, we used the definition (2.2) of d_n to get that for n large enough the term inside the parenthesis is smaller than a constant times $\ell_n (\varepsilon_n)^{(1-\alpha-\delta)/2}$, which can be made arbitrarily small by taking n large.

Going back to (4.5), and using that $\ell_n = o(e^{\frac{1}{2}(\varepsilon_n)^{-(1-\delta')/2}})$ we get that

$$\mathbb{P} \left(\sum_{z=-C_n}^n \bar{R}_z^n > (\varepsilon_n)^{\delta'} d_n \right) \leq c'' e^{-\frac{1}{2}(\varepsilon_n)^{-(1-\delta')/2}}. \quad (4.6)$$

We can treat the second term in (4.4) similarly. Setting $V_z^n := \sum_{k=\ell_n}^{C_n} \rho_z^{(k)}$, we get that

$$\sum_{k=\ell_n}^{C_n} \rho_z^{(k)} \mathbf{1}_{\{\rho_z^{(k)} \leq e^{-\lambda \ell_n/2} d_n\}} \leq V_z^n \mathbf{1}_{\{V_z^n \leq C_n e^{-\lambda \ell_n/2} d_n\}} =: \bar{V}_z^n.$$

Similarly as above, with C_n playing the role of ℓ_n and $e^{-\lambda \ell_n/2}$ playing the role of ε_n (note that $C_n = o(e^{\lambda \ell_n/2})$), we obtain that for n large enough

$$\mathbb{P}\left(\sum_{z=-C_n}^n \bar{V}_z^n > e^{-\delta' \lambda \ell_n/2} d_n\right) \leq c'' e^{-\frac{1}{2} e^{(1-\delta') \lambda \ell_n/4}} \leq c'' e^{-\frac{1}{2} (\varepsilon_n)^{-(1-\delta')/2}}. \quad (4.7)$$

Going back to (4.4), by a union bound we get that for any $1 - \alpha < \delta' < 1$,

$$\mathbb{P}\left(E^\omega\left[T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)\right] > 3(\varepsilon_n)^{\delta'} d_n; \mathcal{H}_n^c\right) \leq 2c'' e^{-\frac{1}{2} (\varepsilon_n)^{-(1-\delta')/2}} \rightarrow 0. \quad (4.8)$$

Using Markov's inequality, we get that on \mathcal{H}_n^c

$$P^\omega\left(\frac{1}{d_n} \left|T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)\right| > \varepsilon_n^{(1-\alpha)/2}\right) \leq \varepsilon_n^{-(1-\alpha)/2} \frac{1}{d_n} E^\omega\left[T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)\right],$$

which thanks to (4.8) is smaller than $\varepsilon_n^{(1-\alpha)/2}$ for n large enough (recall $\delta' > 1 - \alpha$), with \mathbb{P} -probability going to 1. Since $\mathbb{P}(\mathcal{H}_n) \rightarrow 0$ by Lemma 3.7, we get the conclusion of Proposition 4.1. \square

5. CROSSING TIMES OF TRAPPING TRIBLOCKS

Notice that Lemma 3.4 gives that to each trapping block \mathcal{B} can be associated a unique site $x_{\mathcal{B}}$ such that $x_{\mathcal{B}}$ is a n -trap. We denote $\rho_{\mathcal{B}} := \rho_{x_{\mathcal{B}}}$ the depth of the trap associated with $x_{\mathcal{B}}$.

In this section we show that, for a trapping block, the random variable $T(\mathcal{B})/\rho_{\mathcal{B}}$ can be well approximated by another random variable $\tau_{\mathcal{B}}$, given by

$$\tau_{\mathcal{B}} = \tau_{\mathcal{B}}(\omega) := \theta_{\mathcal{B}} \sum_{j=1}^{G(p_{\mathcal{B}})} E_j. \quad (5.1)$$

Let us explain each term appearing in (5.1):

$p_{\mathcal{B}} = P_{x_{\mathcal{B}}+1}^\omega(T_{x_{\mathcal{B}}} > T_{x_{\mathcal{B}}+C_n})$ is the probability, starting from $x_{\mathcal{B}}+1$, of never returning to $x_{\mathcal{B}}$ (this is also the escape probability from $x_{\mathcal{B}}+1$, recall the non-backtracking property of Proposition 2.2). Notice that this only depends on the environment to the right of $x_{\mathcal{B}}+1$ that is in \mathcal{B} .

$G(p_{\mathcal{B}})$ is a geometric random variable of parameter $p_{\mathcal{B}}$. This will be coupled with the random variable that counts the number of times we cross the edge $(x_{\mathcal{B}}+1, x_{\mathcal{B}})$ from the right to the left, that is, the number of times we fall anew in the trap.

$\theta_{\mathcal{B}} = \theta_{\mathcal{B}}(\omega)$ is equal to $E^{\bar{\omega}}[\Theta_x]$, where Θ_x is 1 plus the time that it takes for X_j to go from $x_{\mathcal{B}}-1$ to $x_{\mathcal{B}}$, and $\bar{\omega}$ is ω with all conductances to the left of $x_{\mathcal{B}}-C_n$ replaced by 0 (recall again the non-backtracking property of Proposition 2.2). Notice that $\theta_{\mathcal{B}}$ depends only on the environment to the left of $x_{\mathcal{B}}$ that is in \mathcal{B} .

$(E_j)_{j \geq 1}$ is a sequence of i.i.d. exponential random variables of parameter 1, independent of $G(p_{\mathcal{B}})$ and $\theta_{\mathcal{B}}$.

We have written $\tau_{\mathcal{B}}$ as (5.1) in order to make explicit where the different terms come from, but one realizes (for instance computing the characteristic function) that a geometric sum of independent exponential r.v.s is itself an exponential random variable. We can therefore rewrite $\tau_{\mathcal{B}}$ as

$$\tau_{\mathcal{B}} = \frac{\theta_{\mathcal{B}}}{p_{\mathcal{B}}} \mathbf{e}_{\mathcal{B}}, \quad (5.2)$$

where $\mathbf{e}_{\mathcal{B}} \sim \text{Exp}(1)$ is independent of $p_{\mathcal{B}}$ and $\theta_{\mathcal{B}}$.

Proposition 5.1. *Let $\mathcal{B} = \mathcal{B}_j$ be a good trapping n -triblock (recall the Definition 3.3 of $\bar{\Gamma}_n$). There exists a coupling between $(X_j)_{j \in \mathbb{N}}$ and $\tau_{\mathcal{B}}$ such that for any $\delta \in (0, 1)$*

$$\sup_{\omega \in \bar{\Gamma}_n} P^{\omega} \left(\left| \frac{T(\mathcal{B})}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}} \right| > \delta, A_n \right) \leq 2e^{-e^{q_n}} + c\delta^{-2}e^{50q_n}\rho_{\mathcal{B}}^{-1} \quad (5.3)$$

where A_n is the event that $(X_j)_{j \in \mathbb{N}}$ never backtracks more than C_n .

We proceed in three steps:

Step 1. We decompose $T(\mathcal{B})$ into several pieces, see (5.4): a) the time it takes to arrive at position $x_{\mathcal{B}}$, b) a geometric number $G = G(p_{\mathcal{B}})$ of i.i.d. times to go from $x_{\mathcal{B}}$ to $x_{\mathcal{B}} + 1$ ($G - 1$ corresponds to the geometric number of times we fall anew in the trap), c) the time it takes from $x_{\mathcal{B}} + 1$ to exit the triblock to the right.

Step 2. We control easily the terms a) and c), which bring a small contribution to $T(\mathcal{B})$.

Step 3. We control the main term b), by saying that the time it takes to exit the trap is another geometric r.v. with parameter roughly $1/\rho_{\mathcal{B}}$ (the number of trials one needs to cross the edge $(x_{\mathcal{B}}, x_{\mathcal{B}} + 1)$ from left to right), multiplied by the time it takes between two trials (which is roughly $1 + E_{x_{\mathcal{B}}-1}^{\omega}[T_{x_{\mathcal{B}}}] = \theta_{\mathcal{B}}$ by the law of large numbers).

5.1. Step 1. For simplicity we suppose that $\mathcal{B} = \{-C_n, \dots, 2C_n - 1\}$, that the random walk starts in 0 (we write $P^{\omega} := P_0^{\omega}$) and we simply call $x = x_{\mathcal{B}}$. We notice that under the event A_n the walk never visits points to the left of $-C_n$, so that we can replace each ω in (5.3) by the environment $\bar{\omega}$ where all conductances c_j with $j < -C_n$ are set equal to 0 and the other conductances stay as before. For each $\omega \in \bar{\Gamma}_n$ we will therefore try to bound

$$P^{\omega} \left(\left| \frac{T(\mathcal{B})}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}} \right| > \delta, A_n \right) \leq P^{\bar{\omega}} \left(\left| \frac{T(\mathcal{B})}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}} \right| > \delta \right).$$

Let us introduce some further notation. We call $G := 1 + \#\{\ell : X_{\ell-1} = x + 1, X_{\ell} = x\}$. We notice that G is a geometric random variable of parameter $p_{\mathcal{B}}$ appearing in (5.1). Let $T^{(1)} := T_{x+1} - T_x$ and $t_1 = T_{x+1}$. When $G \geq 2$, we iteratively define, for $2 \leq i \leq G$,

$$\begin{aligned} t_i &:= \inf\{\ell > t_{i-1} : X_{\ell-1} = x, X_{\ell} = x + 1\} \\ T^{(i)} &:= t_i - t_{i-1}. \end{aligned}$$

We interpret $T^{(i)}$ as the time it takes for the walk to escape from the trap for the i -th time. We can now decompose $T(\mathcal{B})$ in the following way:

$$T(\mathcal{B}) = T_x + \sum_{i=1}^G T^{(i)} + \bar{T}_{x+1, 2C_n}, \quad (5.4)$$

where $\bar{T}_{x+1,2C_n}$ is the time it takes for $(X_j)_{j \in \mathbb{N}}$ to go from $x+1$ to $2C_n$ conditioned on never returning to x . In light of (5.4), for any $\omega \in \bar{\Gamma}_n$ we can bound

$$P^{\bar{\omega}}\left(\left|\frac{T(\mathcal{B})}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}}\right| > \delta\right) \leq D_1 + D_2 + D_3, \quad (5.5)$$

where

$$D_1 = P^{\bar{\omega}}\left(\frac{T_x}{\rho_{\mathcal{B}}} > \frac{\delta}{3}\right), \quad D_2 = P^{\bar{\omega}}\left(\left|\frac{\sum_{i=1}^G T^{(i)}}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}}\right| > \frac{\delta}{3}\right), \quad D_3 = P^{\bar{\omega}}\left(\frac{\bar{T}_{x+1,2C_n}}{\rho_{\mathcal{B}}} > \frac{\delta}{3}\right).$$

5.2. Step 2. The first and last term are fairly easy and are treated in the same way. For D_1 , we use Markov's inequality to get that

$$D_1 \leq 3\delta^{-1}\rho_{\mathcal{B}}^{-1}E^{\bar{\omega}}[T_x].$$

Formula (A.8) and (A.4) gives an explicit expression for $E^{\bar{\omega}}[T_x]$. In particular, by Definition 3.3 of a good block (in particular since $c_y \in [e^{-4q_n}, e^{4q_n}]$ for all $y < x-1$ and $c_{x-1} > e^{4q_n}$), we get that $E^{\bar{\omega}}[T_x] \leq c'C_ne^{8q_n}$. Hence, for n large enough,

$$D_1 \leq c\delta^{-1}e^{10q_n}\rho_{\mathcal{B}}^{-1}.$$

For the term D_3 , we use the same idea:

$$D_3 \leq 3\delta^{-1}\rho_{\mathcal{B}}^{-1}E^{\bar{\omega}}[\bar{T}_{x+1,2C_n}] = 3\delta^{-1}\rho_{\mathcal{B}}^{-1}\frac{1}{p_{\mathcal{B}}}E_{x+1}^{\bar{\omega}}[T_{2C_n}\mathbf{1}_{\{T_{2C_n} < T_x\}}] \quad (5.6)$$

Then, we can use that Here we have used that $E_{x+1}^{\bar{\omega}}[T_{2C_n}\mathbf{1}_{\{T_{2C_n} < T_x\}}] \leq E_{x+1}^{\bar{\omega}}[T_{2C_n}]$, where thanks to the indicator function we have replaced $\bar{\omega}$ by $\tilde{\omega}$ (in which the conductances to the left of $x+1$ have been replaced by 0), and then dropped the indicator function. Then, formula (A.8) together with Definition 3.3 of a good block gives that $E_{x+1}^{\tilde{\omega}}[T_{2C_n}] \leq c'C_ne^{8q_n}$. We finally get that

$$D_3 \leq c\delta^{-1}\rho_{\mathcal{B}}^{-1}C_ne^{8q_n}/p_{\mathcal{B}} \leq c\delta^{-1}\rho_{\mathcal{B}}^{-1}e^{20q_n}. \quad (5.7)$$

For the last inequality, we used the formula (A.6) for $p_{\mathcal{B}}$, together with the definition of a good block to get that $p_{\mathcal{B}} \geq ce^{-8q_n}$ (and took n large enough so that $C_n \leq e^{4q_n}$).

5.3. Step 3. It remains to deal with the term D_2 in (5.5), which is the most technical part of this section. For $i = 1, \dots, G$, we decompose $T^{(i)}$ as follows

$$T^{(i)} = \Upsilon_i + \left(\sum_{j=1}^{\mathcal{G}_i-1} \Theta_x(j, i)\right) + 1, \quad (5.8)$$

where $\Upsilon_1 = 0$ and $\Upsilon_i := \inf\{\ell > t_{i-1} : X_\ell = x+1\} - t_{i-1}$ for $i \geq 2$; \mathcal{G}_i is a geometric random variable with parameter $1/(1+\rho_x)$ and $\{\Theta_x(j, i)\}_{j, i \in \mathbb{N}}$ are i.i.d. copies of Θ_x . In fact, we can describe $T^{(i)}$, with $i = 2, \dots, G$, as follows (the discussion is similar for $i = 1$): at time t_{i-1} the random walk is in $x+1$ and in a time Υ_i it reaches x for the first time after t_{i-1} . Then it jumps to the right with probability $1/(1+\rho_x)$ (in which case we either go to iteration $i+1$ or never return to x) or to the left with probability $\rho_x/(1+\rho_x)$. In this second case, $\Theta_x(1, i)$ represents this one step plus the time it takes for (X_ℓ) to go from $x-1$ back to x . Then (X_ℓ) “tries” again to jump to $x+1$ with probability $1/(1+\rho_x)$ (in which case we either go to iteration $i+1$ or never return to x) or goes back to $x-1$ with probability $\rho_x/(1+\rho_x)$, etc... This continues until the random walk manages to reach $x+1$. Since each attempt of jumping from x to $x+1$ is independent from the others, the number of attempts is geometric. The $\Theta_x(j, i)$ are i.i.d. by the Markov property.

Thanks to the representation (5.8), and using i.i.d. exponential r.v. $(E_i)_{i \geq 1}$ of parameter 1 coupled with \mathcal{G}_i (by $\mathcal{G}_i = \lceil E_1 \log \frac{\rho_B}{1+\rho_B} \rceil$), we can bound

$$\begin{aligned} D_2 &\leq P^{\bar{\omega}} \left(\sum_{i=1}^G \left| \frac{\Upsilon_i + \sum_{j=1}^{\mathcal{G}_i-1} \Theta_x(j, i) + 1}{\rho_B} - \theta_B E_i \right| > \frac{\delta}{3} \right) \\ &\leq P^{\bar{\omega}} \left(\sum_{i=1}^G \left| \frac{\Upsilon_i + \sum_{j=1}^{\mathcal{G}_i-1} \Theta_x(j, i) + 1}{\rho_B} - \theta_B E_i \right| > \frac{\delta}{3}, G \leq e^{10q_n} \right) + P^{\bar{\omega}}(G > e^{10q_n}). \end{aligned} \quad (5.9)$$

For the second summand of (5.9), recall that $1 + G$ is a geometric variable with parameter p_B : we have that $P^{\bar{\omega}}(G > e^{10q_n}) \leq \exp(-p_B e^{10q_n})$. Now, we have already seen that $p_B \geq c e^{8q_n}$, so that for n large enough we get

$$P^{\bar{\omega}}(G > e^{10q_n}) \leq \exp(-e^{q_n}). \quad (5.10)$$

We control now the first term in (5.9) via the triangular inequality and a union bound (recall that $\Upsilon_1 = 0$ by definition): we get that it is bounded by

$$e^{10q_n} P^{\bar{\omega}} \left(\frac{\Upsilon_2 + 1}{\rho_B} > \frac{\delta}{6} e^{-10q_n} \right) + e^{10q_n} P^{\bar{\omega}} \left(\left| \frac{1}{\rho_B} \sum_{j=1}^{\mathcal{G}_1-1} \Theta_x(j, 1) - \theta_B E_1 \right| > \frac{\delta}{6} e^{-10q_n} \right). \quad (5.11)$$

We deal with the two summands of (5.11) in the following claims.

Claim 5.2. *There is a constant $c > 0$ such that, for all $\omega \in \bar{\Gamma}_n$, for all $v > 0$, we have*

$$P^{\bar{\omega}} \left(\frac{\Upsilon_2 + 1}{\rho_B} > v \right) < c v^{-1} \rho_B^{-1} e^{10q_n}. \quad (5.12)$$

Claim 5.3. *There is a constant $c > 0$ such that, for all $\omega \in \bar{\Gamma}_n$, for all $v > 0$, we have*

$$P^{\bar{\omega}} \left(\left| \frac{1}{\rho_B} \sum_{j=1}^{\mathcal{G}_1-1} \Theta_x(j, 1) - \theta_B E_1 \right| > v \right) \leq e^{-e^{q_n}} + c e^{20q_n} v^{-2} \rho_B^{-1}. \quad (5.13)$$

We prove these two claims just below. Thanks to Claims 5.2 and 5.3, we obtain, together with estimates (5.9) and (5.10)-(5.11), that (using also that $\delta^{-1} \leq \delta^{-2}$)

$$D_2 \leq 2e^{-e^{q_n}} + c' \delta^{-2} e^{50q_n} \rho_B^{-1},$$

for some constant $c' > 0$. This concludes the proof of Proposition 5.1.

Proof of Claim 5.2 We will use Markov inequality, and therefore proceed to the calculation of the expectation of Υ_2 . By definition, $E^{\bar{\omega}}[\Upsilon_2] = E_{x+1}^{\bar{\omega}}[T_x | T_x < T_{2C_n}] = E_{x+1}^{\bar{\omega}}[T_x \mathbf{1}_{\{T_x < T_{2C_n}\}}] / P_{x+1}^{\bar{\omega}}(T_x < T_{2C_n})$. We use formulas (A.11) and (A.1), which yield

$$E^{\bar{\omega}}[\Upsilon_2] = \frac{1}{c_x^\lambda} \sum_{y=x+1}^{2C_n-1} \pi(y) P_y(T_{x+1} < T_{2C_n}) P_y(T_x < T_{2C_n}) \leq \sum_{y=x+1}^{2C_n-1} \frac{c_y^\lambda}{c_x^\lambda} P_y(T_x < T_{2C_n})$$

For all $y = x+1, \dots, 2C_n-1$, we can use again (A.1) and bound

$$\frac{c_y^\lambda}{c_x^\lambda} P_y(T_x < T_{2C_n}) = \frac{c_y}{c_x} e^{\lambda(y-x)} \frac{\sum_{j=y}^{2C_n-1} c_j^{-1} e^{-\lambda j}}{\sum_{j=x}^{2C_n-1} c_j^{-1} e^{-\lambda j}} \leq \frac{1}{c_x} e^{8q_n} \frac{\sum_{j=0}^{2C_n-1-y} e^{-\lambda j}}{c_x^{-1}} = c' e^{8q_n},$$

for some $c' > 0$, where for the first inequality we have used the fact that in good blocks $e^{-4q_n} < c_j < e^{4q_n}$ for all $x < j < 2C_n$ and in the denominator we simply kept the term $j = x$. We therefore obtain

$$E^\omega[\Upsilon_2 + 1] \leq 2c'C_n e^{8q_n} \leq e^{10q_n}$$

for n large enough. This gives the claim by Markov inequality. \square

Proof of Claim 5.3 We abbreviate $\Theta_x^{(j)} := \Theta_x(j, 1)$, and recall that $\theta_B = E^\omega[\Theta_x^{(1)}]$. By the triangular inequality, we bound the left hand side of (5.13) by

$$P^\omega\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} \left(\Theta_x^{(j)} \rho_B^{-1} - \theta_B \rho_B^{-1}\right)\right| > \frac{v}{2}\right) + P^\omega\left(\theta_B \left|E_1 - (\mathcal{G}_1 - 1) \rho_B^{-1}\right| > \frac{v}{2}\right). \quad (5.14)$$

For the second term in (5.14), recall that we coupled E_1 with \mathcal{G}_1 by $\mathcal{G}_1 = \lceil E_1 \log \frac{\rho_B}{1+\rho_B} \rceil$: this gives that $|E_1 - (\mathcal{G}_1 - 1)/\rho_B| \leq (2 + E_1)/\rho_B$, so that the second term in (5.14) is bounded by $P^\omega(E_1 > cv\rho_B\theta_B^{-1})$ for some constant $c > 0$. Thanks to Lemma 5.4 below, this is smaller than $\exp(-c''v\rho_B e^{-8q_n}) \leq c'''v^{-1}\rho_B^{-1}e^{8q_n}$.

We are left with the first term in (5.14). We study the mean and variance of the i.i.d. random variables $\Theta_x^{(j)}$, which by definition are distributed as the random variable Θ_x , in the following lemma.

Lemma 5.4. *We have, for all $\omega \in \bar{\Gamma}_n$*

$$\theta_B = E^\omega[\Theta_x] \leq c'e^{8q_n}, \quad (5.15)$$

$$\text{Var}^\omega(\Theta_x) \leq c'C_n^2 e^{16q_n}. \quad (5.16)$$

Let us first conclude the proof of Claim 5.3 before we turn to the proof of Lemma 5.4. We have that the first term in (5.14) is

$$\begin{aligned} P^\omega\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} \left(\Theta_x^{(j)} - \theta_B\right)\right| > \frac{v}{2}\rho_B\right) &\leq P^\omega(\mathcal{G}_1 > e^{2q_n}\rho_B) + e^{2q_n}\rho_B \frac{\text{Var}^\omega(\Theta_x)}{v^2\rho_B^2/4} \\ &\leq e^{-e^{q_n}} + c''e^{20q_n}v^{-2}\rho_B^{-1}. \end{aligned} \quad (5.17)$$

In the first line we have used Chebychev inequality and the independence of the $\Theta_x^{(j)}$, while in the second line we used the fact that \mathcal{G}_1 is a geometric of parameter $1/(1+\rho_B)$ for the first term and the bound (5.16) for the second term. \square

Proof of Lemma 5.4 Recall that Θ_x is 1 plus the time it takes for the random walk, starting in $x-1$, to reach x . By formula (A.10) and by the definition of a good triblock we easily obtain bound (5.15).

For studying the second moment of Θ_x , we call N_j , with $j \in \{-C_n, \dots, x-1\}$, the random variable counting the number of visits to point j before touching x , so that $\Theta_x = 1 + \sum_{j=-C_n}^{x-1} N_j$ and

$$(\Theta_x)^2 \leq 2C_n \sum_{j=-C_n}^{x-1} N_j^2. \quad (5.18)$$

Let us describe the distribution of the random variables N_j : (i) N_j is equal to 0 with probability $P_{x-1}^\omega(T_x < T_j)$ (when the random walk reaches x before ever visiting j); (ii) N_j is equal to a random variable $Y(j)$ with the remaining probability, where $Y(j)$ is a geometric random variable with parameter $q(j) = P_j^\omega(T_x < T_j)$ (once j is reached for

the first time, the random walk returns to j before touching x with probability $1 - q(j)$. Hence, we get that $E^\omega[N_j^2] \leq E^\omega[Y(j)^2] \leq 1/q(j)^2$.

We notice that, on good blocks, because of formula (A.5), $q(j) \geq ce^{-8q_n}$ for any $j \leq x-1$. Therefore the second moment of N_j is bounded by $1/q(j)^2 \leq c'e^{16q_n}$, so that taking the expectation in (5.18) we get $\text{Var}^\omega(\Theta_x) \leq c''C_n^2e^{16q_n}$ as desired. \square

6. REDUCTION TO A FINITE NUMBER OF TRAPS

First of all, by a union bound and then Proposition 5.1 with $\delta = \frac{d_n e^{-q_n}}{|\mathcal{J}_n| \rho_{\mathcal{B}_j}} \leq |\mathcal{J}_n|^{-1} < 1$, we get that for a.e. ω , for n large enough (so that all trapping blocks are good, by Lemma 3.6)

$$\begin{aligned} P^\omega \left(\left| \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j) - \sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \right| \geq d_n e^{-q_n} \right) &\leq \sum_{j \in \mathcal{J}_n} P^\omega \left(|T(\mathcal{B}_j) - \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j}| \geq \frac{d_n e^{-q_n}}{|\mathcal{J}_n|} \right) \\ &\leq 2|\mathcal{J}_n| e^{-e^{-q_n}} + c|\mathcal{J}_n| e^{50q_n} \left(\frac{d_n e^{-q_n}}{|\mathcal{J}_n|} \right)^{-2} \max_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j}. \end{aligned}$$

Then, by Lemmas 3.4 and 3.5, we get that $|\mathcal{J}_n| \leq e^{5q_n}$ and $\max_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \leq d_n e^{q_n}$ for n large enough, so that

$$P^\omega \left(\left| \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j) - \sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \right| \geq d_n e^{-q_n} \right) \leq c' d_n^{-1} e^{70q_n}. \quad (6.1)$$

Together with Proposition 4.1, we conclude that with $\mathbb{P} \otimes P^\omega$ -probability going to 1,

$$\frac{1}{d_n} \left| T_n - \sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \right| \leq \varepsilon_n^{(1-\alpha)/2} \xrightarrow{n \rightarrow \infty} 0, \quad (6.2)$$

where we recall that $\varepsilon_n = q_n^{-\delta}$ for some (small) fixed $\delta > 0$.

Now, we show that the main contribution to $\sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j}$ comes from the blocks with the largest $\rho_{\mathcal{B}}$'s (of order d_n).

Proposition 6.1. *For any $\eta, \eta' > 0$, there exists $\varepsilon = \varepsilon(\eta, \eta') > 0$ such that for n sufficiently large*

$$\mathbb{P} \otimes P^\omega \left(\frac{1}{d_n} \sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \mathbf{1}_{\{\rho_{\mathcal{B}_j} \leq \varepsilon d_n\}} > \eta \right) \leq \eta'. \quad (6.3)$$

This proposition, together with (6.2), shows that for any fixed $\eta, \eta' > 0$, we have for n large enough (recall $k_n := n/C_n$)

$$\frac{1}{d_n} \left| T_n - \sum_{i=1}^{k_n} \rho_{\mathcal{B}_i} \tau_{\mathcal{B}_i} \mathbf{1}_{\{\rho_{\mathcal{B}_i} > \varepsilon d_n\}} \right| \leq 2\eta \quad (6.4)$$

with $\mathbb{P} \otimes P^\omega$ -probability at least $1 - \eta'$. We dropped the condition $j \in \mathcal{J}_n$ in the sum since the condition $\rho_{\mathcal{B}_i} > \varepsilon d_n$ ensures that $i \in \mathcal{J}_n$.

The difficulty here is that we are not able to obtain directly the tail of $\rho_{\mathcal{B}} \tau_{\mathcal{B}}$ (they both have heavy-tail and are not independent). Proposition 6.1 is proved in two steps:

Step 1: We decompose the sum according to some ranges of values for $\rho_{\mathcal{B}}$, and dominate each term by a sum of a constant number N_ℓ of i.i.d. random variables.

Step 2: By a union bound, we reduce the proof to controlling each term separately. In particular, an important estimate is Lemma 7.1, which controls the tail of $\tau_{\mathcal{B}}$ conditionally on having $\rho_{\mathcal{B}}$ large (uniformly on the large value of $\rho_{\mathcal{B}}$).

6.1. Step 1. We divide the contribution according to ranges of value of $\rho_{\mathcal{B}_j}$:

$$\mathbb{P} \otimes P^\omega \left(\sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \mathbf{1}_{\{\rho_{\mathcal{B}_j} < \varepsilon d_n\}} > \eta d_n \right) \leq \mathbb{P} \otimes P^\omega \left(\sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell+1} \sum_{j \in \mathcal{J}_n^{(\ell)}} \tau_{\mathcal{B}_j} > \eta \right) \quad (6.5)$$

where we set $\mathcal{J}_n^{(\ell)} = \{j \in \mathcal{J}_n, 2^{-\ell} \leq \rho_{\mathcal{B}_j}/d_n \leq 2^{-\ell+1}\}$ (we also used that $q_n/\log 2 \leq 2q_n$).

Then, we control the number of terms in each sum, *i.e.* we show that $|\mathcal{J}_n^{(\ell)}| \leq N_\ell$, for some well chosen N_ℓ . This is essentially due to the following lemma.

Lemma 6.2. *For any $\delta' > 0$, there is a constant $C_{\delta'}$ such that, for all $n \in \mathbb{N}$ and all $u \in (0, 1)$, setting $i_u := \lceil u^{-\alpha(1+\delta')} \rceil$,*

$$\mathbb{P}(|\{1 \leq j \leq n : \rho_j \geq u d_n\}| \geq i_u) \leq (C_{\delta'} u)^{-\delta' i_u \alpha/4} \leq 2^{-i_u},$$

the last inequality being valid for u small enough. In the case $\alpha_0 = \alpha_\infty$ (recall Assumption “Traps”), we have that there is some $A > 0$ such that for any $u \in (d_n^{-1/2}, 1)$

$$\mathbb{P}(|\{1 \leq j \leq n : \rho_j \geq u d_n\}| \geq A u^{-\alpha}) \leq 2^{-u^{-\alpha}}.$$

Proof The probability we need to control is

$$\begin{aligned} & \mathbb{P}(\exists x_1, \dots, x_{i_u} \in \{1, \dots, n\} : \rho_{x_j} \geq u d_n \text{ for all } j \leq i) \\ & \leq \mathbb{P}(\exists x_1, \dots, x_{i_u/2} \leq n \text{ all odd or all even, s.t. } \rho_{x_j} \geq u d_n \text{ for all } j \leq i_u/2) \\ & \leq \binom{n}{i_u/2} \mathbb{P}(\rho_0 \geq u d_n)^{i_u/2}. \end{aligned}$$

The second inequality comes from a union bound, and from the fact that the ρ_{x_j} are independent. Then we can use that $\mathbb{P}(\rho_0 > t)$ is regularly varying with exponent α , so that by Potter’s bound (cf. [BGT89]) we have a constant C'_δ such that for any $u \in (0, 1)$, $\mathbb{P}(\rho_0 \geq u d_n) \leq C'_\delta u^{-\alpha(1+\delta'/2)} \mathbb{P}(\rho_0 \geq d_n)$. Then, using that d_n has been chosen such that $\mathbb{P}(\rho_0 \geq d_n) \sim 1/n$, together with the fact that $u^{-\alpha(1+\delta'/2)} \leq i_u u^{\delta'\alpha/2}$, we obtain

$$\mathbb{P}(|\{1 \leq j \leq n : \rho_j \geq u d_n\}| \geq i_u) \leq \frac{n^{i_u/2}}{(i_u/2)!} \left(\frac{C'_\delta i_u u^{\delta'\alpha/2}}{n} \right)^{i_u/2} \leq (C''_\delta u)^{-\delta' i_u \alpha/4}. \quad (6.6)$$

The last inequality comes from the fact that there is a constant c such that $(i/2)! \geq (ci)^{i/2}$ for all $i \geq 1$.

In the case where $\alpha_0 = \alpha_\infty$, we have that $g(u) := \psi(e^u)$ is regularly varying, recall Assumption “Traps”. Then we use the same union bound, setting $i_u = \lceil A u^{-\alpha} \rceil$: using that $\mathbb{P}(\rho_0 \geq u d_n) \leq c u^{-\alpha} \psi(u d_n) d_n^{-\alpha}$, and that $\psi(d_n) d_n^{-\alpha} \leq c/n$, we get that

$$\mathbb{P}(|\{1 \leq j \leq n : \rho_j \geq u d_n\}| \geq i_u) \leq \frac{n^{i_u/2}}{(i_u/2)!} \left(\frac{i_u/A \psi(u d_n)}{n \psi(d_n)} \right)^{i_u/2} \leq (c' A^{-1})^{i_u/2}.$$

For the last inequality, we use that $\log d_n \geq \log(u d_n) \geq \frac{1}{2} \log d_n$ for any $u \in (d_n^{-1/2}, 1)$, so that $\psi(u d_n) \leq c \psi(d_n)$ since $t \mapsto \psi(e^t)$ is regularly varying. Having fixed A sufficiently large in the above display gives the conclusion of the Lemma. \square

Now, let $N_\ell := 2^{\alpha\ell(1+\delta')}$ if $\alpha_0 \neq \alpha_\infty$ and $N_\ell := A 2^{\alpha\ell}$ if $\alpha_0 = \alpha_\infty$ (in any case $N_\ell \geq 2^{\alpha\ell}$): thanks to Lemma 6.2, we have that for ℓ large

$$\mathbb{P}(|\mathcal{J}_n^{(\ell)}| \geq N_\ell) \leq \mathbb{P}(|\{j : \rho_{\mathcal{B}_j} \geq 2^{-\ell} d_n\}| \geq N_\ell) \leq 2^{-N_\ell}.$$

Hence, we get that

$$\mathbb{P} \otimes P^\omega \left(\sum_{j \in \mathcal{J}_n^{(\ell)}} \tau_{\mathcal{B}_j} > \mathcal{H}_\ell \right) \leq 2^{-2^{\alpha\ell}} \quad \text{with } \mathcal{H}_\ell := \sum_{i=1}^{N_\ell} \tilde{\tau}_{\mathcal{B}_i}, \quad (6.7)$$

where $\tilde{\tau}_{\mathcal{B}_i} := \tilde{\tau}_{\mathcal{B}_i}^{(\ell)}$ are i.i.d. random variables, distributed as $\tau_{\mathcal{B}_j}$ conditioned on $j \in \mathcal{J}_n^{(\ell)}$.

Going back to (6.5), we therefore get that

$$\mathbb{P} \otimes P^\omega \left(\sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell+1} \sum_{j \in \mathcal{J}_n^{(\ell)}} \tau_{\mathcal{B}_j} > \eta \right) \leq \mathbb{P} \otimes P^\omega \left(\sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell+1} \mathcal{H}_\ell > \eta \right) + \sum_{\ell \geq \log_2(1/\varepsilon)} 2^{-2^{\alpha\ell}}, \quad (6.8)$$

the last sum being bounded by a constant time $2^{-1/\varepsilon^\alpha}$, which goes to 0 as $\varepsilon \rightarrow 0$.

6.2. Step 2. Using a union bound and Markov's inequality, we have

$$\begin{aligned} \mathbb{P} \otimes P^\omega \left(\sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell+1} \mathcal{H}_\ell > \eta \right) &\leq \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} \mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > 2^\ell) \\ &\quad + \frac{2}{\eta} \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell} \mathbb{E} \otimes E^\omega [\mathcal{H}_\ell \mathbf{1}_{\{\mathcal{H}_\ell \leq 2^\ell\}}]. \end{aligned} \quad (6.9)$$

We treat the two terms separately.

First term in (6.9). We need to bound the probability $\mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > 2^\ell)$, with $\mathcal{H}_\ell = \sum_{i=1}^{N_\ell} \tilde{\tau}_{\mathcal{B}_i}$. We can use that $\mathbb{P}(j \in \mathcal{J}_n^{(\ell)})$ is bounded below by a constant times $\mathbb{P}(\rho_{\mathcal{B}_j} > 2^{-\ell} d_n)$, so that we get that $\mathbb{P} \otimes P^\omega (\tilde{\tau}_{\mathcal{B}} > t) \leq c \mathbb{P} \otimes P^\omega (\tau_{\mathcal{B}} > t \mid \rho_{\mathcal{B}} > 2^{-\ell} d_n)$. We now need the following lemma (that we prove in Appendix B.3), to control the distribution on $\tau_{\mathcal{B}}$ conditionally on having $\rho_{\mathcal{B}}$ large.

Lemma 6.3. Define $\xi_{\mathcal{B}} := \frac{\theta_{\mathcal{B}}}{p_{\mathcal{B}}}$, so that $\tau_{\mathcal{B}} = \xi_{\mathcal{B}} \mathbf{e}_{\mathcal{B}}$. For any $\tilde{\delta}, \bar{\delta} > 0$ fixed small enough, there is some n_0 such that for any $n \geq n_0$ and uniformly for $1 \leq t \leq n^{2\bar{\delta}}$, $A_n \geq n^{\frac{1}{\alpha} - \bar{\delta}}$,

$$\mathbb{P}(\xi_{\mathcal{B}} > t \mid \rho_{\mathcal{B}} \geq A_n) \leq ct^{-2\alpha+\tilde{\delta}} + cL_0(t)t^{-\alpha_0}f_\infty(A_n) + cL_\infty(t)t^{-\alpha_\infty}f_0(A_n), \quad (6.10)$$

with $f_\infty(A_n) := \mathbb{P}(c_{-1} > A_n)/\mathbb{P}(\rho_0 > A_n)$ and $f_0(A_n) := \mathbb{P}(1/c_0 > A_n)/\mathbb{P}(\rho_0 > A_n)$. We also have that

$$\mathbb{P} \otimes P^\omega (\tau_{\mathcal{B}} > t \mid \rho_{\mathcal{B}} \geq A_n) \leq ct^{-2\alpha+\tilde{\delta}} + cL_0(t)t^{-\alpha_0}f_\infty(A_n) + cL_\infty(t)t^{-\alpha_\infty}f_0(A_n), \quad (6.11)$$

Let us stress that, in view of Proposition 1.1, if $\mathbb{E}[1/c_0^\alpha] < +\infty$ (resp. $\mathbb{E}[c_{-1}^\alpha] < +\infty$), then $f_\infty(A_n)$ (resp. $f_0(A_n)$) is bounded by a constant, whereas if $\mathbb{E}[1/c_0^\alpha] = +\infty$ (resp. $\mathbb{E}[c_{-1}^\alpha] = +\infty$), then $f_\infty(A_n) \rightarrow 0$ (resp. $f_0(A_n) \rightarrow 0$). Therefore, as a corollary, we get that for any fixed $t > 1$,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{P}(\xi_{\mathcal{B}} > t \mid \rho_{\mathcal{B}} \geq A_n) \\ \leq ct^{-2\alpha+\tilde{\delta}} + cL_0(t)t^{-\alpha_0} \mathbf{1}_{\{\mathbb{E}[1/c_0^\alpha] < +\infty\}} + cL_\infty(t)t^{-\alpha_\infty} \mathbf{1}_{\{\mathbb{E}[c_{-1}^\alpha] < +\infty\}}. \end{aligned} \quad (6.12)$$

With Lemma 6.3 at hand, we decompose the probability $\mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > z)$ for some $z > 1$, according to whether there is some $1 \leq i \leq N_\ell$ with $\tilde{\tau}_{\mathcal{B}_i} > z$ (we will choose $z = 2^\ell$,

but the general bound will be useful). With a union bound and Markov's inequality, we get

$$\mathbb{P} \otimes P^\omega \left(\sum_{i=1}^{N_\ell} \tilde{\tau}_{\mathcal{B}_i} > z \right) \leq N_\ell \mathbb{P} \otimes P^\omega (\tilde{\tau}_{\mathcal{B}} > z) + z^{-1} N_\ell \mathbb{E} \otimes E^\omega [\tilde{\tau}_{\mathcal{B}} \mathbf{1}_{\{\tilde{\tau}_{\mathcal{B}} \leq z\}}]. \quad (6.13)$$

For the second term, we write

$$\begin{aligned} \mathbb{E} \otimes E^\omega [\tilde{\tau}_{\mathcal{B}} \mathbf{1}_{\{\tilde{\tau}_{\mathcal{B}} \leq z\}}] &\leq 1 + \int_1^z \mathbb{P} \otimes P^\omega (\tilde{\tau}_{\mathcal{B}} > t) dt \\ &\leq cz^{1-(\alpha+\tilde{\delta})} + cL_0(z)z^{(1-\alpha_0)}f_\infty(2^{-\ell}d_n) + cL_\infty(z)z^{1-\alpha_\infty}f_0(2^{-\ell}d_n) \end{aligned}$$

where for the second inequality we used Lemma 6.3 and integrated over t using Karamata's Theorem [BGT89, Prop. 1.5.8]. We have exploited the fact that $\alpha_0, \alpha_\infty < 1$ and, for the integral $\int_1^z t^{-2\alpha+\tilde{\delta}}$, we have used that $-2\alpha + \tilde{\delta} \leq -(\alpha + \tilde{\delta})$ with $\alpha + \tilde{\delta} < 1$, for $\tilde{\delta}$ small enough. Using Lemma 6.3 for the first part of (6.13), we get that for any $z > 1$

$$\mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > z) \leq c' N_\ell \left(z^{-(\alpha+\tilde{\delta})} + L_0(z)z^{-\alpha_0}f_\infty(2^{-\ell}d_n) + L_\infty(z)z^{-\alpha_\infty}f_0(2^{-\ell}d_n) \right). \quad (6.14)$$

Now, we set $z = 2^\ell$, and we treat the first term in (6.8) separately the different cases.

- If $\alpha_0 > \alpha_\infty$, then $\alpha = \alpha_\infty$ (the case where $\alpha_\infty > \alpha_0$ is symmetric) and $\mathbb{P}(\rho_0 > t) \sim cL_\infty(t)t^{-\alpha_\infty}$ since $\mathbb{E}[1/c_0^\alpha] < +\infty$ (cf. Proposition 1.1). We therefore get that $f_\infty(A_n)$ is bounded by a constant (uniformly for $A_n \geq 1$), and that $f_0(A_n) \leq cA_n^{\alpha_\infty - \alpha_0 + \tilde{\delta}}$ (using Potter's bound). Recalling that $N_\ell = 2^{\alpha\ell(1+\delta')}$, and setting $z = 2^\ell$, the left-hand side in (6.14) is therefore bounded by a constant times

$$2^{\ell(\alpha(1+\delta')-(\alpha+\tilde{\delta}))} + 2^{\ell(\alpha(1+\delta')-\alpha_0+\tilde{\delta})} + 2^{\ell(\delta'+\tilde{\delta})}(2^{-\ell}d_n)^{\alpha_\infty-\alpha_0+\tilde{\delta}} \leq 2^{-\tilde{\delta}\alpha\ell/2} + 2^\ell d_n^{-(\alpha_0-\alpha_\infty)/2},$$

where the last bound holds provided that $\tilde{\delta}$ then δ' have been fixed sufficiently small. Going back to the first term in (6.9), we have that in the case $\alpha_0 > \alpha_\infty$,

$$\begin{aligned} \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} \mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > 2^\ell) &\leq c \sum_{\ell=\log_2(1/\varepsilon)}^{+\infty} 2^{-\tilde{\delta}\alpha\ell/2} + cd_n^{-(\alpha_0-\alpha_\infty)/2} \sum_{\ell=1}^{2q_n} 2^\ell \\ &\leq c'\varepsilon^{\tilde{\delta}\alpha/2} + c'4^{q_n}d_n^{-(\alpha_0-\alpha_\infty)/2}. \end{aligned} \quad (6.15)$$

This can be made arbitrarily small by taking ε small enough and n large.

- In the case $\alpha_0 = \alpha_\infty = \alpha$ we have $\mathbb{P}(\rho_0 > t) \sim \psi(t)t^{-\alpha}$ (see Proposition 1.1), so that $f_0(t) \sim L_0(t)/\psi(t)$ and $f_\infty(t) \sim L_\infty(t)/\psi(t)$. Now we can use that $L_0(e^t), L_\infty(e^t)$ are regularly varying by Assumption “Traps”, so that $\psi(e^t)$ is regularly varying, and therefore so are $f_0(e^t), f_\infty(e^t)$. Then, using that $d_n 2^{-\ell} \geq d_n^{1/2}$ for all $0 \leq \ell \leq q_n$, we get that there is a constant c such that $f_0(2^{-\ell}d_n) \leq cf_0(d_n)$ and $f_\infty(2^{-\ell}d_n) \leq cf_\infty(d_n)$. Recalling that $N_\ell = A2^{\alpha\ell}$, we therefore get from (6.14) (setting $z = 2^\ell$) that there is a constant c_A depending on A such that

$$\mathbb{P} \otimes P^\omega (\mathcal{H}_\ell > 2^\ell) \leq c_A (2^{-\tilde{\delta}\ell} + L_0(2^\ell)f_\infty(d_n) + L_\infty(2^\ell)f_0(d_n)).$$

Going back to the first term in (6.9), we get that

$$\begin{aligned} \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} \mathbb{P} \otimes P^\omega(\mathcal{H}_\ell > 2^\ell) &\leq c_A \sum_{\ell=\log_2(1/\varepsilon)}^{+\infty} 2^{-\tilde{\delta}\ell} \\ &+ c_A f_\infty(d_n) \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} L_0(2^\ell) + c_A f_0(d_n) \sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} L_\infty(2^\ell). \end{aligned} \quad (6.16)$$

The first term is bounded by a constant times $\varepsilon^{\tilde{\delta}}$, and goes to 0 as $\varepsilon \downarrow 0$. The second and third term can be treated in an analogous way, so we focus on the second one.

a) If $\mathbb{E}[1/c_0^\alpha] < +\infty$, then we already mentioned that $f_\infty(d_n)$ is bounded by a constant (thanks to Proposition 1.1). We also have that

$$\sum_{\ell \geq \log_2(1/\varepsilon)} L_0(2^\ell) \leq c \sum_{\ell \geq \log_2(1/\varepsilon)} (2^\ell)^\alpha \mathbb{P}(1/c_0 > 2^\ell) \leq c' \mathbb{E}[(1/c_0)^\alpha \mathbf{1}_{\{1/c_0 > 1/\varepsilon\}}], \quad (6.17)$$

and hence it goes to 0 as $\varepsilon \rightarrow 0$.

b) If $\mathbb{E}[1/c_0^\alpha] = +\infty$ but $\mathbb{E}[c_1^\alpha] < +\infty$, then we get from Proposition 1.1 that $f_\infty(d_n)$ is bounded by a constant times $L_\infty(d_n)/L_0(d_n) \leq c(\log n)^{\gamma_\infty - \gamma_0} \varphi_\infty(\log n)/\varphi_0(\log n)$ with $\gamma_0 > -1 > \gamma_\infty$ (recall Assumption “Traps”), so

$$f_\infty(d_n) \sum_{\ell=\log_2(1/\varepsilon)}^{q_n} L_0(2^\ell) \leq c \frac{\varphi_\infty(\log n)}{\varphi_0(\log n)} (\log n)^{\gamma_\infty - \gamma_0} \sum_{\ell=1}^{2q_n} \varphi_0(\ell) \ell^{\gamma_0}. \quad (6.18)$$

Using that $2q_n \leq \log n$, the last sum is bounded by a constant times $\varphi_0(\log n)(\log n)^{1+\gamma_0}$, so that the second term in (6.16) goes to 0 as $n \rightarrow \infty$, since $\gamma_\infty < -1$.

All together, we have obtained that, provided n is large enough and ε is small enough (how small depends on η, η'), (6.16) is bounded by $\eta'/4$.

Second term in (6.9). We just adapt the previous argument. Using the bound (6.14) and integrating it over z (recall $\alpha + \tilde{\delta}, \alpha_0, \alpha_\infty < 1$), we have that

$$\begin{aligned} \mathbb{E} \otimes E^\omega[\mathcal{H}_\ell \mathbf{1}_{\{\mathcal{H}_\ell \leq 2^\ell\}}] &\leq 1 + \int_1^{2^\ell} \mathbb{P} \otimes P^\omega(\mathcal{H}_\ell > z) dz \\ &\leq 1 + cN_\ell \left(2^{\ell(1-\alpha-\tilde{\delta})} + L_0(2^\ell) 2^{\ell(1-\alpha_0)} f_\infty(2^{-\ell} d_n) + L_\infty(2^\ell) 2^{\ell(1-\alpha_\infty)} f_0(2^{-\ell} d_n) \right). \end{aligned}$$

Hence we get that

$$\begin{aligned} 2^{-\ell} \mathbb{E} \otimes E^\omega[\mathcal{H}_\ell \mathbf{1}_{\{\mathcal{H}_\ell \leq 2^\ell\}}] &\leq c' N_\ell \left(2^{-\ell(\alpha+\tilde{\delta})} + N_\ell L_0(2^\ell) 2^{-\alpha_0 \ell} f_\infty(2^{-\ell} d_n) + L_\infty(2^\ell) 2^{-\alpha_\infty \ell} f_0(2^{-\ell} d_n) \right). \end{aligned}$$

This is the analogous of (6.14), and the proof concludes identically to what is done above.

All together, going back to (6.9), we have shown that there is a constant C such that

$$\mathbb{P} \otimes P^\omega \left(\sum_{\ell=\log_2(1/\varepsilon)}^{2q_n} 2^{-\ell} \mathcal{H}_\ell > \eta \right) \leq \frac{C}{\eta} (\varepsilon' + \varepsilon_n),$$

with $\varepsilon' \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. This concludes the proof of Proposition 6.1. \square

7. CONCLUSION OF THE PROOF OF THEOREM 2.1

7.1. Convergence of the depths, positions and block-crossing times. Recall we defined $\xi_{\mathcal{B}} := \frac{\theta_{\mathcal{B}}}{p_{\mathcal{B}}}$, so $\tau_{\mathcal{B}} = \xi_{\mathcal{B}} \mathbf{e}_{\mathcal{B}}$ with $\mathbf{e}_{\mathcal{B}}$ an exponential r.v. independent of the environment. In Lemma 7.1 below, we state that under Assumption “Simple Traps”, conditionally on having $\rho_{\mathcal{B}} \rightarrow +\infty$, $\xi_{\mathcal{B}}$ converges in distribution to a r.v. ζ , that we define as

$$\zeta := 2 \left(1 + B \bar{c}_0 V + (1 - B) \frac{1}{\bar{c}_{-1}} W \right), \quad (7.1)$$

where:

* B is a Bernoulli r.v. with parameter $q = 1$ if $\mathbb{E}[c_{-1}^{\alpha}] = +\infty$; $q = 0$ if $\mathbb{E}[1/c_0^{\alpha}] = +\infty$; and $q = \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[c_{-1}^{\alpha}] L_0(t)}{\mathbb{E}[c_{-1}^{\alpha}] L_0(t) + \mathbb{E}[1/c_0^{\alpha}] L_{\infty}(t)}$ if $\mathbb{E}[c_{-1}^{\alpha}] < +\infty$, $\mathbb{E}[1/c_0^{\alpha}] < +\infty$.

* V, W are defined by (see the analogous quantities for n fixed in (B.3)-(B.4))

$$V = V^{\lambda} := \sum_{j \geq 1} \frac{1}{c_j} e^{-\lambda(j+1)}, \quad W = W^{\lambda} := \sum_{j \geq 2} c_{-j} e^{-\lambda(j+1)}. \quad (7.2)$$

* If $\mathbb{E}[c_{-1}^{\alpha}] < +\infty$, \bar{c}_{-1} is a r.v. whose c.d.f. is given by $F_{\bar{c}_{-1}}(u) := \frac{1}{\mathbb{E}[c_{-1}^{\alpha}]} \mathbb{E}[c_{-1}^{\alpha} \mathbf{1}_{\{c_{-1} \leq u\}}]$. If $\mathbb{E}[1/c_0^{\alpha}] < +\infty$, then $1/\bar{c}_0$ is a r.v. with c.d.f. $F_{1/\bar{c}_0}(u) := \frac{1}{\mathbb{E}[1/c_0^{\alpha}]} \mathbb{E}[1/c_0^{\alpha} \mathbf{1}_{\{1/c_0 \leq u\}}]$.

Lemma 7.1. *Under Assumption “Simple Traps”, conditionally on having $\rho_{\mathcal{B}} > \varepsilon d_n$, $\xi_{\mathcal{B}} := \frac{\theta_{\mathcal{B}}}{p_{\mathcal{B}}}$ converges in distribution as $n \rightarrow +\infty$ to the random variable ζ defined above in (7.1). More precisely, we have*

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\xi_{\mathcal{B}} \in \cdot \mid \rho_{\mathcal{B}} > \varepsilon d_n) = \mathbb{P}(\zeta \in \cdot).$$

Additionally, we have that $\mathbb{E}[\zeta^{\alpha}] < +\infty$.

If $B = 1$ it means that the trap is a well (roughly, $\theta_{\mathcal{B}} \rightarrow 2$), while if $B = 0$ it means that the trap is a wall (roughly, $p_{\mathcal{B}} \rightarrow 1$). Hence, if $q = 1$ only wells can occur, while if $q = 0$ only walls can occur. The proof of Lemma 7.1 is provided in Appendix B.3.

Then, we rely on Lemma 7.1 to show the following (recall $k_n = \lfloor n/C_n \rfloor$).

Proposition 7.2. *For any $\varepsilon > 0$, we have the following convergence in distribution:*

$$\left\{ \left(\frac{i}{k_n}, \frac{\rho_{\mathcal{B}_i}}{d_n}, \xi_{\mathcal{B}_i} \right) : 1 \leq i \leq k_n, \rho_{\mathcal{B}_i} > \varepsilon d_n \right\} \implies \mathcal{P}_{\varepsilon},$$

where $\mathcal{P}_{\varepsilon}$ is a PPP on $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \alpha w^{-(1+\alpha)} \mathbf{1}_{\{w > \varepsilon\}} dw \mu(dz)$, and μ is the law of a random variable ζ defined in (7.1).

Proof The first remark we make is that the $(\rho_{\mathcal{B}_i}, \xi_{\mathcal{B}_i})_{1 \leq i \leq k_n}$ are not independent: however $(\rho_{\mathcal{B}_i}, \xi_{\mathcal{B}_i})$ and $(\rho_{\mathcal{B}_j}, \xi_{\mathcal{B}_j})$ are independent as soon as $|j - i| > 2$. But because of Lemma 3.4, the indices i such that $\rho_{\mathcal{B}_i} > \varepsilon d_n$ are a.s. well separated, and we can do as if the $\rho_{\mathcal{B}_i}, \xi_{\mathcal{B}_i}$ appearing in $\{(\frac{i}{k_n}, \frac{\rho_{\mathcal{B}_i}}{d_n}, \xi_{\mathcal{B}_i}) : 1 \leq i \leq k_n, \rho_{\mathcal{B}_i} > \varepsilon d_n\}$ were independent.

We therefore only have to consider the joint distribution of $(\rho_{\mathcal{B}}, \xi_{\mathcal{B}})$ on the event $\rho_{\mathcal{B}} > \varepsilon d_n$. Thanks to Lemma 7.1, conditionally on $\rho_{\mathcal{B}} > \varepsilon d_n$, $\xi_{\mathcal{B}}$ converges in distribution to ζ defined in (7.1), which does not depend on the value of $\rho_{\mathcal{B}}$: on the event $\rho_{\mathcal{B}} > \varepsilon d_n$, $\xi_{\mathcal{B}}$ is asymptotically independent of $\rho_{\mathcal{B}}$. It remains to see that, for any $t > 0$, as $n \rightarrow +\infty$

$$\begin{aligned} \mathbb{P}(\rho_{\mathcal{B}} \geq t d_n) &= \mathbb{P}(\exists x \in \{1, \dots, C_n\}, \rho_x > t d_n) \sim C_n \mathbb{P}(\rho_0 > t d_n) \\ &\sim C_n \psi(t d_n) (t d_n)^{-\alpha} \sim \frac{1}{k_n} t^{-\alpha}. \end{aligned}$$

Here, we used that there is at most one trap larger than $d_n e^{-q_n}$ in a block \mathcal{B} , and then that $\psi(d_n) d_n^{-\alpha} \sim 1/n$ by definition of d_n (recall also that $k_n = n/C_n$ is the number of blocks).

All together, for any $t > \varepsilon$, $v > 0$, and for any $i \geq 1$, $\mathbb{P}(\frac{\rho_{\mathcal{B}_i}}{d_n} > t, \xi_{\mathcal{B}_i} > v)$ is asymptotically equivalent to $\frac{1}{k_n} t^{-\alpha} \mathbb{P}(\zeta > v)$. By the Poisson approximation (as mentioned above, we can do as if the $(\rho_{\mathcal{B}_i}, \xi_{\mathcal{B}_i})_{i \geq 0}$ were independent), this is enough to conclude. \square

7.2. Convergence in distribution of $(T_n/d_n)_{n \geq 1}$. With Proposition 7.2 at hand, and recalling that $\tau_{\mathcal{B}} = \xi_{\mathcal{B}} \mathbf{e}_{\mathcal{B}}$ where $\mathbf{e}_{\mathcal{B}}$ is an exponential random variable independent of ω , we easily have the convergence in distribution, under $\mathbb{P} \otimes P^\omega$,

$$\frac{1}{d_n} \sum_{i=1}^{k_n} \rho_{\mathcal{B}_i} \tau_{\mathcal{B}_i} \mathbf{1}_{\{\rho_{\mathcal{B}_i} > \varepsilon d_n\}} \implies \sum_{(x,w,z,r) \in \tilde{\mathcal{P}}} w z r \mathbf{1}_{\{x \leq 1\}} \mathbf{1}_{\{w > \varepsilon\}}, \quad (7.3)$$

where $\tilde{\mathcal{P}}$ is a PPP on $(\mathbb{R}_+)^4$ with intensity $dx \alpha w^{-(1+\alpha)} dw \mu(dz) e^{-r} dr$.

All together, thanks to (6.2) and Proposition 6.1, and letting $\varepsilon \downarrow 0$ (the right-hand-side of (7.3) is monotone in ε), we obtain the following convergence in distribution

$$\frac{1}{d_n} T_n \implies \sum_{(x,w,z,r) \in \tilde{\mathcal{P}}} w z r \mathbf{1}_{\{x \leq 1\}} \stackrel{(d)}{=} \left(\frac{\pi \alpha}{\sin(\pi \alpha)} \mathbb{E}[\zeta^\alpha] \right)^{1/\alpha} \sum_{(x,v) \in \mathcal{P}} v \mathbf{1}_{\{x \leq 1\}}, \quad (7.4)$$

with \mathcal{P} a PPP on $(\mathbb{R}_+)^2$ of intensity $dx \frac{\alpha}{\Gamma(1-\alpha)} v^{-(1+\alpha)} dv$. Note that a crucial point, shown in Lemma 7.1, is that $\mathbb{E}[\zeta^\alpha] < +\infty$.

For the last identity in distribution in (7.4), we have used that $\{(x, wzr)\}_{(x,w,z,r) \in \tilde{\mathcal{P}}}$ is a PPP on $(\mathbb{R}_+)^2$ with intensity $dx \alpha c_{\alpha,\mu} u^{-(1+\alpha)} du$, where $c_{\alpha,\mu} = \int_0^\infty z^\alpha \mu(dz) \int_0^\infty r^\alpha e^{-r} dr = \mathbb{E}[\zeta^\alpha] \Gamma(1+\alpha)$. In turn, we get that $\{(x, (c_{\alpha,\mu} \Gamma(1-\alpha))^{-1/\alpha} wzr)\}_{(x,w,z,r) \in \tilde{\mathcal{P}}}$ is a PPP on $(\mathbb{R}_+)^2$ with intensity $dx \frac{\alpha}{\Gamma(1-\alpha)} v^{-(1+\alpha)} dv$ (i.e. with the same law as \mathcal{P}): this gives (7.4), using also that $\Gamma(1+\alpha) \Gamma(1-\alpha) = \pi \alpha / \sin(\pi \alpha)$.

Now, the term $\sum_{(x,v) \in \mathcal{P}} v \mathbf{1}_{\{x \leq 1\}}$ on the right-hand-side of (7.4) is a standard α -stable subordinator at time 1, with Laplace exponent t^α (by straightforward calculation, recall $t^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-ts}) s^{-(1+\alpha)} ds$, see [Ber99]).

As a matter of fact, our proof shows that for any fixed $u > 0$, we have the convergence in distribution, as $n \rightarrow +\infty$: under $\mathbb{P} \otimes P^\omega$

$$\mathcal{T}_n(u) := \frac{1}{d_n} T_{[un]} \implies \left(\frac{\pi \alpha}{\sin(\pi \alpha)} \mathbb{E}[\zeta^\alpha] \right)^{1/\alpha} \mathcal{S}_\alpha(u), \quad (7.5)$$

where $(\mathcal{S}_\alpha(u))_{u \geq 0}$ is the α -stable subordinator with Laplace exponent $\log \mathbf{E}[e^{-t \mathcal{S}_\alpha(u)}] = ut^\alpha$; note that $\mathcal{S}_\alpha(u)$ can be written, analogously to (7.4), as $\sum_{(x,w) \in \mathcal{P}} w \mathbf{1}_{\{x \leq u\}}$.

Remark 7.3. We believe that one could also obtain a quenched version of Theorem 2.1, in analogy with Corollary 1 in [ESTZ13]. More precisely, in view of (7.3), we expect that with high \mathbb{P} -probability, the law of T_n/d_n conditionally on ω is close (for instance in Wasserstein distance) to the law of $\sum_{p=1}^\infty \varpi_p \mathbf{e}_p$ conditionally on $(\varpi_p)_{p \geq 1}$, where $(\varpi_p)_{p \geq 1}$ is a Poisson Point Process of intensity $\alpha \mathbb{E}[\zeta^\alpha] \varpi^{-(1+\alpha)}$ coupled with the environment ω , and $(\mathbf{e}_p)_{p \geq 1}$ are i.i.d. exponential r.v. of parameter 1, independent of $(\varpi_p)_{p \geq 0}$. We think, though, that the convergence in distribution of Theorem 1.2 already describes the essence of the limiting behavior of the walk. We prefer not to develop the technical details for the quenched result, avoiding to make the paper heavier.

7.3. Process convergence, conclusion of the proof of Theorem 2.1. First of all, we reduce ourselves to the study of

$$\tilde{\mathcal{T}}_n(u) := \frac{1}{d_n} \sum_{j=1}^{\lfloor uk_n \rfloor} \rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \mathbf{1}_{\{\mathcal{B}_j \in \bar{\Gamma}_n\}}, \quad (7.6)$$

the *coarse* version of $\mathcal{T}_u^{(n)}$ (where we only keep the crossing times of good trapping blocks). Thanks to Propositions 4.1 and 5.1 we know that, with probability going to 1, the M_1 -distance between $(\mathcal{T}_u^{(n)})_{u \in [0,1]}$ and $(\tilde{\mathcal{T}}_u^{(n)})_{u \in [0,1]}$ goes to 0 as $n \rightarrow +\infty$. In view of Lemma 3.4, $\mathbb{P} \otimes P^\omega$ -a.s., for n large enough the trapping blocks are disjoint, and the non-zero terms $\rho_{\mathcal{B}_j} \tau_{\mathcal{B}_j} \mathbf{1}_{\{\mathcal{B}_j \in \bar{\Gamma}_n\}}$ are independent. Hence, we may reduce to studying

$$\hat{\mathcal{T}}_n(u) := \frac{1}{d_n} \sum_{i=1}^{\lfloor uk_n \rfloor} Y_i^{(n)}, \quad (7.7)$$

with $(Y_i^{(n)})_{i \geq 1}$ i.i.d. copies of $\rho_{\mathcal{B}} \tau_{\mathcal{B}} \mathbf{1}_{\{\mathcal{B} \in \bar{\Gamma}_n\}}$. It is then clear that the proof of (7.5) shows that all finite-dimensional marginals of $(\hat{\mathcal{T}}_n(u))_{u \in [0,1]}$ converge to those of the α -stable subordinator $(\mathcal{S}_\alpha(u))_{u \in [0,1]}$, since for $v > u$, $(\hat{\mathcal{T}}_n(v) - \hat{\mathcal{T}}_n(u))$ is independent of $\hat{\mathcal{T}}_n(u)$ for all n .

We can easily upgrade this to a process convergence: thanks to Prohorov's theorem, and since we already have the convergence of the finite-dimensional distributions, we only need to show that $\{\hat{\mathcal{T}}_n, n \in \mathbb{N}\}$ is tight, see e.g. [Bil68, Thm. 15.1]. Showing the tightness is relatively standard, since we are working here with increasing processes $\hat{\mathcal{T}}_n$ (and actually follows from the finite-dimensional convergence): the proof is identical to that in [BABČ08, Sec. 5] (or to that of [FK18, Sec. 11]). Tightness can also be seen as a consequence of Proposition 6.1, which says that with high probability (uniform in n) the main contribution to $\hat{\mathcal{T}}_n$ comes from jumps larger than ε , i.e. $\hat{\mathcal{T}}_n(u) - \frac{1}{d_n} \sum_{i=1}^{\lfloor uk_n \rfloor} Y_i^{(n)} \mathbf{1}_{\{Y_i^{(n)} > \varepsilon d_n\}}$ is uniformly smaller than η with probability at least $1 - \eta'$ (this correspond to the (15.11) tightness criterion of [Bil68, Thm. 15.4], the part (15.10) following from finite-dimensional convergence). This concludes the proof of Theorem 2.1.

8. CASE OF WELL-AND-WALL TRAPS

Throughout this section we will assume Assumption “Well-and-walls”.

8.1. Definition of k -distant traps and their properties. Recall that the maximal depth of a trap between 0 and n will be of order d_n , see the definition (2.2) of d_n . We introduce here the definition of k -traps, or well-and-wall traps, that is, traps formed by a very big conductance c_{x-1} followed by a very small conductance c_{x+k} , making $\rho_x^{(k)} := e^{-\lambda(k+1)} c_{x-1} / c_{x+k}$ large (we include the case $k = 0$). As we shall see, the only k -traps that will appear are such that neither c_{x-1} nor $1/c_{x+k}$ are big enough to give rise to a simple-trap (in the sense of Definition 3.1). Recall that we let $q_n := (\log n)^{1/4}$.

Definition 8.1. For $k \geq 0$, a site x is called a k -distant (well-and-wall) n -trap if

$$\rho_x^{(k)} > d_n e^{-q_n}, \quad c_{x-1} > e^{q_n^2} \quad \text{and} \quad \frac{1}{c_{x+k}} > e^{q_n^2}.$$

We call $\mathcal{W}_x^{(k)} := \{\rho_x^{(k)} > d_n e^{-q_n}\} \cap \{c_{x-1}, \frac{1}{c_{x+k}} > e^{q_n^2}\}$ the event that x is a k -distant trap.

We now extend Definitions 3.2-3.3 to the case of k -distant traps. We use the same notations as in the case of simple traps for simplicity.

Definition 8.2. For a given $n \in \mathbb{N}$, we define a trapping environment as

$$\Gamma_n := \bigcup_{x=0}^{C_n-1} \bigcup_{k=0}^{C_n-1} \mathcal{W}_x^{(k)},$$

that is, an environment with a k -distant n -trap in the first block. A n -triblock \mathcal{B}_j is called a trapping block if $\theta^{jC_n}\omega \in \Gamma_n$. Let $\mathcal{J}_n := \{j \in \{0, \dots, k_n\} : \theta^{jC_n}\omega \in \Gamma_n\}$ be the set of indices of the trapping blocks.

Definition 8.3. A trapping n -triblock with a k -distant n -trap at site x is said to be good if for all $y \neq x-1, x+k$ in the n -triblock, $c_y \leq e^{4q_n}$ and $1/c_y \leq e^{4q_n}$. We denote

$$\bar{\Gamma}_n := \bigcup_{x=0}^{C_n-1} \bigcup_{k=0}^{C_n-1} \left(\mathcal{W}_x^{(k)} \cap \bigcap_{\substack{-C_n \leq y < 2C_n \\ y \neq x-1, x+k}} \{c_y \in [e^{-4q_n}, e^{4q_n}]\} \right).$$

Note that in the definition $\mathcal{W}_x^{(k)}$, we have that $c_{x+k} < e^{-q_n} \leq e^{4q_n}$ and $c_{x-1} > e^{q_n} > e^{-4q_n}$.

8.2. Properties of k -distant traps. In this section we collect some important properties of k -distant traps and of the new trapping blocks. Besides updating to k -distant traps some of the properties of simple traps, we add one new feature (property 4).

1. k -distant traps are isolated, in fact distant by at least ne^{-5q_n} (Lemma 8.4);
2. k -distant traps are not too deep, they cannot be deeper than $d_n e^{q_n}$ (Lemma 8.5);
3. All trapping triblocks are good (Lemma 8.6);
4. k -distant traps cannot have k too large, in fact, $k < \frac{6}{\lambda} q_n$ (Lemma 8.7);
5. k -distant traps are the only annoying parts of the environment (Lemma 8.8).

Lemma 8.4. Let \mathcal{D}_n be the event

$$\mathcal{D}_n := \bigcup_{\substack{-n \leq x < y \leq n \\ |x-y| \leq ne^{-5q_n}}} \bigcup_{k \geq 0} \bigcup_{k' \geq 0} \mathcal{W}_x^{(k)} \cap \mathcal{W}_y^{(k')}.$$

Then, \mathbb{P} -a.s. \mathcal{D}_n occurs for only finitely many n .

The proof of Lemma 8.4 can be found in Section C.1 of the Appendix. As a consequence of Lemma 8.4, we have that $|\mathcal{J}_n| \leq e^{5q_n}$ also in the case of k -distant traps.

Lemma 8.5. \mathbb{P} -a.s., there is some n_0 such that for all $n \geq n_0$

$$M_n := \sup_{-n \leq x \leq n} \sup_{k \geq 0} \rho_x^{(k)} \leq d_n e^{q_n}.$$

The proof of Lemma 8.5 can be found in Section C.2 of the Appendix.

Lemma 8.6. Let \mathcal{G}_n be the event

$$\mathcal{G}_n := \bigcup_{-n \leq x \leq n} \bigcup_{k \geq 0} \{ \rho_x^{(k)} > d_n e^{-q_n} \} \cap \left(\bigcup_{\substack{y=x-2C_n \\ y \neq x-1, x+k}}^{x+2C_n} \{c_y \notin [e^{-4q_n}, e^{4q_n}]\} \right).$$

Then \mathbb{P} -a.s. \mathcal{G}_n occurs only for finitely many n .

The proof of Lemma 8.6 can be found in Section C.3 of the Appendix.

Lemma 8.7. *Let \mathcal{K}_n be the event that there exists a k -distant trap with $k \geq \frac{6}{\lambda}q_n$, i.e.*

$$\mathcal{K}_n := \bigcup_{-n \leq x \leq n} \bigcup_{k \geq \frac{6}{\lambda}q_n} \mathcal{W}_x^{(k)}.$$

Then \mathbb{P} -a.s., \mathcal{K}_n occurs for only finitely many n .

The proof of Lemma 8.7 can be found in Section C.4 of the Appendix.

The last lemma shows that, with high probability, there are no ways to have $\rho_x^{(k)}$ close to d_n without having both $c_{x-1}, \frac{1}{c_{x+k}} > e^{q_n^2}$: it tells that the traps of depth of order d_n can only be of well-and-wall type.

Lemma 8.8. *Let $\varepsilon_n = q_n^{-\delta}$ for some $\delta > 0$, and let*

$$\mathcal{H}_n := \bigcup_{-n \leq x \leq n} \bigcup_{k \geq 0} \{\rho_x^{(k)} > \varepsilon_n e^{-\lambda k/2} d_n\} \cap (\mathcal{W}_x^{(k)})^c.$$

Then, if δ is sufficiently small, we have that $\mathbb{P}(\mathcal{H}_n) \rightarrow 0$ as $n \rightarrow +\infty$.

The proof of Lemma 8.8 can be found in Section C.6 of the Appendix.

8.3. Reduction to large traps. Also under Assumption “Well-and-walls” the result of Proposition 4.1 is valid: with $\varepsilon_n = q_n^{-\delta}$ defined in Lemma 8.8, we have that

$$P^\omega\left(\frac{1}{d_n} \left|T_n - \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j)\right| > \varepsilon_n^{(1-\alpha)/2}\right) \leq \varepsilon_n^{(1-\alpha)/2}, \quad (8.1)$$

with \mathbb{P} -probability going to 1. The proof is identical to the one of Proposition 4.1, and relies on Lemma 8.8 in place of Lemma 3.7, in order to say that with high probability, all $\rho_z^{(k)}$ outside trapping blocks are smaller than $\varepsilon_n e^{-\lambda k/2} d_n$ (so that one gets (4.4)).

8.4. Crossing of large traps. As in the case of simple traps, Lemma 8.4 guarantees that to each trapping block \mathcal{B} can be associated a unique site $x_{\mathcal{B}}$ and a unique $k_{\mathcal{B}} \in \{0, \dots, \frac{6}{\lambda}q_n\}$ such that $x_{\mathcal{B}}$ is a $k_{\mathcal{B}}$ -distant n -trap (Lemma 8.7 ensures that $k_{\mathcal{B}} < \frac{6}{\lambda}q_n$). As before we denote by $\rho_{\mathcal{B}} := \rho_{x_{\mathcal{B}}}^{(k_{\mathcal{B}})}$ the depth of the trap associated with $x_{\mathcal{B}}$. We approximate again $T(\mathcal{B})/\rho_{\mathcal{B}}$ by $\tau_{\mathcal{B}}$, which in the case of well-and-wall traps becomes extremely simple:

$$\tau_{\mathcal{B}} = 2 \mathbf{e}_{\mathcal{B}}, \quad (8.2)$$

where $\mathbf{e}_{\mathcal{B}} \sim \text{Exp}(1)$. Analogously to the case of simple traps define

$$\begin{aligned} p_{\mathcal{B}} &:= P_{x_{\mathcal{B}}+k_{\mathcal{B}}+1}^\omega(T_{x_{\mathcal{B}}} > T_{x_{\mathcal{B}}+k_{\mathcal{B}}+C_n}) \text{ the probability of escaping to the right of } x_{\mathcal{B}} + k_{\mathcal{B}} + 1; \\ \theta_{\mathcal{B}} &:= E^\omega[\Theta_{x_{\mathcal{B}}+k_{\mathcal{B}}}], \text{ where } \Theta_{x_{\mathcal{B}}+k_{\mathcal{B}}} \text{ equal to 1 plus the time that it takes for } X_j \text{ to go from } x_{\mathcal{B}} + k_{\mathcal{B}} - 1 \text{ to } x_{\mathcal{B}} + k_{\mathcal{B}}. \end{aligned}$$

Since we have $c_{x_{\mathcal{B}}+k_{\mathcal{B}}}^{-1} > e^{q_n^2}$, we get that $p_{\mathcal{B}} \rightarrow 1$ as $n \rightarrow +\infty$. On the other hand, also $c_{x_{\mathcal{B}}-1} > e^{q_n^2}$, so that the main contribution to $\theta_{\mathcal{B}}$ comes from the time spent on the edge $(x_{\mathcal{B}} - 1, x_{\mathcal{B}})$, which is approximately 2 if $k_{\mathcal{B}} = 0$ and $2c_{x_{\mathcal{B}}-1}$ if $k_{\mathcal{B}} > 0$. In (5.1), this roughly corresponds to having $p_{\mathcal{B}} = 1$ and $\theta_{\mathcal{B}} = 2$, and this gives a heuristic reason why (8.2) holds.

In analogy with Proposition 5.1, we claim that there is a constant $c > 0$ such that for any $\delta \in (0, 1)$,

$$\sup_{\omega \in \bar{\Gamma}_n} P^\omega\left(\left|\frac{T(\mathcal{B})}{\rho_{\mathcal{B}}} - \tau_{\mathcal{B}}\right| > \delta, A_n\right) \leq c e^{-q_n^2/2} + c \delta^{-2} e^{-q_n^2/2} e^{2\lambda k_{\mathcal{B}}} \leq \delta^{-2} e^{-q_n^2/4}, \quad (8.3)$$

the last inequality holding \mathbb{P} -a.s., for n large enough, since $\lambda k_{\mathcal{B}} \leq 6q_n$ thanks to Lemma 8.7. We follow the three steps of the proof of Proposition 5.1. We use the same notations as in Section 5.1, re-adapted to the case of k -traps. We call $x = x_{\mathcal{B}}$ and $k = k_{\mathcal{B}}$.

Step 1. We let $T^{(1)} := T_{x+k+1} - T_{x+k}$ and $t_1 = T_{x+k+1}$. The random variable $G := 1 + \#\{\ell : X_{\ell-1} = x+k+1, X_{\ell} = x+k\}$ is a geometric r.v. of parameter $p_{\mathcal{B}}$, representing the number of attempts to run away from $x+k+1$ and never come back into $x+k$. When $G \geq 2$, we iteratively define, for $2 \leq i \leq G$,

$$t_i := \inf\{\ell > t_{i-1} : X_{\ell-1} = x+k, X_{\ell} = x+k+1\} \quad \text{and} \quad T^{(i)} := t_i - t_{i-1}.$$

As in (5.4) we rewrite $T(\mathcal{B})$ as $T(\mathcal{B}) = T_{x+k} + \sum_{i=1}^G T^{(i)} + \bar{T}(x+k+1, 2C_n)$, where $\bar{T}(x+k+1, 2C_n)$ is the time it takes for $(X_{\ell})_{\ell \in \mathbb{N}}$ to go from $x+k+1$ to $2C_n$ conditioned on never returning to $x+k$. By the triangular inequality

$$P^{\bar{\omega}}\left(\left|\frac{T(\mathcal{B})}{\rho_{\mathcal{B}}}\right| > \delta\right) \leq D_1 + D_2 + D_3, \quad (8.4)$$

with D_1, D_2 and D_3 that are the obvious homologous of the quantities appearing in (5.5).

Step 2. For D_1 we use Markov inequality. We notice that in the case $k = 0$ we have $E^{\bar{\omega}}[T_{x+k}] \leq c' C_n^2 e^{8q_n}$, while for $k > 0$ we get $E^{\bar{\omega}}[T_{x+k}] \leq c' c_{x-1} C_n e^{8q_n}$ (cf. (A.8), together with Definition 8.3). Hence

$$D_1 \leq \begin{cases} c \delta^{-1} e^{11q_n} \rho_x^{-1} & \text{if } k = 0; \\ c \delta^{-1} e^{11q_n} (c_{x-1})^{-1} e^{\lambda k} \leq c \delta^{-1} e^{-q_n^2/2} e^{\lambda k} & \text{if } k > 0. \end{cases}$$

For the term D_3 , we use the same idea as (5.7). For well-and-wall traps we even have, as can be seen from (8.5) below, $p_{\mathcal{B}} > c > 0$, so

$$D_3 \leq c' \delta^{-1} \rho_{\mathcal{B}}^{-1} E^{\bar{\omega}}[\bar{T}_{x+k+1, 2C_n}] \leq c'' \delta^{-1} \rho_{\mathcal{B}}^{-1} C_n e^{8q_n}.$$

Step 3. It remains to deal with the term D_2 . Following the reasoning after (??) and remembering that A_n happens a.s. for n large enough, we see that for well-and-wall traps, using (A.6), since $c_{x+k} < e^{-q_n^2}$ and all other conductances are in $[e^{-4q_n}, e^{4q_n}]$,

$$p_{\mathcal{B}} = P_{x+k+1}^{\omega}(T_{x+k+C_n} < T_{x+k}) \geq \left(1 + c' e^{-q_n^2/2}\right)^{-1}. \quad (8.5)$$

In light of (8.5), it is sufficient to replace the bound (5.9) by

$$D_2 \leq P^{\bar{\omega}}\left(\left|\frac{T^{(1)}}{\rho_{\mathcal{B}}}\right| - 2 \mathbf{e}_{\mathcal{B}}\right| > \frac{\delta}{3}\right) + P^{\bar{\omega}}(G > 1). \quad (8.6)$$

Note that the second term is equal to $1 - p_{\mathcal{B}} \leq c' e^{-q_n^2/2}$.

In order to control the first term in (8.6), we use decomposition (5.8) with $\Upsilon_1 = 0$, \mathcal{G}_1 a geometric of parameter $1/(1 + \rho_{x+k})$ (with $\mathbf{e}_{\mathcal{B}}$ coupled with \mathcal{G}_1 , $\mathcal{G}_1 = \lceil \mathbf{e}_{\mathcal{B}} \log \frac{\rho_{x+k}}{1 + \rho_{x+k}} \rceil$) and $\{\Theta_{x+k}(j)\}_{j \in \mathbb{N}}$ a collection of i.i.d. copies of Θ_{x+k} . We end up with

$$P^{\bar{\omega}}\left(\left|\frac{T^{(1)}}{\rho_{\mathcal{B}}}\right| - 2 \mathbf{e}_{\mathcal{B}}\right| > \frac{\delta}{3}\right) = P^{\bar{\omega}}\left(\left|\frac{1}{\rho_{\mathcal{B}}} \sum_{j=1}^{\mathcal{G}_1-1} \Theta_{x+k}(j) - 2 \mathbf{e}_{\mathcal{B}}\right| > \frac{\delta}{3}\right). \quad (8.7)$$

Then, the triangular inequality gives the following upper bound, analogously to (5.14),

$$P^{\bar{\omega}}\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} \left(\Theta_{x+k}(j) \rho_{\mathcal{B}}^{-1} - 2 \rho_{x+k}^{-1}\right)\right| > \frac{\delta}{6}\right) + P^{\bar{\omega}}\left(2 \left|\mathbf{e}_{\mathcal{B}} - (\mathcal{G}_1 - 1) \rho_{x+k}^{-1}\right| > \frac{\delta}{6}\right). \quad (8.8)$$

For the second term in (8.8), we use that $|\mathbf{e}_B - (\mathcal{G}_1 - 1)/\rho_{x+k}| \leq (2 + \mathbf{e}_B)/\rho_{x+k}$, so that the second term is bounded by $P^{\bar{\omega}}(\mathbf{e}_B > c'\delta\rho_{x+k}) \leq \exp\{-c'\delta e^{q_n^2/2}\}$ for some constant $c > 0$, since $\rho_{x+k} > e^{q_n^2/2}$ (either $k = 0$ and this is obvious, or $k > 1$ and $c_{x+k-1} > e^{-4q_n}$, $1/c_{x+k} > e^{q_n^2}$ by definition of a good block).

We finally deal with the first term in (8.8). We first split it with the triangular inequality into (recall $\theta_B := E^{\bar{\omega}}[\Theta_{x+k}]$)

$$P^{\bar{\omega}}\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} (\Theta_{x+k}(j) - \theta_B)\right| > \frac{\delta}{12}\rho_B\right) + P^{\bar{\omega}}\left(\mathcal{G}_1\left|\theta_B\rho_B^{-1} - 2\rho_{x+k}^{-1}\right| > \frac{\delta}{12}\right). \quad (8.9)$$

These two terms can be controlled thanks to the estimates on the mean and the variance of Θ_{x+k} given by the following lemma.

Lemma 8.9. *There exists a constant $c' > 0$ such that, for $\omega \in \bar{\Gamma}_n$:*

* if $k = 0$,

$$\theta_B := E^{\bar{\omega}}[\Theta_{x+k}] \in [2, 2 + c'e^{-q_n^2/2}] \quad (8.10)$$

$$\text{Var}(\Theta_x) \leq c'C_n^2 e^{16q_n}. \quad (8.11)$$

* if $k \geq 1$,

$$\theta_B := E^{\bar{\omega}}[\Theta_{x+k}] \in \left[2\frac{c_{x-1}}{c_{x+k-1}}e^{-\lambda k}, 2\frac{c_{x-1}}{c_{x+k-1}}e^{-\lambda k} + c'e^{4q_n}\right] \quad (8.12)$$

$$\text{Var}(\Theta_{x+k}) \leq c'C_n^2 e^{16q_n} c_{x-1}^2. \quad (8.13)$$

Let us first bound (8.9) with the help of this lemma. In the case $k = 0$, we have that $\rho_B = \rho_{x+k}$. Since $|\theta_B - 2| \leq c'e^{-q_n^2/2}$, we get that the second term in (8.9) is bounded by $P^{\bar{\omega}}(\mathcal{G}_1\rho_B^{-1} > c'\delta e^{q_n^2/2}) \leq \exp\{-c''\delta e^{q_n^2/2}\} \leq c\delta^{-1}e^{-q_n^2/2}$ (using that \mathcal{G}_1 is a geometric r.v. with parameter $1/(1 + \rho_{x+k}) > (2\rho_B)^{-1}$). For the first term in (8.9), we get thanks to Chebychev inequality and the variance bound (8.11), as in (5.17):

$$\begin{aligned} P^{\bar{\omega}}\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} (\Theta_{x+k}(j) - \theta_B)\right| > \frac{\delta}{12}\rho_B\right) &\leq P^{\bar{\omega}}(\mathcal{G}_1 > e^{q_n}\rho_B) + e^{q_n}\rho_B \frac{\text{Var}(\Theta_x)}{\rho_B^2(\delta/12)^2} \\ &\leq e^{-ce^{q_n}} + c'\delta^{-2}e^{20q_n}\rho_B^{-1}. \end{aligned}$$

All together, this gives that $D_2 \leq c'e^{-q_n^2/2} + c''\delta^{-2}e^{-q_n^2/2}$.

In the case $k \geq 1$, recall that $\rho_B := \frac{c_{x-1}}{c_{x+k-1}}e^{-\lambda(k+1)}$: by (8.12), we see that $|\theta_B\rho_B^{-1} - 2\rho_{x+k}^{-1}| \leq c'e^{4q_n}\rho_B^{-1}$. Hence, since \mathcal{G}_1 is a geometric r.v. with parameter $1/(1 + \rho_{x+k})$, the second term in (8.9) is bounded by $\exp\{-c\delta e^{-4q_n}\rho_B/\rho_{x+k}\} \leq \exp\{-c'\delta e^{q_n^2/2}e^{-\lambda k}\} \leq c''\delta^{-1}e^{-q_n^2/2}e^{\lambda k}$, where we used that $\rho_B/\rho_{x+k} = e^{-\lambda k}c_{x-1}/c_{x+k-1}$, with $c_{x-1} > e^{q_n^2}$ and $c_{x+k-1} \leq e^{4q_n}$. It remains to bound the first term of (8.9), again thanks to Chebychev inequality and (8.13) as above:

$$\begin{aligned} P^{\bar{\omega}}\left(\left|\sum_{j=1}^{\mathcal{G}_1-1} (\Theta_{x+k}(j) - \theta_B)\right| > \frac{\delta}{12}\rho_B\right) &\leq P^{\bar{\omega}}(\mathcal{G}_1 > e^{q_n}\rho_{x+k}) + e^{q_n}\rho_{x+k} \frac{\text{Var}(\Theta_{x+k})}{\rho_B^2(\delta/12)^2} \\ &\leq e^{-e^{q_n}/2} + c'\delta^{-2}e^{-q_n^2/2}e^{2\lambda k}, \end{aligned}$$

where we used that $\rho_{x+k}\text{Var}(\Theta_{x+k})/\rho_B^2 \leq c'e^{20q_n}e^{2\lambda k}c_{x+k}c_{x+k-1}$, with $c_{x+k} < e^{-q_n^2}$ and $c_{x+k-1} \leq e^{4q_n}$. Collecting all the previous estimates, we get that $D_2 \leq ce^{-q_n^2/2} + c''\delta^{-2}e^{-q_n^2/2}e^{2\lambda k}$, which concludes the proof of (8.3).

Proof of Lemma 8.9 We follow the proof of Lemma 5.4. We have, see (A.10)

$$E^{\bar{\omega}}[\Theta_{x+k}] = 1 + E_{x+k-1}^{\bar{\omega}}[T_{x+k}] = 2 + \frac{2}{c_{x+k-1}} \sum_{\ell=x+k-C_n}^{x+k-2} c_{\ell} e^{-\lambda(x+k-1-\ell)}.$$

In the case $k = 0$, all c_{ℓ} in the sum are smaller than $e^{4q_n} \leq e^{q_n^2/2}$, and $c_{x-1} > e^{q_n^2}$, which gives (8.10). In the case $k \geq 1$, then we separate the term $\ell = x - 1$ in the sum, and we use that all $c_{\ell} < e^{4q_n}$ for $\ell \neq x - 1$ to obtain (8.12) (we also use that $C_n \leq e^{q_n}$).

For the second moment, we bound, analogously to (5.18)

$$(\Theta_{x+k})^2 \leq 2C_n \sum_{j \neq x-1, x} N_j^2 \quad (8.14)$$

with N_j the random variable counting the number of visits to point j before touching $x+k$, for $j \in \{-C_n, \dots, x+k-1\}$. As above, $N_j \leq Y(j)$, where $Y(j)$ is a geometric random variable with parameter $q(j) = P_j^{\bar{\omega}}(T_{x+k} < T_j)$. Thanks to (A.5), we get that if $k = 0$ then $q(j) \geq ce^{-8q_n}$ for all $j < x$, by definition of a good block; on the other hand, if $k > 0$, we get that $q(j) \geq ce^{-8q_n}/c_{x-1}$ for all $j < x+k$ (one could get a better bound in the case $j \neq x-1, x$, but we do not need it). Then, using that $E_{x+k-1}^{\omega}[N_j^2] \leq 1/q(j)^2$, equations (8.11) and (8.13) follow by taking the expectation in (8.14) (and using that $C_n \leq e^{q_n}$). \square

8.5. Reduction to a finite number of traps. First of all, using a union bound and then (8.3) (with $\delta = \frac{d_n e^{-q_n}}{|\mathcal{J}_n| \rho_{\mathcal{B}_j}} \leq |\mathcal{J}_n|^{-1} < 1$), we get that for all a.e. ω , for n large enough

$$\begin{aligned} P^{\omega} \left(\left| \sum_{j \in \mathcal{J}_n} T(\mathcal{B}_j) - \sum_{j \in \mathcal{J}_n} 2\rho_{\mathcal{B}_j} \mathbf{e}_{\mathcal{B}_j} \right| \geq d_n e^{-q_n} \right) &\leq \sum_{j \in \mathcal{J}_n} P^{\omega} \left(|T(\mathcal{B}_j) - 2\rho_{\mathcal{B}_j} \mathbf{e}_{\mathcal{B}_j}| \geq \frac{d_n e^{-q_n}}{|\mathcal{J}_n|} \right) \\ &\leq c |\mathcal{J}_n|^3 e^{-q_n^2/4}. \end{aligned}$$

Then, by Lemma 8.4, a.s. $|\mathcal{J}_n| \leq e^{5q_n}$ for n large, so this goes to 0. Hence, together with (8.1), and using that $\varepsilon_n^{(1-\alpha)/2} \geq e^{-q_n}$ for n large, we have that with $\mathbb{P} \otimes P^{\omega}$ -probability going to 1, as $n \rightarrow +\infty$,

$$\frac{1}{d_n} |T_n - 2 \sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \mathbf{e}_{\mathcal{B}_j}| \leq 2\varepsilon_n^{(1-\alpha)/2} \rightarrow 0. \quad (8.15)$$

Now, the analogous of Proposition 6.1 holds: for any $\eta, \eta' > 0$, there exists $\varepsilon = \varepsilon(\eta, \eta')$ such that (6.3) holds also in the present case, *i.e.* the main contribution to $\sum_{j \in \mathcal{J}_n} \rho_{\mathcal{B}_j} \mathbf{e}_{\mathcal{B}_j}$ comes from the blocks with $\rho_{\mathcal{B}} > \varepsilon d_n$. The proof follows the same scheme as that of Proposition 6.1, with fewer technicalities since $\tau_{\mathcal{B}_j}$ is simply replaced by $2\mathbf{e}_{\mathcal{B}_j}$ (we do not need Lemma 7.1). In particular, Step 1 of the proof is the same (up to (6.8)), but Step 2 is much easier, since \mathcal{H}_{ℓ} is now the sum of N_{ℓ} independent exponential random variables. We do not provide the details here, since they are straightforward.

In the end, (6.3) combined with (8.15) shows that for any fixed $\eta, \eta' > 0$, we can choose ε so that (6.4) holds with $\mathbb{P} \otimes P^{\omega}$ -probability at least $1 - \eta'$.

8.6. Convergence. The analogous of Proposition 7.2 holds and is in fact much simpler, since $\xi_{\mathcal{B}} = 2$ under Assumption “Well-and-walls”. We simply need to use that for any $\varepsilon > 0$, $\{(\frac{i}{k_n}, \frac{1}{d_n} \rho_{\mathcal{B}_i}); 1 \leq i \leq k_n, \rho_{\mathcal{B}_i} > \varepsilon d_n\}$ converges in distribution to a Poisson Point Process $\mathcal{P}_{\varepsilon}$ on $[0, 1] \times \mathbb{R}_+$ with intensity $d\alpha w^{-(1+\alpha)} \mathbf{1}_{\{w > \varepsilon\}} dw$ —this is due to the tail

behavior of ρ_0 and to the definition (2.2) of d_n , see the proof of Proposition 7.2. This easily gives that, analogously to (7.3), we have the following under $\mathbb{P} \otimes P^\omega$

$$2 \sum_{i=1}^{k_n} \rho_{\mathcal{B}_i} \mathbf{e}_{\mathcal{B}_i} \mathbf{1}_{\{\rho_{\mathcal{B}_i} > \varepsilon d_n\}} \implies 2 \sum_{(x,w,r) \in \bar{\mathcal{P}}} w r \mathbf{1}_{\{x \leq 1\}} \mathbf{1}_{\{w > \varepsilon\}},$$

where $\bar{\mathcal{P}}$ is a PPP on $(\mathbb{R}_+)^3$ with intensity $dx \alpha w^{-(1+\alpha)} dw e^{-r} dr$. All together, with (8.15), letting $\varepsilon \downarrow 0$, we get as in (7.4) that

$$\frac{1}{d_n} T_n \implies 2 \left(\frac{\pi \alpha}{\sin(\pi \alpha)} \right)^{1/\alpha} \sum_{(x,w) \in \mathcal{P}} w \mathbf{1}_{\{x \leq 1\}}, \quad (8.16)$$

with \mathcal{P} a PPP on $(\mathbb{R}_+)^2$ of intensity $dx \frac{\alpha}{\Gamma(1-\alpha)} w^{-(1+\alpha)} dw$. The conclusion of the proof of Theorem 2.1, *i.e.* the convergence of the process, is identical to Section 7.3 from that point on.

APPENDIX A. SOME FORMULAS FOR RESISTOR NETWORKS

In this section, we recall some classical formulas for resistor networks, which translate into properties for the hitting times of random walks among random conductances. The first important identity is the following: for $i < x < j$, we have

$$P_x^\omega(T_i < T_j) = \frac{C_{\text{eff}}(\{x\} \leftrightarrow \{i\})}{C_{\text{eff}}(\{x\} \leftrightarrow \{i, j\})}. \quad (\text{A.1})$$

The effective conductance C_{eff} is here (see [LP16] for the general definition)

$$C_{\text{eff}}(\{x\} \leftrightarrow \{i\}) = S(i, x-1)^{-1} \quad (\text{A.2})$$

$$C_{\text{eff}}(\{x\} \leftrightarrow \{i, j\}) = S(i, x-1)^{-1} + S(x, j-1)^{-1}, \quad (\text{A.3})$$

with (recall the definition (3.1) of $\rho_x^{(k)} := e^{-\lambda(k+1)} c_{x-1}/c_{x+k}$)

$$S(i, j) := \sum_{\ell=i}^j \frac{1}{c_\ell^\lambda} = \frac{1}{c_{i-1}^\lambda} \sum_{k=0}^{j-i} \rho_i^{(k)}. \quad (\text{A.4})$$

For the first formula we have used that we have conductances in series, while for the second formula we have two sequences of conductances-in-series that are in parallel. Then, (A.2) and (A.3) together with (A.1) give

$$P_x^\omega(T_i < T_j) = \frac{S(x, j-1)}{S(i, j-1)}. \quad (\text{A.5})$$

This apply for instance to the probability $p_{\mathcal{B}}$ appearing in Section 5: we have that

$$p_x := P_{x+1}^\omega(T_{x+C_n} > T_x) = \left(1 + c_x \sum_{j=1}^{C_n} \frac{1}{c_{x+j}} e^{-\lambda(j+1)} \right)^{-1}. \quad (\text{A.6})$$

Another important identity we use throughout the paper deals with the expectation of the hitting times. We use the following representation, cf. [Bov06, Eq. (3.22)]:

$$E_x^\omega[T_y] = \frac{1}{C_{\text{eff}}(\{x\} \leftrightarrow \{y\})} \sum_{z < y} \pi(z) P_z^\omega(T_x < T_y) \quad (\text{A.7})$$

where $\pi(z) := c_{z-1}^\lambda + c_z^\lambda$ is a reversible measure for $(X_n)_{n \in \mathbb{N}}$. We notice that in (A.7) the quantity $P_z^\omega(T_x < T_y)$ is equal to 1 for $z \leq x$, while for $x < z < y$ we can use (A.5). We therefore get that, after calculation, see [BS19, Eq. (2.10)], for $y > x$

$$\begin{aligned} E_x^\omega[T_y] &= \sum_{z \leq x} (c_z^\lambda + c_{z-1}^\lambda) S(x, y-1) + \sum_{x < z < y} (c_z^\lambda + c_{z-1}^\lambda) S(z, y-1) \\ &= 2 \sum_{z \leq x-1} c_z^\lambda S(x, y-1) + \sum_{x \leq z < y} c_z^\lambda S(z, y-1) + \sum_{x < z < y} c_{z-1}^\lambda S(z, y-1) \\ &= (y-x) + 2 \sum_{z \leq x} c_{z-1}^\lambda S(x, y-1) + 2 \sum_{x < z < y} c_{z-1}^\lambda S(z, y-1) \end{aligned} \quad (\text{A.8})$$

(we used that $c_z S(z, y-1) = 1 + c_z S(z+1, y)$, and that $S(y, y-1) = 0$). We can finally rewrite this as follows:

$$E_x^\omega[T_y] = (y-x) + 2 \sum_{z < x} \sum_{k=x-z}^{y-z} \rho_z^{(k)} + 2 \sum_{z=x}^y \sum_{k=0}^{y-z} \rho_z^{(k)}. \quad (\text{A.9})$$

This applies for instance to quantity $\theta_x = 1 + E_{x-1}^\omega[T_x]$ defined in Section 5:

$$\theta_x = 1 + E_{x-1}^\omega[T_x] = 2 + \frac{2}{c_{x-1}} \sum_{\ell=x-C_n}^{x-2} c_\ell e^{-\lambda(x-1-\ell)}. \quad (\text{A.10})$$

We also need a formula for the expectation of hitting times for a random walk killed at some site z . It is not explicitly stated in [Bov06], but can be obtained in the same way as (A.7) (as mentioned in the sentence before [Bov06, eq. (3.21)]): for $y < x < v$, we have

$$E_x^\omega[T_y \mathbf{1}_{\{T_y < T_v\}}] = \frac{1}{C_{\text{eff}}(\{x\} \leftrightarrow \{y, v\})} \sum_{y < z < v} \pi(z) P_z^\omega(T_x < T_y \wedge T_v) P_z^\omega(T_y < T_v). \quad (\text{A.11})$$

APPENDIX B. ESTIMATES CONDITIONALLY ON HAVING A LARGE TRAP

Let us recall Proposition 1.1, which derives from [Cli86, Corollary 5]. It gives the following sharp asymptotics: under Assumption “Simple Traps” we have as $t \rightarrow +\infty$

$$\mathbb{P}(\rho_0 > t) e^{\lambda\alpha} \sim \mathbb{E}[c_0^\alpha] L_0(t) t^{-\alpha_0} \mathbf{1}_{\{\mathbb{E}[c_0^\alpha] < +\infty\}} + \mathbb{E}[1/c_0^\alpha] L_\infty(t) t^{-\alpha_\infty} \mathbf{1}_{\{\mathbb{E}[1/c_0^\alpha] < +\infty\}}; \quad (\text{B.1})$$

under Assumption “Well-and-walls” we have as $t \rightarrow +\infty$

$$\mathbb{P}(\rho_0 > t) e^{\lambda\alpha} \sim \alpha \frac{\Gamma(1+\gamma_0)\Gamma(1+\gamma_\infty)}{\Gamma(1+\gamma_0+\gamma_\infty)} (\log t) L_0(t) L_\infty(t) t^{-\alpha}. \quad (\text{B.2})$$

B.1. Proof of Propositions 1.4. Recall that when $\mathbb{E}[c_0^\alpha] < +\infty$, we define \bar{c}_{-1} a random variable with c.d.f. $F_{\bar{c}_{-1}}(u) = \frac{1}{\mathbb{E}[c_{-1}^\alpha]} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{c_{-1} \leq u}]$, and similarly for $1/\bar{c}_0$. We consider two cases separately: 1) $\mathbb{E}[c_0^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] = +\infty$ (the case $\mathbb{E}[c_0^\alpha] = +\infty$, $\mathbb{E}[1/c_0^\alpha] < +\infty$ is symmetric); 2) the case $\mathbb{E}[c_0^\alpha], \mathbb{E}[1/c_0^\alpha] = +\infty$.

1) Suppose that $\mathbb{E}[c_0^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] = +\infty$, so the second term in (B.1) is equal to 0. Then for any fixed $0 < a < b$ and $v > 0$ we have as $t \rightarrow +\infty$

$$\begin{aligned} \mathbb{P}(c_{-1} \in (a, b], 1/c_0 > v, \rho_0 > t) &= \mathbb{E}[\mathbb{P}(c_{-1}/c_0 > e^\lambda t \mid c_{-1}) \mathbf{1}_{\{c_{-1} \in (a, b]\}}] \\ &= (1 + o(1)) L_0(t) e^{-\lambda\alpha_0 t^{-\alpha_0}} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} \in (a, b]\}}]. \end{aligned}$$

Here we used that $\mathbb{P}(c_{-1}/c_0 > e^\lambda t/x) = (1 + o(1)) L_0(t) e^{-\lambda\alpha_0 (t/x)^{-\alpha_0}}$ as $t \rightarrow \infty$, uniformly for $x \in (a, b]$. Letting $b \rightarrow \infty$, we get that $\mathbb{P}(c_{-1} > a, 1/c_0 > v, \rho_0 > t) = (1 + o(1)) L_0(t) e^{-\lambda\alpha_0 t^{-\alpha_0}} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} > a\}}]$.

As a consequence, in view of (B.1), we have that

$$\mathbb{P}(c_{-1} > a, 1/c_0 > v \mid \rho_0 > t) \xrightarrow{t \rightarrow +\infty} \frac{1}{\mathbb{E}[c_{-1}^\alpha]} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} > a\}}] = 1 - F_{\bar{c}_{-1}}(a),$$

giving that $(c_{-1}, 1/c_0)$, conditionally on $\rho_0 > t$, converges in distribution to $(\bar{c}_{-1}, +\infty)$ (i.e. $B = 0$ in (1.6)).

2) If $\mathbb{E}[c_0^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] < +\infty$ (necessarily $\alpha = \alpha_0 = \alpha_\infty$), then both \bar{c}_{-1} and $1/\bar{c}_0$ are well defined, and both terms in (B.1) are non-null. The same reasoning as above is still valid: for any fixed $0 < a < b$ and $v > 0$, and any fixed $0 < c < d$ and $v' > 0$, we get as $t \rightarrow +\infty$

$$\mathbb{P}(c_{-1} \in (a, b], 1/c_0 > v, \rho_0 > t) = (1 + o(1)) L_0(t) e^{-\lambda \alpha_0 t^{-\alpha_0}} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} \in (a, b]\}}];$$

$$\mathbb{P}(c_{-1} > v', 1/c_0 \in (c, d], \rho_0 > t) = (1 + o(1)) L_\infty(t) e^{-\lambda \alpha_\infty t^{-\alpha_\infty}} \mathbb{E}[1/c_0^\alpha \mathbf{1}_{\{1/c_0 \in (c, d]\}}].$$

As a consequence, we get that for any $a, c > 0$, as $t \rightarrow +\infty$

$$\mathbb{P}(c_{-1} > a, 1/c_0 > c, \rho_0 > t) \sim L_0(t) e^{-\lambda \alpha t^{-\alpha}} \mathbb{E}[c_{-1}^\alpha \mathbf{1}_{\{c_{-1} > a\}}] + L_\infty(t) e^{-\lambda \alpha t^{-\alpha}} \mathbb{E}[1/c_0^\alpha \mathbf{1}_{\{1/c_0 > c\}}].$$

In view of (B.1), we get that

$$\mathbb{P}(c_{-1} > a, 1/c_0 > c \mid \rho_0 > t) \xrightarrow{t \rightarrow +\infty} q(1 - F_{\bar{c}_{-1}}(a)) + (1 - q)(1 - F_{1/\bar{c}_0}(c)),$$

where $q = \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[c_0^\alpha] L_0(t)}{\mathbb{E}[c_0^\alpha] L_0(t) + \mathbb{E}[1/c_0^\alpha] L_\infty(t)} = \lim_{c \rightarrow +\infty} \lim_{t \rightarrow +\infty} \mathbb{P}(1/c_0 > c \mid \rho_0 > t)$. This concludes the proof.

B.2. Proof of Propositions 1.5. Under Assumption “Well-and-walls”, as above, we get that for any fixed $b, d > 0$, as $t \rightarrow +\infty$

$$\mathbb{P}(c_{-1} \leq a, \rho_0 > t) = (1 + o(1)) L_0(t) e^{-\lambda \alpha_0 t^{-\alpha_0}} \mathbb{E}[c_{-1}^{\alpha_0} \mathbf{1}_{\{c_{-1} \leq b\}}];$$

$$\mathbb{P}(1/c_0 \leq d, \rho_0 > t) = (1 + o(1)) L_\infty(t) e^{-\lambda \alpha_\infty t^{-\alpha_\infty}} \mathbb{E}[1/c_0^{\alpha_\infty} \mathbf{1}_{\{1/c_0 \leq d\}}].$$

Since $L_0(t) = \varphi_0(\log t)(\log t)^{\gamma_0}$, $L_\infty(t) = \varphi_\infty(\log t)(\log t)^{\gamma_\infty}$ with $\gamma_\infty, \gamma_0 > -1$, we have that $L_0(t), L_\infty(t)$ are both negligible compared to $(\log t) L_0(t) L_\infty(t)$. In view of (B.2), for any fixed $b, d > 0$, we get $\mathbb{P}(c_{-1} \leq b \text{ or } 1/c_0 \leq d \mid \rho_0 > t) \rightarrow 0$, which concludes the proof.

B.3. Distribution of τ_B conditionally on having ρ_B large. In this section, we prove Lemma 6.3 and 7.1: we consider the case of “Simple Traps”. For simplicity, we assume that $\rho_B = \rho_0$ (i.e. $x_B = 0$). We recall from (5.2) that τ_B is equal to $\tau_B := \xi_B \mathbf{e}_B = \frac{\theta_B}{p_B} \mathbf{e}_B$ with $\mathbf{e}_B \sim \text{Exp}(1)$, and with

$$p_B = p_0 = P_1^\omega(\tau_0 > \tau_{C_n}) = \left(1 + c_0 V^{(n)}\right)^{-1} \quad \text{with } V^{(n)} := \sum_{j=1}^{C_n} \frac{1}{c_j} e^{-\lambda(j+1)}, \quad (\text{B.3})$$

$$\theta_B = \theta_0 = 1 + E_0^\omega[T_1] = 2 \left(1 + \frac{1}{c_{-1}} W^{(n)}\right) \quad \text{with } W^{(n)} := \sum_{j=2}^{C_n} c_{-j} e^{-\lambda(j+1)}. \quad (\text{B.4})$$

We refer to (A.6), (A.10) for the formulas.

Let us stress right away that $\mathbb{P}(V^{(n)} > t)$ is bounded by a constant times $\mathbb{P}(1/c_0 > t)$. Indeed, denoting $C_\lambda := \sum_{j \geq 1} e^{-\lambda(j+1)}$, a union bound gives that $\mathbb{P}(V^{(n)} > t)$ is bounded by

$$\sum_{j \geq 1} \mathbb{P}\left(\frac{1}{c_j} \geq \frac{e^{\lambda(j+1)} t}{C_\lambda}\right) \leq \sum_{j \geq 1} c'_\lambda e^{-\frac{\alpha}{2} \lambda(j+1)} \mathbb{P}(1/c_j > t) = c \mathbb{P}(1/c_0 > t), \quad (\text{B.5})$$

where we also used Potter's bound. Similarly, $\mathbb{P}(W^{(n)} > t) \leq c \mathbb{P}(c_{-1} > t)$.

Before we prove Lemma 6.3, we prove the following result, which deals with the tail distribution of p_0, θ_0 conditionally on having ρ_0 large.

Lemma B.1. *For any $\tilde{\delta}, \bar{\delta} > 0$ fixed small enough, there is a constant $c > 0$ such that for any $1 \leq t \leq n^{2\bar{\delta}}$ and any $A_n \geq n^{\frac{1}{\alpha} - \bar{\delta}}$,*

$$\mathbb{P}(1/p_0 > t \mid \rho_0 \geq A_n) \leq ct^{-2\alpha_\infty + \tilde{\delta}} + c\mathbb{P}(1/c_0 > t)f_\infty(A_n), \quad (\text{B.6})$$

$$\mathbb{P}(\theta_0 > t \mid \rho_0 \geq A_n) \leq ct^{-2\alpha_0 + \tilde{\delta}} + c\mathbb{P}(c_{-1} > t)f_0(A_n), \quad (\text{B.7})$$

with $f_\infty(A_n) := \mathbb{P}(c_{-1} > A_n)/\mathbb{P}(\rho_0 > A_n)$ and $f_0(A_n) := \mathbb{P}(1/c_0 > A_n)/\mathbb{P}(\rho_0 > A_n)$.

Notice that, as a function of t , these bounds are regularly varying.

Proof We only treat (B.6), the other bound (B.7) being similar. Using (B.3), we write

$$\mathbb{P}(1/p_0 > t \mid \rho_0 > A_n) = \mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n) / \mathbb{P}(\rho_0 > A_n).$$

We split $\mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n)$ into four parts (recall that $\rho_0 = e^{-\lambda} c_{-1}/c_0$)

$$\begin{aligned} & \mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n, c_0 < 1/A_n) + \mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n, c_0 \in [A_n^{-1}, 1]) \\ & + \mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n, c_0 \in [1, t]) + \mathbb{P}(c_0 V^{(n)} > t/2, \rho_0 > A_n, c_0 > t) \end{aligned} \quad (\text{B.8})$$

- The first term is bounded by $\mathbb{P}(V^{(n)} > tA_n/2)\mathbb{P}(1/c_0 > A_n)$, so that, recalling (B.5), it is bounded by a constant times $A_n^{-2\alpha_0 + \tilde{\delta}}$, which is itself bounded by a constant times $A_n^{-\alpha_0 + 2\tilde{\delta}}\mathbb{P}(\rho_0 > A_n)$ (we have $\alpha_0 \geq \alpha$).

- The last term in (B.8) is bounded by $\mathbb{P}(c_{-1} > tA_n)\mathbb{P}(c_0 > t) \leq ct^{-2\alpha_\infty + \tilde{\delta}}\mathbb{P}(c_{-1} > A_n)$, the inequality coming from Potter's bound. Note also that $\mathbb{P}(\rho_0 > A_n) \geq c\mathbb{P}(c_{-1} > A_n)$.

- Using (B.5) and Potter's bound, we get that the second term in (B.8) is bounded by

$$\begin{aligned} & \sum_{l=0}^{\log_2(A_n)-1} \mathbb{P}(V^{(n)} > 2^{l-1}t)\mathbb{P}(c_{-1} > 2^{-(l+1)}A_n)\mathbb{P}(c_0 \in [2^{-(l+1)}, 2^{-l}]) \\ & \leq c \sum_{l=0}^{\log_2(A_n)} (2^l)^{-2\alpha_0 + \alpha_\infty + \tilde{\delta}} \mathbb{P}(1/c_0 > t)\mathbb{P}(c_{-1} > A_n). \end{aligned} \quad (\text{B.9})$$

- Finally, the third term in (B.8) is bounded by

$$\begin{aligned} & \sum_{l=0}^{\log_2(t)-1} \mathbb{P}(V^{(n)} > 2^{l-2}t)\mathbb{P}(c_{-1} > 2^l A_n)\mathbb{P}(c_0 \in [2^l, 2^{l+1}]) \\ & \leq c \sum_{k=0}^{\log_2(t)} (2^l)^{\alpha_0 - 2\alpha_\infty + \tilde{\delta}} \mathbb{P}(1/c_0 > t)\mathbb{P}(c_{-1} > A_n). \end{aligned}$$

Now, if $\alpha_\infty < 2\alpha_0 - \tilde{\delta}$ and $\alpha_0 < 2\alpha_\infty - \tilde{\delta}$ (which is the case for instance if $\alpha_0 = \alpha_\infty = \alpha$ and $\tilde{\delta}$ small), then both sums over k are finite: we get that the second and third term in (B.8) are bounded by $\mathbb{P}(1/c_0 > t)\mathbb{P}(c_{-1} > A_n)$. If on the other hand $\alpha_\infty \geq 2\alpha_0 - \tilde{\delta} > \alpha_0$ (in which case $\alpha_0 < 2\alpha_\infty - \tilde{\delta}$), then we get that $\sum_{l=0}^{\log_2(A_n)} (2^l)^{-2\alpha_0 + \alpha_\infty + \tilde{\delta}} \leq cA_n^{\alpha_\infty - 2\alpha_0 + \tilde{\delta}}$, and (B.9) is bounded by a constant times $A_n^{-2\alpha_0 + 2\tilde{\delta}} \leq cA_n^{-\alpha_0 + 3\tilde{\delta}}\mathbb{P}(\rho_0 > A_n)$ (recall that if $\alpha_0 < \alpha_\infty$ then $\mathbb{P}(\rho_0 > A_n) \sim c'\mathbb{P}(1/c_0 > A_n)$). Finally, if $\alpha_0 \geq 2\alpha_\infty - \tilde{\delta}$, we have that

$\sum_{l=0}^{\log_2(t)} (2^l)^{\alpha_0 - 2\alpha_\infty + \bar{\delta}} \leq ct^{\alpha_0 - 2\alpha_\infty + \bar{\delta}}$, and then the third term in (B.8) is bounded by a constant times $t^{-2\alpha_\infty + 2\bar{\delta}} \mathbb{P}(c_{-1} > A_n)$, with $\mathbb{P}(c_{-1} > A_n) \sim c\mathbb{P}(\rho_0 > A_n)$, since $\alpha_0 > \alpha_\infty$.

All together, we have bounded the four terms in (B.8), so that

$$\mathbb{P}(1/p_0 > t \mid \rho_0 > A_n) \leq cA_n^{-\alpha_0 + 3\bar{\delta}} + ct^{-2\alpha_\infty + 2\bar{\delta}} + \mathbb{P}(1/c_0 > t) \frac{\mathbb{P}(c_{-1} > A_n)}{\mathbb{P}(\rho_0 > A_n)}. \quad (\text{B.10})$$

We then get (B.6) by bounding $A_n^{-\alpha_0 + 3\bar{\delta}}$ by $n^{-1+4\bar{\delta}}$ (recall $A_n \geq n^{\frac{1}{\alpha} - \bar{\delta}}$), and using that $t \leq n^{2\bar{\delta}}$: we get that the first term in (B.10) is negligible. \square

Proof of Lemma 6.3 Let us write $\mathbb{P}_{A_n}(\cdot)$ for $\mathbb{P}(\cdot \mid \rho_0 > A_n)$ for simplicity, and $\xi_0 = \theta_0/p_0$. Recall (5.2). We decompose the probability according to whether $\theta_0 \leq 4$ or $1/p_0 \leq 4$ (or neither), and we write

$$\mathbb{P}_{A_n}(\xi_0 > t) \leq \mathbb{P}_{A_n}\left(\frac{1}{p_0} > t/4\right) + \mathbb{P}_{A_n}(\theta_0 > t/4) + \mathbb{P}_{A_n}(\theta_0 > 4, \frac{1}{p_0} > 4).$$

For the last term, recalling formulas (B.3)-(B.4), we get that

$$\begin{aligned} \mathbb{P}_{A_n}(\theta_0 > 4, 1/p_0 < 4) &= \mathbb{P}_{A_n}\left(\frac{1}{c_{-1}}W^{(n)} > 1, c_0V > 3\right) \leq \mathbb{P}_{A_n}(V^{(n)}W^{(n)} > 3\rho_0) \\ &\leq \mathbb{P}(V^{(n)}W^{(n)} > A_n). \end{aligned}$$

Now, one can easily get that $\mathbb{P}(V^{(n)} > t) = t^{-\alpha_0 + o(1)}$ and $\mathbb{P}(W^{(n)} > t) = t^{-\alpha_\infty + o(1)}$ as $t \rightarrow +\infty$ (uniformly in n , see (B.5)), and hence that $\mathbb{P}(V^{(n)}W^{(n)} > t) = t^{-\alpha + o(1)}$ (see [BS19, Lemma 1.3]). Therefore, we get that $\mathbb{P}(V^{(n)}W^{(n)} > A_n) \leq n^{-1+2\bar{\delta}}$, for n large enough and $\bar{\delta}$ small enough (recall $\alpha < 1$ and $A_n \geq n^{\frac{1}{\alpha} - \bar{\delta}}$). Hence

$$\mathbb{P}_{A_n}(\xi_0 > t) \leq \mathbb{P}_{A_n}(1/p_0 > t/4) + \mathbb{P}_{A_n}(\theta_0 > t/4) + n^{-1+2\bar{\delta}},$$

which together with Lemma B.1 concludes the proof of the first part of the lemma (note that $n^{-1+2\bar{\delta}}$ is negligible compared to $t^{-2\alpha + \bar{\delta}}$, uniformly over $t \leq n^{\bar{\delta}}$, provided $\bar{\delta}$ is small).

For the second part of the lemma, recall that $\tau_B := \xi_B \mathbf{e}_B$, with $\mathbf{e}_B \sim \text{Exp}(1)$ independent of ξ_B : we therefore get, conditioning first on ξ_B

$$\mathbb{P} \otimes P^\omega(\xi_B \mathbf{e}_B > t \mid \rho_B > A_n) = \mathbb{E}[e^{-t/\xi_B} \mid \rho_B > A_n] \leq e^{-t} + \mathbb{P}_{A_n}(\xi_0 > t),$$

which concludes the proof. \square

Proof of Lemma 7.1 The proof follows essentially from Proposition 1.4. Again, for simplicity of notations, we reduce to the case where $\rho_B = \rho_0$, i.e. $x_B = 0$. Recall the definition of ξ_B , and the formulas (B.3)-(B.4) for p_B, θ_B ,

$$\xi_B := \frac{\theta_B}{p_B} = 2\left(1 + c_0V^{(n)}\right)\left(1 + \frac{1}{c_{-1}}W^{(n)}\right). \quad (\text{B.11})$$

Notice that $V^{(n)}, W^{(n)}$ are independent of c_{-1}, c_0 , hence of ρ_0 . We clearly have that $V^{(n)}, W^{(n)} \rightarrow V, W$ as $n \rightarrow +\infty$, by monotone convergence. We now simply use Proposition 1.4: it gives that conditionally on $\rho_0 > t$, $(\frac{1}{c_{-1}}, c_0)$ converges in distribution as $t \rightarrow +\infty$ to

1. $(\frac{1}{\bar{c}_{-1}}, 0)$ if $\mathbb{E}[c_{-1}^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] = +\infty$;
2. $(0, \bar{c}_0)$ if $\mathbb{E}[c_{-1}^\alpha] = +\infty$, $\mathbb{E}[1/c_0^\alpha] < +\infty$;
3. $((1-B)\frac{1}{\bar{c}_{-1}}, B\bar{c}_0)$ if $\mathbb{E}[c_{-1}^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] < +\infty$.

The distributions of \bar{c}_{-1} , \bar{c}_0 and $B \sim \text{Bern}(q)$ are those given in the statement of Lemma 7.1. Hence, in view of (B.11), we get that conditionally on $\rho_0 > n$, $\xi_{\mathcal{B}}$ converges in distribution as $n \rightarrow +\infty$ to

$$\zeta := 2\left(1 + B \bar{c}_0 V\right)\left(1 + (1 - B) \frac{1}{\bar{c}_{-1}} W\right) = 2\left(1 + B \bar{c}_0 V + (1 - B) \frac{1}{\bar{c}_{-1}} W\right),$$

with $B = 1$ if $\mathbb{E}[c_{-1}^\alpha] < +\infty$, $\mathbb{E}[1/c_0^\alpha] = +\infty$ and $B = 0$ if $\mathbb{E}[c_{-1}^\alpha] = +\infty$, $\mathbb{E}[1/c_0^\alpha] < +\infty$.

Moreover, in view of Lemma 6.3 (see in particular (6.12)), we get that there is a constant $c > 0$ such that for any $t > 1$,

$$\mathbb{P}(\zeta > t) \leq cL_0(t)t^{-\alpha_0} \mathbf{1}_{\{\mathbb{E}[1/c_0^\alpha] < +\infty\}} + cL_\infty(y)t^{-\alpha_\infty} \mathbf{1}_{\{\mathbb{E}[c_{-1}^\alpha] < +\infty\}}.$$

This implies in particular that $\mathbb{E}[(\zeta)^\alpha] < +\infty$. \square

APPENDIX C. TRAP PROPERTIES: PROOFS OF THE LEMMAS

In this section we collect the proofs of some technical properties of both simple and k -distant traps. Often the proofs deal with both cases at once. We recall that $\rho_x^{(k)} = e^{-\lambda(k+1)\frac{c_{x-1}}{c_{x+k}}} = \rho_x \cdots \rho_{x+k}$, and $\rho_x = \rho_x^{(0)}$.

C.1. Traps are isolated: Proof of Lemmas 3.4 and 8.4. The case $k, k' = 0$ will include both the case of simple traps of Lemma 3.4 and the case of 0-distant well-and-wall traps of Lemma 8.4. We start by estimating the probability $\mathbb{P}(\mathcal{W}_x^{(k)} \cap \mathcal{W}_y^{(k')})$, for all $-n \leq x < y \leq n$ and $k, k' \geq 0$. Note that $\mathcal{W}_x^{(k)} \subset \{\rho_x^{(k)} > d_n e^{-q_n}\}$.

• First, if $y - 1 \neq x + k$ and $y + k' \neq x + k$, then $\rho_x^{(k)}$ and $\rho_y^{(k')}$ are independent, so that by (4.2) (recall also the definition (2.2) of d_n)

$$\begin{aligned} \mathbb{P}(\rho_x^{(k)}, \rho_y^{(k')} > d_n e^{-q_n}) &\leq \mathbb{P}(\rho_x^{(k)} > d_n e^{-q_n}) \mathbb{P}(\rho_y^{(k')} > d_n e^{-q_n}) \\ &\leq c e^{-\lambda \alpha k/2} e^{-\lambda \alpha k'/2} \psi(d_n)^2 d_n^{-2\alpha} e^{4\alpha q_n} \leq \frac{c'}{n^2} e^{4q_n} e^{-\lambda \alpha (k+k')/2}. \end{aligned}$$

Here, we used Potter's bound to get that $\psi(d_n e^{-q_n}) \leq c \psi(d_n) e^{\alpha q_n}$ (and the fact that $\alpha \leq 1$).

• If $y = x + k + 1$, then we have that $\rho_x^{(k)} \rho_y^{(k')} = \rho_x^{(k+k'+1)}$, and

$$\begin{aligned} \mathbb{P}(\rho_x^{(k)}, \rho_y^{(k')} > d_n e^{-q_n}) &\leq \mathbb{P}(\rho_x^{(k+k'+1)} > d_n^2 e^{-2q_n}) \\ &\leq c e^{-\lambda \alpha (k+k'+1)/2} \psi(d_n^2 e^{-2q_n}) d_n^{-2\alpha} e^{2\alpha q_n} \leq c' e^{-\lambda \alpha (k+k')/2} n^{-3/2}. \end{aligned}$$

The last inequality uses once again Potter's bound (the expression is regularly varying in n , with index -2).

• If $y + k' = x + k$, then necessarily $k > 0$. Note that c_{y-1} is independent of $\rho_x^{(k)}$, so recalling Definition 8.1, $\mathbb{P}(\mathcal{W}_x^{(k)} \cap \mathcal{W}_y^{(k')})$ is bounded by

$$\begin{aligned} \mathbb{P}(\rho_x^{(k)} > d_n e^{-q_n}; c_{y-1} > e^{q_n^2}) &\leq c e^{-\lambda k/2} \psi(d_n e^{-q_n}) d_n^{-\alpha} e^{\alpha q_n} e^{-\alpha_\infty q_n^2/2} \\ &\leq \frac{c}{n} e^{-\alpha_\infty q_n^2/4} e^{-\lambda k/2}. \end{aligned}$$

For the last inequality, we used Potter's bound and the definition (2.2) of d_n , and then took n large enough.

Therefore, by a union bound, we have that

$$\begin{aligned} \mathbb{P}(\mathcal{D}_n) &\leq Cn^2 e^{-5q_n} \sum_{k,k' \geq 0} \frac{1}{n^2} e^{4q_n} e^{-\lambda\alpha(k+k')/2} \\ &\quad + c'n \sum_{k,k'=0}^{+\infty} e^{-\lambda\alpha(k+k')/2} n^{-3/2} + cn \sum_{k=1}^{+\infty} \frac{c}{n} e^{-\alpha_\infty q_n^2/4} e^{-\lambda k/2}. \end{aligned}$$

All together, we obtain that $\mathbb{P}(\mathcal{D}_n)$ is bounded by a constant times e^{-q_n} and therefore goes to 0 as $n \rightarrow +\infty$. To upgrade this to an almost sure statement, we use a monotonicity trick. We need to introduce

$$\tilde{\mathcal{D}}_\ell := \bigcup_{\substack{-2\ell \leq x < y \leq 2\ell \\ |x-y| \leq 2\ell e^{-5q_\ell}}} \bigcup_{k \geq 0} \bigcup_{k' \geq 0} \tilde{\mathcal{W}}_x^{(k)} \cap \tilde{\mathcal{W}}_y^{(k')}, \quad (\text{C.1})$$

with $\tilde{\mathcal{W}}_x^{(k)} := \{\rho_x > \tilde{c} d_\ell e^{-5q_\ell}\} \cap \{c_{x-1}, \frac{1}{c_{x+k}} > \tilde{c} e^{q_\ell^2}\}$ for $k \geq 0$ (for simple traps the second condition is absent). The constant \tilde{c} is chosen large enough so that for all $n \in \{\ell, \dots, 2\ell\}$ we have $\tilde{c} d_\ell e^{-5q_\ell} \leq d_n e^{-q_n}$, $\tilde{c} e^{q_\ell^2} \leq d_n e^{-q_n^2}$. With this definition, we have that $\mathcal{D}_n \subset \tilde{\mathcal{D}}_\ell$ for all $\ell \leq n \leq 2\ell$. Then, as above, we obtain that $\mathbb{P}(\tilde{\mathcal{D}}_\ell) \leq c e^{-5q_\ell}$. Setting $n_\ell = \exp((2 \log \ell)^4)$ so that $q_{n_\ell} = 2 \log \ell$, we therefore have that $\sum_\ell \mathbb{P}(\tilde{\mathcal{D}}_{n_\ell}) < +\infty$, and by Borel-Cantelli there is some ℓ_0 such that $\tilde{\mathcal{D}}_{n_\ell}$ does not occur for $\ell \geq \ell_0$. Then, we realize that $n_{\ell+1} \sim n_\ell$ as $\ell \rightarrow +\infty$, so $n_{\ell+1} \leq 2n_\ell$ for all ℓ large enough, say $\ell \geq \ell_1$. Therefore, for any $\ell \geq \tilde{\ell} := \max(\ell_0, \ell_1)$, \mathcal{D}_{n_ℓ} does not occur, and additionally $\mathcal{D}_n \subset \tilde{\mathcal{D}}_{n_\ell}$ for any $n_\ell \leq n \leq 2n_\ell \leq n_{\ell+1}$: we conclude that \mathcal{D}_n does not occur for any $n \geq n_{\tilde{\ell}}$.

C.2. Traps are not too deep: Proof of Lemmas 3.5 and 8.5. The case $k = 0$ will include both the case of simple traps of Lemma 3.5 and the case of 0-distant well-and-wall traps of Lemma 8.5. First of all, by (4.2) (and using Potter's bound), we have that

$$\mathbb{P}(\rho_x^{(k)} > d_n e^{q_n/2}) \leq c e^{-\alpha\lambda k/2} \psi(d_n) d_n^{-\alpha} e^{-\frac{\alpha}{4} q_n} \leq \frac{c'}{n} e^{-\lambda\alpha k/2} e^{-\alpha q_n/4}.$$

We therefore get thanks to a union bound that $\mathbb{P}(M_n > d_n e^{q_n}) \leq c'' e^{-\alpha q_n/4}$. Hence, setting $n_\ell := \exp\{(\frac{8}{\alpha} \log \ell)^4\}$ (so that $q_{n_\ell} = \frac{8}{\alpha} \log n$), we get by Borel-Cantelli that if ℓ is large enough, $M_{n_\ell} \leq d_{n_\ell} e^{q_{n_\ell}/2}$. Then, since M_n is increasing, we get that for all $n_\ell \leq n \leq n_{\ell+1}$ we have $M_n \leq M_{n_{\ell+1}} \leq 2d_n e^{q_n/2}$ for ℓ large enough, since $n_{\ell+1} \sim n_\ell$ as $\ell \rightarrow +\infty$. We get Lemma 8.5 by using that $2 \leq e^{q_n/2}$ for n large enough.

C.3. All triblocks are good: Proof of Lemmas 3.6 and 8.6. The case $k = 0$ will include both the case of simple traps of Lemma 3.5 and the case of 0-distant well-and-wall traps of Lemma 8.5. Let us estimate the probability, for any $x, k \geq 0$ and $y \neq x-1, x+k$,

$$\begin{aligned} \mathbb{P}(\rho_x^{(k)} > d_n e^{-q_n}, c_y \notin [e^{-4q_n}, e^{4q_n}]) &\leq \mathbb{P}(\rho_x^{(k)} > d_n e^{-q_n}) (\mathbb{P}(c_y > e^{4q_n}) + \mathbb{P}(\frac{1}{c_y} > e^{4q_n})) \\ &\leq c e^{-\lambda\alpha k/2} \psi(d_n) d_n^{-\alpha} e^{2\alpha q_n} e^{-3\alpha q_n}. \end{aligned}$$

Here, we used that c_y and $\rho_x^{(k)}$ are independent, and Potter's bound to get that $\psi(d_n e^{q_n}) \leq c \psi(d_n) e^{\alpha q_n}$ and that $\mathbb{P}(c_y > e^{4q_n}) \leq c e^{-3\alpha q_n}$ (and similarly for $\mathbb{P}(\frac{1}{c_y} > e^{4q_n})$). For Lemma 3.6, we additionally have to bound

$$\begin{aligned} \mathbb{P}(\rho_x > d_n e^{-q_n}, c_x > e^{4q_n}) &\leq \mathbb{P}(c_{x-1} > d_n e^{3q_n}, c_x > e^{4q_n}) \\ &\leq c L_\infty(d_n) d_n^{-\alpha_\infty} e^{-\alpha q_n} \leq c \psi(d_n) d_n^{-\alpha} e^{-\alpha q_n}. \end{aligned}$$

A similar bound holds for $\mathbb{P}(\rho_x > d_n e^{-q_n}, c_{x-1} < e^{-4q_n})$.

Therefore, since by definition of d_n we have $\psi(d_n)d_n^{-\alpha} \leq c'/n$, we get by a union bound

$$\mathbb{P}(\mathcal{G}_n) \leq c' C_n \sum_{k \geq 0} e^{-\lambda \alpha k/2} e^{-\alpha q_n} \leq c'' e^{-\alpha q_n/2},$$

so $\mathbb{P}(\mathcal{G}_n) \rightarrow 0$ as $n \rightarrow +\infty$.

We can easily upgrade this to an a.s. statement, in the same manner as in Section C.1, by introducing some appropriate event $\tilde{\mathcal{G}}_\ell$ (analogously to (C.1), by using the events $\{\rho_x^{(k)} > \tilde{c} d_n e^{-q_n}\}$ and $\{c_y \notin [\tilde{c} e^{-4q_n}, \tilde{c} e^{4q_n}]\}$ for some appropriate constant \tilde{c}), in such a way that $\mathcal{G}_n \subset \tilde{\mathcal{G}}_\ell$ for all $\ell \leq n \leq 2\ell$. Then, \mathbb{P} -a.s., $\tilde{\mathcal{G}}_\ell$ occurs finitely many times along the subsequence $n_\ell = \exp((\frac{4}{\alpha} \log \ell)^4)$ (so $q_{n_\ell} = \frac{4}{\alpha} \log n$), and we get the conclusion of the lemma since $n_{\ell+1} \leq 2n_\ell$ for ℓ large enough).

C.4. k is small: Proof of Lemma 8.7. Let us notice that for any x , and any $k \geq 0$, recalling (4.2)

$$\mathbb{P}(\mathcal{W}_x^{(k)}) \leq \mathbb{P}(\rho_x^{(k)} > d_n e^{-q_n}) \leq c e^{-\lambda \alpha k/2} \psi(d_n) d_n^{-\alpha} e^{2\alpha q_n},$$

where we used Potter's bound to get that $\psi(d_n e^{-q_n}) \leq c \psi(d_n) e^{\alpha q_n}$. Since $\psi(d_n) d_n^{-\alpha} \leq c/n$ by definition of d_n , we get by a union bound that

$$\mathbb{P}(\mathcal{K}_n) \leq c' e^{2\alpha q_n} \sum_{k \geq \frac{6}{\lambda} q_n} e^{-\lambda \alpha k/2} \leq c'' e^{2\alpha q_n} e^{-3\alpha q_n} = c'' e^{-\alpha q_n},$$

so $\mathbb{P}(\mathcal{K}_n) \rightarrow 0$ as $n \rightarrow +\infty$. This is easily upgraded to an a.s. statement, as in Section C.1.

C.5. Traps are the only annoying parts of the environment (“Simple Traps”): Proof of Lemma 3.7. In this subsection, we work with Assumption “Simple Traps”. Let us estimate, for any $x \in \mathbb{Z}$ and $k \geq 1$, the probability

$$\mathbb{P}(\rho_x^{(k)} > \varepsilon_n e^{-\lambda \alpha k/2} d_n; \rho_x, \rho_{x+k} \leq d_n e^{-q_n}).$$

We assume that we are in the case where $\mathbb{P}(\rho_0 > d_n) \sim c L_0(d_n) d_n^{-\alpha_0}$: in particular $\alpha_\infty \geq \alpha_0 = \alpha$ (and $\gamma_\infty \leq \gamma_0$ if $\alpha_\infty = \alpha_0$), and we have $\mathbb{E}[c_0^\alpha] < +\infty$. The case $\mathbb{P}(\rho_0 > t) \sim c L_\infty(t) t^{-\alpha_\infty}$ is symmetric.

Since $\{\rho_x < d_n e^{-q_n}\} \subset \{\{c_x > e^{\frac{q_n}{2}-\lambda}\} \cup \{c_{x-1} < d_n e^{-q_n/2}\}\}$, it is enough to control two contributions. The first contribution is

$$\begin{aligned} \mathbb{P}(\rho_x^{(k)} > \varepsilon_n e^{-\lambda k/2} d_n; c_x > e^{\frac{q_n}{2}-\lambda} \text{ or } \frac{1}{c_{x+k-1}} > e^{\frac{q_n}{2}-\lambda}) &\leq 2\mathbb{P}\left(\frac{c_{x-1}}{c_{x+k}} > \varepsilon_n e^{\lambda k/2} d_n\right) e^{-\alpha q_n/4} \\ &\leq c e^{-\lambda \alpha k/4} \psi(d_n) d_n^{-\alpha} \varepsilon_n^{-2\alpha} e^{-\alpha q_n/4} \leq \frac{c'}{n} e^{-\lambda \alpha k/4} e^{-\alpha q_n/5}. \end{aligned} \quad (\text{C.2})$$

Here, we used that c_x, c_{x+k-1} are independent of $\rho_x^{(k)}$, together with Potter's bound and the definition of d_n . We also used the fact that $\varepsilon_n^{-2\alpha} = e^{o(q_n)}$ from our choice of ε_n .

The second contribution we need to control is, using $c_{x-1} \leq \rho_x c_x$ and $\frac{1}{c_{x+k}} \leq \rho_{x+k} \frac{1}{c_{x+k-1}}$ (recall that in \mathcal{H}_n we have that $\rho_x, \rho_{x+k} \leq d_n e^{-q_n}$),

$$\begin{aligned} \mathbb{P}\left(\rho_x^{(k)} \geq \varepsilon_n e^{-\lambda k/2} d_n; c_{x-1}, \frac{1}{c_{x+k}} \leq d_n e^{-\frac{1}{2} q_n}\right) \\ \leq \mathbb{P}\left(\frac{1}{c_{x+k}} > \frac{1}{c_{x-1}} e^{\lambda k/2} \varepsilon_n d_n; e^{\frac{1}{4} q_n} \leq c_{x-1} \leq d_n e^{-\frac{1}{2} q_n}\right). \end{aligned} \quad (\text{C.3})$$

Indeed, we used that $c_{x-1} = c_{x+k} e^{\lambda(k+1)} \rho_x^{(k)}$, so that $c_{x-1} \geq e^{\lambda k/2} \varepsilon_n e^{q_n/2} \geq e^{q_n/4}$ (for n large enough). Then, the idea is similar to the proof of Proposition 1.4, but here we need a

more quantitative estimate, which bring several technicalities. Conditioning with respect to c_{x-1} , we get that (C.3) is bounded by a constant times

$$\begin{aligned} & \mathbb{E} \left[L_0 \left(e^{\lambda k/2} d_n \varepsilon_n / c_{x-1} \right) e^{-\lambda \alpha k/2} d_n^{-\alpha} \varepsilon_n^{-\alpha} c_{x-1}^\alpha \mathbf{1}_{\{e^{q_n/4} \leq c_{x-1} \leq d_n e^{-q_n/2}\}} \right] \\ & \leq c' e^{-\lambda \alpha k/4} \varepsilon_n^{-2\alpha} d_n^{-\alpha} \mathbb{E} \left[L_0 \left(d_n / c_{x-1} \right) c_{x-1}^\alpha \mathbf{1}_{\{e^{q_n/4} \leq c_{x-1} \leq d_n e^{-q_n/2}\}} \right], \end{aligned} \quad (\text{C.4})$$

where we used Potter's bound.

- If $\alpha = \alpha_0 < \alpha_\infty$, we get by Potter's bound that for any $\eta > 0$ (small enough)

$$\begin{aligned} \mathbb{E} \left[L_0(d_n/c_0) c_0^\alpha \mathbf{1}_{\{e^{q_n/4} \leq c_0 \leq d_n e^{-q_n/2}\}} \right] & \leq c L_0(d_n) \mathbb{E}[(c_0)^{\alpha+\eta} \mathbf{1}_{\{c_0 \geq e^{q_n/4}\}}] \\ & \leq c L_0(d_n) e^{-\eta' q_n}, \end{aligned}$$

with $\eta' = (\alpha - \alpha_\infty + 2\eta)/4 > 0$ (in particular $\alpha + \eta < \alpha_\infty$). Plugged into (C.4), we therefore get that

$$\mathbb{P} \left(\rho_x^{(k)} \geq \varepsilon_n d_n; c_{x-1}, \frac{1}{c_{x+k}} \leq d_n e^{-q_n/2} \right) \leq \frac{c}{n} \varepsilon_n^{-2\alpha} e^{-\lambda \alpha k/4} e^{-\eta' q_n}, \quad (\text{C.5})$$

where we used that $L_0(d_n) d_n^{-\alpha} \leq c/n$ (recall that $\mathbb{P}(\rho_0 > d_n) \sim c L_0(d_n) d_n^{-\alpha_0}$).

- If $\alpha_0 = \alpha_\infty$, then we split

$$\begin{aligned} \mathbb{E} \left[L_0(d_n/c_0) c_0^\alpha \mathbf{1}_{\{e^{q_n/4} \leq c_0 \leq d_n e^{-q_n/2}\}} \right] & = \mathbb{E} \left[L_0(d_n/c_0) c_0^\alpha \mathbf{1}_{\{e^{q_n/4} \leq c_0 \leq \sqrt{d_n}\}} \right] \\ & + \mathbb{E} \left[L_0(d_n/c_0) c_0^\alpha \mathbf{1}_{\{\sqrt{d_n} \leq c_0 \leq d_n e^{-q_n/2}\}} \right]. \end{aligned} \quad (\text{C.6})$$

For the first term, we use that $L_0(e^t)$ is regularly varying to get that $L_0(d_n/c_0) \leq c L_0(d_n)$ for $e^{q_n/4} \leq c_0 \leq \sqrt{d_n}$ (since $\frac{1}{2} \log d_n \leq \log(d_n/c_0) \leq \log d_n$). Therefore, the first term in (C.6) is bounded by a constant times $L_0(d_n) \mathbb{E} \left[c_0^\alpha \mathbf{1}_{\{c_0 > e^{q_n/4}\}} \right]$, with

$$\begin{aligned} \mathbb{E} \left[c_0^\alpha \mathbf{1}_{\{c_0 > e^{q_n/4}\}} \right] & = e^{\alpha q_n/4} \mathbb{P}(c_0^\alpha > e^{\alpha q_n/4}) + \int_{e^{\alpha q_n/4}}^{+\infty} \mathbb{P}(c_0^\alpha > t) dt \\ & \sim L_\infty(e^{q_n/4}) + \alpha \int_{q_n/4}^{+\infty} L_\infty(e^u) du, \end{aligned} \quad (\text{C.7})$$

where for the second line we used that $\mathbb{P}(c_0^\alpha > t) \sim L_0(t^{1/\alpha}) t^{-1}$ and then a change of variable $t = e^{\alpha u}$. Now, we use that $L_\infty(e^u) = \varphi_\infty(u) u^{\gamma_\infty}$ with $\gamma_\infty > -1$ to get that the last integral is bounded by a constant times $\varphi_\infty(q_n) q_n^{1+\gamma_\infty}$ (and so is the first term).

For the second term in (C.6), we write

$$\begin{aligned} \mathbb{E} \left[L_0(d_n/c_0) c_0^\alpha \mathbf{1}_{\{\sqrt{d_n} \leq c_0 \leq d_n e^{-q_n}\}} \right] & \leq \sum_{j=q_n}^{\frac{1}{2} \log d_n} L_0(e^j) (e^{-j} d_n)^\alpha \mathbb{P}(c_0 \in [e^{-(j+1)} d_n, e^{-j} d_n]) \\ & \leq c L_\infty(d_n) \sum_{j=q_n}^{\frac{1}{2} \log d_n} L_0(e^j). \end{aligned} \quad (\text{C.8})$$

For the last inequality, we used that $\mathbb{P}(c_0 \in [e^{-(j+1)} d_n, e^{-j} d_n]) \leq c L_\infty(e^{-j} d_n) (e^{-j} d_n)^{-\alpha}$, together with the fact that $L_\infty(e^{-j} d_n) \leq c L_\infty(d_n)$ for the range of j considered (since $L_\infty(e^t)$ is regularly varying). Then, since $L_0(e^j) \sim \varphi_0(j) j^{\gamma_0}$ with $\gamma_0 \neq 1$, we get that $\sum_{j=q_n}^{\frac{1}{2} \log d_n} L_0(e^j)$ is bounded by a constant times $\varphi_0(\log d_n) (\log d_n)^{1+\gamma_0} \leq c(\log n) L_0(d_n)$ if $\gamma_0 > -1$; by $\varphi_0(q_n) q_n^{1+\gamma_0}$ if $\gamma_0 < -1$.

All together, we get that (C.6) is bounded by a constant times

$$L_0(d_n)\varphi_\infty(q_n)q_n^{1+\gamma_\infty} + \begin{cases} L_0(d_n)\infty \times (\log n)L_\infty(d_n) & \text{if } \gamma_0 > -1; \\ L_\infty(d_n) \times \varphi_0(q_n)q_n^{1+\gamma_0} & \text{if } \gamma_0 < -1. \end{cases}$$

In the case $\gamma_0 > -(\mathbb{E}[1/c_0^\infty] = +\infty)$, then necessarily $\gamma_\infty < -1$ (we need to have $\mathbb{E}[c_0^\infty] < +\infty$): we may bound $(\log n)L_\infty(d_n) \leq c\varphi_\infty(\log n)(\log n)^{1+\gamma_\infty} \leq c'gp_\infty(q_n)q_n^{1+\gamma_\infty}$ (recall $q_n = (\log n)^{1/4}$). In the case $\gamma_0 < -1$, since we assume that $\mathbb{P}(\rho_0 > d_n) \sim cL_0(d_n)d_n^{-\alpha}$ it means in particular that $L_\infty(d_n) \leq L_0(d_n)$. Overall, using again that $L_0(d_n)d_n^{-\alpha} \leq c/n$, we get from (C.4) that

$$\mathbb{P}\left(\rho_x^{(k)} \geq \varepsilon_n d_n; c_{x-1}, \frac{1}{c_{x+k}} \leq d_n e^{-q_n/2}\right) \leq \frac{c}{n} \varepsilon_n^{-2\alpha} e^{-\lambda\alpha k/4} q_n^{-c}, \quad (\text{C.9})$$

for some constant $c > 0$; $c = -(1 + \gamma_\infty)/2$ if $\gamma_0 > -1$ and $c = -(1 + \gamma_0)/2$ if $\gamma_0 < -1$ (recall $\gamma_\infty \leq \gamma_0$).

Finally, by a union bound we get that, thanks to (C.2) and (C.5)-(C.9)

$$\mathbb{P}(\mathcal{H}_n) \leq c\varepsilon_n^{-2\alpha} e^{-\eta' q_n} \quad \text{if } \alpha_0 < \alpha_\infty; \quad \mathbb{P}(\mathcal{H}_n) \leq c\varepsilon_n^{-2\alpha} q_n^{-c} \quad \text{if } \alpha_0 = \alpha_\infty.$$

This concludes the proof if we had chosen $\varepsilon_n = q_n^{-\delta}$ with $\delta > 0$ small enough.

C.6. Traps are the only annoying parts of the environment (“Well-and-walls”):

Proof of Lemma 8.8. We work under Assumption “Well-and-walls”. In this case, because of (B.2) and of the definition (2.2) of d_n , we have that (recall Assumption “Traps”)

$$\varphi_0(\log n)\varphi_\infty(\log n)(\log n)^{1+\gamma_0+\gamma_\infty}d_n^{-\alpha} \sim c n^{-1}, \quad (\text{C.10})$$

with $\gamma_0, \gamma_\infty > -1$. We want to estimate $\mathbb{P}(\rho_x^{(k)} > \varepsilon_n e^{-\lambda k/2} d_n; c_{x-1} < e^{q_n^2}$ or $c_{x+k}^{-1} < e^{q_n^2}$).

We will deal with $\mathbb{P}(\rho_x^{(k)} > \varepsilon_n e^{-\lambda k/2} d_n; c_{x+k}^{-1} < e^{q_n^2})$, the remaining part being similar. First of all, using (C.10), we see that, using Potter’s bound

$$\begin{aligned} \mathbb{P}\left(\frac{c_{x-1}}{c_{x+k}} > \varepsilon_n e^{\lambda k/2} d_n; c_{x+k}^{-1} < 1\right) &\leq \mathbb{P}(c_{x-1} > \varepsilon_n e^{\lambda k/2} d_n) \\ &\leq \frac{c}{n} \varphi_0(\log n)^{-1} (\log n)^{-(1+\gamma_0)} \varepsilon_n^{-2\alpha} e^{-\alpha\lambda k/4}. \end{aligned}$$

Provided that δ has been fixed small enough, summing this quantity over $x \in [-n, n]$ and over $k \geq 0$ we obtain something that goes to 0 as $n \rightarrow \infty$ (recall $\varepsilon_n = q_n^{-\delta} = (\log n)^{-\delta/4}$). We can therefore restrict to the event $c_{x+k}^{-1} > 1$. We bound

$$\begin{aligned} \mathbb{P}\left(\frac{c_{x-1}}{c_{x+k}} > \varepsilon_n e^{\lambda k/2} d_n; 1 < c_{x+k}^{-1} < e^{q_n^2}\right) &= \mathbb{E}\left[\mathbb{P}\left(\frac{c_{x-1}}{c_{x+k}} > \varepsilon_n e^{\lambda k/2} d_n \mid c_{x+k}\right) \mathbf{1}_{\{1 < c_{x+k}^{-1} < e^{q_n^2}\}}\right] \\ &\leq d_n^{-\alpha} \varepsilon_n^{-2\alpha} e^{-\alpha\lambda k/4} \mathbb{E}\left[L_\infty(d_n c_{x+k}) c_{x+k}^{-\alpha} \mathbf{1}_{\{1 < c_{x+k}^{-1} < e^{q_n^2}\}}\right]. \end{aligned} \quad (\text{C.11})$$

Since $L_\infty(d_n c_{x+k}) = \varphi_\infty(\log(d_n c_{x+k}))(\log(d_n c_{x+k}))^{\gamma_\infty} \leq c' \varphi_\infty(\log n)(\log n)^{\gamma_\infty}$ under the condition that $1 < c_{x+k}^{-1} < e^{q_n^2}$, we are left to control the last expectation in (C.11). Analogously to (C.7):

$$\begin{aligned} \mathbb{E}\left[c_{x+k}^{-\alpha} \mathbf{1}_{\{1 < c_{x+k}^{-1} < e^{q_n^2}\}}\right] &\leq 1 + \int_1^{e^{\alpha q_n^2}} \mathbb{P}(c_{x+k}^{-\alpha} > t) dt \\ &\leq 1 + c \int_0^{q_n^2} L_0(e^u) du \leq c' \varphi_0(q_n^2) q_n^{2(1+\gamma_0)}. \end{aligned}$$

Putting this back into (C.11) (and using Potter's bound), we obtain

$$\begin{aligned} \mathbb{P}\left(\frac{c_x-1}{c_x+k} > \varepsilon_n e^{\lambda k/2} d_n; 1 < c_{x+k}^{-1} < e^{q_n^2}\right) &\leq c'' d_n^{-\alpha} \varepsilon_n^{-2\alpha} e^{-\alpha \lambda k/4} \varphi_\infty(\log n) (\log n)^{\gamma_\infty} q_n^{2(1+\gamma_0)} \varphi_0(q_n^2) \\ &\leq \frac{c'''}{n} e^{-\alpha \lambda k/4} (\log n)^{-\frac{1}{2}(1+\gamma_0)} \frac{\varphi_0(q_n^2)}{\varphi_0(\log n)} \varepsilon_n^{-2\alpha}, \end{aligned}$$

where we have used (C.10) and the fact that $q_n = (\log n)^{1/4}$. Provided that δ has been fixed small enough (recall $\varepsilon_n = q_n^{-\delta}$), summing the last quantity over $x \in [-n, n]$ and over $k \geq 0$ we obtain again something that goes to 0 as $n \rightarrow \infty$, concluding the proof.

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