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Infinite-time observability of the wave equation with time-varying observation domains under a geodesic recurrence condition

Cyril LETROUIT^{*†}

April 19, 2019

Abstract

Our goal is to relate the observation (or control) of the wave equation on observation domains which evolve in time with some dynamical properties of the geodesic flow. In comparison to the case of static domains of observation, we show that the observability of the wave equation in any dimension of space can be improved by allowing the domain of observation to move. We first prove that, for any domain Ω satisfying a geodesic recurrence condition (GRC), it is possible to observe the wave equation in infinite time on a ball of radius ε moving in Ω at finite speed v , where $\varepsilon > 0$ and $v > 0$ can be taken arbitrarily small, whereas the wave equation in Ω may not be observable on any static ball of radius ε . We comment on the recurrence condition: we give examples of Riemannian manifolds (Ω, g) for which (GRC) is satisfied, and, using a construction inspired by the Birkhoff-Smale homoclinic theorem, we show that there exist Riemannian manifolds (Ω, g) for which (GRC) is not satisfied. Then we prove that on the 2-dimensional torus and on Zoll manifolds, it is possible to observe the wave equation in finite time with moving balls. Finally, we establish a result of spectral observability (or of concentration of eigenfunctions) on time-dependent domains.

Contents

1	Introduction and main results	2
1.1	Infinite-time observability: main result	6
1.2	Comments on (GRC)	6
1.2.1	Examples of domains Ω satisfying (GRC)	6
1.2.2	(GRC) is not always verified	7
1.3	Comments on infinite-time observability	7
1.4	Exact observability in finite time	8
1.5	Spectral observability	9
2	Proofs	10
2.1	Proof of Theorem 1	10
2.2	Proof of Proposition 1	11
2.3	Proof of Proposition 2	11
2.4	Proof of Proposition 3	11
2.5	Proof of Proposition 4	12
2.6	Proof of Theorem 2	15
2.7	Proof of Proposition 5	16
2.8	Proof of Theorem 3	17

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1 Introduction and main results

Observation of waves. The study of controllability and observability properties for the wave equation goes back at least to the work of Russell [Rus71a], [Rus71b]. By exact controllability in time $T > 0$ for the wave equation

$$\partial_{tt}^2 u - \Delta u = \chi_\omega f \quad (1)$$

on a Riemannian manifold (Ω, g) with or without boundary controlled in an open subset $\omega \subset \Omega$, we mean that given an initial state (u_0, u_1) and a final state (u_0^F, u_1^F) , it is possible to find a control f such that the solution of (1) with initial datum $(u_{t=0}, \partial_t u_{t=0}) = (u_0, u_1)$ verifies $(u_{t=T}, \partial_t u_{t=T}) = (u_0^F, u_1^F)$. Typically, in case of Dirichlet boundary conditions, we take the initial datum (u_0, u_1) in the energy space $H_0^1(\Omega) \times L^2(\Omega)$ and we seek $f \in L^2((0, T) \times \Omega)$.

By duality, exact controllability of the wave equation is equivalent to the observability inequality

$$C(\|u|_{t=0}\|_{H^1(\Omega)}^2 + \|\partial_t u|_{t=0}\|_{L^2(\Omega)}^2) \leq \int_0^T \|\partial_t u\|_{L^2(\omega)}^2 dt \quad (2)$$

for any solution u of the free wave equation $\partial_{tt}^2 u - \Delta u = 0$ where $C > 0$ does not depend on u . This last inequality means that it is possible, from the observation of u in the region ω , to recover u in the whole manifold Ω , with an accuracy which is measured by the best possible constant C such that (2) is satisfied.

If the open set ω satisfies the so-called geometric control condition (GCC) in time T , which roughly means that all rays of geometric optics in Ω meet ω in time at most T , then the results of [RTP74] and [BLR92] show that the infimum of all possible times T such that (2) is verified coincides with the infimum of the times T such that the geometric control condition is verified in time T . The geometric control condition is "almost" necessary and sufficient (see [HPT]).

The time-dependent geometric control condition (t -GCC). It is natural to generalize GCC to a time-dependent setting, i.e. for a domain of observation ω that is allowed to move in time. In other words, the domain of observation is now a measurable subset Q of $(0, T) \times \bar{\Omega}$, which is not necessarily a cylinder $(0, T) \times \omega$ as in the time-independent setting.

The search for time-varying observation domains is motivated for instance by seismic exploration, in order to address situations in which all sensors cannot be active at the same time. In many practical examples, it is also possible to move sensors in order to get better precision in the inverse problems which arise while trying to recover data.

The observability inequality (2) is now replaced by

$$C(\|u|_{t=0}\|_{H^1(\Omega)}^2 + \|\partial_t u|_{t=0}\|_{L^2(\Omega)}^2) \leq \iint_Q |\partial_t u|^2 dt dx. \quad (3)$$

The corresponding generalization of the geometric control condition is intuitive: it says that for any ray of geometric optics $t \mapsto y(t)$ in Ω , there exists a time $t \in (0, T)$ such that $(t, y(t)) \in Q$. This condition, denoted t -GCC, is called the time-dependent geometric control condition. The main result of [LRLTT17] says that if the t -GCC is verified for an open set Q , then (3) holds for any solution of the free wave equation $\partial_{tt}^2 u - \Delta u = 0$ with Dirichlet boundary conditions.

Main problem. According to the results mentioned above, it is possible to find some domains Ω and $\omega \subset \Omega$ for which the wave equation in Ω is not observable on ω , mostly in the case where ω does not satisfy GCC (we give an explicit example below). However, by making ω time-dependent, we wonder whether it is always possible or not to make the wave equation observable on $\omega(t)$ (i.e., find $Q \subset (0, T) \times \Omega$ which satisfies t -GCC), at least in infinite time. In

other words, we wonder whether observability can be improved by considering time-varying observation domains, may it be in infinite time.

This issue was raised as an open problem in [LRLTT17, Section 3E] in the following form. Let Ω be a bounded domain of \mathbb{R}^n (i.e., a bounded connected open set), let $x_0 \in \Omega$ be a point and let $\varepsilon > 0$ be a (small) positive real number. Let $v > 0$ be arbitrary. Given a path $t \mapsto x(t)$ in Ω , we define $\omega(t) = B(x(t), \varepsilon)$ the geodesic ball of radius ε centered at $x(t)$, with $x(0) = x_0$. We say that the path $x(\cdot)$ is admissible if $\omega(t) \subset \Omega$ for every time t . The question reads as follows: do there exist $T \in (0, +\infty]$ and an admissible C^1 path $t \mapsto x(t)$ in Ω , with speed less than or equal to v , such that (Q, T) satisfies the t -GCC, where $Q = \{(t, x) \mid t \in (0, T), x \in B(x(t), \varepsilon)\}$?

In this paper, we consider this problem more generally in the case of Ω being a manifold with or without boundary, so that the geodesic flow which has to be considered in the GCC or the t -GCC is the (generalized) bicharacteristic flow.

Moreover, to encompass the case of domains Ω with a boundary which is smooth but folded at a scale smaller than ε (making for example small meanders), we relax the condition of $\omega(t)$ being a ball and consider domains of observation of the form $\omega(t) = B(x(t), \varepsilon) \cap \Omega$, where $B(x(t), \varepsilon)$ is the open ball of center $x(t) \in \Omega$ and radius ε with respect to the geodesic distance, so that $\omega(t)$ is not repelled by the frontiers of Ω . Our goal is to find $\omega(t)$ of this form which meets all geodesic rays of Ω when time ranges over $[0, T]$. Note that in our setting, the observation set Q is not any measurable set of $(0, T) \times \bar{\Omega}$, since it has to take the form of a moving open ball.

An example of a manifold Ω such that the wave equation is not observable on any ball of sufficiently small radius ε is the two-dimensional torus $\Omega = \mathbb{T}^2$ with the flat metric on it. For any sufficiently small ball, there is a periodic geodesic which does not meet this ball, and thus GCC is not verified. Although GCC is not in general a necessary condition for observability, it is possible to infer in this case (see for example [HPT]) that the wave equation is not observable on such a small ball. In Theorem 1, we will however show that if we allow the small ball to move (even with very low speed) in \mathbb{T}^2 , it is always possible to make the wave equation observable in infinite time on this moving domain.

We now give precise definitions and recall the results which will be used in the sequel.

Setting. We adopt the same setting as in [LRLTT17]. We recall it here for the sake of completeness. Let (M, g) be a smooth n -dimensional Riemannian manifold with $n \geq 1$. Let Ω be a bounded open connected subset of M , with a smooth boundary if $\partial\Omega \neq \emptyset$. We consider the wave equation

$$\partial_{tt}^2 u - \Delta_g u = 0 \tag{4}$$

in $\mathbb{R} \times \Omega$, where Δ_g denotes the Laplace-Beltrami operator on (M, g) . If the boundary $\partial\Omega$ of Ω is nonempty, then we consider boundary conditions of the form

$$Bu = 0 \quad \text{on } \mathbb{R} \times \partial\Omega \tag{5}$$

where the operator B is either

- the Dirichlet trace operator, $Bu = u|_{\partial\Omega}$;
- or the Neumann trace operator, $Bu = \partial_n u|_{\partial\Omega}$, where ∂_n is the outward normal derivative along $\partial\Omega$.

In the case of a manifold without boundary or in the case of homogeneous Neumann boundary conditions, the Laplace-Beltrami operator is not invertible on $L^2(\Omega)$ but is invertible in

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega) \mid \int_{\Omega} u(x) dx_g = 0 \right\}.$$

In what follows, we set $X = L_0^2(\Omega)$ in the boundaryless case or in the Neumann case, and $X = L^2(\Omega)$ in the Dirichlet case (in both cases, the norm on X is the usual L^2 -norm). We denote by $A = -\Delta_g$ the operator defined on the domain

$$D(A) = \{u \in X \mid Au \in X \text{ and } Bu = 0\}$$

with one of the above boundary conditions whenever $\partial\Omega \neq \emptyset$. We refer to [LRLTT17] for an explicit description of $D(A)$, $D(A^{1/2})$ and of $D(A^{1/2})'$.

For all $(u^0, u^1) \in D(A^{1/2}) \times X$, there exists a unique solution $u \in C^0(\mathbb{R}; D(A^{1/2})) \cap \mathcal{C}(\mathbb{R}; X)$ of (4)-(5) such that $u|_{t=0} = u^0$ and $\partial_t u|_{t=0} = u^1$. Such solutions of (4)-(5) are understood in the weak sense.

Let Q be an open subset of $\mathbb{R} \times \bar{\Omega}$. We set

$$\omega(t) = \{x \in \Omega \mid (t, x) \in Q\}.$$

Let $T \in (0, +\infty]$ be arbitrary. We say that (4)-(5) is observable on Q in time T if there exists $C > 0$ such that

$$C \| (u|_{t=0}, \partial_t u|_{t=0}) \|_{D(A^{1/2}) \times X}^2 \leq \int_0^T \int_{\omega(t)} |\partial_t u(t, x)|^2 dx_g dt \quad (6)$$

for any solution u of (4)-(5). The integral at the right-hand side of (6) is allowed to be infinite, when $T = +\infty$.

When $\partial\Omega \neq \emptyset$, the usual notions of geodesics and of bicharacteristics have to be generalized in order to take into account the reflections on $\partial\Omega$. Generalized geodesics are usual geodesics in Ω , and they reflect on $\partial\Omega$ according to the laws of geometric optics. This generalization is called the generalized bicharacteristic flow of Melrose and Sjöstrand, see [MS78]. We do not recall the construction but simply mention that, setting $Y = \mathbb{R} \times \bar{\Omega}$, a generalized bicharacteristic ${}^b\gamma : \mathbb{R} \rightarrow {}^bT^*Y$ is a continuous map which is uniquely determined if it has no point in G^∞ , the set of cotangent vectors with contact of infinite order. Using t as a parameter, generalized geodesics for Ω , traveling at speed 1, are then the projection on M of generalized bicharacteristics.

The time-dependent GCC is defined as follows.

Definition 1. *Let Q be an open subset of $\mathbb{R} \times \bar{\Omega}$, and let $T \in (0, +\infty]$. We say that (Q, T) satisfies the time-dependent geometric condition (in short, t -GCC) if every generalized bicharacteristic ${}^b\gamma : \mathbb{R} \rightarrow {}^bT^*Y$, $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ is such that there exists $s \in \mathbb{R}$ such that $t(s) \in (0, T)$ and $(t(s), x(s)) \in Q$. We say that Q satisfies the t -GCC if there exists $T \in (0, +\infty]$ such that (Q, T) satisfies the t -GCC. When $T = +\infty$, we speak of t -GCC in infinite time.*

If there exists $0 < T < +\infty$ such that (Q, T) satisfies t -GCC, the control time $T_0(Q, \Omega)$ is defined by

$$T_0(Q, \Omega) = \inf\{T \in (0, +\infty) \mid (Q, T) \text{ satisfies the } t\text{-GCC}\}.$$

The main theorem of [LRLTT17] states:

Let Q be an open subset of $\mathbb{R} \times \bar{\Omega}$ that satisfies the t -GCC. Let $T \in (T_0(Q, \Omega), +\infty)$. When $\partial\Omega \neq \emptyset$, we assume moreover that no generalized bicharacteristic has a contact of infinite order with $(0, T) \times \partial\Omega$, that is, $G^\infty = \emptyset$. Then the observability inequality (6) holds.

In Appendix A, we extend this result to the case where $T = +\infty$, i.e., to the case of infinite-time observability. Indeed, the main result of [LRLTT17] has been established for finite-time observability, but actually, following their argument and using microlocal defect measures on the compressed cotangent bundle, one can show that t -GCC in infinite time implies infinite-time observability. This fact is not obvious when one thinks of a proof based on the Egorov theorem, but there is actually no problem in extending [LRLTT17], as we briefly show in Appendix A. The notion of infinite-time observability is important in the sequel and we will comment further on it in Section 1.3.

With these results at hand, our goal is therefore to construct a moving ball ω which captures all geodesics traveling at speed 1 in Ω .

Remark 1. If $\partial\Omega$ is non-empty and not smooth, the generalized bicharacteristic flow is not necessarily well defined since there may be no uniqueness of a bicharacteristic at the points where $\partial\Omega$ is not smooth. However, when uniqueness is ensured (for example in the case of a rectangle), the above result is still true (see [LRLTT17, Remark 1.9] for more comments on this issue).

Class of moving domains. The definition of a moving domain that we adopt here is physical and adapted to our main goal, which is to show that it is possible to improve observability by making a domain ω move in time.

Definition 2. Let $\varepsilon > 0$ be a real number, $T \in (0, +\infty]$ be a fixed time and $x : [0, T] \rightarrow \Omega$ be a C^1 path (if $T = +\infty$, we take $x : [0, +\infty) \rightarrow \Omega$). The moving domain $\omega(t)$ associated with the path $x(t)$ is defined for every $t \in [0, T]$ (or $t \in [0, +\infty)$ if $T = +\infty$) by $\omega(t) = B(x(t), \varepsilon) \cap \Omega$, where $B(x(t), \varepsilon)$ is the open ball of center $x(t)$ and of radius ε with respect to the geodesic distance in Ω .

In fact, all our results work in the context of translated domains, meaning that given open bounded sets $\omega \subset \Omega \subset \mathbb{R}^n$, a point $x_0 \in \mathbb{R}^n$ and a C^1 -path $x : [0, T] \rightarrow \Omega$, we define $\omega(t)$ for every $t \in [0, T]$ as $\omega(t) = (x(t) - x_0 + \omega) \cap \Omega$. It is then possible to find appropriate formulations for our results to encompass this slight generalization. For the sake of simplicity we keep Definition 2 for our moving domains.

Note also that we require all our paths $t \mapsto x(t)$ to be C^1 because we need to define paths with bounded speed.

Role of the speed v . In our results, the speed $v = \sqrt{g(\dot{x}(t), \dot{x}(t))}$ plays a key role. Let us first remark that if v is allowed to be very large, then it is easy to construct a moving domain $\omega(t)$ on which the wave equation is observable (even in finite time). For example, if we take a path $x(t)$ which, within very short time (so that the geodesic rays do not have time to move much), passes near any point in Ω , then the associated moving domain $\omega(t) = B(x(t), \varepsilon) \cap \Omega$ meets any geodesic ray. More precisely if $\varepsilon > 0$ is fixed, and the C^1 path $x : [0, \varepsilon/2] \rightarrow \Omega$ verifies that for any $x \in \Omega$, there exists $t \in [0, \varepsilon/2]$ such that $|x(t) - x| < \varepsilon/4$, then the moving domain $\omega(t) = B(x(t), \varepsilon) \cap \Omega$ meets any geodesic ray. Of course, in this case, the speed $|\dot{x}(t)|$ is very large (of the order of ε^{-1}) and is therefore not comparable with the speed of the geodesic rays (which is fixed to 1).

Therefore, our results have to be established for a speed v bounded independently of ε , or, even better, for any speed $v > 0$. This is the case in our theorems.

Bibliography. There is not much literature about observation of the wave equation on moving domains. The first paper to address this question (in one dimension of space) seems to be [Kha95]. More recently, in [LRLTT17] the authors have proved that the t -GCC condition is a necessary and sufficient condition for controllability of n -dimensional waves by a moving domain. In [CCM14] the authors proved the same result for the one-dimensional wave equation, and then characterized the minimal norm controls. Finally, the paper [Cas13] gives sufficient conditions on the trajectory of a moving interior point of an interval to ensure the controllability of the wave equation.

Organization of the paper. The paper is organized as follows. In Section 1, we state the main results. Namely, in Section 1.1, we state a theorem on observation of the wave equation in Ω on moving domains $\omega(t)$ in infinite time under a certain geodesic recurrence condition (GRC) on Ω . In Section 1.2 we give examples of manifolds Ω satisfying (GRC) and, using a construction inspired by the Birkhoff-Smale homoclinic theorem, we build an example of manifold Ω which does not satisfy (GRC). In Section 1.3 we comment on the notion of infinite-time observability. In Section 1.4 we build specific bounded domains Ω for which we can construct a moving domain $\omega(t)$ on which the wave equation is observable even in finite time. In Section 1.5, we prove a result of spectral observability (or of concentration

of eigenfunctions) which is an analog of our previous results in a quantum setting. Section 2 is devoted to proving our results.

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1.1 Infinite-time observability: main result

Our first result deals with observability in infinite time. It roughly says that if for each geodesic trajectory $t \mapsto y(t)$ in Ω , there exists a small open (time-independent) set ω where the trajectory spends (asymptotically) some positive part of its time, then we can construct a ball-shaped domain $\omega(t)$ moving at a speed v as small as we want and on which the wave equation is observable.

Definition 3. Let $\varepsilon > 0$ be a (small) positive real number. We say that the generalized geodesic $t \mapsto y(t)$ in Ω satisfies the Geodesic Recurrence Condition (GRC) for ε if

(GRC) there exists an open set $\omega \subset \Omega$ (depending on $y(\cdot)$) contained in a ball of radius $< \varepsilon$ such that

$$\liminf_{T \rightarrow +\infty} \frac{|\{t \in [0, T], y(t) \in \omega\}|}{T} > 0. \quad (7)$$

Theorem 1. Let $\varepsilon > 0$ be a positive real number such that any generalized geodesic $t \mapsto y(t)$ in Ω satisfies (GRC) for ε . Let $v > 0$ be a fixed speed. Then there exists a C^1 path $x : [0, +\infty) \rightarrow \Omega$ with speed bounded by v (meaning that for every $t \geq 0$, $|\dot{x}(t)| \leq v$) such that the moving domain $\omega(t)$ defined by $\omega(t) = B(x(t), \varepsilon) \cap \Omega$ satisfies the t -GCC for $T = +\infty$.

As a corollary of Theorem 1 and Appendix A, we have:

Corollary 4. The wave equation is infinite-time exactly observable on $\omega(\cdot)$, i.e., (6) is satisfied for $T = +\infty$.

Determining which domains Ω satisfy (GRC) seems to be a difficult question in general. In Section 1.2 we give some examples of domains Ω for which (GRC) is verified.

Remark 2. Remark that a given geodesic $t \mapsto y(t)$ verifies (GRC) if and only if for any $s \in \mathbb{R}$, the geodesic $t \mapsto y(t+s)$ satisfies (GRC): the fact that (GRC) is verified depends only the trace of the geodesic since it is invariant by translations in time. We note that our proof of Theorem 1 can easily be extended to the case where all traces of geodesics but a countable number satisfy (GRC). For the sake of simplicity, we did not include this extension in the statement of the theorem.

1.2 Comments on (GRC)

We first give some examples of bounded domains Ω for which all geodesics satisfy (GRC), so that Theorem 1 applies. Then, we show that (GRC) is not always satisfied, by constructing an explicit counterexample. Of course, this does not mean that on this domain the wave equation cannot be observed in infinite-time on any moving domain, but only that our construction does not cover this case.

1.2.1 Examples of domains Ω satisfying (GRC)

We start by presenting a class of domains Ω for which (GRC) is satisfied.

Definition 5. A bounded domain (Ω, g) satisfies the dichotomy property if each of its geodesics is either periodic or uniformly distributed, meaning that for every open set $\omega \subset \Omega$,

$$\lim_{T \rightarrow +\infty} \frac{|\{t \in [0, T], y(t) \in \omega\}|}{T} = \frac{\text{vol}_g(\omega)}{\text{vol}_g(\Omega)}.$$

Typical examples are the square and the rectangles with the flat metric. More generally, any polygon that tiles the plane by reflection has the dichotomy property. In fact, this property is satisfied by all the "lattice examples" (see [Smi00]).

Proposition 1. If (Ω, g) satisfies the dichotomy property, then all its geodesics satisfy (GRC).

If (Ω, g) satisfies a weaker property, namely that each geodesic is either periodic or uniformly distributed in some open subset $\Omega' \subset \Omega$, the same proof shows that each geodesic of Ω also satisfies (GRC), so that Theorem 1 also applies. With the same argument, we can prove the following proposition.

Proposition 2. Any geodesic of the two-dimensional disk satisfies (GRC) but the two-dimensional disk with flat metric does not satisfy the dichotomy property.

1.2.2 (GRC) is not always verified

A natural question is to wonder whether (GRC) is always verified. We will see that it is not the case : using ideas coming from the Birkhoff-Smale homoclinic theorem, we construct an example in which an uncountable number of geodesics do not satisfy this condition. We start by giving a property which is always satisfied and which is somewhat weaker than (GRC) (although not exactly because the parameter ε is fixed in (GRC) and arbitrary in Proposition 3).

Proposition 3. For any geodesic $t \mapsto y(t)$ and any $\varepsilon > 0$, there exists a ball $B \subset \Omega$ of radius ε and an increasing sequence of times $(T_n)_{n \in \mathbb{N}^*}$ tending to $+\infty$ such that

$$\liminf_{n \rightarrow +\infty} \frac{|\{t \in [0, T_n], y(t) \in B \cap \Omega\}|}{T_n} > 0.$$

This proposition roughly means that (GRC) is verified up to a subsequence. However, the following proposition shows that (GRC) is not always satisfied.

Proposition 4. There exist $\varepsilon > 0$ and a bounded open subset Ω of \mathbb{R}^2 with smooth boundary $\partial\Omega$ such that an uncountable number of its geodesics do not satisfy (GRC).

Note that, since (GRC) in Theorem 1 is only a sufficient condition for infinite-time observability of the wave equation on a time-varying domain, Proposition 4 does not mean that the wave equation in a domain Ω satisfying Proposition 4 is not observable on any ball-shaped moving domain $\omega(t)$.

To sum up, (GRC) fails for some domains, and in this case it does not seem easy to construct a moving domain $\omega(t)$ on which the wave equation is observable. We are not able to use a weaker property than (GRC) (like Proposition 3) to adapt the proof of Theorem 1 to a larger context.

1.3 Comments on infinite-time observability

We now comment on the notion of infinite-time observability used in Theorem 1, which is equivalent to the strict positivity of the Gramian matrix. This notion appears for example in [TW09, Section 6.5] (see also [TW09, Section 5.1]), with the following statement:

For an exponentially stable semigroup \mathbb{S} , infinite-time observability is equivalent to exact observability in finite time. But this equivalence fails in general when the semigroup is not exponentially stable.

What is meant in this context by exponentially stable semigroup is that $\mathbb{S} = (\mathbb{S}_t)_{t \geq 0}$ is a strongly continuous semigroup with growth bound $\omega_0(\mathbb{S}) = \inf_{t \in (0, \infty)} \frac{1}{t} \log \|\mathbb{S}_t\| < 0$.

The semigroup associated to the wave equation is clearly not exponentially stable, and therefore the above statement does not apply. We conjecture that in the case of the wave equation, infinite-time observability is not equivalent to finite-time observability.

Conjecture 1. *There exists a compact two-dimensional manifold with smooth boundary Ω and a (time-independent) open subset $\omega \subset \Omega$ such that the wave equation in Ω is exactly observable in infinite time on ω but not exactly observable on ω in time T for any finite $T > 0$.*

More precisely, let Ω be the Sinai billiard, that is, a torus with a circular obstacle at its center (see Figure 1 below). We consider two vertical lines in Ω , the first one being the trace of a periodic geodesic γ in Ω and the second one (at its right on Figure 1) being close to it. We consider a domain of observation ω (depicted by the region with red lines on Figure 1) delimited by these two vertical lines. It is easy to see that there exists a geodesic γ' in Ω which does not meet ω for any positive time T , and that the periodic geodesic γ is asymptotic to γ' . From that, one can deduce that the wave equation in Ω is not observable on ω in any finite time $T > 0$. However since γ' comes closer and closer to ω as time goes to $+\infty$, we conjecture that the wave equation in Ω is observable in infinite time on ω .

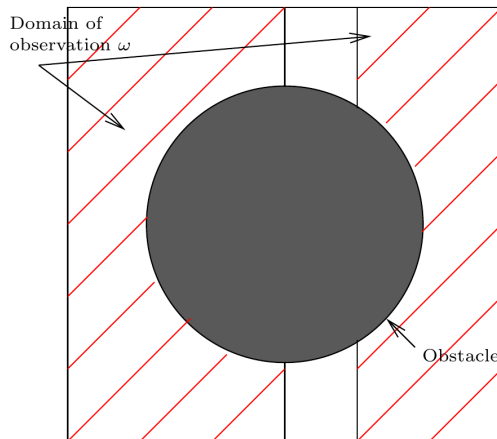


Figure 1: The conjectured domain of observation ω in the Sinai billiard

Conjecture 1, if true, would mean in particular that we cannot reduce easily the infinite time needed for observability in Theorem 1 to a finite-time observability result.

Remark 3. Note that there are other possible definitions of infinite-time observability, such as time-asymptotic observability (see [PTZ16]).

1.4 Exact observability in finite time

As already said, an example of a manifold Ω such that the wave equation is not observable on any ball of sufficiently small radius ε is the two-dimensional torus $\Omega = \mathbb{T}^2$. In Theorem 1, we proved that if we allow the small ball to move, it is possible to construct such a moving domain on which the wave equation is observable in infinite time. In the following theorem, we improve Theorem 1 by proving finite time observability for two different (types of) Riemannian manifolds Ω (namely the 2-dimensional torus and the Zoll manifolds). It means that we construct a domain moving in Ω with finite speed such that the wave equation is exactly observable in finite time. We recall that a Zoll manifold is a manifold all of which geodesics are periodic.

Theorem 2. *Let $\varepsilon > 0$ and $v > 0$ be arbitrary and let (Ω, g) be either a Zoll manifold or the 2-dimensional torus \mathbb{T}^2 with the flat metric. Then there exist a time $T > 0$ and a C^1 path $x : [0, T] \rightarrow \Omega$ with speed at most v (i.e., $\forall t \geq 0, |\dot{x}(t)| \leq v$) such that the moving domain $\omega(t)$ defined by $\omega(t) = B(x(t), \varepsilon)$ satisfies the t -GCC for $t = T$, meaning that the wave equation is observable on $\omega(t)$ (in finite time T).*

1.5 Spectral observability

Let Ω be a bounded open connected subset of M , with a smooth boundary if $\partial\Omega \neq \emptyset$. Let $(\phi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of normalized eigenfunctions of the opposite of the laplacian $-\Delta_g$ on Ω , associated to the eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$. For a measurable domain of observation $\omega \subset \Omega$, the quantity

$$C_{T,spec}(\omega) = \inf_{j \in \mathbb{N}^*} \int_0^T \int_{\omega} \phi_j(x)^2 dx dt = T \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 dx \quad (8)$$

is a spectral analog of the observability constant.

For a moving domain $\omega(t)$, we define analogously

$$C_{T,spec}(\omega(t)) = \inf_{j \in \mathbb{N}^*} \int_0^T \int_{\omega(t)} \phi_j(x)^2 dx dt. \quad (9)$$

The constants defined in (8) and (9) are called spectral observability constants of ω and $\omega(t)$ respectively.

Remark 4. The constant (8) is a well-known quantity (up to a factor T), see [BZ04] and [HHM09] for example. It measures the concentration of eigenfunctions on ω . In the work [PTZ16] reviewed in [Tré18] it was interpreted as a "randomized observability constant" since it naturally appears when one randomizes the initial datum of the wave equation.

Our goal is to prove for these spectral constants results of the same kind as the ones proved in paragraphs 1.1 and 1.4. Again, we will only consider ball-shaped domains of observation ω (or more precisely balls intersected with Ω) and for $\omega(t)$, we require them to satisfy definition 2, which means that the $\omega(t)$'s are moving balls (again intersected with Ω).

From now, we fix a (small) $\varepsilon > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. We need the following definitions.

Definition 6. *For $T > 0$ and $v > 0$, we set*

$$A_{fix} = \{\omega \subset \Omega \mid \exists x \in \Omega, \omega = B(x, \varepsilon) \cap \Omega\}$$

and

$$A_{mov}^{T,v} = \{\omega(t) \subset \Omega \mid \exists x : [0, T] \rightarrow \Omega \text{ a } C^1 \text{ path of speed } |\dot{x}(t)| \leq v, \omega(t) = B(x(t), \varepsilon) \cap \Omega\}.$$

We want to know whether the observation on a well-chosen moving domain can be better than the observation on any fixed domain. Therefore, according to (8) and (9), if we fix $T > 0$ and $v > 0$, we want to compare

$$C_{fix}^T = \sup_{\omega \in A_{fix}} \inf_{j \in \mathbb{N}^*} T \int_{\omega} \phi_j(x)^2 dx \quad (10)$$

and

$$C_{mov}^{T,v} = \sup_{\omega(t) \in A_{mov}^{T,v}} \inf_{j \in \mathbb{N}^*} \int_0^T \int_{\omega(t)} \phi_j(x)^2 dx dt. \quad (11)$$

The inequality $C_{fix}^T \leq C_{mov}^{T,v}$ is obvious. Our goal is in some sense to understand in which cases $C_{fix}^T < C_{mov}^{T,v}$.

Proposition 5. *The supremum in ω (resp. $\omega(t)$) in (10) (resp. in (11)) is reached, but the infima in j in (10) and (11) may not be reached.*

Our main result in this section is that for T sufficiently large compared to v , the constant $C_{mov}^{T,v}$ is strictly positive, although the constant C_{fix}^T may be equal to 0.

Theorem 3. *There exists a compact Riemannian manifold Ω such that for any sufficiently small $\varepsilon > 0$ and any $T > 0$, we have $C_{fix}^T = 0$. Moreover, there also exists a compact Riemannian manifold Ω verifying that for any sufficiently small $\varepsilon > 0$, there exists $D > 0$ such that if $vT \leq D$, we have $C_{mov}^{T,v} = C_{fix}^T$. Lastly, for any Riemannian manifold (M, g) , any bounded open connected subset Ω of M with a smooth boundary if $\partial\Omega \neq \emptyset$, and for any $\varepsilon > 0$, there exists $C > 0$ such that if $vT \geq C$, then $C_{mov}^{T,v} > 0$.*

2 Proofs

2.1 Proof of Theorem 1

Let $\varepsilon > 0$ and $v > 0$. We consider a dense sequence of points $(x_i)_{i \in \mathbb{N}^*}$ in Ω , we set $B_i = B(x_i, \varepsilon)$ for $i \in \mathbb{N}^*$ and we define $A_i = B_i \cap \Omega$ for $i \in \mathbb{N}^*$. Then we construct $x : \mathbb{R}^+ \rightarrow \Omega$ a map of speed $|\dot{x}(t)| \leq v$ in the following way. The point $x(t)$ will successively stay on the points x_i in the following order (we will precise the time it stays on each point later):

$$x_1, x_2, x_1, x_2, x_3, x_4, x_1, x_2, \dots, x_8, x_1, x_2, \dots, x_{16}, \dots, x_1, \dots, x_{2^n}, x_1, \dots$$

and this sequence continues until infinity. We call each of these positions a "step": for example, at step 1, $x(t) = x_1$, at step 2, $x(t) = x_2$, at step 3, $x(t) = x_1$, etc. Of course, between two steps, there is a smooth transition: the point $x(t)$ goes from one x_i to the following. Then we have to specify how much time $x(t)$ stays on each x_j (i.e., the time duration of each step): we require that if $x(t)$ arrives at step j at time t , then step j lasts $t2^j$ seconds (much more time than all the time already passed). Lastly, we take $\omega(t) = B(x(t), \varepsilon) \cap \Omega$. This construction implies that for each $i \in \mathbb{N}^*$, the following assertion is true:

(B) : For every constant $0 < K < 1$ and every $T > 0$, there exist $T'' > T' > T$ such that $\frac{T'' - T'}{T''} > K$ and $x(t) = x_i$ for $t \in [T', T'']$.

Now consider a geodesic $t \mapsto y(t)$ in Ω . We will show that there exists $t \geq 0$ such that $y(t) \in \omega(t)$. Let ω satisfy (GRC) for the trajectory $t \mapsto y(t)$. Since ω is contained in a ball of radius $< \varepsilon$, there exists $j \in \mathbb{N}^*$ such that $\omega \subset B_j = B(x_j, \varepsilon)$ with the above notations. (GRC) implies that

$$\liminf_{T \rightarrow +\infty} \frac{|\{t \in [0, T], y(t) \in B_j\}|}{T} \geq 3C > 0$$

for some $C > 0$. This means that the trajectory $y(t)$ spends at least a fraction $3C$ of time in B_j when time goes to infinity. Let T be such that

$$\forall t \geq T, \quad \frac{|\{s \in [0, t], y(s) \in B_j\}|}{t} \geq 2C. \quad (12)$$

By assertion (B) for $K = 1 - C$, there exist $T'' > T' > T$ such that

$$(T'' - T')/T'' > 1 - C \quad \text{and} \quad x(t) = x_j \quad \text{for } t \in [T', T'']. \quad (13)$$

If we take $t = T''$ in (12), we get

$$\frac{|\{s \in [0, T''], y(s) \in B_j = B(x_j, \varepsilon)\}|}{T''} \geq 2C. \quad (14)$$

Combining (13) et (14) we see that there exists a time $t \leq T''$ such that $y(t) \in \omega(t)$. Therefore, any trajectory meets $\omega(t)$ when time goes to infinity and t -GCC is verified in infinite time.

2.2 Proof of Proposition 1

We first note that all closed geodesics in Ω satisfy (GRC). To see it, just fix a periodic geodesic $t \mapsto y(t)$, let $T_1 > 0$ be its minimal period and set $\omega = B(y(0), \varepsilon/2) \cap \Omega$. Then the liminf appearing in (7) is $\geq 1/T_1$ and hence is positive. Secondly, for the uniformly distributed geodesics, any open set $\omega \subset \Omega$ can be used in (GRC). This concludes the proof of Proposition 1.

2.3 Proof of Proposition 2

Let D be an open two-dimensional disk. If we take a sufficiently short chord of the disk which intercepts an angle α which is not commensurable to π , then the geodesic which follows this chord is not periodic (since $\frac{\alpha}{\pi} \notin \mathbb{Q}$) and it is not uniformly distributed (since it does not meet a small disk with the same center as D if the chord is sufficiently short).

However, it is possible to verify that any geodesic of D is either periodic or there exists $0 < r < 1$ (depending on the geodesic) such that any open subset ω of the annulus $D \setminus rD$ satisfies (GRC). To see it, take a geodesic $t \mapsto y(t)$, $t \geq 0$ in D which is not periodic. Its trace on ∂D is a sequence of points x_1, x_2, \dots which are always separated by the same distance. If one sees ∂D as the quotient $\mathbb{R}/\pi\mathbb{Z}$, then the points x_i form an arithmetic sequence which is dense in $\mathbb{R}/\pi\mathbb{Z}$. Moreover, since the oriented angle α made by the forward trajectory $t \mapsto y(t)$ at each x_i with the tangent to ∂D is always the same, it is straightforward to see that there exists $0 < r < 1$ such that $\Omega' = \overline{D} \setminus rD$ is the closure of the trajectory $t \mapsto y(t)$. This oriented angle α also determines, for each point $x \in D$, a unique point $\pi_\alpha(x)$ on ∂D which is the only point for which the segment with ends $\pi_\alpha(x)$ and x makes an oriented angle α with the tangent at $\pi_\alpha(x)$. Let ω be a small ball in Ω' and let $\pi_\alpha^{-1}(\omega)$ be its preimage by π_α . It is an interval on the boundary ∂D . It means that if $x_i \in \pi_\alpha^{-1}(\omega)$, the trajectory $t \mapsto y(t)$ falls in ω while running from x_i to x_{i+1} . Proposition 2 then follows from the uniform distribution of the sequence $(x_i)_{i \in \mathbb{N}^*}$ in ∂D , which is due to $\frac{\alpha}{\pi} \notin \mathbb{Q}$.

2.4 Proof of Proposition 3

Let $t \mapsto y(t)$ be a (generalized) geodesic in Ω (or in $\overline{\Omega}$ if Ω has a boundary). For the sake of simplicity, we assume in what follows that Ω has no boundary, but the proof also works in the case with boundary. We set, for $n \in \mathbb{N}$,

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{y(i)}$$

where δ_x is the Dirac measure in Ω located at x . The measure μ_n is a probability measure on Ω . By Prokhorov's theorem, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ and a probability measure μ on Ω such that $\mu_{n_k} \rightharpoonup \mu$ in the weak-* topology of measures. There exists ω_1 an open ball of radius $\varepsilon/4$ such that $\mu(\omega_1) > 0$. We denote by ω_2 the ball of radius $\varepsilon/2$ with the same center as ω_1 . Let $f : \Omega \rightarrow [0, 1]$ be a continuous function equal to 1 on ω_1 and to 0 outside of ω_2 . Then

$$\int_{\Omega} f d\mu_{n_k} \rightarrow \int_{\Omega} f d\mu > 0$$

as $k \rightarrow +\infty$. But $\mu_{n_k}(\omega_2) \geq \int_{\omega_2} f d\mu_{n_k} = \int_{\Omega} f d\mu_{n_k}$ since f vanishes outside of ω_2 . Therefore

$$\liminf_{k \rightarrow +\infty} \mu_{n_k}(\omega_2) > 0. \tag{15}$$

Note that if $y(i) \in \omega_2$, then $y(t) \in \omega$ for any t satisfying $|t - i| \leq \varepsilon/2$, where ω denotes the ball with the same center as ω_1 and ω_2 . Combining this remark with (15), we get

$$\liminf_{k \rightarrow +\infty} \frac{|\{t \in [0, n_k], y(t) \in \omega\}|}{n_k} > 0,$$

which finishes the proof of Proposition 3.

2.5 Proof of Proposition 4

We consider three circles $\mathcal{C}^0, \mathcal{C}^1$ and \mathcal{C}^2 of the same radius in the plane, whose centers form an equilateral triangle Δ and which do not intersect. These circles will be part of the boundary $\partial\Omega$ of the domain Ω . More precisely, we take Ω to be a domain with smooth boundary $\partial\Omega = \mathcal{C}^0 \cup \mathcal{C}^1 \cup \mathcal{C}^2 \cup \Gamma$ where Γ is a smooth curve which encloses the three circles. Ω is therefore connected but $\partial\Omega$ has four connected components. The precise form of Γ does not matter at all since the geodesic trajectories we will construct only bounce on the obstacles $\mathcal{C}^0, \mathcal{C}^1$ and \mathcal{C}^2 . The billiard Ω is shown in Figure 2 below.

On each side of Δ , there is a periodic trajectory which bounces only on two of the circles $\mathcal{C}^0, \mathcal{C}^1$ and \mathcal{C}^2 . We denote by p the trajectory which bounces only on \mathcal{C}^0 and \mathcal{C}^1 , and by q the trajectory which only bounces on \mathcal{C}^0 and \mathcal{C}^2 . Let finally A be the point located in the middle of the small arc of \mathcal{C}^0 whose extremities are the feet of p and q on \mathcal{C}^0 (see Figure 2).

We will show that there exist $\varepsilon > 0$ and an uncountable number of geodesics of Ω which do not satisfy (GRC). Recall that in our terminology, a geodesic is uniquely determined by a given point $x \in \Omega$ and a given direction $v \in S^1$. Such a datum uniquely determines a geodesic $\gamma : [0, +\infty) \rightarrow \Omega$ with $\gamma(0) = x$.

In our proof, we only consider geodesics starting at A , that is $\gamma(0) = A$. For such a geodesic, we can encode its trajectory by a sequence $(a_n)_{n \in \mathbb{N}}$ on the alphabet $\{0, 1, 2\}$ in an obvious manner, with $a_0 = 0$ since $\gamma(0) = A \in \mathcal{C}^0$. If the i -th circle hit by $\gamma(t)$ is \mathcal{C}^j , then we set $a_i = j$.

From now on, we only consider geodesics whose encoding sequence in the $\{0, 1, 2\}$ alphabet described above verifies $\forall n \in \mathbb{N}, a_{2n} = 0$. It corresponds to geodesics which hit the circle \mathcal{C}^0 alternatively with circles \mathcal{C}^1 and \mathcal{C}^2 . For all these geodesics, we can forget the 0's and just write their coding in the $\{1, 2\}$ alphabet. Said differently, to any such sequence $(a_n)_{n \in \mathbb{N}}$, we associate the sequence $(b_n) \in \{1, 2\}^{\mathbb{N}}$ defined by $b_n = a_{2n+1}$.

The key point of our proof, which uses this new coding, is the following fact.

Fact 1. *For any sequence $(b_n) \in \{1, 2\}^{\mathbb{N}}$, there exists a geodesic whose coding is (b_n) .*

We postpone the proof of Fact 1 to the end of this section. Let us first show how this result can be used in order to prove Proposition 4.

Let γ be a given geodesic which has a coding sequence $(b_n)_{n \in \mathbb{N}}$ in the $\{1, 2\}$ alphabet. We can associate to it a sequence $(c_n)_{n \in \mathbb{N}}$ which counts the number of consecutive 1's at the beginning of (b_n) (which means that $b_k = 1$ for $0 \leq k \leq c_0 - 1$), then the number of consecutive 2's which follow (which means that $b_k = 2$ for $c_0 \leq k \leq c_0 + c_1 - 1$), then the number of consecutive 1's which follow (meaning that $b_k = 1$ for $c_0 + c_1 \leq k \leq c_0 + c_1 + c_2 - 1$), etc.

Definition 7. *We call "alternating geodesic" any geodesic γ for which the sequence $(c_n)_{n \in \mathbb{N}}$ verifies*

$$\lim_{n \rightarrow +\infty} \frac{c_{2n}}{c_0 + c_1 + \dots + c_{2n}} = \lim_{n \rightarrow +\infty} \frac{c_{2n+1}}{c_0 + c_1 + \dots + c_{2n+1}} = 1. \quad (16)$$

We denote by \mathcal{A} the set of all alternating geodesics.

Under Fact 1, we have:

Lemma 6. *\mathcal{A} is uncountable.*

Proof. Thanks to Fact 1, we forget about geodesics and just think in terms of encoding sequences (b_n) . Let us denote by \mathcal{D} the set of all increasing sequences $(d_n) \in \mathbb{N}^{\mathbb{N}}$. The set \mathcal{D} is uncountable. We show that there exists a one-to-one function from \mathcal{D} to \mathcal{A} , which will prove Lemma 6. Let $(d_n) \in \mathcal{D}$. We associate to it an element $(b_n)_{n \in \mathbb{N}} \in \mathcal{A}$ (which we describe via its coding in the $\{1, 2\}$ alphabet) in the following way. We set the first $10^{10^{d_0}}$ elements $b_0, \dots, b_{10^{10^{d_0}} - 1}$ to be equal to 1, then the next $10^{10^{d_1}}$ to be equal to 2, then the next $10^{10^{d_2}}$ are equal to 1, and so on we alternate between a very long sequence of 1's and an even longer sequence of 2's until infinity. One can easily verify that the obtained sequence (b_n) is in \mathcal{A} . \square

Lemma 7. *There exists $\varepsilon > 0$ such that (GRC) fails for any alternating geodesic.*

Proof. Let us first fix the dimensions of the billiard Ω we consider. These dimensions will determine the (maximal) size of ε . We assume that the equilateral triangle Δ has side-length 1 and that the three circles $\mathcal{C}^0, \mathcal{C}^1$ and \mathcal{C}^2 have radius $1/4$. Let $\gamma \in \mathcal{A}$ and denote by $\omega_\varepsilon \subset \Omega$ an open set contained in a ball of radius $\varepsilon = 1/100$. We prove that

$$\liminf_{T \rightarrow +\infty} \frac{|\{t \in [0, T], \gamma(t) \in \omega_\varepsilon\}|}{T} = 0, \quad (17)$$

which in turn immediately implies Lemma 7.

We split Ω into three parts. Let B_1 be a thin strip joining \mathcal{C}^0 and \mathcal{C}^1 , with one side being the periodic trajectory p and the other one being parallel to p and on the right of p . Similarly let B_2 be a thin strip joining \mathcal{C}^0 and \mathcal{C}^2 , with one side being the periodic trajectory q and the other one being parallel to q and on the left of q . We choose them sufficiently thin so that the distance between them is strictly greater than $1/100$. Finally, we set $B = \Omega \setminus (B_1 \cup B_2)$. Figure 2 below summarizes the notations.

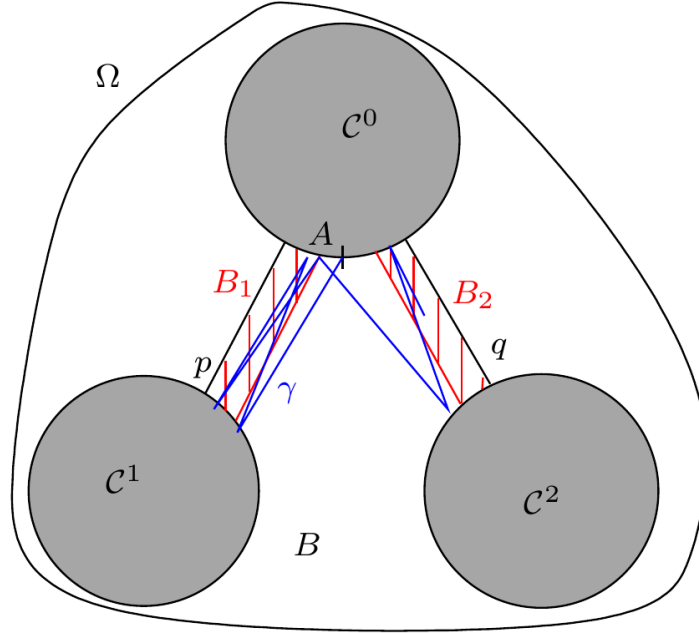


Figure 2: The billiard Ω in which (GRC) breaks down

The key point is the following. If the encoding sequence (b_n) has a very long sequence of m consecutive ones (resp. twos), this means that γ spends a long time interval, which lasts l seconds, in B_1 (resp. B_2). Moreover, since the lengths of p and q are $1/2$, one can check that the difference $|l - m|$ is bounded above by a constant C in the limit $m \rightarrow +\infty$.

Since γ is alternating, it spends a big amount of time in B_1 (which differs from the constant c_0 of Definition 7 at most by C), then an even bigger amount of time in B_2 (which differs from c_1 at most by C), then back to B_1 (which differs from c_2 at most by C), etc, and the transition time where $\gamma \in B$ between two of these is bounded above. Hence, if ω_ε is fully contained in B , then (17) is satisfied. If ω_ε intersects B_1 , then it does not intersect B_2 since B_1 and B_2 were chosen sufficiently thin. The equalities (16) imply that after every period where γ spent a big amount of time in B_2 , the ratio $|\{t \in [0, T], \gamma(t) \in \omega_\varepsilon\}|/T$ is becoming smaller and goes to 0 when time goes to infinity. Note that this ratio may become again greater when γ returns to B_1 , but this does not matter since we consider the \liminf

in (17). A similar reasoning using now the long periods of time that γ spends in B_1 proves that if ω_ε intersects B_2 , (17) also holds. This finishes the proof of Lemma 7. \square

Lemmas 6 and 7 imply Proposition 4: there exists an uncountable number of alternating geodesics, and for any of them, (GRC) breaks down.

Lastly, it remains to prove Fact 1.

Proof of Fact 1. This proof is strongly inspired by the ideas underlying the Smale-Birkhoff homoclinic theorem [KH95, Theorem 6.5.5]. However, in order to keep the paper as readable as possible, we decided to build a fully elementary proof of Fact 1 which does not require this theorem. We include nonetheless in Remark 5 explanations on how our particular billiard Ω enters the more general setting given by the Smale-Birkhoff homoclinic theorem.

We now start the proof, during which we will refer constantly to Figure 2. Let $(b_n) \in \{1, 2\}^{\mathbb{N}}$. Recall that we only consider geodesics starting at A . Our goal is to find an initial angle θ such that the coding sequence of the geodesic starting at time 0 at A and making an initial angle θ with the horizontal axis is (b_n) . Intuitively, this will be done step by step, trying to progressively adjust θ so that the geodesic first hits \mathcal{C}^{b_0} (which is the case for plenty of geodesics starting at A), then restricting this set of geodesics to those then hitting \mathcal{C}^0 and \mathcal{C}^{b_1} , etc. At each step, the set of possible directions is nonempty, closed and contained in the preceding one. The desired geodesic will then be picked in the intersection of this infinite number of nested sets. In the following paragraphs, we make this intuition precise.

The set of admissible initial velocities (i.e., pointing outwards \mathcal{C}^0) is a connected subset of \mathbb{S}^1 . In the sequel, the set of velocities \mathbb{S}^1 will be identified to $[0, 2\pi)$, where all angles considered are taken with respect to the horizontal axis. In particular, the set of admissible initial velocities is of the form $[\alpha, \beta] \subset [0, 2\pi)$. Given $\eta \in [\alpha, \beta]$, we denote by γ_η the geodesic starting at A with initial angle η .

For $i \geq 0$, we set $A_{b_0 b_1 \dots b_i} \subset [0, 2\pi)$ the set of all angles η such that the encoding sequence of γ_η starts with b_0, b_1, \dots, b_i . Our goal is to prove that

$$\bigcap_{i \geq 0} A_{b_0 b_1 \dots b_i} \neq \emptyset \quad (18)$$

since the encoding sequence of any geodesic in this set is $(b_n)_{n \in \mathbb{N}}$.

When η runs over $[\alpha, \beta]$, we see by continuity of the flow that there exists a non-empty set $[\alpha_0, \beta_0] \subset [\alpha, \beta]$ such that $\eta \in [\alpha_0, \beta_0]$ if and only if the encoding sequence of γ_η starts with b_0 . The set $[\alpha_0, \beta_0]$ is what we called A_{b_0} and we now know that it is non-empty. Moreover, remark that γ_{α_0} hits \mathcal{C}^{b_0} tangentially on its left and γ_{β_0} hits \mathcal{C}^{b_0} tangentially on its right.

When η now runs over $[\alpha_0, \beta_0]$, because of this last remark, by continuity of the flow, there exists $[\alpha_1, \beta_1] \subset [\alpha_0, \beta_0]$ such that $\eta \in [\alpha_1, \beta_1]$ if and only if γ_η hits successively $\mathcal{C}^{b_0}, \mathcal{C}^0$ and \mathcal{C}^{b_1} . The set $[\alpha_1, \beta_1]$ is what we called $A_{b_0 b_1}$ and we now know that it is non-empty. Moreover, remark again that γ_{α_2} necessarily hits \mathcal{C}^{b_1} tangentially on its left and γ_{β_2} necessarily hits \mathcal{C}^{b_1} tangentially on its right.

Iterating this construction, we define successively the closed sets $A_{b_0 b_1 \dots b_i}$ for $i \geq 0$ and we remark that

$$\forall i \geq 0, \quad A_{b_0 b_1 \dots b_i} \neq \emptyset, \quad A_{b_0 b_1 \dots b_i b_{i+1}} \subset A_{b_0 b_1 \dots b_i}. \quad (19)$$

At step i , for each $\eta \in A_{b_0 \dots b_i}$, we know that the geodesic γ_η hits successively $\mathcal{C}^{b_0}, \mathcal{C}^0, \mathcal{C}^{b_1}, \mathcal{C}^0, \dots$ until \mathcal{C}^{b_i} . Moreover, we know that the "extreme" geodesic γ_{α_i} (resp., γ_{β_i}) hits \mathcal{C}^{b_i} tangentially on its left (resp., on its right). For each $\eta \in [\alpha_i, \beta_i]$, we can look at γ_η at the moment just after it hits \mathcal{C}^{b_i} . It defines a point x_η^i on \mathcal{C}^{b_i} and a velocity v_η^i pointing outwards \mathcal{C}^{b_i} . The key point which makes the argument work is that $x_{\alpha_i}^i$ is on the left of \mathcal{C}^{b_i} and $v_{\alpha_i}^i$ points towards left, whereas $x_{\beta_i}^i$ is on the right of \mathcal{C}^{b_i} and $v_{\beta_i}^i$ points towards right. The set $\{(x_\eta^i, v_\eta^i), \eta \in [\alpha_i, \beta_i]\}$ defines a connected submanifold of dimension 1 of the phase space with footpoint in \mathcal{C}^{b_i} . Therefore, by continuity of the flow, when η runs over $[\alpha_i, \beta_i]$, it is necessary that there exists $[\alpha_{i+1}, \beta_{i+1}] \subset [\alpha_i, \beta_i]$ such that for any $\eta \in [\alpha_{i+1}, \beta_{i+1}]$, the couple (x_η^i, v_η^i) defines a geodesic which continues its path by hitting \mathcal{C}^0 and $\mathcal{C}^{b_{i+1}}$.

Using (19), we immediately get (18), which concludes the proof of Fact 1. \square

Remark 5. In this remark, we explain the links of our proof with the Smale-Birkhoff homoclinic theorem [KH95, Theorem 6.5.5]. For any geodesic γ with an infinite encoding sequence $(b_n)_{n \in \mathbb{N}}$ and for any $n \in \mathbb{N}$, we define x_n and v_n to be respectively the point of \mathcal{C}^{b_n} and the velocity in \mathbb{S}^1 (pointing outwards \mathcal{C}^{b_n}) at the moment when γ hits \mathcal{C}^{b_n} . We define the function f by $f(x_n, v_n) = (x_{n+1}, v_{n+1})$.

Recalling that p and q are periodic trajectories (see Figure 2), we denote by $W^s(p)$ (resp. $W^u(p)$) the stable (resp. unstable) manifold of p , and we define similarly $W^s(q)$ and $W^u(q)$. It is not difficult to prove that $W^s(p)$ and $W^u(q)$ intersect transversely at (A, θ_1) in the phase space for a well-chosen θ_1 and that $W^s(q)$ and $W^u(p)$ intersect transversely at (A, θ_2) in the phase space for a well-chosen θ_2 . Therefore, by [KH95, Theorem 6.5.5], there is a horseshoe map hidden in Figure 2. Said precisely, there exists an integer $m > 0$ such that f^m has an hyperbolic invariant set on which f^m is topologically conjugated to the shift on two symbols. It *nearly* means that for any sequence $(b_n) \in \{1, 2\}^{\mathbb{N}}$, it is possible to find a geodesic γ with encoding sequence (b_n) . It would exactly mean so if we were able to prove that $m = 1$, but we did not find any easy proof of this fact.

Lastly, remark that the family of sets $A_{b_0 b_1 \dots b_i}$ where (b_i) runs over all sequences $\{1, 2\}^{\mathbb{N}}$ defines a Cantor structure similar to those appearing in the construction of the horseshoe map [KH95, Section 2.5].

Remark 6. The billiard Ω described above is not the only one for which there exist geodesics not satisfying (GRC). For example, it is possible to construct such a billiard with three obstacles which are circles whose centers form an isosceles right-angled triangle, instead of an equilateral triangle. Then, folding it into a torus, we see that in the Sinai billiard (described in Section 1.3), there also exist geodesics not satisfying (GRC). This remark is related to Conjecture 1.

2.6 Proof of Theorem 2

Zoll manifolds. We start with the proof for Zoll manifolds. Assume that Ω is a Zoll manifold. By a result due to Wadsley (see [Bes12]) all closed geodesics share a least common period $T_1 > 0$. Let x_1, \dots, x_n be points in Ω such that $\cup_i B(x_i, \varepsilon) = \Omega$. Then we take $0 < T < +\infty$ and $x : [0, T] \rightarrow \Omega$ a C^1 path of speed at most v which spends a time at least T_1 on each x_i . We define as usual $\omega(t) = B(x(t), \varepsilon) \cap \Omega$. It is immediate to verify that each geodesic of Ω hits $\omega(t)$ during the interval $[0, T]$. Therefore, t -GCC is verified in time $T < +\infty$.

The 2d torus. The proof for the 2-dimensional torus is quite technical. We represent the 2-dimensional torus as the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ of side length 1 with opposite edges identified one to another. We will describe the motion of the moving point $x(t)$ (which is the center of $\omega(t) = B(x(t), \varepsilon)$) step by step. Remark that each geodesic (traveling at speed 1) in the torus can be described by its coordinates $y(t) = (y_1(t), y_2(t))$ with $y_1, y_2 \in [-\frac{1}{2}, \frac{1}{2}]$. Note also that, because of the particular structure of the torus, each of these geodesics is:

- either "mostly horizontal", meaning that for almost every $t \in \mathbb{R}$ we have $|\dot{y}_1(t)| \geq \sqrt{2}/2$, in which case the geodesic cuts the axis $y_1 = 0$ very regularly, at least every $\sqrt{2}$ period of time;
- or "mostly vertical", meaning that for almost every $t \in \mathbb{R}$ we have $|\dot{y}_2(t)| \geq \sqrt{2}/2$, in which case it is mostly vertical and the geodesic cuts the axis $y_2 = 0$ very regularly, at least every $\sqrt{2}$ period of time.

We will use this splitting to first catch all the "mostly vertical" geodesics with a domain centered on a point $x(t)$ moving only on the horizontal axis (which is regularly cut by the mostly vertical geodesics) and then we catch all the "mostly horizontal" geodesics by making $x(t)$ move only along the vertical axis. In fact, by symmetry of the situation, we see that it is sufficient to catch with $\omega(t)$ all the geodesics which are "mostly vertical". With a similar argument of symmetry, we can even restrict to the case of "mostly vertical" geodesics which travel from left to right, that is $\dot{y}_1(t) \geq 0$. To catch them with $\omega(t)$, we use a center $x(t)$ moving only on the x -axis from left to right. Then, to catch the "mostly vertical" geodesics which travel from right to left (meaning that $\dot{y}_1(t) \leq 0$), we do the same by making $x(t)$

move only on the x -axis from right to left. To sum up, we are now reduced to showing that it is possible to choose a finite-time trajectory $t \mapsto x(t)$ on the x -axis such that the ball $B(x(t), \varepsilon)$ meets all "mostly vertical" geodesics which travel from left to right. We denote by G_v this set of geodesics. In the sequel, we assume that $\varepsilon \leq v$ and that $\varepsilon < 1/10$. We do not lose any generality by making this assumption since reducing ε makes the construction of the moving domain ω even harder.

We now construct the trajectory $t \mapsto x(t)$. We set $x(0) = 0$. We will construct times $t_2 > t_1 > 0$ and a C^1 path $x : [0, t_2] \rightarrow \Omega$ such that $|\dot{x}(t)| = 0$ between times 0 and t_1 and $0 \leq |\dot{x}(t)| < v$ for almost every time t between t_1 and t_2 . We also require that $x(t)$ is a C^1 path. First of all, we set $t_1 = 2\varepsilon^{-5}$, which means that $x(t)$ stays at first during a time $2\varepsilon^{-5}$ at 0. Then, it moves along the x -axis at speed at most v in the following way. Fix z_1, \dots, z_m points on the x -axis such that for every point z on the x -axis, there exists $1 \leq i \leq m$ verifying $|z - z_i| < \varepsilon/2$ (in the distance of the torus). We can choose $m \leq 3/\varepsilon$. We require for $x(t)$ to spend a large time on each z_i between times t_1 and t_2 . More precisely, we construct $x(t)$ for $t \in [t_1, t_2]$ a C^1 path of speed at most v which spends, for each $1 \leq i \leq m$, a time-interval of length at least $2\varepsilon^{-1}$ on z_i . Clearly, it is possible for some finite time t_2 verifying $t_2 - t_1 < 6\varepsilon^{-2} + \frac{1}{\varepsilon v} \leq 7\varepsilon^{-2}$ since $\varepsilon \leq v$.

We now prove that such a moving ball $\omega(t) = B(x(t), \varepsilon)$ meets all geodesics between times 0 and t_2 .

Fix $t \mapsto y(t) \in \mathbb{T}^2$ a geodesic in G_v . We have to show that there exists $t \in [0, t_2]$ such that (in the distance of the torus) $|y(t) - x(t)| < \varepsilon$. Since $y(t)$ is mostly vertical, it crosses periodically the x -axis at most every $\sqrt{2}$ period of time. We consider the trace of this geodesic $y(t)$ on the x -axis. By the pigeonhole principle, in the time interval $[0, 2/\varepsilon]$, there are two points of this trace which are separated by a distance $< \varepsilon$. Let us call them $y(s_1)$ and $y(s_2)$ with $0 \leq s_1 < s_2 \leq 2/\varepsilon$. Then for every $k \in \mathbb{N}$, we have $y(s_1 + k(s_2 - s_1)) = y(s_1) + k(y(s_2) - y(s_1))$ in the torus, and the sequence $(y(s_1 + k(s_2 - s_1)))_{k \in \mathbb{N}}$ describes an arithmetic sequence on the x -axis of the torus with step $< \varepsilon$.

If $\varepsilon > |y(s_1) - y(s_2)| \geq \varepsilon^5$, then this step is large enough to reach a neighborhood of 0 before time t_1 . More precisely, there exists $t \in [0, t_1]$ such that $y_2(t) = 0$ and $|y_1(t)| < \varepsilon$, so that $|y(t) - x(t)| < \varepsilon$.

Otherwise, $0 \leq |y(s_1) - y(s_2)| < \varepsilon^5$. Then, in a way, $y(t)$ is "very close" to a periodic geodesic of period at most $2/\varepsilon$. We will prove that each periodic geodesic of period at most $2/\varepsilon$ comes very close to $x(t)$ between times t_1 and t_2 and deduce from it that it is also the case for $y(t)$. Since $0 \leq |y(s_1) - y(s_2)| < \varepsilon^5$, there exists a periodic geodesics $t \mapsto y_p(t)$ of period at most $2/\varepsilon$ and a time t'_1 verifying $t_1 \leq t'_1 \leq t_1 + 2$ such that $y_p(t'_1) = y(t'_1)$ is on the x -axis, and for all $t \in [t_1, t_2]$, $|y(t) - y_p(t)| \leq \varepsilon^2$. Take $i \leq m$ such that $|y_p(t'_1) - z_i| \leq \varepsilon/2$. By construction, there exist times t_3 and t_4 satisfying $t_1 \leq t_3 \leq t_4 \leq t_2$, $t_4 - t_3 \geq 2\varepsilon^{-2}$ and $x(t) = z_i$ for $t \in [t_3, t_4]$. Since y_p is periodic of period at most $2/\varepsilon$, we can pick $t \in [t_3, t_4]$ such that $y_p(t) = y_p(t'_1)$. For this time t , we have

$$|y(t) - x(t)| = |y(t) - z_i| \leq |y(t) - y_p(t)| + |y_p(t) - z_i| \leq \varepsilon^2 + \frac{\varepsilon}{2} < \varepsilon,$$

and therefore $y(t) \in \omega(t)$.

This concludes the proof of Theorem 2.

2.7 Proof of Proposition 5

We recall that $\varepsilon > 0, T > 0$ and $v > 0$ are fixed once for all. We define $f : A_{fix} \rightarrow \mathbb{R}$ by

$$f(\omega) = \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 dx$$

and we will prove that f is upper semi-continuous. Let $x \in \Omega$ and $(x_i)_{i \in \mathbb{N}}$ a sequence of points of Ω which converges to x . We set $\omega = B(x, \varepsilon)$ and $\omega_i = B(x_i, \varepsilon)$. We want to show that $f(\omega) \geq \limsup_i f(\omega_i)$. Assume by contradiction that there exists $j \in \mathbb{N}$ such that $\int_{\omega} \phi_j^2(x) dx < \limsup_i f(\omega_i)$. In particular we have $\int_{\omega} \phi_j^2(x) dx < \limsup_i \int_{\omega_i} \phi_j(x)^2 dx =$

$\int_{\omega} \phi_j(x)^2 dx$ by the regularity properties of the eigenfunction ϕ_j . This is a contradiction. Therefore, f is upper semi-continuous. In particular, since A_{fix} is compact, f reaches its maximum on it, and the supremum in (10) is in fact a maximum.

Similarly, we can prove that the supremum in (11) is reached by just adapting the above argument to a time-dependent setting (using the uniform topology on $(0, T) \times \Omega$ and the fact that $v > 0$ is fixed).

Finally, we show that the infimum in j in (10) and (11) is not necessarily reached. We set $\Omega = \mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere of the three-dimensional space. We recall that a quantum limit is by definition a weak limit of the sequence of measures $|\phi_j|^2 d\mu$, where μ is the Lebesgue measure on Ω . It is known that any equator (or great circle) is a quantum limit (see [JZ99]) and, since for any $\varepsilon < \pi$ and any $x \in \mathbb{S}^2$ it is always possible to find a great circle which does not intersect $B(x, \varepsilon) \cap \mathbb{S}^2$, it follows that $C_{fix}^T = 0$. However, for any $j \in \mathbb{N}$ and any $x \in \mathbb{S}^2$, $\int_{B(x, \varepsilon)} \phi_j(x)^2 dx > 0$ since ϕ_j is a non-zero analytic function. It means that the infimum in (10) (and therefore in (11)) is not necessarily reached.

2.8 Proof of Theorem 3

By the proof of Proposition 5, we already know that for $\Omega = \mathbb{S}^2$, for any sufficiently small $\varepsilon > 0$ and for any $T > 0$, we have $C_{fix}^T = 0$. This proves the first point of the theorem.

Since for $T > 0$ and $v > 0$, the quantity vT is the maximal distance that a point moving with speed v in Ω can cover within time T , it is clear that if $vT < \pi - \varepsilon$, for $\Omega = \mathbb{S}^2$, we have $C_{mov}^{T, v} = 0$, which proves the second point of the theorem.

For the last point of the theorem, fix $\varepsilon > 0$. We show that there exists a C^1 path $x : [0, 1] \rightarrow \Omega$ and a constant $C > 0$ such that

$$\forall y \in \Omega, \quad |\{t \in [0, 1], y \in B(x(t), \varepsilon) \cap \Omega\}| \geq C. \quad (20)$$

We fix a $\varepsilon/2$ -net x_1, \dots, x_n in Ω , which means that for any $x \in \Omega$, there exists $1 \leq j \leq n$ such that $|x - x_j| \leq \varepsilon/2$ where $|\cdot|$ is the Euclidean distance in \mathbb{R}^n . We take a C^1 path $x : [0, 1] \rightarrow \Omega$ with bounded velocity v_1 (which can be very large) which stays at least a time $1/(2n)$ on each x_i . Clearly (20) is satisfied.

Now for $T > 0$ and $t \in [0, T]$, we set $x_T(t) = x(\frac{t}{T})$, and the speed of the C^1 path x_T is bounded by v_1/T . Moreover, the path x_T spends a time at least $T/(2n)$ on each x_i . Therefore

$$C_{mov}^{T, v_1/T} \geq \inf_{j \in \mathbb{N}} \int_0^T \int_{B(x_T(t), \varepsilon)} \phi_j(x)^2 dx dt \geq \inf_{j \in \mathbb{N}} \frac{T}{2n} \int_{\Omega} \phi_j(x)^2 dx = \frac{T}{2n} > 0.$$

This inequality proves the last point of the theorem.

A t -GCC in infinite time implies observability

In [LRLTT17], the fact that (Q, T) verifies t -GCC implies the observability inequality (6) is proved only for $T < +\infty$. For Theorem 1, we need to establish this implication for $T = +\infty$.

Theorem 4. *Let Q be an open subset of $\mathbb{R} \times \bar{\Omega}$ that satisfies the t -GCC in infinite time. When $\partial\Omega \neq \emptyset$, we assume moreover that no generalized bicharacteristic has a contact of infinite order with $(0, T) \times \partial\Omega$, that is, $G^\infty = \emptyset$. Then the observability inequality (6) holds for $T = +\infty$.*

Proof. The proof follows the same line as the proof of [LRLTT17, Theorem 1.8], and we adopt in the sequel the same notations. The only point which needs to be adapted is [LRLTT17, Lemma 2.1], which establishes a weakened observability inequality:

There exists $C > 0$ such that

$$C \|(u^0, u^1)\|_{D(A^{1/2}) \times X}^2 \leq \|\chi_Q \partial_t u\|_{L^2((0, +\infty) \times \Omega)}^2 + \|(u^0, u^1)\|_{X \times D(A^{1/2})}^2,$$

for all $(u^0, u^1) \in D(A^{1/2}) \times X$, where u is the corresponding solution of (4)-(5) with $u|_{t=0} = u^0$ and $\partial_t u|_{t=0} = u^1$.

We prove the result by contradiction. We assume that there exists a sequence $(u_n^0, u_n^1)_{n \in \mathbb{N}}$ in $D(A^{1/2}) \times X$ such that

$$\|(u_n^0, u_n^1)\|_{D(A^{1/2}) \times X} = 1 \quad \forall n \in \mathbb{N}, \quad (21)$$

$$\|(u_n^0, u_n^1)\|_{X \times D(A^{1/2})'} \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (22)$$

$$\|\chi_Q \partial_t u_n\|_{L^2((0, +\infty) \times \Omega)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (23)$$

where u_n is the solution of (4)-(5) satisfying $u_n|_{t=0} = u_n^0$ and $\partial_t u_n|_{t=0} = u_n^1$. From (21), the sequence $(u_n^0, u_n^1)_{n \in \mathbb{N}}$ is bounded in $D(A^{1/2}) \times X$, and using (22) we deduce that the only possible closure point for the weak topology of $D(A^{1/2}) \times X$ is $(0, 0)$. Therefore the sequence (u_n^0, u_n^1) converges to $(0, 0)$ for the weak topology of $D(A^{1/2}) \times X$. By continuity of the flow with respect to initial data, it follows that for any $0 < T < +\infty$, the sequence $(u_n)_{n \in \mathbb{N}}$ of corresponding solutions converges to 0 for the weak topology of $H^1((0, T) \times \Omega)$; in particular it is bounded in any of these spaces.

Set $Y = \mathbb{R} \times \Omega$. Let bTY be the fiber bundle of rank $\dim Y$ whose sections are the vector fields which are tangent to ∂Y , and ${}^bT^*Y$ be its dual fiber bundle (the compressed cotangent fiber bundle of Melrose). We denote by j the natural projection of T^*Y on ${}^bT^*Y$, by Σ the image by j of the characteristic manifold of the wave equation (of equation $\tau^2 = |\xi|^2$), $\hat{\Sigma} = \Sigma \cup j(T^*Y|_{\partial Y})$ and $S\hat{\Sigma} = (\hat{\Sigma} \setminus Y)/\mathbb{R}_+^*$ the quotient space by the natural action of \mathbb{R}_+^* . The space $S\hat{\Sigma}$ is a locally compact metric space.

According to [Leb96, Section 2.1], there exists an increasing function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and a microlocal defect measure μ on $S\hat{\Sigma}$ such that for every $R \in \Psi_{\text{comp}}^0(\mathbb{R} \times \bar{\Omega})$ (see definition in [LRLTT17, Appendix A]),

$$(Ru_{\varphi(n)}, u_{\varphi(n)}) \rightarrow \int \kappa(\sigma(R)) d\mu \quad \text{as } n \rightarrow +\infty.$$

It follows from (23) that μ vanishes in $j(T^*Q) \cap S\hat{\Sigma}$. It is well-known (see [Leb96]) that the measure μ is invariant under the compressed generalized bicharacteristic flow. Therefore, the t -GCC assumption in infinite time implies that μ vanishes identically. Hence, for any $0 < T < +\infty$, $(u_n)_{n \in \mathbb{N}}$ strongly converges to 0 in $H^1((0, T) \times \Omega)$.

As in [LRLTT17], this last fact contradicts the conservation of energy. \square

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