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The localisation of low-temperature interfaces in d dimensional Ising model

Wei Zhou*

January 21, 2019

Abstract

We study the Ising model in a box Λ in \mathbb{Z}^d (not necessarily parallel to the directions of the lattice) with Dobrushin boundary conditions at low temperature. We couple the spin configuration with the configurations under $+$ and $-$ boundary conditions and we define the interface as the edges whose endpoints have the same spins in the $+$ and $-$ configurations but different spins with the Dobrushin boundary conditions. We prove that, inside the box Λ , the interface is localized within a distance of order $\ln^2 |\Lambda|$ of the set of the edges which are connected to the top by a $+$ path and connected to the bottom by a $-$ path.

1 Introduction

At the macroscopic level, the dynamics of the interface between two pure phases in the Ising model seem to be deterministic. In fact, the interface tends to minimize the surface tension between the two phases. The microscopic justification of this fact in the context of 2D Ising model was achieved in [4]. In the limit where the size of the system grows to infinity, the two types of spins, at low temperature, form two regions separated by the interfaces. After a suitable spatial rescaling, these interfaces converge to deterministic shapes. However, the interfaces remain random and their geometric structure is extremely complex. In two dimensions, the fluctuations of the

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interfaces are well analysed in [3] using the cluster expansions techniques. Recently, Ioffe and Velenik gave a geometric description of the interfaces and their scaling limits with the help of the Ornstein–Zernike theory in [10]. In higher dimensions, the famous result of Dobrushin in [5] says that at low temperature, the interface in a straight box is localised around the middle hyperplane of the box when the temperature is low. One of the difficulties to study the interfaces is to define them properly. The usual way is to consider the Dobrushin boundary conditions. More precisely, the vertices on the upper boundary of the box are pluses and those on the lower boundary are minuses. It is a geometric fact that, with such a boundary condition, the spin configurations present an interface separating a region of plus spins containing the upper boundary and a region of minus spins containing the lower boundary. However, for several reasons, it is still not obvious to define an interface in this setting. For example, there are more than one separating set between the pluses and minuses in a typical configuration with Dobrushin boundary conditions.

Remark While I was finalizing this paper, Gheissari and Lubetzky completed a very interesting paper [6] on the large deviations of the interface in 3D Ising model. They study the height of the interface in a straight box using a decomposition of the pillars and they obtain a localisation result at an order $\ln |\Lambda|$ at low temperature. Our localisation result is clearly weaker in the case of a straight box, however it holds also for a tilted box.

The first goal of this study is to adapt to the Ising model the definition of the interfaces, introduced in [1] for the percolation model. The second goal is to progress in the geometric description of these interfaces for a box not necessarily straight in dimensions $d \geq 2$ at low temperature. In [1], we constructed a coupling between the dynamical percolation process and a conditioned process. The interface was defined as the difference between the two processes. We showed a localisation result for the interface around the pivotal edges for the disconnection event. For the Ising model, a coupling can be realised by the Glauber dynamics, yet there is no corresponding notion for the pivotal edges. However, the objects introduced for the percolation model in [1] are defined naturally for the FK-percolation model. With the help of the Edwards-Sokal coupling, we can define and localise the interfaces using the results obtained in the FK-percolation model. To realise our first goal, we construct three spin configurations $(\sigma^+, \sigma^-, \sigma^D)$, corresponding to the plus, minus and Dobrushin boundary conditions, and a probability measure $\pi_{\Lambda, \beta}$ on this triplet, whose marginals are the Ising measures with the corresponding boundary conditions. We consider a box $\Lambda = (V, E)$ and we define the

interface as follows:

Definition 1.1. *The set \mathcal{P}_I is the set of the edges $\langle x, y \rangle \in E$ such that*

$$\begin{aligned} \sigma^D(x) &= +1 \text{ and } x \text{ is connected to } T \text{ by a path of vertices with } +1 \\ \sigma^D(y) &= -1 \text{ and } y \text{ is connected to } B \text{ by a path of vertices with } -1. \end{aligned}$$

The set \mathcal{I}_I is the set of the edges $\langle x, y \rangle \in E$ such that

$$\sigma^+(x) = \sigma^+(y), \quad \sigma^-(x) = \sigma^-(y), \quad \sigma^D(x) \neq \sigma^D(y).$$

The interface \mathcal{I}_I is the set of the edges whose endpoints have different spins in σ^D but have the same spins in the other two configurations. The set \mathcal{P}_I corresponds to the edges of \mathcal{I}_I connected to the boundary in σ^D . As for the second goal, we show the following result:

Theorem 1.2. *There exist $0 < \tilde{\beta} < \infty$ and $\kappa \geq 0$, such that for $\beta \geq \tilde{\beta}$, $c > 0$ and any Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$, we have*

$$\pi_{\Lambda, \beta} \left(\exists e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

The key to obtain this result is to construct a coupling (X, Y) , where X is a standard FK-percolation configuration and Y is a configuration where the top side T and the bottom side B of the box Λ are disconnected. We denote this event by $\{T \leftrightarrow B\}$. The localisation of the interface in the Ising model is induced by a control of the distance between the interface \mathcal{I} and the set of the pivotal edges \mathcal{P} of the coupling (X, Y) . In this paper, we consider the FK-percolation model with a parameter p close to 1 and q larger than 1. Interfaces in a box Λ are naturally created when the configuration is conditioned to stay in the set $\{T \leftrightarrow B\}$. The interface \mathcal{I} is defined as

$$\mathcal{I} = \{ e \subset \Lambda : X(e) \neq Y(e) \}$$

and we denote by \mathcal{P} the set of the pivotal edges for the event $\{T \leftrightarrow B\}$ in Y . Our main result for the FK model is the following.

Theorem 1.3. *For any $q \geq 1$, there exist $\tilde{p} < 1$ and $\kappa > 0$, such that, for $p \geq \tilde{p}$, any $c \geq 1$ and any box Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$,*

$$\mu_{\Lambda, p, q} \left(\exists e \in \mathcal{P} \cup \mathcal{I}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

Let us explain briefly how we build the measures $\mu_{\Lambda,p,q}$ and $\pi_{\Lambda,\beta}$ as well as the strategy for proving theorem 1.2. With the help of a Gibbs sampler algorithm (see section 8.4 of [9]), we construct a coupling between two Markov chains $(X_t, Y_t)_{t \in \mathbb{N}}$ on the space of the percolation configurations in a box Λ . The measure $\mu_{\Lambda,p,q}$ is the unique invariant measure of the process $(X_t, Y_t)_{t \in \mathbb{N}}$. Starting from a coupled configuration (ω, ω') under the measure $\mu_{\Lambda,p,q}$, we put spins on the vertices in the box Λ using an adaptation of the Edwards-Sokal coupling. By construction, the configuration ω dominates ω' . We put spins at first on the vertices according to the configuration ω' , under the Dobrushin boundary condition, to obtain a spin configuration σ^D . Then, we put spins according to ω , under the plus (respectively minus) boundary condition to obtain the configuration σ^+ (respectively σ^-) with the restriction that an open cluster in ω also appearing in ω' has the same spin as in σ^D . The measure $\pi_{\Lambda,\beta}$ is the measure of $(\sigma^+, \sigma^-, \sigma^D)$ obtained from $\mu_{\Lambda,p,q}$ and the colouring. Each of its marginals is an Ising measure in the box with the corresponding boundary conditions.

As for the proof of theorem 1.3, we follow the ideas presented in [1]. We control the distance between two pivotal edges by identifying a cut and a closed path disjoint from the cut. However, due to the correlations between all the edges in the FK-percolation model, we can not use the BK inequality which holds for a product space and which is a key ingredient in [1]. To solve this difficulty, we explore adequately the open clusters and we identify a sub-graph in Λ containing a long closed path and outside of which we can find a cut. The configurations in this sub-graph can be compared to a Bernoulli configuration. To study the case where the distance between an edge of the interface and the pivotal edges is big, we show that the interface edge can not have been created a long time ago. Moreover, at the time when it is created, it must be a pivotal edge. Therefore, the set of the pivotal edges must move rather fast. We obtain a control over the speed of the pivotal edges. This estimate relies on the study of specific space-time paths, which describes how the cut sets move.

These results answer the question 4 raised in [1], and gives some information to the subsequent question 5. However, we would like to obtain more information about the structure of the set \mathcal{P}_I .

This paper is organised as follows. In section 2, we give the definitions of the objects and the notations which we will use in this article. In section 3, we show the estimate on the distance between two pivotal edges in the FK-percolation model. In section 4, we control the speed of the pivotal edges. In section 5, we show theorem 1.3 and we prove theorem 1.2 in section 6.

2 The notations

In this section, we present the FK-percolation model which we study and we recall some fundamental tools which we will use in the rest of this paper.

2.1 Geometric definitions

We start with some geometric definitions.

The lattice \mathbb{L}^d . For an integer $d \geq 2$, the lattice \mathbb{L}^d is the graph $(\mathbb{Z}^d, \mathbb{E}^d)$, where the set \mathbb{E}^d is the set of pairs $\langle x, y \rangle$ of points in \mathbb{Z}^d which are at Euclidean distance 1.

The usual paths. We say that two edges e and f are neighbours if they have one endpoint in common. A usual path is a sequence of edges $(e_i)_{1 \leq i \leq n}$ such that for $1 \leq i < n$, e_i and e_{i+1} are neighbours.

The *-paths. In order to study the cuts in any dimension $d \geq 2$, we use *-connectedness on the edges as in [2]. We consider the supremum norm on \mathbb{R}^d :

$$\forall x = (x_1, \dots, x_d) \in \mathbb{R}^d \quad \|x\|_\infty = \max_{i=1, \dots, d} |x_i|.$$

For e an edge in \mathbb{E}^d , we denote by m_e the center of the unit segment associated to e . We say that two edges e and f of \mathbb{E}^d are *-neighbours if $\|m_e - m_f\|_\infty \leq 1$. A *-path is a sequence of edges (e_1, \dots, e_n) such that, for $1 \leq i < n$, the edge e_i and e_{i+1} are *-neighbours. For a path γ , we denote by $\text{support}(\gamma)$ the set of the edges of γ . We say that a path is simple if the cardinal of its support equals to its length.

The box Λ . We will mostly work in a box Λ centred at the origin (not necessarily straight) as illustrated in the figure 1. More precisely, we will consider a d -cube K centred at origin. The box $\Lambda = (V, E)$ is the sub-graph of \mathbb{L}^d whose vertices are included in K . The boundary of Λ , denoted by $\partial\Lambda$, is defined as,

$$\partial\Lambda = \{x \in V : \exists y \notin K, \langle x, y \rangle \in \mathbb{E}^d\}.$$

We will distinguish two disjoint non-empty subsets of $\partial\Lambda$, denoted by T and B . The results presented in our study do not rely on the choices of T and B . For conventional reasons, we can consider a $(d - 1)$ dimensional plane containing the origin and parallel to a side of K . This plane separates K

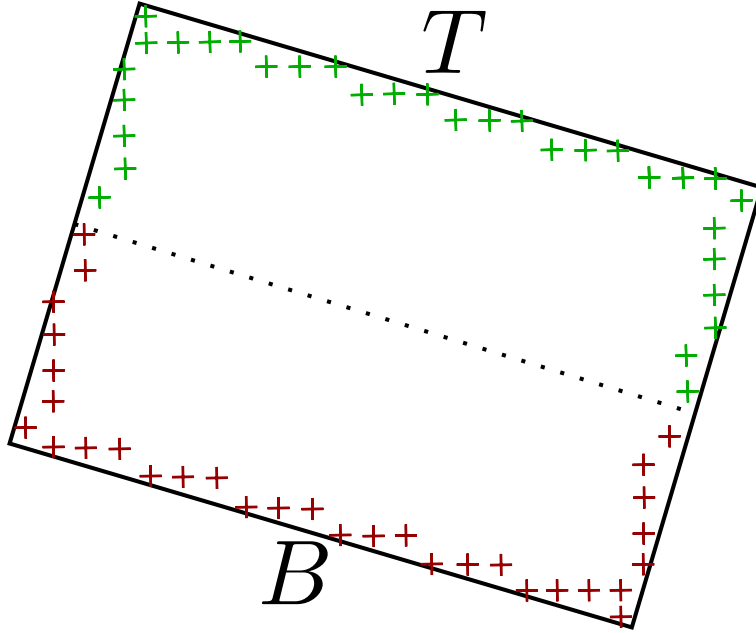


Figure 1: The box Λ and its boundary (the crosses). The green crosses form the side T and the red ones form the side B .

into two parts K^+ and K^- (we suppose that K^+ contains the plane). The set T is the subset of $\partial\Lambda$ included in K^+ and B the one included in K^- .

The separating sets. Let A, B be two subsets of Λ . We say that a set of edges $S \subset \Lambda$ separates A and B if no connected subset of $\Lambda \cap \mathbb{E}^d \setminus S$ intersects both A and B . Such a set S is called a separating set for A and B . We say that a separating set is minimal if there does not exist a strict subset of S which separates A and B .

The cuts. We say that S is a cut if S separates T and B , and S is minimal for the inclusion.

2.2 The Ising model

Let $G = (V, E)$ be a finite graph. We associate to each vertex $x \in V$ a random spin $\sigma(x)$ which can either be $+1$ or -1 . The spin values are chosen according to a certain probability measure λ_β , known as a Gibbs state, which is defined with a parameter $\beta \in [0, +\infty[$ as

$$\lambda_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_I}, \quad \sigma \in \{+1, -1\}^V,$$

where

$$H(\sigma) = - \sum_{\langle x,y \rangle \in E} \sigma(x)\sigma(y)$$

is the Hamiltonian and Z_I is the normalisation constant called the partition function.

2.3 The FK-percolation model

Also known as the random-cluster model, the FK-percolation model is a generalisation of the Bernoulli percolation model, in which we introduce correlations between edges by taking into account the number of open clusters in a configuration. On a finite graph (V, E) , a random cluster-measure is a member of a certain class of probability measures on the space set $\{0, 1\}^E$. Let ω belongs to $\{0, 1\}^E$, we say that an edge e is open if $\omega(e) = 1$ and closed if $\omega(e) = 0$, and we set

$$\eta(\omega) = \{ e \in E : \omega(e) = 1 \}.$$

Let $k(\omega)$ be the number of connected components (or the open clusters) of the graph $(V, \eta(\omega))$, and note that $k(\omega)$ includes the count of the isolated vertices, that is, of vertices incident to no open edge. For two parameters $p \in [0, 1]$ and $q > 0$, the *random-cluster measure* $\Phi_{\Lambda,p,q}$ is defined as

$$\Phi_{\Lambda,p,q}[\omega] = \frac{1}{Z_{RC}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \{0, 1\}^E,$$

where the partition function Z_{RC} is given by

$$Z_{RC} = \sum_{\omega \in \{0,1\}^E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}.$$

In our study, we will consider only the case where $q \geq 1$.

Boundary conditions. We will consider different boundary conditions. Let $\xi \in \{0, 1\}^{\mathbb{E}^d}$ and let $G = (V, E)$ be the box Λ . Let Ω^ξ denote the subset of $\{0, 1\}^{\mathbb{E}^d}$ consisting of all the configurations ω satisfying $\omega(e) = \xi(e)$ for $e \in \mathbb{E}^d \setminus E$. We shall write $\Phi_{\Lambda,p,q}^\xi$ for the random-cluster measure on Λ with boundary condition ξ , given by

$$\Phi_{\Lambda,p,q}^\xi[\omega] = \begin{cases} \frac{1}{Z^\xi} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)} & \text{if } \omega \in \Omega^\xi, \\ 0 & \text{otherwise,} \end{cases}$$

where $k(\omega, \Lambda)$ is the number of components of the graph $(\mathbb{Z}^d, \eta(\omega))$ that intersect Λ , and Z^ξ is the appropriate normalizing constant. In particular, we will consider three special boundary conditions in this paper:

- The 0-boundary condition corresponds to the case where all the edges of ξ are closed. This condition is also called the *free* boundary condition.
- The 1-boundary condition corresponds to the case where all the edges of ξ are open. We can also see this condition as adding one vertex which is connected to all the vertices of $\partial\Lambda$, therefore, this boundary condition is called the *wired* boundary condition.
- The TB -boundary condition corresponds to the Dobrushin boundary condition for the Ising model introduced in [5]. This boundary condition corresponds to a configuration ξ where all the vertices of T are connected and all the vertices of B are connected by open paths of edges outside of Λ , but there is no open path which connects a vertex of T to a vertex of B . We can also see this condition as adding two vertices to the graph Λ , one of which is connected to all the vertices of T and the other one connected to all the vertices of B .

In order to simplify the notations, we omit Λ, p, q, ξ in $\Phi_{\Lambda, p, q}^\xi$ if it doesn't create confusions. We will use two fundamental properties of the FK-percolation model called the spatial Markov property (see chapter 4.2 of [9]) and the comparison between different values of p and q stated in chapter 3.4 of [9].

2.4 Coupled dynamics of FK-percolation

We will use a special Glauber process called the Gibbs sampler to study the conditioned FK-measure. Consider a finite graph $G = (V, E)$. To simplify the notations, we omit the G, p, q in $\Phi_{G, p, q}$ in this section. The Gibbs sampler is a Markov chain $(X_t)_{t \in \mathbb{N}}$ on the state space $\Omega = \{0, 1\}^E$. At a time t , we choose an edge e uniformly in E , and we set the status of e according to the current states of the other edges. More precisely, let $(U_t)_{t \in \mathbb{N}}$ be a sequence of uniform variables on $[0, 1]$ and $(E_t)_{t \in \mathbb{N}}$ be a sequence of uniform variables in E . For $\omega \in \Omega$ and $e \in E$, we denote by ω^e the configuration obtained by opening the edge e and by ω_e the one where e is closed. At time t , we suppose $X_{t-1} = \omega$ and we set

$$X_t(e) = \begin{cases} \omega(e) & \text{if } E_t \neq e \\ 1 & \text{if } E_t = e \text{ and } U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \\ 0 & \text{if } E_t = e \text{ and } U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases} .$$

We also define a coupled process $(Y_t)_{t \in \mathbb{N}}$ which stays in $\{T \leftrightarrow B\}$. We use the same sequences $(E_t)_{t \in \mathbb{N}}$ and $(U_t)_{t \in \mathbb{N}}$. At time t , we suppose $Y_{t-1}(e) = \omega$ and we change the status of the edge e in Y_t as follows:

$$Y_t(e) = \begin{cases} \omega(e) & \text{if } E_t \neq e \\ 1 & \text{if } E_t = e, U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \text{ and } T \xrightarrow{\omega^e} B \\ 0 & \text{if } E_t = e, U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \text{ and } T \xleftarrow{\omega^e} B \\ 0 & \text{if } E_t = e, U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases} .$$

Before opening a closed edge e at time t , we verify whether this will create a connexion between T and B in Y_t . If it is the case, the edge e stays closed in Y_t but can be opened in X_t , otherwise the edge e is opened in both X_t and Y_t . On the contrary, the two processes behave similarly for the edge closing events since we cannot create a new connexion by closing an edge. The set of the configurations satisfying $\{T \leftrightarrow B\}$ is irreducible and the process $(X_t)_{t \in \mathbb{N}}$ is reversible. By the lemma 1.9 of [11], there exists a unique stationary distribution $\Phi^{\mathcal{D}}$ for the process $(Y_t)_{t \in \mathbb{N}}$ and $\Phi^{\mathcal{D}}$ is equal to the probability Φ^{TB} conditioned by the event $\{T \leftrightarrow B\}$, i.e.,

$$\Phi^{\mathcal{D}}(\cdot) = \Phi^{TB}(\cdot | T \leftrightarrow B).$$

Suppose that we start from a configuration (X_0, Y_0) belonging to the set

$$\mathcal{E} = \{ (\omega_1, \omega_2) \in \{0, 1\}^{\mathbb{E}^d \cap \Lambda} \times \{T \leftrightarrow B\} : \forall e \in \Lambda \quad \omega_1(e) \geq \omega_2(e) \}.$$

The set \mathcal{E} is irreducible and aperiodic. In fact, each configuration of \mathcal{E} communicates with the configuration where all edges are closed. The state space \mathcal{E} is finite, therefore the Markov chain $(X_t, Y_t)_{t \in \mathbb{N}}$ admits a unique equilibrium distribution μ_p . We denote by P_μ the law of the process $(X_t, Y_t)_{t \in \mathbb{N}}$ starting from a random initial configuration (X_0, Y_0) with distribution $\mu_{\Lambda, p, q}$. We define the following objects using the previous coupling.

Definition 2.1. *The interface at time t between T and B , denoted by \mathcal{I}_t , is the set of the edges in Λ that differ in the configurations X_t and Y_t , i.e.,*

$$\mathcal{I}_t = \{ e \in \Lambda : X_t(e) \neq Y_t(e) \}.$$

The edges of \mathcal{I}_t are open in X_t but closed in Y_t and the configuration X_t is above the configuration Y_t . We define next the set \mathcal{P}_t of the pivotal edges for the event $\{T \leftrightarrow B\}$ in the configuration Y_t .

Definition 2.2. *The set \mathcal{P}_t of the pivotal edges in Y_t is the collection of the edges in Λ whose opening would create a connection between T and B , i.e.,*

$$\mathcal{P}_t = \{ e \in \Lambda : T \xleftrightarrow{Y_t^e} B \}.$$

We define finally the set \mathcal{C}_t of the cuts in Y_t .

Definition 2.3. *The set \mathcal{C}_t of the cuts in Y_t is the collection of the cuts in Λ at time t .*

2.5 The classical Edwards-Sokal coupling

We wish to gain insight into the interface in the Ising model with the help of our previous results and the classical coupling of Edwards and Sokal (see chapter 1 of [9] for more details on this coupling). Let $G = (V, E)$ be a graph. We consider the product space $\Sigma \times \Omega$ where $\Sigma = \{-1, 1\}^V$ and $\Omega = \{0, 1\}^E$. We define a probability ν on $\Sigma \times \Omega$ by

$$\nu(\sigma, \omega) \propto \prod_e \left\{ (1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma) \right\}, \quad (\sigma, \omega) \in \Sigma \times \Omega,$$

where $\delta_e(\sigma) = \delta_{\sigma(x),\sigma(y)}$ for $e = \langle x, y \rangle \in E$. The constant of proportionality is the one which ensures the normalization

$$\sum_{(\sigma, \omega) \in \Sigma \times \Omega} \nu(\sigma, \omega) = 1.$$

For $q = 2$, $p = 1 - e^{-\beta}$ and $\omega \in \Omega$, the conditional measure $\nu(\cdot|\omega)$ on Σ is obtained by colouring randomly the clusters of ω . More precisely, conditionally on a percolation configuration ω , the spins are constant on the clusters of ω and they are independent between the clusters. With the help of this coupling, we can transport results in FK-percolation to the Ising model.

2.6 The coupling of spin configurations

We construct a coupling of the Ising configurations $(\sigma^+, \sigma^-, \sigma^D)$ with different boundary conditions from a pair of percolation configurations

$$(\omega, \omega') \in \Omega \times \{T \leftrightarrow B\}$$

satisfying $\omega \geq \omega'$. The configuration σ^+ (resp. σ^-) will correspond to the spin configuration with the $+$ boundary condition (resp. $-$ boundary condition) and the configuration σ^D will correspond to the Dobrushin boundary

condition. We will put spins on the vertices in Λ as follows. We start by putting spins on the vertices of the clusters of ω' using the Edwards-Sokal coupling with the Dobrushin boundary conditions. This way we obtain a spin configuration, which we denote by σ^D . Notice that the open clusters of ω are unions of the open clusters of ω' . For an open cluster in ω which touches the boundary of Λ , we color its vertices with $+1$ in the configuration σ^+ and with -1 in σ^- . For an open cluster C which does not touch the boundary of Λ , if C is also an open cluster in ω' , we set

$$\forall x \in C \quad \sigma^+(x) = \sigma^-(x) = \sigma^D(x).$$

For an open cluster C of ω which does not touch the boundary of Λ and which is the union of several open clusters of ω' , we set

$$\forall x \in C \quad \sigma^+(x) = \sigma^-(x) = N(C),$$

where $N(C)$ is equal to 1 with probability $1/2$ and -1 with probability $1/2$. Of course, the random variables $N(C)$ associated to the clusters of ω are independent and also independent from the configuration outside C . For $q = 2, p = 1 - e^{-\beta}$ and (ω, ω') distributed under $\mu_{\Lambda, p, 2}$, the configurations σ^+, σ^- and σ^D obtained are distributed according to the Gibbs state at inverse temperature β with boundary conditions $+, -$ and Dobrushin. We denote by $\pi_{\Lambda, \beta}$ the distribution of the triple $(\sigma^+, \sigma^-, \sigma^D)$

3 Localising a cut around the pivotal edges

The following proposition controls the distance between an edge belonging to a cut and the set of the pivotal edges.

Proposition 3.1. *For any $q \geq 1$, there exist $\tilde{p} < 1$ and $\kappa > 1$ such that, for $p \geq \tilde{p}$, and for any $c \geq 1$ and any box Λ satisfying $|\Lambda| > 3^{6d}$, we have*

$$\Phi_{\Lambda, p, q}^{TB} [\exists C \in \mathcal{C}, \exists e \in C, d(e, \mathcal{P} \cup \Lambda^c \setminus \{e\}) \geq \kappa c \ln |\Lambda| \mid T \leftrightarrow B] \leq \frac{1}{|\Lambda|^c}.$$

The strategy of the proof follows that of proposition 1.4 of [1]. We can still observe a closed path which is disjoint from a cut. However, one key ingredient of the proof in [1] is the BK inequality (see [8]) which doesn't hold for the FK-percolation model. The following lemma gives us an inequality which plays the role of the BK inequality in the proof.

Lemma 3.2. *Let $n \geq 2$, $q \geq 1$, $p \in [0, 1]$, and let e be an edge in Λ . We define the event*

$$\Gamma(e, n, T) = \left\{ \begin{array}{l} \text{there exists a closed path of length } n \text{ starting from } e \\ \text{and there exists a cut which separates this path from } T \end{array} \right\}.$$

The following inequality holds:

$$\Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T)] \leq \alpha^n(d) (1 - f(p, q))^n \Phi_{\Lambda}^{TB}[T \leftrightarrow B],$$

where

$$f(p, q) = \frac{p}{p + q - pq}$$

and $\alpha(d)$ is the number of $*$ -neighbours of an edge in the lattice \mathbb{L}^d .

Notice that this inequality also holds if we consider the symmetric event $\Gamma(e, n, B)$.

Proof. Let us start with the construction of a random graph in Λ in which we can find a path starting from e and the complementary of which contains a cut. The construction is inspired by the proof of theorem 5.3 in [7]. For the topological complications in the construction, we refer to the related passage of [12, Sect. 2]. We define the open cluster of T as

$$O(T) = \{x \in \Lambda : x \longleftrightarrow T\}.$$

We consider a configuration satisfying $\{T \leftrightarrow B\}$. Then we define the set C^+ as the set of the edges f satisfying

1. f has exactly one endpoint in $O(T)$;
2. there exists a geometric path in \mathbb{L}^d which connects B to f and which does not use vertices of $O(T)$.

Notice that the path in point 2 is only geometric, there is no requirement on the status of the edges of this path. We note three facts about the set C^+ :

- a. the edges of C^+ are closed,
- b. the set C^+ contains a cut,
- c. C^+ is measurable with respect to the configuration of the edges which have at least one endpoint in $O(T)$.

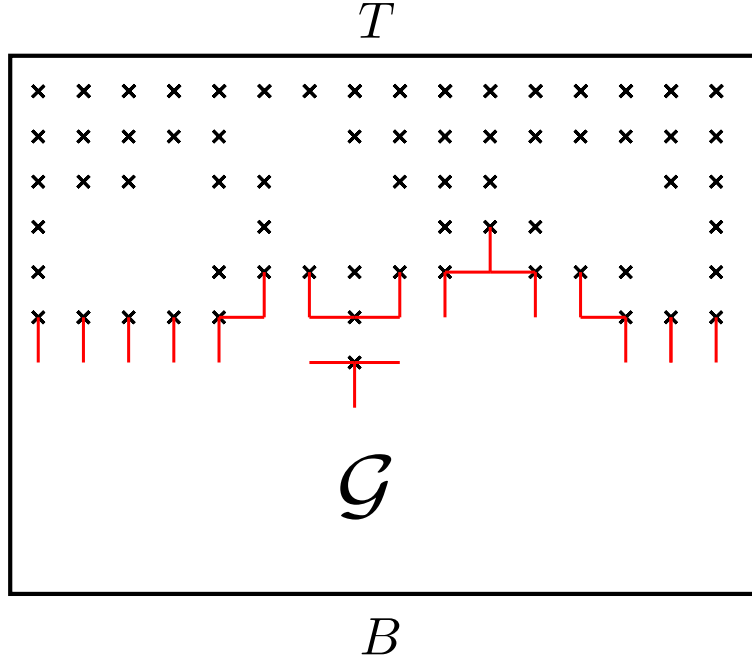


Figure 2: The crosses are the vertices of $O(T)$ and the red edges form the set C^+ , the graph \mathcal{G} remains unexplored.

Let us now consider a configuration in the event $\Gamma(e, n, T)$. The cut in C^+ separates e from T . We now construct a sub-graph \mathcal{G} of Λ . We define V' as the set of the vertices included in $V \setminus O(T)$ which are connected to B without using an edge of C^+ . We define the sub-graph \mathcal{G} as

$$\mathcal{G} = (V', \{\langle x, y \rangle : x, y \in V'\}).$$

The figure 2 illustrates the construction of the graph \mathcal{G} . From the construction, it follows that the cut in C^+ does not contain an edge of the graph \mathcal{G} . For a fixed sub-graph $G = (V_1, E_1)$ of $\Lambda = (V, E)$, we denote by G' the graph (V', E') , where $E' = E \setminus E_1$ and V' is the set of the endpoints of the edges in E' . We decompose the set of the edge configurations Ω in Λ as

$$\Omega = \Omega_G \times \Omega_{G'},$$

where Ω_G (respectively $\Omega_{G'}$) is the set of the configurations of the edges in G (respectively G'). We notice that the event $\mathcal{G} = G$ is entirely determined

by the configurations $\omega_{G'} \in \Omega_{G'}$. We obtain that

$$\begin{aligned}
\Phi_{\Lambda,p,q}^{TB}[\Gamma(e,n,T)] &= \sum_G \Phi_{\Lambda,p,q}^{TB}[\Gamma(e,n,T), \mathcal{G} = G] \\
&= \sum_G \sum_{\eta \in \Omega_{G'}: \mathcal{G}=G} \Phi_{\Lambda,p,q}^{TB}[\Gamma(e,n,T), \omega_{G'} = \eta] \\
&= \sum_G \sum_{\eta \in \Omega_{G'}: \mathcal{G}=G} \Phi_{\Lambda,p,q}^{TB}[\Gamma(e,n,T) \mid \omega_{G'} = \eta] \Phi_{\Lambda,p,q}^{TB}[\omega_{G'} = \eta]. \quad (3.1)
\end{aligned}$$

We define the event $\Gamma(e,n)$ as

$$\Gamma(e,n) = \{ \text{there exists a closed } * \text{-path of length } n \text{ starting from } e \}.$$

By the spatial Markov property, for a fixed graph G containing all the edges intersecting B and η such that $\mathcal{G} = G$, we have

$$\Phi_{\Lambda,p,q}^{TB}[\Gamma(e,n,T) \mid \omega_{G'} = \eta] = \Phi_{G,p,q}^{\xi(\eta)}[\Gamma(e,n)] \leq \Phi_{G,p,q}^0[\Gamma(e,n)],$$

where $\xi(\eta)$ is the boundary condition on G induced by η . The last inequality holds because the event $\Gamma(e,n)$ is decreasing. As for the configurations in the graph G , we can compare the measure $\Phi_{G,p,q}^0$ with the Bernoulli percolation measure with parameter

$$f(p,q) = \frac{p}{p+q-pq}.$$

Since the event $\Gamma(e,n)$ is decreasing, by the comparison inequalities stated in chapter 3.4 of [9], we have

$$\Phi_{G,p,q}^0[\Gamma(e,n)] \leq \Phi_{G,f(p,q),1}^0[\Gamma(e,n)].$$

By a standard Peierls argument, we have then

$$\Phi_{G,f(p,q),1}^0[\Gamma(e,n)] \leq \alpha^n(d)(1-f(p,q))^n,$$

where $\alpha(d)$ is the number of $*$ -neighbours of an edge in dimension d . We use this upper bound in the sum (3.1) and we obtain

$$\begin{aligned}
\Phi_{\Lambda}^{TB}[\Gamma(e,n,T)] &\leq \alpha^n(d)(1-f(p,q))^n \sum_G \sum_{\eta: \mathcal{G}=G} \Phi_{\Lambda}^{TB}[\omega_{G'} = \eta] \\
&= \alpha^n(d)(1-f(p,q))^n \sum_G \Phi_{\Lambda}^{TB}[\mathcal{G} = G].
\end{aligned}$$

We now calculate the last sum. We have

$$\sum_G \Phi_\Lambda^{TB}[\mathcal{G} = G] = \Phi_\Lambda^{TB} \left[\begin{array}{l} \exists \mathcal{G} = (V, E) \text{ sub-graph of } \Lambda \\ V \cap O(T) = \emptyset, \quad B \subset V \end{array} \right].$$

By construction, the existence of such a graph \mathcal{G} implies the existence of a cut. We obtain

$$\sum_G \Phi_\Lambda^{TB}[\mathcal{G} = G] \leq \Phi_\Lambda^{TB}[\exists C \in \mathcal{C}].$$

The existence of a cut implies the disconnection between T and B , thus we have

$$\Phi_\Lambda^{TB}[\Gamma(e, n, T)] \leq \alpha^n(d) (1 - f(p, q))^n \Phi_\Lambda^{TB}[T \leftrightarrow B],$$

which is the desired inequality. \square

We now prove proposition 3.1 with the help of the previous lemma.

Proof of proposition 3.1. Notice that every edge of a cut is connected to the set of the pivotal edges \mathcal{P} or to the boundary of Λ by a closed path. Let us fix an edge e which belongs to a cut and which is at distance more than $\kappa c \ln |\Lambda|$ from $\mathcal{P} \cup \Lambda^c \setminus \{e\}$. There is a closed path starting from the edge e and which is of length $\kappa c \ln |\Lambda| / 4d$. This path is disjoint from a cut since there is no pivotal edge on this path. We refer to the section 3 of [1] for the detailed geometric justifications. The probability appearing in the proposition is less than

$$\Phi_\Lambda^{TB}[A \mid T \leftrightarrow B],$$

where A is the event

$$A = \left\{ \begin{array}{l} \exists \gamma \text{ closed path of length } \frac{\kappa c \ln |\Lambda|}{4d} \text{ starting from } e \\ \exists C \in \mathcal{C} \quad C \text{ disjoint from } \gamma \end{array} \right\}.$$

Since the existence of a cut implies the disconnection between T and B , we can rewrite the conditioned probability as

$$\Phi_\Lambda^{TB}[A \mid T \leftrightarrow B] = \frac{\Phi_\Lambda^{TB}[A]}{\Phi_\Lambda^{TB}[T \leftrightarrow B]}.$$

We distinguish two cases according to the positions of the path γ and the cut C . Since the cut C splits the box Λ into two parts and the path γ starting from e is disjoint from the cut, then it is included in one of the two parts. Therefore, the path γ is either separated from T or from B by C . We obtain

$$\Phi_\Lambda^{TB}[A] \leq \Phi_\Lambda^{TB}[\Gamma(e, T)] + \Phi_\Lambda^{TB}[\Gamma(e, B)],$$

where we reuse the notation Γ introduced in the previous lemma by setting $n = \kappa c \ln |\Lambda| / 4d$ and we omit n in the notation as follows:

$$\Gamma(e, T) = \left\{ \begin{array}{l} \exists \gamma \text{ closed path of length } \frac{\kappa c \ln |\Lambda|}{4d} \text{ starting from } e \\ \exists C \in \mathcal{C}, C \text{ separates } \gamma \text{ from } T \end{array} \right\}$$

and $\Gamma(e, B)$ is defined similarly, with B instead of T . By lemma 3.2, we have

$$\Phi_{\Lambda}^{TB}[\Gamma(e, T)] \leq (\alpha(d)(1 - f(p, q)))^{\kappa c \ln |\Lambda| / 4d} \Phi_{\Lambda}^{TB}[T \leftrightarrow B]$$

and the same holds for $\Phi_{\Lambda}^{TB}[\Gamma(e, B)]$. Thus, we have

$$\frac{\Phi_{\Lambda}^{TB}[A \circ \{\exists C \in \mathcal{C}\}]}{\Phi_{\Lambda}^{TB}[T \leftrightarrow B]} \leq 2(\alpha(d)(1 - f(p, q)))^{\kappa c \ln |\Lambda| / 4d}.$$

For $q \geq 1$, there exist a $\tilde{p} < 1$ and $\kappa \geq 1$, such that, for $p \geq \tilde{p}$ and $c \geq 1$, we have

$$(\alpha(d)(1 - f(p, q)))^{\kappa c \ln |\Lambda| / 4d} \leq \frac{1}{2|\Lambda|^c},$$

which yields the desired result. \square

4 The speed estimate of the pivotal edges

We now study the difference between a cut at two different times t and $t + s$. We begin with a key lemma which controls the number of closing events which can be realised on an interval of length s .

Lemma 4.1. *Let Γ be a simple $*$ -path and t a time and $s \in \mathbb{N}$. For any $k \in \{0, \dots, |\text{support}(\Gamma)|\}$ and any configuration y such that exactly k edges of Γ are closed, we have the following inequality:*

$$P_{\mu}(\Gamma \text{ closed in } Y_{t+s} \mid Y_t = y) \leq \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{|\text{support}(\Gamma)|-k}.$$

Proof. Let us denote by n the cardinal of $\text{support}(\Gamma)$. If an edge e is visited, then it must be closed at time t or become closed at a time after t . Therefore, there exist $n - k$ different edges e_1, \dots, e_{n-k} of Γ , and $n - k$ different times t_1, \dots, t_{n-k} strictly bigger than t such that

$$\forall i \in \{1, \dots, n - k\} \quad E_{t_i} = e_i \quad \text{and} \quad U_{t_i} \leq \frac{1-p}{1-p+p/q}.$$

These events are independent of the configuration Y_t and we obtain an upper bound on the desired probability as follows:

$$P_\mu (\Gamma \text{ closed in } Y_{t+s} \mid Y_t = y) \leq P \left(\begin{array}{l} \forall i \in \{1, \dots, n-k\} \\ \exists t_i \in]t, t+s] \quad E_{t_i} = e_i \\ U_{t_i} \leq \frac{1-p}{1-p+p/q} \end{array} \right). \quad (4.1)$$

This last probability depends on the i.i.d. sequence of couples $(E_t, U_t)_{t \in \mathbb{N}}$. Moreover, the sets

$$J(e) = \{s < j \leq t : E_j = e\}, \quad e \in \{e_1, \dots, e_{n-k}\}$$

are pairwise disjoint subsets of \mathbb{N} . For an edge $e \in \{e_1, \dots, e_{n-k}\}$, the event

$$\left\{ \begin{array}{l} \exists r \in]s, t] \quad E_r = e \\ U_r \leq \frac{1-p}{1-p+p/q} \end{array} \right\}$$

is entirely determined by $(E_t, U_t)_{t \in J(e)}$. Therefore, by the BK inequality (see [8]) applied with the random variables $(E_r, U_r)_{s < r \leq t}$, the second probability in (4.1) is less than

$$\prod_{e \in \{e_1, \dots, e_{n-k}\}} P \left(\begin{array}{l} \exists r \in]t, t+s] \quad E_r = e \\ U_r \leq \frac{1-p}{1-p+p/q} \end{array} \right). \quad (4.2)$$

For each $e \in \{e_1, \dots, e_{n-k}\}$, we have

$$P \left(\begin{array}{l} \exists r \in]t, t+s] \quad E_r = e \\ U_r \leq \frac{1-p}{1-p+p/q} \end{array} \right) \leq \frac{s(1-p)}{|\Lambda|(1-p+p/q)}.$$

An upper bound for (4.2) is

$$\left(\frac{s(1-p)}{|\Lambda|(1-p+p/q)} \right)^{n-k}.$$

Combined with (4.1), we have the inequality stated in the lemma. \square

In order to apply the previous result to control the speed of a cut, we will consider s satisfying $0 < s \leq |\Lambda|$ and we study the probability that there exists a pivotal edge e at time $t+s$ which is at distance ℓ from the pivotal edges at time t . The following proposition gives an upper bound on the speed of a cut.

Proposition 4.2. *There exist $\tilde{p} < 1$ and $\kappa \geq 1$, such that for $p \geq \tilde{p}$, $t \in \mathbb{N}$, $s \in \{1, \dots, |\Lambda|\}$, $\ell \geq 1$ and any edge $e \subset \Lambda$ at distance more than ℓ from Λ^c ,*

$$P_\mu\left(e \in \mathcal{P}_{t+s} \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right) \leq e^{-\ell}.$$

The proof follows the ideas in the proof of proposition 4.1 of [1].

Proof. We start by rewriting the conditioned probability as

$$\frac{P_\mu\left(\{e \in \mathcal{P}_{t+s}\} \cap \{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\}\right)}{P_\mu\left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right)}. \quad (4.3)$$

Since a pivotal edge is in a cut, there exists a closed simple $*$ -path γ which connects e to an edge intersecting the boundary of Λ at time $t + s$. This $*$ -path travels a distance at least ℓ and the length of γ is at least ℓ/d . Therefore, by taking the first n edges such that the support of this sequence is equal to $\ell/2d$, the numerator is bounded from above by

$$P_\mu\left(\left\{\begin{array}{l} \exists \gamma \text{ simple closed } * \text{-path of length } \\ n \text{ starting at } e \text{ at time } t + s \end{array}\right\} \cap \left\{\begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array}\right\}\right) \leq \\ \sum_{\Gamma} P_\mu\left(\left\{\begin{array}{l} \Gamma \text{ simple closed } * \text{-path of length } \\ n \text{ starting at } e \text{ at time } t + s \end{array}\right\} \cap \left\{\begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array}\right\}\right). \quad (4.4)$$

For a $*$ -path Γ of length n starting from e , we consider the set $M(k)$ of the configurations defined as

$$M(k) = \left\{ \omega : \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad \omega(f) = 0 \\ \forall f \in \text{support}(\Gamma) \setminus F \quad \omega(f) = 1 \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right\}.$$

For k fixed, the probability of having exactly k edges in Γ which are closed at time t is less than

$$\sum_{y \in M(k)} P_\mu\left(\begin{array}{l} \Gamma \text{ simple } * \text{-path} \\ \text{closed at time } t + s \end{array} \mid Y_t = y\right) P_\mu(Y_t = y).$$

By lemma 4.1, each term of the sum is less than

$$\left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)}\right)^{n-k} P_\mu(Y_t = y).$$

We obtain an upper bound for (4.4) as

$$\sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{n-k} P_{\mu}(Y_t \in M(k)).$$

Let us calculate the probability $P_{\mu}(Y_t \in M(k))$. Notice that this event is determined by the configuration Y_t . This probability is less than

$$\begin{aligned} \Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \middle| T \leftrightarrow B \right] \\ = \frac{\Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right]}{\Phi_{\Lambda}^{TB}[T \leftrightarrow B]}. \end{aligned} \quad (4.5)$$

Denote by Δ the set of the edges at distance less than $\ell - 1$ from e , then the event

$$\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\}$$

depends on the edges inside Δ , whereas the event

$$D = \{ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \}$$

depends on the edges in $\Lambda \setminus \Delta$. Denote by $\Omega_{\Lambda \setminus \Delta}$ the set of the configurations of the edges in $\Lambda \setminus \Delta$. By the spatial Markov property, we have

$$\begin{aligned} \Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right] \\ = \sum_{\omega \in \Omega_{\Lambda \setminus \Delta} \cap D} \Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \middle| \omega \right] \Phi_{\Lambda}^{TB}[\omega] \\ = \sum_{\omega \in \Omega_{\Lambda \setminus \Delta} \cap D} \Phi_{\Delta}^{\xi(\omega)} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \Phi_{\Lambda}^{TB}[\omega]. \end{aligned} \quad (4.6)$$

Since the event

$$\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\}$$

is decreasing, for any boundary condition $\xi(\omega)$ on Δ induced by ω , we have

$$\Phi_{\Delta}^{\pi(\omega)} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \leq \Phi_{\Delta}^0 \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right].$$

We compare this last probability to the one under Bernoulli percolation with parameter

$$f(p, q) = \frac{p}{p + q - pq}.$$

By the comparison inequalities between different values of p, q , we have

$$\Phi_{\Delta}^0 \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \leq \binom{\ell/2d}{k} (1 - f(p, q))^k.$$

We use this inequality in (4.6) and we obtain

$$\Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right] \leq \binom{\ell/2d}{k} (1 - f(p, q))^k \Phi_{\Lambda}^{TB}[D].$$

Combined with (4.5), we have

$$P_{\mu}(Y_t \in M(k)) \leq \binom{\ell/2d}{k} (1 - f(p, q))^k \Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B].$$

We replace $P_{\mu}(Y_t \in M(k))$ in the previous upper bound for (4.4) and we have another upper bound as follows:

$$\Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B] \sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{n-k} \binom{\ell/2d}{k} (1 - f(p, q))^k.$$

We calculate the sum and we get

$$\begin{aligned} & \sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{n-k} \binom{\ell/2d}{k} (1 - f(p, q))^k \\ & \leq \left(\frac{1-p}{p/q} \right)^n \sum_{\Gamma} \sum_k \binom{\ell/2d}{k} \left(\frac{s}{|\Lambda|} \right)^{n-k} \\ & \leq \sum_{\Gamma} \left(\frac{1-p}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d} \leq \left(\frac{\alpha(d)(1-p)}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d}. \end{aligned}$$

Since $s \leq |\Lambda|$, there exists a constant $\tilde{p} < 1$ such that

$$\left(\frac{\alpha(d)(1-p)}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d} \leq e^{-\ell}.$$

We obtain the following upper bound for (4.3):

$$\frac{P_\mu\left(\{e \in \mathcal{P}_{t+s}\} \cap \{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\}\right)}{P_\mu\left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right)} \leq e^{-\ell} \frac{\Phi_\Lambda^{TB}[D \mid T \leftrightarrow B]}{P_\mu\left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right)}.$$

Finally, we notice that

$$\Phi_\Lambda^{TB}[D \mid T \leftrightarrow B] = P_\mu\left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right)$$

and we obtain the desired inequality. \square

5 The interface in the FK-percolation model

We now prove the main result stated in theorem 1.3. We follow the main steps of the proof of theorem 1.1 in [1]. We start with the definition of d_H^ℓ , a pseudo-distance, similar to the Hausdorff distance, between two subsets A, B of Λ ,

$$d_H^\ell(A, B) = \inf \left\{ r \geq 0 : \begin{array}{l} A \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(B, r) \\ B \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(A, r) \end{array} \right\}.$$

We show a lemma which controls the speed of the pivotal edges.

Lemma 5.1. *We have the following result:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 1 \quad \forall p \geq \tilde{p} \quad \forall c \geq 1 \quad \forall \Lambda \quad |\Lambda| \geq 4 \quad \forall t \geq 0 \\ P_\mu\left(\exists s \leq |\Lambda| \quad d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda|\right) \leq \frac{8d}{|\Lambda|^c}. \end{aligned}$$

Proof. We fix $s \in \{1, \dots, |\Lambda|\}$. By the definition of the distance d_H^ℓ , we have, for any $\kappa > 1$,

$$\begin{aligned} P_\mu\left(d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda|\right) \leq \\ P_\mu\left(\mathcal{P}_{t+s} \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_t, \kappa c \ln |\Lambda|)\right) \\ + P_\mu\left(\mathcal{P}_t \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_{t+s}, \kappa c \ln |\Lambda|)\right). \quad (5.1) \end{aligned}$$

Since the two probabilities in the sum depend only on the process Y , which is reversible, they are in fact equal to each other. We shall estimate the first probability. By proposition 4.2, for any $\ell \geq 1$, we have

$$P_\mu\left(\begin{array}{l} e \in \mathcal{P}_{t+s} \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq \ell \end{array}\right) \leq e^{-\ell}.$$

In order to replace c_t by \mathcal{P}_t in the last probability, we use proposition 3.1 for the configuration Y_t . We introduce another constant κ' and we have

$$P_\mu \left(\begin{array}{c} \exists C \in \mathcal{C}_t \quad \exists f \in C \\ d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

Therefore, by distinguishing two cases for the configuration Y_t , we have

$$\begin{aligned} & P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|) \leq \\ & P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s}, \quad d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|, \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda| \end{array} \right) \\ & + P_\mu(\exists C \in \mathcal{C}_t, \exists f \in C, d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda|). \end{aligned}$$

For $\kappa > \kappa'$, we have

$$\begin{aligned} & P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \quad d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda| \end{array} \right) \leq \\ & P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq (\kappa - \kappa') c \ln |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^{(\kappa - \kappa')c}}. \end{aligned}$$

We choose now $\kappa = \kappa' + 1$, and we sum over e in Λ and $s \in \{1, \dots, |\Lambda|\}$ to get

$$P_\mu \left(\begin{array}{c} \exists s \leq |\Lambda|, \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{4d}{|\Lambda|^{c-2}}.$$

This is the first probability in (5.1) and we conclude that

$$P_\mu \left(\begin{array}{c} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{8d}{|\Lambda|^{c-2}}.$$

Finally, we replace c by $c + 2$ and we have

$$P_\mu \left(\begin{array}{c} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq 3\kappa c \ln |\Lambda| \end{array} \right) \leq \frac{8d}{|\Lambda|^c}.$$

This is the desired inequality. \square

The rest of the proof of theorem 1.3 is exactly the same as the proof of theorem 1.1 in [1] which relies essentially on lemma 5.1 in [1] independently from the model. We distinguish the case $e \in \mathcal{P}_t$ and the case $e \in \mathcal{I}_t \setminus \mathcal{P}_t$. For the first case, we apply proposition 3.1. For the second case, we notice that this configuration is due to a movement of pivotal edges of distance $\ln^2 |\Lambda|$ during a time interval of order $|\Lambda|$. We then apply lemma 5.1 to prove that this event is unlikely.

6 The interface in the Ising model

We recall the definition of the two sets \mathcal{P}_I and \mathcal{I}_I :

$$\mathcal{P}_I = \left\{ \langle x, y \rangle \in E : \begin{array}{l} \sigma^D(x) = +1 \quad \sigma^D(y) = -1 \\ x \text{ connected to } T \text{ by a path of spin } +1 \text{ in } \sigma^D \\ y \text{ connected to } B \text{ by a path of spin } -1 \text{ in } \sigma^D \end{array} \right\}.$$

and

$$\mathcal{I}_I = \left\{ \langle x, y \rangle \in E : \begin{array}{l} \sigma^+(x) = \sigma^+(y) \\ \sigma^-(x) = \sigma^-(y) \\ \sigma^D(x) \neq \sigma^D(y) \end{array} \right\}.$$

We now show our main result on the Ising model.

Proof of theorem 1.2. We fix a constant c and we let κ be a constant which will be determined later. Let us consider the FK configurations (ω, ω') associated to $(\sigma^+, \sigma^-, \sigma^D)$. We consider the set of the pivotal edges \mathcal{P} and the set of the interface \mathcal{I} in the couple (ω, ω') . We claim that $\mathcal{P} \subset \mathcal{P}_I$. In fact, for an edge $\langle x, y \rangle \in \mathcal{P}$, one of the endpoint x is connected to T and the other B is connected to B in ω' . Therefore, we have $\sigma^D(x) = 1$ and $\sigma^D(y) = -1$. The configuration ω dominates ω' , so both of them are also connected to the boundary of Λ in ω . We conclude that

$$\sigma^+(x) = \sigma^+(y) = 1 \text{ and } \sigma^-(x) = \sigma^-(y) = -1.$$

Let e be an edge in \mathcal{I}_I . Since the endpoints of e have different spins in σ^D , then the edge e must be closed in ω' and its endpoints belong to two distinct open clusters of ω' . We denote by x, y the two endpoints of e and by C_x, C_y the two open clusters of ω' such that $x \in C_x$ and $y \in C_y$. The vertices x and y have the same spin in σ^+ and in σ^- , therefore one of them, as well as its cluster, has different spins in σ^+ and σ^D . Suppose it is C_x . Suppose that e belongs to $\mathcal{I}_I \setminus \mathcal{P}_I$. We have then $\sigma^+(x) = \sigma^-(x)$ and x is not connected to the boundary of Λ in ω' . By construction of σ^+ , C_x is contained in an open cluster of ω which is the union of at least two different open clusters of ω' . Therefore, the boundary of C_x contains at least one edge of ω' in \mathcal{I} and the edge e is $*$ -connected to this edge of \mathcal{I} in the configuration ω' via a closed $*$ -path. Let us fix the edge e at distance more than $\kappa c \ln^2 |\Lambda|$ from Λ^c . We obtain that

$$\pi_\beta \left(e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \mu_{p,2} \left(\begin{array}{l} \exists f \in \mathcal{I}, \exists \gamma \text{ closed } * \text{-path in } Y \\ \gamma \text{ connects } e \text{ to } f \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right).$$

We distinguish two cases according to the position of the edge $f \in \mathcal{I}$. Let κ' be the constant given by theorem 1.3. The last probability is less than

$$\begin{aligned} & \mu_{p,2} \left(\exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) \geq \kappa' c^2 \ln |\Lambda| \right) \\ & + \mu_{p,2} \left(\begin{array}{l} \exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) < \kappa' c^2 \ln |\Lambda| \\ \exists \gamma \text{ closed } * \text{-path in } Y \\ \gamma \text{ connects } e \text{ to } f \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right). \end{aligned} \quad (6.1)$$

By theorem 1.3, there exists a p_1 such that for $p \geq p_1$, we have

$$\mu_{p,2} \left(\exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) \geq \kappa' c^2 \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}. \quad (6.2)$$

We consider the second case where the edge f is at distance less than $\kappa' c^2 \ln^2 |\Lambda|$ from $\Lambda^c \cup \mathcal{P}$. Notice that the path γ travels a distance at least $(\kappa - \kappa') c^2 \ln^2 |\Lambda|$. By taking $\kappa/2 \geq \kappa'$, the path γ contains a subpath of length exactly $(\kappa c^2 \ln^2 |\Lambda|)/2d$. So the second probability in (6.1) is less than

$$\mu_{p,2} \left(\begin{array}{l} \exists \gamma \text{ closed } * \text{-path in } Y \text{ starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right).$$

We distinguish two further cases according to the position of the cuts. Let κ'' be the constant given by proposition 3.1. We write

$$\begin{aligned} & \mu_{p,2} \left(\begin{array}{l} \exists \gamma \text{ closed path in } Y \text{ starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \\ & \mu_{p,2} \left(\begin{array}{l} \exists C \in \mathcal{C} \quad \exists f \in C \\ d(f, \mathcal{P}) \geq \kappa'' c \ln |\Lambda| \end{array} \right) \\ & + \mu_{p,2} \left(\begin{array}{l} \exists \gamma \text{ closed } * \text{-path in } Y \text{ starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \\ \forall C \in \mathcal{C} \quad \forall f \in C \\ d(f, \mathcal{P}) \leq \kappa'' c \ln |\Lambda| \end{array} \right). \end{aligned}$$

We consider the first probability where there exists a cut which goes far away from the set \mathcal{P} . The event in this probability is determined by the configuration Y , we can replace $\mu_{p,2}$ by $\Phi_{p,2}^{TB}(\cdot \mid T \leftrightarrow B)$. By proposition 3.1, there exists $p_2 < 1$ such that for $p \geq p_2$, we have

$$\Phi_{\Lambda,p,2}^{TB} [\exists C \in \mathcal{C}, \exists f \in C, d(f, \mathcal{P}) \geq \kappa'' c \ln |\Lambda| \mid T \leftrightarrow B] \leq \frac{1}{|\Lambda|^c}. \quad (6.3)$$

We consider the last case where all the cuts are at distance less than $\kappa''c \ln |\Lambda|$ from \mathcal{P} , as illustrated in the figure 3. Since the edge e is at distance more than $\kappa c^2 \ln^2 |\Lambda|$ from $\Lambda^c \cup \mathcal{P}$ and the path γ is of length $(\kappa c^2 \ln^2 |\Lambda|)/2d$, for a κ such that

$$\frac{\kappa c^2 \ln^2 |\Lambda|}{2} + \kappa''c \ln |\Lambda| < \kappa c^2 \ln^2 |\Lambda|,$$

the path γ is disjoint from all the cuts. We obtain

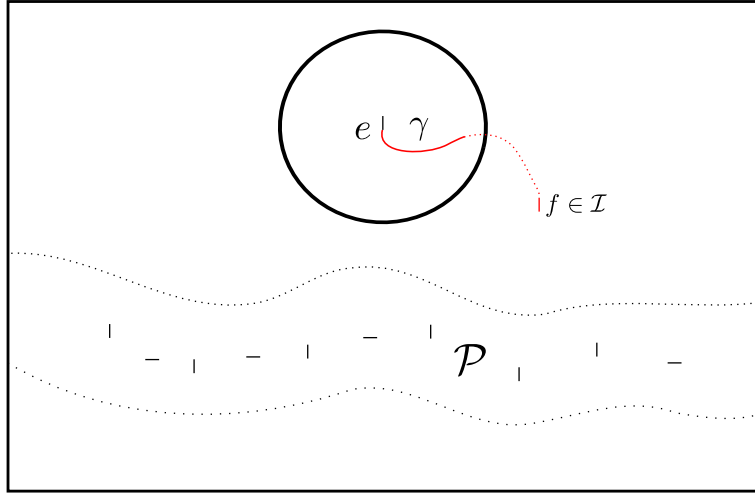


Figure 3: The path γ (in red) is of length $(\kappa c^2 \ln^2 |\Lambda|)/2d$ and is contained in a $*$ -closed path connecting e to an edge $f \in \mathcal{I}$. The set \mathcal{P} is at distance more than $\kappa c^2 \ln^2 |\Lambda|$ from e and all the cuts are at distance less than $\kappa''c \ln |\Lambda|$ from \mathcal{P}

$$\mu_{p,2} \left(\begin{array}{l} \exists \gamma \text{ closed path in } Y \text{ starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ d(e, \Lambda^c \cup \mathcal{P}) \geq \kappa c^2 \ln^2 |\Lambda| \\ \forall C \in \mathcal{C}, \forall f \in C, d(f, \mathcal{P}) \leq \kappa''c \ln |\Lambda| \end{array} \right) \leq \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \exists \gamma \text{ closed path starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ \exists C \in \mathcal{C}, C \cap \gamma = \emptyset \end{array} \middle| T \leftrightarrow B \right].$$

To calculate this last probability, we will use lemma 3.2. Since the cut C is

disjoint from γ , either it separates γ from T or from B . Thus

$$\begin{aligned} \Phi_{\Lambda,p,2}^{TB} & \left[\begin{array}{l} \exists \gamma \text{ closed path starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ \exists C \in \mathcal{C}, C \cap \gamma = \emptyset \end{array} \middle| T \leftrightarrow B \right] \leq \\ & \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \exists \gamma \text{ closed path starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ \exists C \in \mathcal{C}, C \text{ separates } \gamma \text{ from } T \end{array} \middle| T \leftrightarrow B \right] + \\ & \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \exists \gamma \text{ closed path starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ \exists C \in \mathcal{C}, C \text{ separates } \gamma \text{ from } B \end{array} \middle| T \leftrightarrow B \right]. \quad (6.4) \end{aligned}$$

We rewrite the conditioned probabilities as the quotient of the probability of the intersection, and the probability of $\{T \leftrightarrow B\}$. Notice that the existence of a cut implies the disconnection event. By lemma 3.2, we have

$$\begin{aligned} \Phi_{\Lambda,p,2}^{TB} & \left[\begin{array}{l} \exists \gamma \text{ closed path starting at } e \\ \text{length}(\gamma) = (\kappa c^2 \ln^2 |\Lambda|)/2d \\ \exists C \in \mathcal{C}, C \text{ separates } \gamma \text{ from } T \end{array} \middle| T \leftrightarrow B \right] \\ & \leq (\alpha(d)(1 - f(p, q)))^{(\kappa c^2 \ln^2 |\Lambda|)/2d}. \end{aligned}$$

There exist $p_3 < 1$ and $\kappa \geq 1$ such that for $p \geq p_3$,

$$(\alpha(d)(1 - f(p, q)))^{(\kappa c^2 \ln^2 |\Lambda|)/2d} \leq \frac{1}{|\Lambda|^c}.$$

The same is true for the case where C separates γ from B . Combined with (6.1), (6.2), (6.3), (6.4), we have

$$\pi_\beta \left(e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{4}{|\Lambda|^c}.$$

We then sum over the edges e in Λ and we get

$$\pi_\beta \left(\exists e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{4}{|\Lambda|^{c-1}}.$$

For $|\Lambda| \geq 4$, we can replace $4/|\Lambda|^{c-1}$ by $1/|\Lambda|^{c-2}$. By taking κ big enough, we can replace $c-2$ by c and we obtain the result announced in the theorem. \square

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