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Samy Skander Bahoura. A boundedness result for a Liouville type equation with interior singularity. 2019. hal-02100099

**HAL Id: hal-02100099**

**<https://hal.science/hal-02100099>**

Preprint submitted on 15 Apr 2019

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# A boundedness result for a Liouville type equation with interior singularity.

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## Abstract

We consider a variational problem in dimension 2 with Dirichlet condition and exponential nonlinearity and having interior singularity. We derive a local boundedness result for solutions of this Liouville type equation.

Keywords: blow-up, interior singularity, maximum principle, a priori estimate.

MSC: 35J60, 35B44, 35B45, 35B50

## 1 Introduction and Main Results

We set  $\Delta = \partial_{11} + \partial_{22}$  on open set  $\Omega$  of  $\mathbb{R}^2$  with a smooth boundary.

We consider the following equation:

$$(P) \begin{cases} -\Delta u = -\log \frac{|x|}{2d} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here:

$$0 \leq V \leq b, \quad \int_{\Omega} -\log \frac{|x|}{2d} e^u dx \leq C, \quad u \in W_0^{1,1}(\Omega),$$

and,

$$d = \text{diam}(\Omega), \quad 0 \in \Omega$$

Equations of the previous type were studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-18], where one can find some existence and compactness results.

Among other results, we can see in [11] the following important Theorem

**Theorem.** (Brezis-Merle [11]). *If  $(u_i)$  is a sequence of solutions of problem (P) with  $(V_i)$  satisfying  $0 < a \leq V_i \leq b < +\infty$  and without the term  $-\log \frac{|x|}{2d}$ , then, for any compact subset  $K$  of  $\Omega$ , it holds:*

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$$\sup_K u_i \leq c,$$

with  $c$  depending on  $a, b, K, \Omega$

If we assume  $V$  with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [17], that, if  $(u_i)_i$  is a sequence of functions solutions of the previous equation without assumption on the boundary with  $V_i$  satisfying  $0 < a \leq V_i \leq b < +\infty$ , then we have a sup + inf inequality.

Here, we have:

**Theorem** For sequences  $(u_i)_i$  and  $(V_i)_i$  of the previous Problem (P) we have:

$$\|u_i\|_{L^\infty(K)} \leq c(b, C, K, \Omega),$$

Note that by the argument of "moving-plane", near the boundary, of Chen-Li, we have the boundedness of the solutions of (P), near the boundary if we add the assumption;  $\|\nabla V\|_\infty \leq A$ .

## 2 Proof of the Theorem

We have:

$$u_i \in W_0^{1,1}(\Omega), \text{ and } e^{u_i} \in L^1(\Omega).$$

Thus, by corollary 1 of Brezis and Merle we have:

$$e^{u_i} \in L^k(\Omega), \forall k > 2.$$

Using the elliptic estimates and the Sobolev embedding, we have:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

By the maximum principle  $u_i \geq 0$ .

Also, by a duality theorem or a result of Brezis-Strauss, we have:

$$\|\nabla u_i\|_q \leq C_q, \quad 1 \leq q < 2.$$

Since,

$$\int_\Omega -\log \frac{|x|}{2d} V_i e^{u_i} dx \leq C,$$

We have a convergence to a nonnegative measure  $\mu$ :

$$\int_\Omega -\log \frac{|x|}{2d} V_i e^{u_i} \phi dx \rightarrow \int_\Omega \phi d\mu, \quad \forall \phi \in C_c(\Omega).$$

We set  $S$  the following set:

$$S = \{x \in \Omega, \exists (x_i) \in \Omega, x_i \rightarrow x, u_i(x_i) \rightarrow +\infty\}.$$

We say that  $x_0$  is a regular point of  $\mu$  if there function  $\psi \in C_c(\Omega)$ ,  $0 \leq \psi \leq 1$ , with  $\psi = 1$  in a neighborhood of  $x_0$  such that:

$$\int \psi d\mu < 4\pi. \quad (1)$$

We can deduce that a point  $x_0$  is non-regular if and only if  $\mu(x_0) \geq 4\pi$ . A consequence of this fact is that if  $x_0$  is a regular point then:

$$\exists R_0 > 0 \text{ such that one can bound } (u_i) = (u_i^+) \text{ in } L^\infty(B_{R_0}(x_0)). \quad (2)$$

We deduce (2) from corollary 4 of Brezis-Merle paper, because we have:

$$\|u_i^+\|_1 = \|u_i\|_1 \leq \int_\Omega e^{u_i} dx \leq C_1 \int_\Omega -\log \frac{|x|}{2d} e^{u_i} dx \leq C,$$

or, by the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u_i^+\|_1 = \|u_i\|_1 \leq c_q \|u_i\|_{q^*} \leq C'_q \|\nabla u_i\|_q \leq C_q, \quad 1 \leq q < 2.$$

We denote by  $\Sigma$  the set of non-regular points.

Step 1:  $S = \Sigma$ .

We have  $S \subset \Sigma$ . Let's consider  $x_0 \in \Sigma$ . Then we have:

$$\forall R > 0, \lim \|u_i^+\|_{L^\infty(B_R(x_0))} = +\infty. \quad (3)$$

Suppose contrary that:

$$\|u_i^+\|_{L^\infty(B_{R_0}(x_0))} \leq C.$$

Then:

$$\|e^{u_{i_k}}\|_{L^\infty(B_{R_0}(x_0))} \leq C, \text{ and}$$

$$\int_{B_R(x_0)} -\log \frac{|x|}{2d} V_{i_k} e^{u_{i_k}} = o(1).$$

For  $R$  small enough, which imply (1) for a function  $\psi$  and  $x_0$  will be regular, contradiction. Then we have (3). We choose  $R_0 > 0$  small such that  $B_{R_0}(x_0)$  contain only  $x_0$  as non-regular point.  $\Sigma$ . Let's  $x_i \in B_R(x_0)$  such that:

$$u_i^+(x_i) = \max_{B_R(x_0)} u_i^+ \rightarrow +\infty.$$

We have  $x_i \rightarrow x_0$ . Else, there exists  $x_{i_k} \rightarrow \bar{x} \neq x_0$  and  $\bar{x} \notin \Sigma$ , i.e.  $\bar{x}$  is a regular point. It is impossible because we would have (2).

Since the measure is finite, if there are blow-up points, or non-regular points,  $S = \Sigma$  is finite.

Step 2:  $\Sigma = \{\emptyset\}$ .

Now: suppose contrary that there exists a non-regular point  $x_0$ . We choose a radius  $R > 0$  such that  $B_R(x_0)$  contain only  $x_0$  as non-regular point. Thus outside  $\Sigma$  we have local uniform boundedness of  $u_i$ , also in  $C^1$  norm. Also, we have weak \*-convergence of  $V_i$  to  $V \geq 0$  with  $V \leq b$ .

Let's consider (by a variational method):

$$z_i \in W_0^{1,2}(B_R(x_0)),$$

$$-\Delta z_i = f_i = -\log \frac{|x|}{2d} V_i e^{u_i} \text{ in } B_R(x_0), \text{ et } z_i = 0 \text{ on } \partial B_R(x_0).$$

By a duality theorem:

$$z_i \in W_0^{1,q}(B_R), \|\nabla z_i\|_q \leq C_q.$$

By the maximum principle,  $u_i \geq z_i$  in  $B_R(x_0)$ .

$$\int -\log \frac{|x|}{2d} e^{z_i} \leq \int -\log \frac{|x|}{2d} e^{u_i} \leq C. \quad (4)$$

On the other hand,  $z_i \rightarrow z$  a.e. (uniformly on compact sets of  $B_R(x_0) - \{x_0\}$ ) with  $z$  solution of :

$$-\Delta z = \mu \text{ in } B_R(x_0), \text{ et } z = 0 \text{ on } \partial B_R(x_0).$$

Also, we have up to a subsequence,  $z_i \rightarrow z$  in  $W_0^{1,q}(B_R(x_0))$ ,  $1 \leq q < 2$  weakly, and thus  $z \in W_0^{1,q}(B_R(x_0))$ .

Then by Fatou lemma:

$$\int -\log \frac{|x|}{2d} e^z \leq C. \quad (5)$$

As  $x_0 \in S$  is not regular point we have  $\mu(\{x_0\}) \geq 4\pi$ , which imply that,  $\mu \geq 4\pi\delta_{x_0}$  and by the maximum principle in  $W_0^{1,1}(B_R(x_0))$  (obtained by Kato's inequality)

$$z(x) \geq 2 \log \frac{1}{|x - x_0|} + O(1) \text{ if } x \rightarrow x_0.$$

Because,

$$z_1 \equiv 2 \log \frac{1}{|x - x_0|} + 2 \log R \in W_0^{1,s}(B_R(x_0)), \quad 1 \leq s < 2.$$

Thus,

$$-\log \frac{|x|}{2d} e^z \geq C \frac{\log \frac{|x|}{2d}}{-|x - x_0|^2}, \quad C > 0.$$

Both in the cases  $x_0 = 0$  and  $x_0 \neq 0$  we have:

$$\int_{B_R(x_0)} -\log \frac{|x|}{2d} e^z = \infty.$$

But, by (5):

$$\int -\log \frac{|x|}{2d} e^z \leq C.$$

which a contradiction.

### 3 Example

Here,  $\Delta = \partial_{11} + \partial_{22}$ .

Let's consider  $\mu \geq 0$  such that:

$$\mu = \inf \left\{ \int_{\Omega} |\nabla u|^2 dx, \int_{\Omega} -\log \frac{|x|}{2d} V e^u dx = 1, u \in H_0^1(\Omega) \right\}$$

Here,  $\Omega = B_1(0)$ , the unit ball and  $d = \text{diameter}(\Omega)$ .

1) We prove that this set is not empty:

We consider  $u_{\beta}(r) = \beta(1 - r^2)$  and  $0 < \epsilon \leq V \leq 2\epsilon$ . First we take the null function  $u \equiv 0$ . We choose  $\epsilon > 0$  small such that:

$$\int_{\Omega} -\log \frac{|x|}{2d} V e^u dx \leq 2\epsilon \int_{\Omega} -\log \frac{|x|}{2d} dx < 1.$$

We use  $u_{\beta}$ , we use polar coordinates:

$$\int_{\Omega} -\log \frac{|x|}{2d} V e^u dx = \int_{B_1(0)} -\log \frac{|x|}{2d} V e^{u_{\beta}} dx \geq 2\pi\epsilon \int_0^1 -\log(r/2d) r e^{\beta(1-r^2)} dr$$

For  $\beta$  big enough:

$$\int_{\Omega} -\log \frac{|x|}{2d} V e^u dx \geq (1/\beta)\pi\epsilon \log(2d) e^{\beta} (1 - e^{-\beta}) \rightarrow +\infty.$$

Thus, there is  $\beta_0 \leq \beta_{max} = \beta_{max}(\epsilon, \alpha)$  such that:

$$\int_{\Omega} -\log \frac{|x|}{2d} V e^{u_{\beta_0}} dx = 1.$$

Thus  $\mu$  well defined.

2) We use a minimisant sequence and the Moser-Trudinger embedding in  $L^1$ ,  $\mu$  is achieved and by a well known Euler-Lagrange multiplier we have the existence of  $u$  and  $\lambda$  such that:

$$-\Delta u = -\lambda \log \frac{|x|}{2d} V e^u, u \in H_0^1, \int_{\Omega} -\log \frac{|x|}{2d} V e^u dx = 1.$$

$u$  is regular  $C^{1,\theta}$ ,  $\theta > 0$ . If  $\lambda \leq 0$  by the maximum principle  $u \leq 0$  and then:

$$\int_{\Omega} -\log \frac{|x|}{2d} V e^u dx < 1.$$

It is not possible.

Thus  $\lambda > 0$  and  $u > 0$ .

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