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# Supplement to "Market Selection with Differential Financial Constraints"\*

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## Abstract

In this Supplement we provide foundations for the asset structures used in the main part of the paper, as well as in Appendix A. We use results by Choquet (1966), Kendall (1962) and Polyrakis (1999) to demonstrate how these asset structures can be generated from a general set of assets available in the economy and a general set of financial constraints. A sufficient condition called "internal completeness" is for the set of assets to contain an appropriate set of put and call options so that the implied set of payoffs is a sublattice of the Euclidean space.

## 1 Foundations for the Partition-Based Structure of Financial Constraints

### 1.1 The Case of Period-0 Trade

In the main part of the paper, we consider the set of assets available to an individual trader to be the set of generalized unit securities paying on a partition of the state space. In this Supplement, we show how such a structure can be derived from a more general structure of incomplete markets and financial constraints.

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For a given period  $t \geq 1$  (and a corresponding set of nodes  $\Omega_t$ ), we consider a set of assets  $A_t^0$ , with typical element  $a : \Omega_t \rightarrow \mathbb{R}_+$  defined as a state-contingent vector of payoffs. We will assume that  $A_t^0$  consists of  $|A_t^0|$  linearly independent assets  $a_t$ , denoted  $a_{t,1} \dots a_{t,|A_t^0|}$  and that asset  $a_{t,1}$  is a bond that pays 1 at every  $\sigma_t \in \Omega_t$ . However, we will not assume that  $A_t^0$  is complete relative to  $\Omega_t$ .  $A^0 = \cup_{t=1}^{\infty} A_t^0$  is the set of all assets in the economy.

A vector  $\theta = (\theta_1 \dots \theta_{|A_t^0|}) \in \mathbb{R}^{|A_t^0|}$  corresponds to a portfolio<sup>1</sup> or to a "marketed security" with payoffs given by  $T_t(\theta)(\sigma_t) = \sum_{j=1}^{|A_t^0|} \theta_j a_{t,j}(\sigma_t)$ . Let  $\Theta_t^0$  stand for the set of all portfolios at time  $t$ . The range of  $T_t : \Theta_t^0 \rightarrow \mathbb{R}^{|\Omega_t|}$  is the asset span of  $A_t^0$ . Note that  $T_t(\theta + \theta') = T_t(\theta) + T_t(\theta')$ .

We will endow the set of marketed securities with the following structure: for any  $\theta$  and  $\theta' \in \Theta_t^0$ , there exist portfolios  $\theta \vee \theta'$  and  $\theta \wedge \theta'$  with payoffs:

$$T_t(\theta \vee \theta') = \sup \{T_t(\theta); T_t(\theta')\} = (\max \{T_t(\theta)(\sigma_t); T_t(\theta')(\sigma_t)\})_{\sigma_t \in \Omega_t}$$

$$T_t(\theta \wedge \theta') = \inf \{T_t(\theta); T_t(\theta')\} = (\min \{T_t(\theta)(\sigma_t); T_t(\theta')(\sigma_t)\})_{\sigma_t \in \Omega_t}$$

In other words, the set of marketed securities generates a payoff space which is a sublattice of  $\mathbb{R}^{|\Omega_t|}$ , denoted  $M_t^0$ . This space is linear and, as shown by Polyrakis (1999), a minimal such space exists. Hence,  $M_t^0$  is uniquely specified. We will call such a payoff space "internally complete", (see Assumption: Internal Completeness below for a formal definition).

A set of marketed securities with a sublattice structure can be generated by an arbitrary set of assets enriched by a set of options on the marketed securities. To understand this, consider  $\theta$  and  $\theta' \in \Theta_t^0$ . The portfolio  $T_t(\theta) - T_t(\theta')$  is generated by buying 1 unit of  $\theta$  and selling 1 unit of  $\theta'$  and is thus also a marketed security with payoff  $T_t(\theta) - T_t(\theta')$ . In order to obtain the payoff structure  $T_t(\theta \vee \theta')$ , the agent would have to buy  $\theta'$  and a call option on the portfolio  $\theta - \theta'$  with an exercise price of 0, so that for each  $\sigma_t \in \Omega_t$ ,

$$T_t(\theta'(\sigma_t)) + \max \{T_t(\theta(\sigma_t) - \theta'(\sigma_t)); 0\} = \max \{T_t(\theta(\sigma_t)); T_t(\theta'(\sigma_t))\}.$$

Similarly, in order to obtain the payoff structure  $T_t(\theta \wedge \theta')$ , the agent would have to buy  $\theta'$  and a put option on the portfolio  $\theta - \theta'$  with an exercise price of 0, so that:

$$T_t(\theta'(\sigma_t)) + \min \{T_t(\theta(\sigma_t) - \theta'(\sigma_t)); 0\} = \min \{T_t(\theta(\sigma_t)); T_t(\theta'(\sigma_t))\}.$$

Hence, if such options can be written with respect to any two traded securities, the asset span is a sublattice of  $\mathbb{R}^{|\Omega_t|}$ .

Each agent  $i$  has access to a nonempty set  $A_t^i \subseteq A_t$  of assets with cardinality  $|A_t^i|$ .  $A^i = \cup_{t=1}^{\infty} A_t^i$  is the set of all assets available to  $i \in I$ . Let  $\Theta_t^i$  stand for the set of all portfolios available to  $i$  at time  $t$ . We assume:

<sup>1</sup>Portfolio holdings can be negative and thus can include short sales. However, the existence of a positive basis for the set of positive payoffs spanned by the set of assets as shown below ensures that every feasible consumption stream can be obtained without recurring to short sales.

**Assumption (Internal Completeness)** For all  $i \in \{0; 1 \dots n\}$  and all  $t \geq 1$ ,

- (i)  $a_{t,1} \in A_t^i$ . That is, the economy, as well as each agent has access to the bond at each period;
- (ii) the payoff space generated by  $A_t^i$ ,  $M_t^i$  is internally complete, that is, a sublattice of  $\mathbb{R}^{|\Omega_t|}$ .

Thus, every agent can be thought of as having access to options written on those marketed securities he can trade in. Furthermore, for each agent  $i \in \{1 \dots n\}$ ,  $M_t^i$  is a sublattice of  $M_t^0$ .

We now make use of results by Choquet (1956), Kendall (1962) and Polyraakis(1999), who show:

**Theorem 1** (Choquet–Kendall, Polyraakis) *A finite dimensional ordered vector space is a vector lattice if and only if it has a positive basis.*

Since for  $i \in \{0; 1 \dots n\}$ , and any  $t \geq 1$ ,  $M_t^i$  are finite dimensional vector spaces ordered by  $\geq$  and also, by definition, vector lattices, we conclude that each of them has a positive basis, that is, sets of linearly independent vectors  $B_t^i = \left( b_{t,1}^i \dots b_{t,|A_t^i|}^i \right) \subset M_t^i$  such that the positive cone of  $M_t^i$ ,  $M_{t+}^i = \mathbb{R}_+^{|\Omega_t|} \cap M_t^i$  are given by  $M_{t+}^i = \left\{ \sum_{k=1}^{|A_t^i|} \lambda_k b_k^i \mid \lambda_k \in \mathbb{R}_+ \text{ for all } k \right\}$ . That is, a basis for  $M_t^i$  is positive if any positive element of  $M_t^i$ , has positive coefficients<sup>2</sup> for base  $B_t^i$ .

Furthermore, the positive basis of  $M_t^i$  ( $i \in \{0; 1 \dots n\}$ ,  $t \geq 1$ ) is unique up to a multiplication of each of the basis vectors by a strictly positive number (Kountzakis and Polyraakis 2006, pp. 8-9). In general,  $|A_t^0| \neq |A_t^i|$  for  $i \in \{1 \dots n\}$ . Further,  $|A_t^0|$  (and thus  $|A_t^i|$ ) will in general be smaller than the number of states,  $|\Omega_t|$ . That is, in general neither the original set of assets, nor that assigned to each agent need be complete. Proposition 4 in Kountzakis and Polyraakis (2006, p. 9) further demonstrates that for any two vectors in  $B_t^i$ , the supports of the vectors are disjoint, that is, for any two  $b_{t,k}^i, b_{t,k'}^i$  with  $k \neq k'$ ,  $\text{supp}(b_{t,k}^i) \cap \text{supp}(b_{t,k'}^i) = \emptyset$ . Finally, by Proposition 6 in Kountzakis and Polyraakis (2006, p. 9), since the bond  $a_{t,1} \in A_t^i$ , the supports of the vectors in  $B_t^i$  form a partition of the state space  $\Omega_t$  and can be chosen such that  $b_{t,k}^i(\sigma_t) = 1$  whenever  $\sigma_t \in \text{supp}(b_{t,k}^i)$  and  $b_{t,k}^i(\sigma_t) = 0$ , else. We will refer to the partition of  $\Omega_t$  generated by the vectors in  $B_t^i$ ,  $i \in \{0; 1 \dots n\}$  as  $\Omega_t^i$ . We will show below that if  $A_t^i \subset A_t^j$ , then  $\Omega_t^i$  is a coarsening of  $\Omega_t^j$ .

Finally, Theorem 9 in Kountzakis and Polyraakis (2006, p. 10) provides an explicit formula for deriving the set of vectors  $B_t^i$  for a given set of assets  $A_t^i$ .

We summarize this discussion in the following Proposition

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<sup>2</sup>Hence, replacing the original set of assets  $A_t^i$  by the basis  $B_t^i$  allows obtaining any non-negative consumption profile without resorting to short sales.

**Proposition 1** (based on Kountzakis and Polyrakis, 2006) *Suppose that Assumption Internal Completeness holds. The space of payoffs spanned by the assets in  $A_t^i$  for some  $i \in \{0; 1 \dots n\}$  and  $t \geq 1$  is spanned by a set of  $|A_t^i|$  vectors  $B_t^i = \left\{ \left( b_{t,k}^i \right)_{k=1}^{|A_t^i|} \right\}$ . There is a partition  $\Omega_t^i$  of  $\Omega_t$  with representative element  $\omega_t^i$  such that  $|\Omega_t^i| = |A_t^i|$  and  $b_{t,k}^i = \mathbf{1}_{\omega_{t,k}^i}$  for all  $k \in \{1 \dots |A_t^i|\}$ . Finally, if  $A_t^i \subset A_t^j$ , the partition  $\Omega_t^j$  is a refinement of  $\Omega_t^i$ .*

**Proof of Proposition 1:**

The proofs of all statements but the last are in Polyrakis (1999). We will now prove that if  $A_t^i \subset A_t^j$ , then  $\Omega_t^j$  is a refinement of  $\Omega_t^i$ . To do so, start by determining the partition corresponding to  $A_t^j$ ,  $\Omega_t^j$  using the algorithm suggested by Polyrakis (1999). Since the payoffs of all assets in  $A_t^j$  are nonnegative, and since the base for  $M_t^j$  is given by  $B_t^j$ , there exist coefficients  $\left( \lambda_1(a_{t,k}) \dots \lambda_{|A_t^j|}(a_{t,k}) \right)$  such that

$$a_{t,k}(s) = \sum_{l=1}^{|A_t^j|} \lambda_l(a_{t,k}) b_{t,l}^j(\sigma_t) \text{ for all } \sigma_t \in \Omega_t \text{ and all } a_{t,k} \in A_t^j.$$

Note, however, that each  $b_{t,l}^j$  is measurable with respect to  $\Omega_t^j$ . Hence, if  $\sigma_t, \sigma'_t \in \omega_t^j \in \Omega_t^j$ , then

$$a_{t,k}(\sigma_t) = \sum_{l=1}^{|A_t^j|} \lambda_l(a_{t,k}) b_{t,l}^j(\sigma_t) = a_{t,k}(\sigma'_t) = \sum_{l=1}^{|A_t^j|} \lambda_l(a_{t,k}) b_{t,l}^j(\sigma'_t) \text{ for all } a_{t,k} \in A_t^j$$

and hence, the payoffs of the assets in  $A_t^j$  are also measurable with respect to  $\Omega_t^j$ . Hence, define a state space given by  $\Omega_t^j$  and consider the construction of  $\Omega_t^i$  with respect to this state space. By the previous part of the Proposition, we have that for  $B^i$  defined with respect to  $\Omega_t^j$ , there is a partition of  $\Omega_t^j$ ,  $\Omega_t^i$  with representative element  $\omega_{t,k}^i$  such that  $b_{t,k}^i = \mathbf{1}_{\omega_{t,k}^i}$  for  $k \in \{1 \dots |A_t^i|\}$ . We have thus shown that  $\Omega_t^i$  is a coarsening of  $\Omega_t^j$  and hence,  $\Omega_t^j$  is a refinement of  $\Omega_t^i$ . Since  $|A_t^i| < |A_t^j|$ , these relations are strict.

Given the result of Proposition 1, we can restrict attention to what we will call generalized unit securities, that is, securities that pay 1 unit on a subset of the state space  $\omega_t \subset \Omega_t$  and nothing else. Since for each  $t \geq 1$ ,  $\Omega_t$  is finite, so is the maximal number of all such possible securities, given by  $2^{|\Omega_t|} - 1$ . Hence, we can now define the set of period  $t$  securities in the economy to be  $\tilde{A}_t^0$  and that available to an agent  $i \in \{1 \dots n\}$  as  $\tilde{A}_t^i$  given by:

$$\tilde{A}_t^i = \left\{ a_{\omega_t^i} \text{ with } a_{\omega_t^i}(\sigma_t) = \mathbf{1}_{\omega_t^i} \text{ for all } \omega_t^i \in \Omega_t^i \right\},$$

for  $i \in \{0; 1 \dots n\}$  where  $\Omega_t^i$  is the partition of  $\Omega_t$  identified in Proposition 1.

Note that while in general  $\tilde{A}_t^i \not\subset \tilde{A}_t^0$ , we have  $\tilde{M}_t^i \subseteq \tilde{M}_t$  for all  $i \in I$  and, similarly,  $\tilde{M}_t^j \subset \tilde{M}_t^i$ , whenever  $\Omega_t^j$  is coarser than  $\Omega_t^i$ , which is true in particular when  $A_t^j \subset A_t^i$ .

We next make some assumptions about the sets of assets available in the economy and to the individual investors at different time periods. Note that for  $i \in \{0; 1 \dots n\}$  and  $t \geq 1$ ,  $\theta \in \Theta_t^i$  iff  $T_t(\theta) \in M_t^i$ .

**Assumption (Intertemporal Asset Structure)** For all  $t \geq 1$ , and any  $i \in \{0; 1 \dots n\}$

- (i) if for some  $\theta \in \Theta_t^i$ , then there is a  $\theta' \in \Theta_{t+1}^i$  such that  $T_{t+1}(\theta')(\sigma_t; s) = T_t(\theta)(\sigma_t)$  holds for all  $\sigma_t \in \Omega_t$  and all  $s \in S$ ;
- (ii) for any  $\sigma_{t-1} \in \Omega_{t-1}$  and  $s' \in S$ , if for some  $\theta \in \Theta_t^i$ , then there is a  $\theta' \in \Theta_{t+1}^i$  such that  $T_{t+1}(\theta')(\sigma_{t-1}; s'; s) = T_t(\theta)(\sigma_{t-1}; s)$  holds for all  $s \in S$ ;
- (iii) if for some  $s, s' \in S$ ,  $a(s) = a(s')$  for all  $a \in A_1^i$ , then  $a(\sigma_{t-1}; s) = a(\sigma_{t-1}; s')$  holds for all  $\sigma_{t-1} \in \Omega_{t-1}$  and all  $a \in A_t^i$ ;
- (iv) if for some  $\sigma_t \in \Omega$ ,  $a(\sigma_t) = a(\sigma'_t)$  for all  $a \in A_t^i$ , then  $a(\sigma_t; s) = a(\sigma'_t; s)$  holds for all  $s \in S$  and all  $a \in A_{t+1}^i$ .

Conditions (i) and (ii) say that the set of payoffs that can be generated is (weakly) expanding with time. This appears natural given the tree-event structure of the uncertainty, that is, the fact that the number of contingencies on which trade is potentially possible grows over time. In particular, according to (i), if a specific payoff structure could be obtained over the nodes in  $\Omega_t$ , then the same payoff structure can also be obtained over the nodes in  $\Omega_{t+1}$ . Furthermore, according to (ii), if the market (or an agent  $i$ ) can trade across two states  $s$  and  $s'$  at time  $t$ , he can also do so at time  $t + 1$ . Assumption (iii) establishes that if the market (or an agent) cannot trade between two states at time 1, then neither can they do so at any future time period. For example, if states  $s$  and  $s'$  relate to uncertainty in a foreign country, then the fact that an agent does not have access to foreign equity, precludes trades between  $s$  and  $s'$  in all periods of time. Finally, assumption (iv) says that, if it was impossible to trade on certain contingencies, the realization of which was revealed at time  $t$ , then no assets permitting trade on these contingencies will be available after period  $t$ , either. These assumptions apply to an economy, in which financial constraints do not change over time in the sense made precise in Proposition 2 below and clearly do not hold when financial constraints are relaxed over time, as, e.g., in Section 6 of the main paper, or for the case of sequential trade as in Section 2 of this Supplement.

**Proposition 2** *Suppose that the Assumptions Internal Completeness and Intertemporal Asset Structure hold. Then for  $i \in \{0; 1 \dots n\}$  there exist partitions of  $S$ ,  $W^i$  such that for each  $t \geq 1$ ,  $\Omega_t^i = \prod_{\tau=1}^t W^i$ .*

**Proof of Proposition 2:**

We will make the argument for  $\Omega_t^0$ . The argument for  $\Omega_t^i$ ,  $i \in I$  is analogous.

We proceed by induction on  $t \geq 1$ . Note that for  $t = 1$ , we can set the partition derived in Proposition 1  $\Omega_1^0 = W^0$ . Consider  $t > 1$  and assume that  $\Omega_{\tilde{t}}^0 = \prod_{\tau=1}^{\tilde{t}} W^0$  for all  $\tilde{t} \leq t$ . We will show that  $\Omega_{t+1}^0 = \prod_{\tau=1}^{t+1} W^0$ . Indeed, by assumption we have that for all  $w, s, s' \in w$ , and any  $\theta' \in \Theta_1^0$ ,  $T_1(\theta'(s)) = T_1(\theta'(s'))$ . Furthermore, for every  $\theta'' \in \Theta_t^0$  any  $\omega_t \in \Omega_t^0$ , all  $\sigma_t, \sigma'_t \in \omega_t$ ,  $T_t(\theta''(\sigma_t)) = T_t(\theta''(\sigma'_t))$ . Thus, combining (iii) and (iv) of Assumption Intertemporal Asset Structure, we conclude that for all  $\theta \in \Theta_{t+1}^0$ , any  $w \in W^0$ , any  $\omega_t \in \Omega_t^0$ , all  $\sigma_t, \sigma'_t \in \omega_t$  and all  $s, s' \in w$ ,

$$T_{t+1}(\theta)(\sigma_t; s) = T_{t+1}(\theta)(\sigma_t; s') = T_{t+1}(\theta)(\sigma'_t; s) = T_{t+1}(\theta)(\sigma'_t; s').$$

It follows that the finest partition that is spanned by  $M_{t+1}^0$  is  $\prod_{\tau=1}^{t+1} W^0$ . Now suppose that there are two distinct elements of  $\prod_{\tau=1}^{t+1} W^0$ , call them  $(\omega_t; w)$  and  $(\omega'_t; w')$  such that for all  $\theta' \in \Theta_{t+1}^0$ ,  $T_{t+1}(\theta')(\omega_t; w) = T_{t+1}(\theta')(\omega'_t; w')$ . If  $\omega_t \neq \omega'_t$ , this contradicts (i), there is a  $\theta \in \Theta_t^0$  with payoffs  $T_t(\theta)(\omega_t) \neq T_t(\theta)(\omega'_t)$ . If,  $\omega_t = \omega'_t$  and  $w \neq w'$ , this contradicts (ii), since then for all  $\sigma_t \in \omega_t$  and some  $s \in w, s' \in w'$ ,  $T_{t+1}(\theta')(\sigma_t; s) = T_{t+1}(\theta')(\sigma_t; s')$  for all  $\theta' \in \Theta_{t+1}^0$ , whereas there is a portfolio  $\theta \in \Theta_t^0$  such that  $T_t(\theta)(\sigma_{t-1}; s) \neq T_t(\theta)(\sigma_{t-1}; s')$ . Hence,  $\prod_{\tau=1}^{t+1} W^0$  is also the coarsest partition with respect to which all elements of  $M_{t+1}^0$  are measurable and  $\Omega_t^0 = \prod_{\tau=1}^{t+1} W^0$ .

## 1.2 The Case of Sequential Trade

### 1.2.1 Partition Structures with Short Lived Assets

For a given node  $\sigma_t \in \Omega_t$ , we consider a set of assets independent of  $\sigma_t$ ,  $= A^0$ , with typical element  $a_{\sigma_t} : S \rightarrow \mathbb{R}_+$ . I.e., an asset  $a_{\sigma_t}$  is born and traded at  $\sigma_t$ , lives for a single period and pays a nonnegative amount at each successor of  $\sigma_t$ ,  $(\sigma_t; s)$ ,  $s \in S$ . We will assume that at each  $\sigma_t$ ,  $A_{\sigma_t}^0$  consists of  $|A^0|$  linearly independent assets  $a$ , denoted  $a_{\sigma_t;1} \dots a_{\sigma_t;|A^0|}$  and that asset  $a_{\sigma_t;1}$  is a bond that pays 1 at every  $(\sigma_t; s) \in \Omega_{t+1}$ . However, we will not assume that  $A^0$  (and thus,  $A_{\sigma_t}^0$ ) is complete relative to  $S$ .

A vector  $\theta = (\theta_1 \dots \theta_{|A^0|}) \in \mathbb{R}^{|A^0|}$  corresponds to a portfolio or to a "marketed security" with payoffs given by  $T_{\sigma_t}(\theta)(s) = \sum_{j=1}^{|A^0|} \theta_j a_{\sigma_t;j}(s)$ . Let  $\Theta_{\sigma_t}^0 = \Theta^0$  stand for the set of all portfolios at node  $\sigma_t$ . The range of  $T_{\sigma_t} : \Theta_{\sigma_t}^0 \rightarrow \mathbb{R}^{|S|}$  is the asset span of  $A^0$ .

We will endow the set of marketed securities with the following structure: for any  $\theta$  and  $\theta' \in \Theta^0$ , there exist portfolios  $\theta \vee \theta'$  and  $\theta \wedge \theta'$  with payoffs:

$$T_{\sigma_t}(\theta \vee \theta') = \sup \{T_{\sigma_t}(\theta); T_{\sigma_t}(\theta')\} = (\max \{T_{\sigma_t}(\theta)(s); T_{\sigma_t}(\theta')(s)\})_{s \in S}$$

$$T_{\sigma_t}(\theta \wedge \theta') = \inf \{T_{\sigma_t}(\theta); T_{\sigma_t}(\theta')\} = (\min \{T_{\sigma_t}(\theta)(s); T_{\sigma_t}(\theta')(s)\})_{s \in S}$$

In other words, the set of marketed securities generates a payoff space which is a sublattice of  $\mathbb{R}^{|S|}$ , denoted  $M^0$ . This space is linear and, as shown by Polyrakis (1999), a minimal such space exists. Hence,  $M^0$  is uniquely specified and the corresponding payoff space is "internally complete".

As above, the set of agents is given by  $I$  and has cardinality  $n$ . At each  $\sigma_t$ , agent  $i$  has access to a nonempty set  $A_{\sigma_t}^i \equiv A^i$ ,  $A_{\sigma_t}^i \subseteq A_{\sigma_t}^0$  of assets with cardinality  $|A^i|$ . As above, we assume:

**Assumption (Internal Completeness)** For all  $i \in \{0; 1 \dots n\}$  and all  $\sigma_t \in \Omega$ ,

- (i)  $a_{\sigma_t;1} \in A_{\sigma_t}^i$ , i.e., the economy, as well as each agent has access to the bond at each period;
- (ii) the payoff space generated by  $A_{\sigma_t}^i = A^i$ ,  $M^i$ , is internally complete, that is, a sublattice of  $\mathbb{R}^{|S|}$ .

Thus, every agent can be thought of as having access to options written on those marketed securities he can trade in. Furthermore, for each agent  $i$ ,  $M^i$  is a sublattice of  $M^0$ .

The discussion in Section 1 of this Supplement, as well as the results of Theorem 1 apply in this case and we can restate Proposition 2 as:

**Proposition 3** (based on Kountzakis and Polyrakis, 2006) Suppose that Assumption Internal Completeness holds. For  $i \in \{0; 1 \dots n\}$ , there exist partitions of  $S$ ,  $W^i$  with elements  $w_1^i \dots w_{|A^i|}^i$  such that the space of payoffs spanned by the assets in  $A_{\sigma_t}^i = A^i$  is spanned by  $B_{\sigma_t}^i = B^i = \left\{ b_{\sigma(t);k}^i = \mathbf{1}_{w^i}, k = 1 \dots |A^i| \right\}$ . Finally, if  $A^i \subset A^j$ , the partition  $W^j$  is a refinement of  $W^i$ .

## 1.2.2 Asset Structure with Nested Partitions

We construct the sets  $\tilde{A}_{\sigma_t}^i$  for a given  $\sigma_t$ . To do so, we proceed by induction on the set of agents, endowing agent  $n$  with assets corresponding to the first  $|W^n| - 1$  elements of  $n$ 's partition and the bond, i.e.,

$$\tilde{A}_{\sigma_t}^n = \left\{ \mathbf{1}_{w_1^n} \dots \mathbf{1}_{w_{|W^n|-1}^n}; \mathbf{1}_S \right\}.$$

For agent  $n - 1$  and for an element of the partition of agent  $n$ ,  $w_k^n \in W^n$ , let  $W_{k,n}^{n-1} \subseteq W^{n-1}$  denote the set of elements of the partition of agent  $n - 1$  which are subsets of  $w_k^n$ , i.e., for every  $w^{n-1} \in W_{k,n}^{n-1}$ ,  $w^{n-1} \subseteq w_k^n$ . For every  $k \in \{1 \dots |W^n|\}$ , choose arbitrarily an element of  $W_{k,n}^{n-1}$ ,  $\tilde{w}_{k,n}^{n-1}$ . Then, the set of assets of agent  $n - 1$  is given by:

$$\tilde{A}_{\sigma_t}^{n-1} = \tilde{A}_{\sigma_t}^n \cup_{k=1}^{|W^n|} \left\{ \mathbf{1}_{w^{n-1}} \right\}_{w^{n-1} \in W_{k,n}^{n-1} \setminus \{\tilde{w}_{k,n}^{n-1}\}}.$$

By induction, let the set of assets available to agent  $i$  be  $\tilde{A}_{\sigma_t}^i$ . For agent  $i - 1$  and element of the partition of agent  $i$ ,  $w_k^i \in W^i$ , let  $W_{k,i}^{i-1} \subseteq W^{i-1}$  denote the



set of elements of the partition of agent  $i - 1$  which are subsets of  $w_k^i$ , i.e., for every  $w^{i-1} \in W_{k,i}^{i-1}$ ,  $w^{i-1} \subseteq w_k^i$ . For every  $k \in \{1 \dots |W^i|\}$ , choose arbitrarily an element of  $W_{k,i}^{i-1}$ ,  $\tilde{w}_{k,i}^{i-1}$ . Then, the set of assets of agent  $i - 1$  is given by:

$$\tilde{A}_{\sigma_t}^{i-1} = \tilde{A}_{\sigma_t}^i \cup_{k=1}^{|W^i|} \{\mathbf{1}_{w^{i-1}}\}_{w^{i-1} \in W_{k,i}^{i-1} \setminus \{w_{k,i}^{i-1}\}}.$$

Finally, since  $W^1 = W^0$  set  $\tilde{A}_{\sigma_t}^0 = \tilde{A}_{\sigma_t}^1$ . The obtained asset structure satisfies requirements (i), (ii) and (iii) imposed in Section 8.1 of the main part of the paper.

## 2 References

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