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An energy approach of electromagnetism and gravitation

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I - Introduction

The steady progress in the knowledge and understanding of the world that surrounds us have led physicists to develop theoretical models more and more general, partly thanks to more efficient tools developed by mathematicians. Among the major success of these models, are prominently and in chronological order, the equations of electromagnetism, gravitation, and quantum mechanics.

This fragmented treatment of different areas of physics regularly led physicists to wonder about the ability to define a possible most general scheme of description of physics.

This goal led to reflections often extremely complex on the way in which we could give an overview of the whole range of physical phenomena. Among the most impressive are those of Albert EINSTEIN, who devoted much of his life trying to build a theory that would allow to account both for the electromagnetic and gravitational phenomena. It was followed, accompanied or criticized in its work by specialists of tensorial analysis who include, Walter MAYER, Wolfgang PAULI, Marie-Antoinette TONNELAT, Hermann WEIL, Theodor KALUZA, Oskar KLEIN…

In the wake of the success of the equations of gravitation, these works sought to prolong or extend the formalism of the curvature of space-time which had led to tensorial equations so fertile.

Many and intense efforts developed in this way have failed so far to provide the framework expected

When EINSTEIN has built his theory of gravitation, he was supported, from the outset of his thinking, by a strong physical principle: locally, there is nothing to distinguish between what is happening in an accelerated frame to what happens in a frame subject to a uniform gravitational field. He knew while working on a theory of accelerated frames, it should be possible to show the effects of gravitation. It was a long and difficult work, but whose complex mathematical aspects could be linked to the physical aspects by the principle of equivalence.

The stunning success of his theory of gravitation then led him, accompanied in this by several other researchers, looking for an extension that would include the equations of electromagnetism in a natural way. These extensions have taken varied and extremely complex forms, of which the highlight is usually the mathematical rigour of the developments which are presented.

When we're interested in these developments, we see that they have lost this strong link with the physics that has led to the success of general relativity. It is only after building mathematically their theory that physicists will look for possible examples of validations of their work.

We will try in this brief to explore another way to move towards a coherent vision of physics, from the quantum world to the macroscopic world. You'll find no results really new compared to current knowledge of the major areas of physics, except perhaps with regard to the electromagnetic energy that can be associated with a single charge. But there is a common thread in which the conservation of energy and the invariance of the laws of physics by change or frame are assumptions that are admitted without any reservations. There is also highlighting of a formalism that is common to the equations of gravitation and electromagnetism, as long as you stand in a linearized framework.

The solutions of the DIRAC equation in the form of standing waves show that we can interpret these solutions as exchanges of energy between the energy of the vacuum and the energy of mass or impulse of the infinitely small particles. On the basis of this interpretation, all forms of quantifications that are observed in the infinitely small world fall from classic quantification conditions of stationary phenomena

By coming together in the form of atoms, then molecules, then of macroscopic objects, these particles will transmit their mass energy and impulse energy to the world which is familiar to us. We have to add the electromagnetic energy of charged particles in previous exchanges.

In an interpretation of this nature, the origin of all macroscopic energy action is rooted in the energy of the vacuum. When we adopt this vision, we are led to admit that all physics boils down to be able to describe the various transformations of energies, since their origin issue of the vacuum, until their macroscopic view.

In other words, we will focus in this paper on the way in which we can reflect on a consistent base for all of physics, by taking as basis, an evolution of the energy contained in the vacuum.

We must first clarify what are the assumptions that we need. By nature, these assumptions must be verified by all the known physical laws of nature.

The basic postulate is that vacuum contains energy, which is generally characterised by a volumic density. If the quantum and cosmological visions diverge in orders of catastrophic magnitude, it still true that both theories admit one and the other that the vacuum must have energy. The vast majority of physicists seem to agree on this point.

The second postulate is the conservation of energy. It will be assumed that energy is a physical property that cannot be created or destroyed, but only transformed. To the knowledge of the author, there is no known physical experiences questioning this assumption.

To complete the guideline we set, we must be able to describe the evolution in time and space of this energy: let us assume that this evolution is given to us, in the general case, by the principle of least action. It appears that this principle applies to all areas of physics, and that it leads to equations of gravitation, equations of electromagnetism, and equations of quantum mechanics. It stands in a natural way as a unifying element of all the laws that describe physics and we will give some general reminders in the next chapter.

On the basis of these considerations, we will develop some arguments for an energy vision of physics.

II - The Lagrangians

The principle of least action itself can be likened to an empty shell: it requires for its implementation the prior definition of a Lagrangian. This Lagrangian depends on the field of physics considered, and exchanges of energy that are considered.

In this chapter, we briefly describe the main Lagrangiens used in mechanics, electromagnetism, relativity, and quantum mechanics, as well as some properties that result from application of the equations of LAGRANGE.

We do not cover the elements of reflection which led to the development of these Lagrangiens. This information can be found in most courses related to variational methods. They can also be consulted on the following link (in french).

In particular, note exchanges of energy that appear explicitly or implicitly in the heart of the Lagrangiens. It is these exchanges that are the common backbone to the whole of physics that we wish to highlight.

I – The mechanics of NEWTON

Consider a material point of mass m and speed v along the x axis, for which we adopt the notation:

$$v = \frac{dx}{dt} = \dot{x} \qquad \text{(II-1)}$$

This material point moves in a portion of the space where it is possible to associate a potential energy, a classic example is the potential energy of gravitation.

It is recognized that kinetic energy only depends on speed and considered time, while the potential energy depends only on position and considered time. In these conditions, at any time, the difference between kinetic energy and potential energy may be written in the form of a function of the position, speed and time:

$$L(x, \dot{x}, t) = E_c(\dot{x}, t) - E_p(x, t) \qquad \text{(II-2)}$$

where L is called the Lagrangian of the system.

The energy evolution system sets up exchanges between kinetic energy and potential energy. Between two times t_A and t_B, the sum of these exchanges is given by the following integral, which is called "action":

$$S_A^B = \int_{t_A}^{t_B} \left(E_c(\dot{x}, t) - E_p(x, t)\right)dt = \int_{t_A}^{t_B} L(x, \dot{x}, t)\,dt \qquad \text{(II-3)}$$

We then proceed with the following reasoning:

At time t_A, the mass is located at position A, and at time t_B, the mass is located at position B. Consider all the possible paths between A and B: that is to say that in every time the position and the velocity of the mass can be any. From a mathematical point of view, this property can be transposed by saying that the position and speed variables are independent variables. In these conditions, LAGRANGE showed that the path that makes the stationary action imposes the following relationship:

$$\frac{\partial L(x, \dot{x}, t)}{\partial x} - \frac{d}{dt}\frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}} = 0 \qquad \text{(II-4)}$$

By applying this relationship to the Lagrangian of the classical mechanics (II-2), we obtain, recalling that position and speed variables must be considered as independent variables:

$$\frac{\partial\{E_c(\dot{x},t) - E_p(x,t)\}}{\partial x} - \frac{d}{dt}\frac{\partial\{E_c(\dot{x},t) - E_p(x,t)\}}{\partial \dot{x}} = -\frac{\partial E_p(x,t)}{\partial x} - \frac{d}{dt}\frac{E_c(\dot{x},t)}{\partial \dot{x}} = 0 \qquad \text{(II-5)}$$

Recalling that strength derives from a potential energy according to the relationship:

$$F_x = -\frac{\partial E_p(x,t)}{\partial x} \qquad \text{(II-6)}$$

and that the kinetic energy is expressed in non-relativistic mechanics, by the expression:

$$E_c(\dot{x},t) = \frac{1}{2}m\dot{x}^2 \qquad \text{(II-7)}$$

We deduce from the relationship of LAGRANGE (II-5):

$$F_x - \frac{d}{dt}(m\dot{x}) = F_x - m\ddot{x} = 0 \qquad \text{(II-8)}$$

Generalizing to the three dimensions of space, it thus comes to the conclusion that pure energy reasoning, based on exchanges of energy described by the principle of least action, allows to build the whole of physics that arises from the fundamental principle of dynamics.

II – Special relativity

The Lagrangian of special relativity is given by the relationship:

$$L = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{(II-9)}$$

and the action therefore takes the form:

$$S = \int_{t_1}^{t_2} - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} dt = -m_0 c^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt = -m_0 c^2 \int_{\tau_1}^{\tau_2} d\tau \qquad \text{(II-10)}$$

Since m_0 which represents the mass at rest is a constant, and c also, it is concluded that make the maximum action between the initial and final moments is equivalent to minimize (because of the sign less) proper time between these two events.

The quadri-dimensional length element is written:

$$ds = c\sqrt{1 - \frac{v^2}{c^2}} dt \qquad \text{(II-11)}$$

So that an other expression of the commonly used action is one that minimizes the four-dimensional trajectory:

$$S = -mc_0^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt = -m_0 c \int_{s_1}^{s_2} ds \qquad \text{(II-12)}$$

According to this definition, the action will be maximum when the four-dimensional length traveled by the object is minimum: that is to say when between two events, the object will move following a geodesic of space-time.

III – MAXWELL's equations in vacuum

It is known that electromagnetic energy comes in two different aspects: energy provided by the electric field E and energy provided by the magnetic field H. In an energy approach, the Lagrangian that is associated with the electromagnetic field which is present in a volume Ω is given by the relationship:

$$L = \iiint_{\Omega} \left(\frac{1}{2} \varepsilon_0 E^2 - \frac{1}{2} \mu_0 H^2 \right) d\Omega \qquad \text{(II-13)}$$

It expresses, at every time, exchange between electric energy and magnetic energy within the volume Ω.

Quantity ΔL, which depends only on the electromagnetic field, is designated by Lagrangian density: it is homogeneous with a volumic energy density.

$$\Delta L = \frac{1}{2} \varepsilon_0 E^2 - \frac{1}{2} \mu_0 H^2 \qquad \text{(II-14)}$$

By applying LAGRANGE equations to the electromagnetic Lagrangian (II-13), we get MAXWELL's equations in vacuum.

$$\vec{\nabla}\Lambda\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$
$$\vec{\nabla}\Lambda\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0 \qquad \text{(II-15)}$$
$$\vec{\nabla}.\vec{E} = 0$$
$$\vec{\nabla}.\vec{B} = 0$$

Density ΔL may be expressed also, to a multiplicative constant close, as the « norm » of the electromagnetic field tensor $F_{\mu\nu}$. This tensor contains all the components of the electromagnetic field, in a sequence that depends on the metric used.

Its « norm » provides a quantity that does not depend on the frame in which is expressed the electromagnetic field, making it an invariant quantity by changing frame, in the same way that the norm of a vector is invariant by change of frame.

$$\Delta L = \frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2 = -\frac{1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} \qquad \text{(II-16)}$$

IV – Electromagnetic interaction

One designates by electromagnetic interaction, the action of an electromagnetic field represented by its scalar potential ϕ and its vector potential $\vec{A}$, on a particle of point charge equal to q and animated with a speed v. We know that in this situation, the electromagnetic field communicates a part of its energy to the charge, and we can define the Lagrangian of interaction in the form:

$$L = q\left(-\phi + \vec{v}.\vec{A}\right) \qquad \text{(II-17)}$$

Application of the LAGRANGE equations led to the expression of the LORENTZ force, expressed in terms of the scalar and vector potentials:

$$\vec{F} = q\left(-\frac{\partial\vec{A}}{\partial t} - \vec{A}.\left(\vec{\nabla}.\vec{v}\right)\right) + q\left(-\vec{\nabla}.\varphi + \vec{\nabla}.\left(\vec{v}.\vec{A}\right)\right) = q\left(-\frac{\partial\vec{A}}{\partial t} - \vec{A}.\left(\vec{\nabla}.\vec{v}\right)\right) + q\vec{\nabla}\left(-\varphi + \left(\vec{v}.\vec{A}\right)\right) \qquad \text{(II-18)}$$

The motion of a relativistic charge of rest mass m_0 is obtained using the Lagrangian of special relativity, associated with the Lagrangian of the electromagnetic interaction:

$$L = -m_0 c^2\sqrt{1 - \frac{v^2}{c^2}} + q\left(-\varphi + \left(\vec{v}.\vec{A}\right)\right) \qquad \text{(II-19)}$$

V – MAXWELL's equations in the presence of charges and currents

By combining the electromagnetic interaction Lagrangian with the Lagrangian of the electromagnetic field, we get:

$$L = q\left(-\varphi + \left(\vec{v}.\vec{A}\right)\right) + \iiint_\Omega \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2\right) d\Omega \qquad \text{(II-20)}$$

In order to homogenize this expression, we consider that the charge q is distributed in the volume Ω with a density ρ, which leads to the following expression:

$$L = \iiint_{\Omega} \rho\left(-\varphi + \left(\vec{v}.\vec{A}\right)\right)d\Omega + \iiint_{\Omega} \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2\right)d\Omega \qquad \text{(II-21)}$$

$$L = \iiint_{\Omega} \left(-\rho\varphi + \left(\vec{J}.\vec{A}\right)\right)d\Omega + \iiint_{\Omega} \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2\right)d\Omega \qquad \text{(II-22)}$$

where $\vec{J}$ represents the volumic current density. We can deduce the electromagnetic Lagrangian density in the presence of charges and currents:

$$\Delta L = \left(-\rho\varphi + \vec{J}.\vec{A}\right) + \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2\right) \qquad \text{(II-23)}$$

Application of the LAGRANGE equations then leads to the formulation of MAXWELL's equations in the presence of charges and currents:

$$\begin{aligned}
&\vec{\nabla}\Lambda\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \\
&\vec{\nabla}\Lambda\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \\
&\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \\
&\vec{\nabla}.\vec{B} = 0
\end{aligned} \qquad \text{(II-24)}$$

VI – General Relativity: interaction of the gravitation field with a massive object

In general relativity, the metric is characterized by g_{ij} coefficients, and the element of space-time squared is written:

$$ds^2 = g_{ij}dx^i dx^j \qquad \text{(II-25)}$$

By analogy with relativity, we can define an action by the following relationship:

$$S = -m_0 c\int_{s_1}^{s_2} ds = -m_0 c\int_{s_1}^{s_2}\sqrt{g_{ij}dx^i dx^j} \qquad \text{(II-26)}$$

The application of the principle of least action leads to show that the trajectory of the particle of mass m_0 is a geodesic of space-time with the equation:

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \qquad \text{(II-27)}$$

The expression of CHRISTOFFEL coefficients is given by:

$$\Gamma^{r}_{ij} = \frac{1}{2} g^{rk} \left(\frac{\partial g_{kj}}{\partial x^{i}} + \frac{\partial g_{ki}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}} \right) \qquad \text{(II-28)}$$

VII – General Relativity: EINSTEIN equations in vacuum

It returned to the mathematician David HILBERT to have shown that we can obtain the equations of gravitation from a variational principle. The action is a quantity which is invariant by changing frame. We define an action built from the curvature scalar R, as it is a quantity which is invariant by change of frame:

$$S = -\frac{1}{\chi c} \int\iiint (R) \sqrt{|g|} dx^0 dx^1 dx^2 dx^3 = -\frac{1}{\chi c} \int\iiint \left(g^{ij} R_{ij}\right) \sqrt{|g|} dx^0 dx^1 dx^2 dx^3 \qquad \text{(II-29)}$$

In this expression, one can groupe under the appellation of dΩ, the invariant four-dimensional volume element by changing frame:

$$d\Omega = \sqrt{|g|} dx^0 dx^1 dx^2 dx^3 \qquad \text{(II-30)}$$

where g is the determinant of the matrix of g_{ij}, and where x^0 represents the time variable equal to ct.

A full variational calculation allows to show that the variation of the action can be put in the form:

$$\delta S = -\frac{1}{\chi c} \int\iiint \left\{ R_{ij} - \frac{1}{2} g_{ij} R \right\} \sqrt{-g}\, \delta\left(g^{ij}\right) dx^0 dx^1 dx^2 dx^3 \qquad \text{(II-31)}$$

To obtain a stationary action, it is needed the bracketed term is zero, which leads to the EINSTEIN equations in vacuum:

$$R_{ij} - \frac{1}{2} g_{ij} R = 0 \qquad \text{(II-32)}$$

VIII – General Relativity: EINSTEIN equations in the presence of matter

The action of the previous paragraph is amended to take into account the momentum-energy tensor of matter T_{ij}:

$$S = -\frac{1}{\chi c} \int\iiint (R - \chi T) \sqrt{-g} dx^0 dx^1 dx^2 dx^3 = -\frac{1}{\chi c} \int\iiint \left(g^{ij} R_{ij} - \chi g^{ij} T_{ij}\right) \sqrt{-g} dx^0 dx^1 dx^2 dx^3$$

(II-33)

We can deduce the variation of the action:

$$\delta S = -\frac{1}{\chi c}\int\iiint\left\{R_{ij} - \frac{1}{2}g_{ij}R - \chi T_{ij}\right\}\sqrt{-g}\,\delta\left(g^{ij}\right)dx^0dx^1dx^2dx^3 \qquad \text{(II-34)}$$

The equations of gravitation in the presence of matter can be highlighted:

$$R_{ij} - \frac{1}{2}g_{ij}R = \chi T_{ij} \qquad \text{(II-35)}$$

It is possible to introduce the cosmological constant in these demonstrations in order to obtain the more general equations.

IX – Quantum mechanics: DIRAC equation

An expression commonly used to define the Lagrangian density of DIRAC is:

$$\Delta L = \overline{\psi}\left(j\hbar c\gamma^{\mu}\partial_{\mu} - m_0c^2\right)\psi \qquad \text{(II-36)}$$

In which ψ represents the DIRAC bispinor, $\overline{\psi}$ the adjoint bispinor and γ^{μ} DIRAC matrixes.

The Lagrangian is obtained in the same way as in electromagnetism or general relativity, by integrating this density density on a volume Ω:

$$L = \iiint_{\Omega}\overline{\psi}\left(j\hbar c\gamma^{\mu}\partial_{\mu} - m_0c^2\right)\psi d\Omega$$

By choosing for independent variables, the bispinor and its adjoint, the application of LAGRANGE relations allows for establishing the DIRAC equation for each of them.

For a particle of charge q, placed in an electromagnetic field that is defined by its scalar potential ϕ and its vector potential $\vec{A}$, both being gathered under the form of a four-vector A_{μ}, the lagrangian density becomes :

$$\Delta L = \overline{\psi}\left(j\hbar c\gamma^{\mu}\left(\partial_{\mu} + jqA_{\mu}\right) - m_0c^2\right)\psi \qquad \text{(II-37)}$$

Finally, the Lagrangian of quantum electrodynamics density is obtained by adding interaction with the electromagnetic field:

$$\begin{aligned}\Delta L &= \overline{\psi}\left(j\hbar c\gamma^{\mu}\left(\partial_{\mu} + jqA_{\mu}\right) - m_0c^2\right)\psi + \left(\frac{1}{2}\varepsilon_0E^2 - \frac{1}{2}\mu_0H^2\right)\\ \Delta L &= \overline{\psi}\left(j\hbar c\gamma^{\mu}\left(\partial_{\mu} + jqA_{\mu}\right) - m_0c^2\right)\psi - \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\end{aligned} \qquad \text{(II-38)}$$

It is from this Lagrangian density that the most accurate comparisons between theory and experimentation are obtained, in the whole history of physics.

X – Conclusion

It turns out that the equations that describe the main areas of physics such as electromagnetism, classical mechanics including gravitation, special and general relativity, quantum mechanics, can be expressed using the principle of least action.

In a general manner, this principle minimizes exchanges of energy in its various forms, in a given time interval.

If we admit that the universe vacuum is filled with infinitely fluid energy base to quantum exchanges, these last itself used for macroscopic energy exchanges, so we see appear a deep unity of the whole of physics around the notion of energy and the principle of least action.

However, several basic equations, although deriving from the principle of least action, do not describe explicitly an energy evolution of the considered physical area.

A simple example is constituted by MAXWELL's equations. These equations describe the evolution of electromagnetic fields based on the charges and currents that are at their origin, without explicit reference to exchanges of energy.

We propose to consider, later in this document, the possibility of an energy meaning of the equations that govern key areas of physics, and in particular the two basic areas that are gravity and electromagnetism.

III - Energy aspect of the EINSTEIN equations

Taking as guideline of his thought, the equivalence between a frame that is placed in a uniform gravitational field and a uniformly accelerated frame, EINSTEIN showed that the equations of gravitation could be expressed in the form:

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} = \chi T_{ij} \qquad \text{(III-1)}$$

In this expression:

- R_{ij} is the RICCI curvature tensor, obtained by contraction of the RIEMANN-CHRISTOFFEL tensor. Each term of the tensor R_{ij} has dimension of m^{-2}.
- R is called the scalar curvature obtained by contraction of the tensor of curvature, and has dimension of m^{-2}.
- g_{ij} is the metric tensor, each term is without dimension.
- G is the gravitational constant, it has dimension of $m^3.kg^{-1}.s^{-2}$.
- c is the speed of light, it is expressed in $m.s^{-1}$.
- T_{ij} is called the energy momentum tensor. Each term has the dimension of a volumetric energy density. It is expressed in joules per cubic meter, or $m^{-1}.kg.s^{-2}$.
- χ is the EINSTEIN constant. It is expressed as $m.kg.s^{-2}$, or as meter/joule.

The curvature of space-time manifests through the coefficients of the metric g_{ij}, and their derivatives which are present in the tensor R_{ij}. It is apparent that this curvature is imposed by the presence of the energy contained in the T_{ij} momentum-energy tensor. Therefore, the interpretation of this equation generally accepted is:

In the absence of matter, space-time is « flat », it is described by special relativity. The corresponding metric is called the metric of MINKOWSKY. Later in this document, it will be represented by the following matrix:

$$\eta_{ij} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(III-2)}$$

All terms are zero outside the main diagonal, i.e. for i different from j. This property will often be used to simplify calculations involving a linearized metric which will be developed in the following chapters.

The presence of matter manifests through the pulse energy tensor T_{ij} and modifies the metric of MINKOWSKY space-time: it is now designated by the tensor g_{ij}. Changes that come to be added to the flat space-time metric are directly related to the gravitational potential generated by the presence of the masses.

This effect is described by saying that the presence of matter changes the MINKOWSKY space-time metric. The new metric is so representative of a so called curved space.

If now we place a test mass in this space-time, which the modified metric is characterized by the g_{ij}, then this test mass will evolve following a path that corresponds to a geodesic of space-time represented by the tensor g_{ij}.

This effect is described by asserting that the curvature of space-time due to the gravitational field imposed the trajectory of the test mass.

The perfect fit of the experimental results with this theory would lead us to admit its validity without any reserve.

The interpretation of the EINSTEIN equations in terms of curvature of space-time opened a horizon in which mathematicians fully filled at ease. Time and space variables are always variables that are the basis of mathematical constructions extremely elaborate and extremely fertile.

For the physicist, this interpretation creates some frustration, because it is difficult to see an underlying physical phenomenon to the notion of space-time. The notion of force has disappeared, even though it was a key element in the understanding of physical phenomena and their evolution, and in particular the gravitation of NEWTON. After more than a century of reflections, we don't see where the notion of force hides in the EINSTEIN equations, while it would be an element for progress in the understanding of the physical phenomena described by these tensorial equations.

We mentioned in the introduction that when trying to have a global vision of physics, there is an interest to bring out its energy aspects. It immediately follows the question: can we give an energy interpretation to the EINSTEIN equations?

It appears that this interpretation can be done effortlessly. Starting from the formulation usually used:

$$R_{ij} - \frac{1}{2} g_{ij} R = \chi T_{ij} \qquad \text{(III-3)}$$

simply carry the EINSTEIN constant across equality to get:

$$\frac{R_{ij}}{\chi} - \frac{1}{2\chi} R g_{ij} = T_{ij} \qquad \text{(III-4)}$$

In this equality, each of the main terms now has the dimension of a volumic energy density.

In vacuum, and in the absence of matter, the volumic energy density is uniform and constant.

The presence of matter will change this uniform distribution of energy. This imposed change is given by the matter energy-momentum tensor T_{ij}. The terms on the left will then describe the evolution of the volume energy density of the vacuum, under temporal and spatial constraint dictated by T_{ij}.

We can make further progress in the definition of the dimensions of the variables that make up the RICCI tensor:

$$R_{ij} = \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik} + \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} \qquad \text{(III-5)}$$

If we put:

$$R'_{ij} = \frac{R_{ij}}{\chi} \qquad R' = \frac{R}{\chi} \qquad \text{(III-6)}$$

The equations of gravitation are written:

$$R'_{ij} - \frac{1}{2}R' g_{ij} = T_{ij} \qquad \text{(III-7)}$$

With:

$$R'_{ij} = \frac{\Gamma^k_{mk}\Gamma^m_{ij}}{\chi} - \frac{\Gamma^k_{mj}\Gamma^m_{ik}}{\chi} + \frac{\partial \Gamma^k_{ij}}{\chi \partial x^k} - \frac{\partial \Gamma^k_{ik}}{\chi \partial x^j}$$

$$R'_{ij} = \frac{\Gamma^k_{mk}}{\sqrt{\chi}}\frac{\Gamma^m_{ij}}{\sqrt{\chi}} - \frac{\Gamma^k_{mj}}{\sqrt{\chi}}\frac{\Gamma^m_{ik}}{\sqrt{\chi}} + \frac{\partial \frac{\Gamma^k_{ij}}{\sqrt{\chi}}}{\partial\left(\sqrt{\chi}x^k\right)} - \frac{\partial \frac{\Gamma^k_{ik}}{\sqrt{\chi}}}{\partial\left(\sqrt{\chi}x^j\right)} \qquad \text{(III-7)}$$

$$R'_{ij} = \Gamma'^k_{mk}\Gamma'^m_{ij} - \Gamma'^k_{mj}\Gamma'^m_{ik} + \frac{\partial \Gamma'^k_{ij}}{\partial x'^k} - \frac{\partial \Gamma'^k_{ik}}{\partial x'^j}$$

The coefficients of CHRISTOFFEL changed (with a prime) now have as dimension the square root of a volumic energy density. There is an element of consistency with the energy interpretation of the DIRAC equation in which the wave function also has the dimension of the square root of a volumic energy density.

As a result, the variables of space changed (with a prime) have the dimension of the inverse of the square root of a volumic energy density.

The evolution of the modified RICCI tensor R'_{ij}, which is proportional to the unmodified R_{ij} RICCI tensor in a report χ, can be interpreted only in terms of volumic energy density variations.

It comes from this interpretation that the notion of time and space arises directly from an energy evolution. In a system without energy evolution, time and space do not exist, because there is no way to highlight them.

For equations of gravitation, the transition from a classic interpretation to an energy interpretation is so immediate. It does not really add, for now, additional lighting.

IV - MAXWELL's equations

In a comprehensive energy approach, we would like the tensor formalism which describes the behavior of energy attached to a mass can be applied to other areas of physics, and first and foremost to electromagnetism. The static analogy between gravitational and electrostatic is so perfect that it incentives to explore this path.

Special relativity allowed to show the perfect consistency between the force issue from area of electromagnetism, and the force issue from area of gravitation. These two forces are transformed in the same way between two frames in uniform translation movement.

However the notion of energy can be defined by the result of a force that has experienced or generated a movement. The resulting consistency of relativity suggests that if there is a general law of energy conversion, this law must apply both to the gravitational energy and electromagnetic energy. This law can only be a law which, by nature, is independent of the frame in which it is described: the one that comes to mind is given by the EINSTEIN equations.

I – Reminder on MAXWELL's equations

Electromagnetism is one of the major areas of physics. It is described very precisely by MAXWELL's equations. In vacuum, and in the presence of charges and currents, these equations are written in temporal regime:

$$
\begin{aligned}
&\vec{\nabla}\Lambda\vec{E} = -\frac{\partial\vec{B}}{\partial t}\\
&\vec{\nabla}\Lambda\vec{B} = \mu_0.\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\\
&\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}\\
&\vec{\nabla}.\vec{B} = 0
\end{aligned}
\qquad \text{(IV-1)}
$$

No energy wording appears directly in these equations. They can however be deducted from following Lagrangians by using the principle of least action:

$$
\begin{aligned}
&L = \iiint_\Omega \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2}\mu_0 H^2\right)d\Omega\\
&L = q\left(-\phi + \vec{v}.\vec{A}\right) = \iiint_\Omega \rho\left(-\phi + \vec{v}.\vec{A}\right)d\Omega = \iiint_\Omega \left(-\rho\phi + \vec{J}.\vec{A}\right)d\Omega
\end{aligned}
\qquad \text{(IV-2)}
$$

This observation indicates that we can assign them an energy origin.

The potential and scalar vectors, which appear in the Lagrangian (VI-2) close to the sources of the electromagnetic field, are solutions of the wave equation:

$$\vec{\nabla}^2\phi - \frac{\partial^2\phi}{\partial(ct)^2} = -\frac{\rho}{\varepsilon_0}$$
$$\vec{\nabla}^2\vec{A} - \frac{\partial^2\vec{A}}{\partial(ct)^2} = -\mu_0.\vec{J} \qquad \text{(IV-3)}$$

These wave equations must be associated with the choice of gauge, which allowed their determination, i.e. the LORENZ gauge:

$$\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \qquad \text{(IV-4)}$$

As previously, there appears no obvious energy aspect in these equations.

II – Trial of definition of a energy-momentum tensor of the electromagnetic field radiation sources

For a volumic material density ρ_m, animated with speed with components v_x, v_y, v_z, we know how to define a momentum-energy tensor. In the metric (-, +, +, +), we give it the following form:

$$T_{ij} = -\rho_m \begin{pmatrix} c^2 & -cv_x & -cv_y & -cv_z \\ -cv_x & v_x v_x & v_x v_y & v_x v_z \\ -cv_y & v_y v_x & v_y v_y & v_y v_z \\ -cv_z & v_z v_x & v_z v_y & v_z v_z \end{pmatrix} \qquad \text{(IV-5)}$$

This energy momentum tensor is responsible for the modification of the volumic energy density that surrounds each mass. In static mode, it reduces to the term T_{00}, which integrated on the volume containing the mass provides the energy mass at rest: m_0c^2.

This tensor can be described as an intrinsic tensor to a volumic energy density of mass in motion, because it contains no terms from the gravitational field.

The problem with electromagnetism, is that we do not know how to set an intrinsic momentum-energy tensor relative to a volumic charge density ρ. In other words, we do not know to assign a purely electromagnetic energy to an electrostatic charge q.

It is a fundamental problem: without this energy tensor relative to the sources of electromagnetic radiation, it is unclear how it would be possible to consider a reconciliation of electromagnetism with EINSTEIN equations of gravitation.

To propose the structure of such a tensor, we will draw on Ockham Razor: the simplest sufficient assumptions should be preferred.

We conjecture that the evolution of the energy must be identical; for the mass energy or electromagnetic energy.

Its evolution must be described by the same equations. We can consider that it is a consequence of the basic assumptions that we require in the introduction.

Therefore, we adopt the following tensor as momentum-energy sources tensor of electromagnetic field:

$$T_{ij}=\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}=-\lambda\rho\begin{pmatrix} c^2 & -cv_x & -cv_y & -cv_z \\ -cv_x & v_xv_x & v_xv_y & v_xv_z \\ -cv_y & v_yv_x & v_yv_y & v_yv_z \\ -cv_z & v_zv_x & v_zv_y & v_zv_z \end{pmatrix}=\lambda\begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & -J_xv_x & -J_xv_y & -J_xv_z \\ cJ_y & -J_yv_x & -J_yv_y & -J_yv_z \\ cJ_z & -J_zv_x & -J_zv_y & -J_zv_z \end{pmatrix} \quad \text{(IV-6)}$$

Quantity ρ represents the volumic charge density, while the J_x, J_y, J_z quantities represent volumic current densities in each direction of space.

The multiplicative constant λ that appears must give to this tensor the dimension of a volumic energy density, and must ensure the coherence of this tensor with the equations of gravitation. We shall subsequently justify its expression:

$$\lambda=\frac{1}{\sqrt{4\pi\varepsilon_0 G}} \quad \text{(IV-7)}$$

In this expression, ε_0 is the vacuum permittivity, and G the gravitational constant. We can deduce a value close to λ:

$$\lambda=\frac{1}{\sqrt{4\pi\varepsilon_0 G}}=\frac{1}{\sqrt{4\pi*8.85*10^{-12}*6.67*10^{-11}}}\approx 1.16*10^{10} \quad A^{-1}s^{-1}Kg \quad \text{(IV-8)}$$

In a linearized metric, the twice contravariant tensor is given by changing the signs of row 0 and column 0:

$$T^{ij}=\begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}=-\lambda\rho\begin{pmatrix} c^2 & cv_x & cv_y & cv_z \\ cv_x & v_xv_x & v_xv_y & v_xv_z \\ cv_y & v_yv_x & v_yv_y & v_yv_z \\ cv_z & v_zv_x & v_zv_y & v_zv_z \end{pmatrix}=-\lambda\begin{pmatrix} \rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & J_xv_x & J_xv_y & J_xv_z \\ cJ_y & J_yv_x & J_yv_y & J_yv_z \\ cJ_z & J_zv_x & J_zv_y & J_zv_z \end{pmatrix} \quad \text{(IV-9)}$$

For a system isolated from external influences, the divergence of this tensor must be zero:

$$\frac{\partial T^{ij}}{\partial x^j}=\frac{\partial T^{i0}}{\partial x^0}+\frac{\partial T^{i1}}{\partial x^1}+\frac{\partial T^{i2}}{\partial x^2}+\frac{\partial T^{i3}}{\partial x^3}=0 \quad \text{(IV-10)}$$

We get to the first line (i = 0):

$$\frac{\partial T^{00}}{\partial x^0}+\frac{\partial T^{01}}{\partial x^1}+\frac{\partial T^{02}}{\partial x^2}+\frac{\partial T^{03}}{\partial x^3}=0 \qquad \text{(IV-11)}$$

Or still:

$$\frac{\partial(\rho c)}{\partial(ct)}+\frac{\partial(J_x)}{\partial x}+\frac{\partial(J_y)}{\partial y}+\frac{\partial(J_z)}{\partial z}=\frac{\partial \rho}{\partial t}+\mathrm{Div}(\vec{J})=0 \qquad \text{(IV-12)}$$

The divergence is zero because it checks the charge conservation equation. To a multiplicative constant close, it is identical to the law of conservation of energy which is imposed to the momentum-energy tensor T^{ij}.

On the basis of this momentum-energy tensor, we will seek to show that MAXWELL's equations are included in an approached way in the equations of gravitation. Approximations that are needed consist, in part, by the linearization of the EINSTEIN equations: it is this aspect that will be discussed in the following chapters.

V - Linearization of the EINSTEIN equations

The EINSTEIN equations recalled below are expressed with the energy-momentum tensor in the right member, the g_{ij} metric coefficients, and their first and second partial derivatives in left side.

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} \qquad \text{(V-1)}$$

The RICCI tensor is recalled with more details below:

$$\begin{aligned} R_{ij} &= \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik} + \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} \\ \Gamma^r_{ij} &= \frac{1}{2} g^{rk} \left\{ \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right\} \\ R &= g^{ij} R_{ij} \end{aligned} \qquad \text{(V-2)}$$

These equations are non-linear: if we multiply the energy-momentum tensor T_{ij} by 2, the g_{ij} which are solutions are not multiplied by 2. This is apparent in the passage of the covariant metric coefficients to contravariant coefficients metric: the reversal of the matrix introduced a denominator that is formed by the determinant of the g_{ij}, which removes any linearity to the coefficients of CHRISTOFFEL.

In contrast, MAXWELL's equations are linear: If we multiply by 2 charges to the origin of electromagnetic fields, these fields are multiplied by 2. Furthermore, we know their complete coherence with special relativity, which is described by the metric of MINKOWSKI, and we have already adopted the signature:

$$\eta_{00}=\eta^{00} = -1, \quad \eta_{11}=\eta^{11} = 1, \quad \eta_{22}=\eta^{22} = 1, \quad \eta_{33}=\eta^{33} = 1 \qquad \text{(V-3)}$$

This metric is characteristic of a flat space-time, because it makes coefficients of CHRISTOFFEL and the RICCI tensor equal to 0.

It is accepted that MAXWELL's equations are based on the MINKOWSKY metric in an exact way. Then, we can wonder about how a rigorously flat space-time can allow the electromagnetic field to spread. From this point of view, it is more satisfactory for the mind to imagine that this space-time is subject to infinitely small fluctuations that spread in the manner of any fluctuation in its medium of propagation.

Such a representation has the advantage of reconciling the consistency of MAXWELL's equations with the metric of MINKOWSKY, because if the disturbances are low enough, they can be considered as negligible in the metric coefficients of MINKOWSKY. We also ensures consistency with a spread through space-time that can be described through the EINSTEIN equations.

We come to the conclusion that if the hypothesis according to which electromagnetic fields propagate through a distortion of space-time is valid, it must be possible to specify this deformation by introduction of the tensor energy-momentum of electromagnetic sources in the EINSTEIN equation. In an energy approach, we recall that a small distortion of space-time is seen as a small perturbation of the vacuum energy.

The idea of an exact resolution being abandoned because of difficulties, it remains that we can work within the EINSTEIN equations, when they are written for the g_{ij} infinitely close to η_{ij} that characterize the metric of MINKOWSKY. We work then in the frame of the linearized EINSTEIN equations.

The first stage of this work is to rewrite these equations by introducing simplifications made, considering the infinitesimal deviation from the metric of MINKOWSKY. This work has been explored by EINSTEIN himself, and led gradually to a rigorous formalism adopted to describe the propagation of gravitational waves. The outline of this linearization are described in the following paragraphs.

Subsequently, all the indices used in tensorial notation vary from 0 to 3. When this is not the case, this will be explicitly mentioned.

I – The metric tensor which propagates the electromagnetic perturbation

A flat space-time is represented by the MINKOWSKY metric which we will adopt the following representation:

$$\eta_{ij} = \eta^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(V-4)}$$

It is assumed that this metric is the object of small perturbations h_{ij} who check the condition:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} \quad \text{avec} \quad |h_{ij}| << 1 \qquad \text{(V-5)}$$

The coefficients of the perturbed metric g_{ij}, that is to say the metric which is attached to the electromagnetic wave, are written as the sum of the coefficients of the metric of MINKOWSKY, plus the small perturbations of h_{ij}.

$$g_{ij} = \eta_{ij} + h_{ij} = \begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \qquad \text{(V-6)}$$

We assume perturbations h_{ij} small enough so that we can consider as linear effects on the metric g_{ij}: we shall justify this assumption later. In these conditions, we can initially ignore the terms of the second order that correspond to products of h_{ij}.

The first step is to establish the expression of the contravariant metric tensor g^{ij}. We know that the matrix of the contravariant g^{ij} is the inverse matrix of the covariant g_{ij}. We might proceed to a formal inversion of the matrix (V-6), but if we assume the terms of the second order as negligible, we can simplify the inversion and immediately check that the product of the following matrices gives the identity matrix, to the specified approximation:

$$\begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \begin{pmatrix} -1-h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1-h_{11} & -h_{12} & -h_{13} \\ h_{20} & -h_{21} & 1-h_{22} & -h_{23} \\ h_{30} & -h_{31} & -h_{32} & 1-h_{33} \end{pmatrix} \qquad \text{(V-7)}$$

Since the covariant η_{ij} and contravariant η^{ij} tensor of the MINKOWSKY metrics is identical, we deduce that the elements of the contravariant tensor of the perturbed metric can be written:

$$g^{ij} = \eta^{ij} - h^{ij} \qquad \text{(V-8)}$$

with :

$$h^{ij} = \begin{pmatrix} h_{00} & -h_{01} & -h_{02} & -h_{03} \\ -h_{10} & h_{11} & h_{12} & h_{13} \\ -h_{20} & h_{21} & h_{22} & h_{23} \\ -h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} \qquad \text{(V-9)}$$

We can draw attention to the fact that in this linearized approach, h^{ij} which are the contravariant elements of tensor h_{ij} may be deducted from the latter thanks to the MINKOWSKY metric:

$$h^{ij} = \eta^{im}\eta^{jn}h_{mn} \qquad \text{(V-10)}$$

II - The coefficients of CHRISTOFFEL

They are constructed from the perturbed metric tensor following the relationship recalled to memory:

$$\Gamma^r_{ij} = \frac{1}{2} g^{rk} \left\{ \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right\} \tag{V-11}$$

where the expression of the perturbed contravariant metric tensor has been established previously:

$$g^{ij} = \eta^{ij} - h^{ij} \tag{V-12}$$

Since the η_{ij} are constant, their derivative is null, and it remains:

$$\Gamma^r_{ij} = \frac{1}{2} \left(\eta^{rk} - h^{rk} \right) \left\{ \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k} \right\} \tag{V-13}$$

The condition $\left| h^{rk} \right| << \left| \eta^{rk} \right| = 1$ allows a further simplification:

$$\Gamma^r_{ij} \approx \frac{1}{2} \eta^{rk} \left\{ \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k} \right\} \tag{V-14}$$

and since the only not null η^{rk} are the η^{rr}, it remains finally only the following terms:

$$\Gamma^r_{ij} \approx \frac{1}{2} \eta^{rr} \left\{ \frac{\partial h_{ir}}{\partial x^j} + \frac{\partial h_{jr}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^r} \right\} \tag{V-15}$$

III - The RICCI tensor

We use the expression of the RICCI tensor below:

$$R_{ij} = \Gamma^k_{mk} \Gamma^m_{ij} - \Gamma^k_{mj} \Gamma^m_{ik} + \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} \tag{V-16}$$

whose writing developed in a 4-dimensional space takes the following form:

$$\begin{aligned} R_{ij} = {} & \Gamma^1_{ij} \left\{ \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} + \Gamma^0_{10} \right\} + \Gamma^2_{ij} \left\{ \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23} + \Gamma^0_{20} \right\} \\ & + \Gamma^3_{ij} \left\{ \Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33} + \Gamma^0_{30} \right\} + \Gamma^0_{ij} \left\{ \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03} + \Gamma^0_{00} \right\} \\ & - (\Gamma^1_{1j}\Gamma^1_{i1} + \Gamma^1_{2j}\Gamma^2_{i1} + \Gamma^1_{3j}\Gamma^3_{i1} + \Gamma^1_{0j}\Gamma^0_{i1}) - (\Gamma^2_{1j}\Gamma^1_{i2} + \Gamma^2_{2j}\Gamma^2_{i2} + \Gamma^2_{3j}\Gamma^3_{i2} + \Gamma^2_{0j}\Gamma^0_{i2}) \\ & - (\Gamma^3_{1j}\Gamma^1_{i3} + \Gamma^3_{2j}\Gamma^2_{i3} + \Gamma^3_{3j}\Gamma^3_{i3} + \Gamma^3_{0j}\Gamma^0_{i3}) - (\Gamma^0_{1j}\Gamma^1_{i0} + \Gamma^0_{2j}\Gamma^2_{i0} + \Gamma^0_{3j}\Gamma^3_{i0} + \Gamma^0_{0j}\Gamma^0_{i0}) \\ & + \left\{ \frac{\partial \Gamma^1_{ij}}{\partial x^1} + \frac{\partial \Gamma^2_{ij}}{\partial x^2} + \frac{\partial \Gamma^3_{ij}}{\partial x^3} + \frac{\partial \Gamma^0_{ij}}{\partial x^0} \right\} - \left\{ \frac{\partial \Gamma^1_{i1}}{\partial x^j} + \frac{\partial \Gamma^2_{i2}}{\partial x^j} + \frac{\partial \Gamma^3_{i3}}{\partial x^j} + \frac{\partial \Gamma^0_{i0}}{\partial x^j} \right\} \end{aligned} \tag{V-17}$$

It is apparent that without additional simplifying assumption, search for an exact solution using the full expression of the RICCI tensor is a probably an insurmountable work.

Since we place ourselves in an infinitesimal gap compared to the MINKOWSKY metric, we infer that the coefficients of CHRISTOFFEL are very similar to those of a flat space-time, that is to say very close to 0. In these circumstances, the product of the coefficients of CHRISTOFFEL can be considered to be an infinitely small of the second order, and may therefore be neglected.

Under these assumptions, we are led to evaluate the simplified RICCI tensor:

$$R_{ij} \approx \frac{\partial \Gamma^r_{ij}}{\partial x^r} - \frac{\partial \Gamma^r_{ir}}{\partial x^j} = \left\{ \frac{\partial \Gamma^1_{ij}}{\partial x^1} + \frac{\partial \Gamma^2_{ij}}{\partial x^2} + \frac{\partial \Gamma^3_{ij}}{\partial x^3} + \frac{\partial \Gamma^0_{ij}}{\partial x^0} \right\} - \left\{ \frac{\partial \Gamma^1_{i1}}{\partial x^j} + \frac{\partial \Gamma^2_{i2}}{\partial x^j} + \frac{\partial \Gamma^3_{i3}}{\partial x^j} + \frac{\partial \Gamma^0_{i0}}{\partial x^j} \right\} \qquad \text{(V-18)}$$

It comes, by introducing the approximate expression of CHRISTOFFEL coefficients (V-14) recalled to memory:

$$\Gamma^r_{ij} \approx \frac{1}{2} \eta^{rk} \left\{ \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k} \right\} \qquad \text{(V-19)}$$

$$R_{ij} = \frac{\partial \Gamma^r_{ij}}{\partial x^r} - \frac{\partial \Gamma^r_{ir}}{\partial x^j} = \frac{1}{2} \eta^{rk} \left\{ \frac{\partial^2 h_{ik}}{\partial x^j \partial x^r} + \frac{\partial^2 h_{jk}}{\partial x^i \partial x^r} - \frac{\partial^2 h_{ij}}{\partial x^r \partial x^k} \right\} - \frac{1}{2} \eta^{rk} \left\{ \frac{\partial^2 h_{ik}}{\partial x^r \partial x^j} + \frac{\partial^2 h_{rk}}{\partial x^i \partial x^j} - \frac{\partial^2 h_{ir}}{\partial x^j \partial x^k} \right\} \qquad \text{(V-20)}$$

$$R_{ij} = \frac{1}{2} \eta^{rk} \left\{ \frac{\partial^2 h_{jk}}{\partial x^i \partial x^r} - \frac{\partial^2 h_{ij}}{\partial x^r \partial x^k} - \frac{\partial^2 h_{rk}}{\partial x^i \partial x^j} + \frac{\partial^2 h_{ir}}{\partial x^j \partial x^k} \right\} \qquad \text{(V-21)}$$

We can simplify this expression by developing products by the metric tensor η^{rk}. This latter behaves as a constant and can pass under the derivation sign, which allows to raise or lower an indices, when this is relevant:

$$R_{ij} = \frac{1}{2} \left\{ \frac{\partial^2 h^r_j}{\partial x^i \partial x^r} - \eta^{rk} \frac{\partial^2 h_{ij}}{\partial x^r \partial x^k} - \eta^{rk} \frac{\partial^2 h_{rk}}{\partial x^i \partial x^j} + \frac{\partial^2 h^k_i}{\partial x^j \partial x^k} \right\} \qquad \text{(V-22)}$$

In the MINKOWSKY metric, only coefficients not null are the η^{rr}, and are recalled to memory:

$\eta^{00} = -1 \qquad \eta^{11} = 1 \qquad \eta^{22} = 1 \qquad \eta^{33} = 1 \qquad$ (V-23)

So that the term:

$$\eta^{rk} \frac{\partial^2 h_{ij}}{\partial x^r \partial x^k} = \eta^{rr} \frac{\partial^2 h_{ij}}{\partial x^r \partial x^r} = -\frac{\partial^2 h_{ij}}{\partial (ct)^2} + \frac{\partial^2 h_{ij}}{\partial x^2} + \frac{\partial^2 h_{ij}}{\partial y^2} + \frac{\partial^2 h_{ij}}{\partial z^2} = \Box (h_{ij}) \qquad \text{(V-24)}$$

represents the d'Alembertian of the perturbation in the metric (- + + +), while the term:

$$\eta^{rk}h_{rk} = \eta^{rr}h_{rr} = -h_{00} + h_{11} + h_{22} + h_{33} = h \qquad \text{(V-25)}$$

represents the trace of the perturbation. With these writing conventions, the linearized RICCI tensor stands in the form:

$$R_{ij} = \frac{1}{2}\left\{\frac{\partial^2 h^r_j}{\partial x^i \partial x^r} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h^k_i}{\partial x^j \partial x^k} - \Box(h_{ij})\right\} = \frac{1}{2}\left\{\frac{\partial^2 h^k_j}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h^k_i}{\partial x^j \partial x^k} - \Box(h_{ij})\right\}$$

(V-26)

IV - The curvature scalar R

It is obtained from the RICCI tensor R_{ij} and from the metric tensor η^{ij} by the contraction:

$$R = g^{ij}R_{ij} = \left(\eta^{ij} - h^{ij}\right)R_{ij} \approx \eta^{ij}R_{ij} = \frac{1}{2}\eta^{ij}\left\{\frac{\partial^2 h^k_j}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h^k_i}{\partial x^j \partial x^k} - \Box(h_{ij})\right\} \qquad \text{(V-27)}$$

The metric tensor η^{ij} allows to lower or raise indices of terms between brackets:

$$R = \frac{1}{2}\left\{\frac{\partial^2 h^{ik}}{\partial x^i \partial x^k} - \eta^{ij}\frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h^{jk}}{\partial x^j \partial x^k} - \Box\left(\eta^{ij}h_{ij}\right)\right\} \qquad \text{(V-28)}$$

The sums described in the terms derived from the two tensors twice contravariant h^{ik} and h^{jk} are identical, which allows to combine them:

$$R = \frac{1}{2}\left\{2\frac{\partial^2 h^{ij}}{\partial x^i \partial x^j} - \eta^{ij}\frac{\partial^2 h}{\partial x^i \partial x^j} - \Box\left(\eta^{ij}h_{ij}\right)\right\} \qquad \text{(V-29)}$$

As previously, we use the fact that all terms of the metric tensor η^{ij} are null, except for i = j: in these conditions, as stated above, the last two terms between brackets correspond to the d'Alembertian of the trace h, and we get after this ultimate contraction:

$$R = \frac{\partial^2 h^{ij}}{\partial x^i \partial x^j} - \Box(h) \qquad \text{(V-30)}$$

V – Linearized EINSTEIN equation

The field equations are written based on the momentum-energy tensor describing the electromagnetic sources, and which is recalled to memory:

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi G}{c^4}T_{ij} \qquad \text{(V-31)}$$

By reporting in this expression, the partial results for the expression of R_{ij} (V-26) and R (V-30), we get the equation of propagation of electromagnetic waves in a linearized metric:

$$\frac{1}{2}\left\{\frac{\partial^2 h_j^k}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h_i^k}{\partial x^j \partial x^k} - \Box\left(h_{ij}\right)\right\} - \frac{1}{2} g_{ij}\left\{\frac{\partial^2 h^{pq}}{\partial x^p \partial x^q} - \Box\left(h\right)\right\} = \frac{8\pi G}{c^4} T_{ij} \qquad \text{(V-32)}$$

and since the g_{ij} are substantially equal to the η_{ij}, the fields equations with small perturbations h_{ij} takes the very general form:

$$\frac{\partial^2 h_j^k}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h_i^k}{\partial x^j \partial x^k} - \Box\left(h_{ij}\right) - \eta_{ij} \frac{\partial^2 h^{pq}}{\partial x^p \partial x^q} + \eta_{ij}\Box\left(h\right) = \frac{16\pi G}{c^4} T_{ij} \qquad \text{(V-33)}$$

We can get another formulation, starting with an equivalent form of the EINSTEIN equation (V-31) deducted by using the relationship: $-R = \chi T$ (obtained by multiplying EINSTEIN equation by the contravariant metric tensor g^{ij}):

$$R_{ij} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T\right) \qquad \text{(V-34)}$$

Only the linearized expression of R_{ij} (V-25) is necessary, and we get:

$$\frac{1}{2}\left\{\frac{\partial^2 h_j^k}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h_i^k}{\partial x^j \partial x^k} - \Box\left(h_{ij}\right)\right\} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T\right) \qquad \text{(V-35)}$$

or still :

$$\frac{\partial^2 h_j^k}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h_i^k}{\partial x^j \partial x^k} - \Box\left(h_{ij}\right) = \frac{16\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T\right) \qquad \text{(V-36)}$$

The two obtained equations (V-33) and (V-36) do not yet allow direct interpretation in terms of propagation of electromagnetic fields and require simplifications we need to justify: it will be the subject of the next chapter.

VI - Harmonic coordinates and harmonic gauge condition

The gauge problem arises in electromagnetism when we try to write MAXWELL's equations in terms of potential rather than in terms of electric and magnetic field components. We can see that, for a given electromagnetic field, potentials which represent this field are not defined in a unique way. We have to draw a choice among the infinite number of possible potential, and this choice is called choice of gauge. It appears finally that there is only a choice of gauge that preserves the invariance of MAXWELL's equations by change of frame: this is the LORENZ gauge.

The problem arises on similar terms with the fields equations. In-depth considerations, including through a better understanding of gravitational waves and their equation formatting, helped establish a rigorous framework of choice of gauge, which highlights are presented below.

We obtained, in the previous chapter, two expressions of the equation of propagation of a small perturbation of the metric of MINKOWSKY, by linearizing the one or other of the EINSTEIN equations:

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} \qquad \text{(VI-1)}$$

$$R_{ij} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T \right) \qquad \text{(VI-2)}$$

To do this, we have considered a metric of MINKOWSKY η_{ij} where we injected a small perturbation h_{ij}. We concluded that the perturbed metric g_{ij} was of the form:

$$g_{ij} = \eta_{ij} + h_{ij} \qquad \text{(VI-3)}$$

We then calculated the RICCI tensor R_{ij} and the curvature scalar R of modified space with perturbation h_{ij}.

The indeterminacy of this method comes from the fact that if we place ourselves in a space with a slightly different metric (represented by the sign ~ in the following paragraphs), we have:

$$\tilde{g}_{ij} = \eta_{ij} + \tilde{h}_{ij} \qquad \text{(VI-4)}$$

and that, to the approximations imposed, g_{ij} and $\tilde{g}_{ij}$ can lead to the same Rij RICCI tensor and the same curvature scalar R. There are so many h_{ij} which lead to the same wave equation, and, by analogy with electromagnetism, the choices that will be required to operate among the infinite will be designated by choice of gauge.

To clarify this property, we proceed in two steps:

- One places oneself in a space with a slightly different metric by considering a small change of coordinate and we show that the original perturbation h_{ij} turns into a new perturbation $\tilde{h}_{ij}$.

- It is shown that, under certain assumptions, this new perturbation $\tilde{h}_{ij}$ leads to the same linearized RICCI tensor as h_{ij} and therefore to the same equations of the fields.

I – Invariance of the linearized EINSTEIN equation under a small variation of the metric.

We consider that each x^i coordinate experiences a small change ε^i and becomes $\tilde{x}^i$ in a new coordinate system:

$$\tilde{x}^i = x^i + \varepsilon^i \qquad \text{(VI-5)}$$

It will be assumed in this transformation that ε^i is an infinitely small amount of first order (like h_{ij}), and furthermore that this change of coordinates is slowly variable, which allows to consider that the quantity $\frac{\partial \varepsilon^i}{\partial x^i}$ is also an infinitely small of the first order. In these conditions, we deduce from the processing (VI-5):

$$\frac{\partial \tilde{x}^i}{\partial x^i} = 1 + \frac{\partial \varepsilon^i}{\partial x^i} \qquad \text{et} \qquad \frac{\partial x^i}{\partial \tilde{x}^i} = 1 - \frac{\partial \varepsilon^i}{\partial x^i} \qquad \text{(VI-6)}$$

The tensor g_{ij} bringing the perturbed metric will undergo the change of coordinates (VI-5), and we know what a covariant tensor of the second order is transformed by change of coordinates according to the relationship:

$$\tilde{g}_{ij} = \frac{\partial x^i}{\partial \tilde{x}^i}\frac{\partial x^j}{\partial \tilde{x}^j} g_{ij} \qquad \text{(VI-7)}$$

In cases where we are, this transformation is growing in function of ε^i :

$$\tilde{g}_{ij} = \frac{\partial x^i}{\partial \tilde{x}^i}\frac{\partial x^j}{\partial \tilde{x}^j} g_{ij} \approx \left(1 - \frac{\partial \varepsilon^i}{\partial x^i}\right)\left(1 - \frac{\partial \varepsilon^j}{\partial x^j}\right) g_{ij} \approx \left(1 - \frac{\partial \varepsilon^i}{\partial x^i} - \frac{\partial \varepsilon^j}{\partial x^j}\right) g_{ij} \qquad \text{(VI-8)}$$

By making explicit the g_{ij}, we do appear the MINKOWSKY metric as well as its small perturbations h_{ij}:

$$\tilde{\eta}_{ij} + \tilde{h}_{ij} \approx \left(1 - \frac{\partial \varepsilon^i}{\partial x^i} - \frac{\partial \varepsilon^j}{\partial x^j}\right)\left(\eta_{ij} + h_{ij}\right) \qquad \text{(VI-9)}$$

At first order, $\tilde{\eta}_{ij} = \eta_{ij}$ and it remains:

$$\tilde{h}_{ij} \approx h_{ij} - \eta_{ij}\frac{\partial \varepsilon^i}{\partial x^i} - \eta_{ij}\frac{\partial \varepsilon^j}{\partial x^j} \qquad \text{(VI-10)}$$

Because the metric tensor of the metric of MINKOWSKY η_{ij} is constant, it can be included in the partial derivative, and we get the relationship that links the small perturbations h_{ij} under a small transformation of coordinates ε_i:

$$\tilde{h}_{ij} \approx h_{ij} - \frac{\partial\left(\eta_{ij}\varepsilon^i\right)}{\partial x^i} - \frac{\partial\left(\eta_{ij}\varepsilon^j\right)}{\partial x^j} = h_{ij} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{\partial \varepsilon_i}{\partial x^j} \qquad \text{(VI-11)}$$

It was noted that this relationship is consistent with the hypothesis of slowly varying coordinates in (VI-5) and (VI-6), because changes in coordinates should be of the order of magnitude of the h_{ij} to ensure the homogeneity of the relationship (VI-11).

In the previous chapter, we showed that the linearized RICCI tensor is written based on the small perturbations h_{ij}:

$$R_{ij} = \frac{1}{2}\eta^{rk}\left\{\frac{\partial^2 h_{jk}}{\partial x^i \partial x^r} - \frac{\partial^2 h_{ij}}{\partial x^r \partial x^k} - \frac{\partial^2 h_{rk}}{\partial x^i \partial x^j} + \frac{\partial^2 h_{ir}}{\partial x^j \partial x^k}\right\} \qquad \text{(VI-12)}$$

If we alters the h_{ij} to a small quantity which is an infinitesimal change in coordinates ε_i, the h_{ij} become, according to the above relationship:

$$\tilde{h}_{ij} = h_{ij} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{\partial \varepsilon_i}{\partial x^j} \qquad \text{(VI-13)}$$

This modification of the h_{ij} introduced a change in the RICCI tensor δR_{ij} which is evaluated by puting the modified h_{ij} (VI-13) in expression (VI-12) :

$$R_{ij} + \delta R_{ij} = \frac{1}{2}\eta^{rk}\left\{\begin{array}{l}\dfrac{\partial^2\left(h_{jk} - \dfrac{\partial \varepsilon_k}{\partial x^j} - \dfrac{\partial \varepsilon_j}{\partial x^k}\right)}{\partial x^i \partial x^r} - \dfrac{\partial^2\left(h_{ij} - \dfrac{\partial \varepsilon_j}{\partial x^i} - \dfrac{\partial \varepsilon_i}{\partial x^j}\right)}{\partial x^r \partial x^k} \\ - \dfrac{\partial^2\left(h_{rk} - \dfrac{\partial \varepsilon_k}{\partial x^r} - \dfrac{\partial \varepsilon_r}{\partial x^k}\right)}{\partial x^i \partial x^j} + \dfrac{\partial^2\left(h_{ir} - \dfrac{\partial \varepsilon_r}{\partial x^i} - \dfrac{\partial \varepsilon_i}{\partial x^r}\right)}{\partial x^j \partial x^k}\end{array}\right\} \qquad \text{(VI-14)}$$

or still for the only variation δR_{ij} :

$$\delta R_{ij} = \frac{1}{2}\eta^{rk}\left\{\begin{array}{l}\dfrac{\partial^2\left(-\dfrac{\partial\varepsilon_k}{\partial x^j}-\dfrac{\partial\varepsilon_j}{\partial x^k}\right)}{\partial x^i\partial x^r}-\dfrac{\partial^2\left(-\dfrac{\partial\varepsilon_j}{\partial x^i}-\dfrac{\partial\varepsilon_i}{\partial x^j}\right)}{\partial x^r\partial x^k}\\ -\dfrac{\partial^2\left(-\dfrac{\partial\varepsilon_k}{\partial x^r}-\dfrac{\partial\varepsilon_r}{\partial x^k}\right)}{\partial x^i\partial x^j}+\dfrac{\partial^2\left(-\dfrac{\partial\varepsilon_r}{\partial x^i}-\dfrac{\partial\varepsilon_i}{\partial x^r}\right)}{\partial x^j\partial x^k}\end{array}\right\} \qquad \text{(VI-15)}$$

Developing the expression between braces, we get:

$$\delta R_{ij} = \frac{1}{2}\eta^{rk}\left\{\begin{array}{l}-\dfrac{\partial^3\varepsilon_k}{\partial x^i\partial x^j\partial x^r}-\dfrac{\partial^3\varepsilon_j}{\partial x^i\partial x^k\partial x^r}+\dfrac{\partial^3\varepsilon_j}{\partial x^i\partial x^k\partial x^r}+\dfrac{\partial^3\varepsilon_i}{\partial x^j\partial x^k\partial x^r}\\ +\dfrac{\partial^3\varepsilon_k}{\partial x^i\partial x^j\partial x^r}+\dfrac{\partial^3\varepsilon_r}{\partial x^i\partial x^j\partial x^k}-\dfrac{\partial^3\varepsilon_r}{\partial x^i\partial x^j\partial x^k}-\dfrac{\partial^3\varepsilon_i}{\partial x^j\partial x^k\partial x^r}\end{array}\right\}=0 \qquad \text{(VI-16)}$$

We obtain then $\delta R_{ij} = 0$.

The modification of the h_{ij} (VI-13) does not change the equation of electromagnetic waves since it does not change the RICCI tensor R_{ij}.

In electromagnetism, we can find an infinite number of potential that represent the same electromagnetic field.

Similarly, ε_i being a small amount at first glance any (but respecting the above assumptions), the relationship (VI-13) allows to find an infinite number of h_{ij} representing the same gravitational field propagating through space-time.

II – harmonic coordinates.

The propagation equations of the h_{ij} obtained in the previous chapter are recalled to memory:

$$\frac{\partial^2 h^k_j}{\partial x^i\partial x^k}-\frac{\partial^2 h}{\partial x^i\partial x^j}+\frac{\partial^2 h^k_i}{\partial x^j\partial x^k}-\eta_{ij}\frac{\partial^2 h^{pq}}{\partial x^p\partial x^p}-\Box(h_{ij})-\Box(h)=\frac{16\pi G}{c^4}T_{ij} \qquad \text{(VI-17)}$$

$$\frac{\partial^2 h^r_j}{\partial x^i\partial x^r}-\frac{\partial^2 h}{\partial x^i\partial x^j}+\frac{\partial^2 h^k_i}{\partial x^j\partial x^k}-\Box(h_{ij})=\frac{16\pi G}{c^4}\left(T_{ij}-\frac{1}{2}g_{ij}T\right) \qquad \text{(VI-18)}$$

Despite the linearized framework, they are partial derivatives equations of great complexity and practically impossible to solve as it stands. An idea is to search, among the infinite number of coordinate systems that represents tensorial writing, if there is no one who can simplify the writing of these equations.

It's a problem that we face in other situations, for example in variational analysis to relieve calculations from the expression:

$$\delta(R_{ij}) = \delta\left(\Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik} + \frac{\partial\Gamma^k_{ij}}{\partial x^k} - \frac{\partial\Gamma^k_{ik}}{\partial x^j}\right) \qquad \text{(VI-19)}$$

One stand in a local geodetic coordinate system for which:

$$\frac{\partial g_{ij}}{\partial x^k} = 0 \qquad \text{(VI-20)}$$

In these conditions, the CHRISTOFFEL coefficients are zero (but not their derivative), which simplifies the expression (VI-19).

This simplification remove nothing from the generality of the demonstration, provided you ensure to always work with variables having tensorial quality: We know that in those circumstances, the form of the obtained relationships is independent of the chosen coordinate system.

In the present case, the use of local geodetic coordinates brings no significant simplification, and the fact that we see a propagation equation in the above expressions probably contributed to suggest the idea of trying to use a harmonic coordinate system. We will see later that these coordinates are needed in a natural way because they reflect on the metric coefficients, the properties of null divergence which are imposed on the electromagnetic potential.

II-1 Notion of harmonic function:

A function f of four variables is called harmonic if:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = 0 \qquad \text{(VI-21)}$$

In a metric of MINKOWSKY, this relationship represents the d'Alembertian which is written according to the spatial and temporal variables:

$$\Box(f) = -\frac{\partial^2 f(x,y,z,ct)}{\partial (ct)^2} + \frac{\partial^2 f(x,y,z,ct)}{\partial x^2} + \frac{\partial^2 f(x,y,z,ct)}{\partial y^2} + \frac{\partial^2 f(x,y,z,ct)}{\partial z^2} = 0 \qquad \text{(VI-22)}$$

It is an equation of propagation. In the case of a propagation along a dimension x, any function f(x + ct) or f(x – ct) is a solution, the first expressing the spread towards the x > 0, the second towards the < 0 x. The fact that the function f is any expresses that the shape of the wave that spreads can be a priori any, and we find at a distance x and at a time t = x/c, the shape of the wave at x = 0 and at t = 0, which is the physical translation of the phenomenon of propagation.

Going back to the linearized equations of the fields (VI-17, 18), we can speculate that if one places locally in a coordinate system which evolves according to the shape of the wave that spreads, this should lead to a simplification of these equations.

II-2 Harmonic coordinate system:

So let's choose a coordinate system x^μ such as x^μ be harmonic coordinates, within the meaning of the tensorial analysis: the Laplacian is written in covariant derivatives, whose definition for a tensor of order 1 is recalled below:

$$\frac{DA^i}{\partial x^j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jm}A^m \qquad \frac{DA_i}{\partial x^j} = \frac{\partial A_i}{\partial x^j} - \Gamma^m_{ij}A_m \tag{VI-23}$$

The Laplacian of a scalar function Φ is built according to the classical definition:

$$\nabla^2\Phi = \mathrm{Div}\left(\overrightarrow{\mathrm{Grad}}(\Phi)\right) \tag{VI-24}$$

For a scalar function, gradient, in covariant derivative, is reduced to a covariant tensor of order 1 according to the relationship:

$$A_k = \overrightarrow{\mathrm{Grad}}(\Phi) = \frac{\partial\Phi}{\partial x^k} \tag{VI-25}$$

Applied to this tensor, the covariant derivative (VI-23) provides a covariant tensor of order 2:

$$A_{jk} = \frac{DA_k}{\partial x^j} = \frac{\partial A_k}{\partial x^j} - \Gamma^m_{kj}A_m = \frac{\partial^2\Phi}{\partial x^j \partial x^k} - \Gamma^m_{kj}\frac{\partial\Phi}{\partial x^m} \tag{VI-26}$$

The Laplacian is obtained by taking the associated tensor of order 0, obtained by contraction of the tensor twice covariant, the result being a scalar:

$$\nabla^2\Phi = g^{jk}A_{jk} = g^{jk}\left(\frac{\partial^2\Phi}{\partial x^j \partial x^k} - \Gamma^m_{kj}\frac{\partial\Phi}{\partial x^m}\right) \tag{VI-27}$$

By applying this treatment to each coordinate x^μ, and when looking for a coordinate system such that the Laplacian is zero, we must impose the condition:

$$\nabla^2\left(x^\mu\right) = g^{jk}A_{jk} = g^{jk}\left(\frac{\partial^2 x^\mu}{\partial x^j \partial x^k} - \Gamma^m_{kj}\frac{\partial x^\mu}{\partial x^m}\right) = 0 \tag{VI-28}$$

The coordinates being independent variables, we can deduce, based on the KRONECKER symbol δ_a^b equal to 1 for a = b, and equal to 0 for a different from b :

$$\frac{\partial x^\mu}{\partial x^k} = \delta^\mu_k \qquad \text{et} \qquad \frac{\partial}{\partial x^j}\left(\frac{\partial x^\mu}{\partial x^k}\right) = 0 \tag{VI-29}$$

The relationship that defines the harmonic coordinates (VI-28) is reduced finally after change of indices, in:

$$\nabla^2\left(x^\mu\right) = -g^{ij}\Gamma^{\cdot\mu}_{ij} = 0 \tag{VI-30}$$

It can be expressed according to the metric coefficients in developing the CHRISTOFFEL symbol:

$$\nabla^2\left(x^\mu\right) = -g^{ij}\Gamma^\mu_{ij} = -\frac{1}{2}g^{ij}g^{\mu k}\left\{\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right\} = 0 \qquad \text{(VI-31)}$$

In the weak field approximation broken down into a small perturbation h_{ij} on the metric η_{ij} of the space of MINKOWSKY, according the sum recalled below:

$$g_{ij} = \eta_{ij} + h_{ij} \qquad \text{(VI-32)}$$

the relationship (VI-31) becomes:

$$\nabla^2\left(x^\mu\right) = -\frac{1}{2}\eta^{ij}\eta^{\mu k}\left\{\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right\} = 0 \qquad \text{(VI-33)}$$

The coefficients η^{ij} of the metric of MINKOWSKY are constants, and so can pass under the derivation sign:

$$\nabla^2\left(x^\mu\right) = -\frac{1}{2}\eta^{\mu k}\left\{\frac{\partial\left(\eta^{ij}h_{ik}\right)}{\partial x^j} + \frac{\partial\left(\eta^{ij}h_{jk}\right)}{\partial x^i} - \frac{\partial\left(\eta^{ij}h_{ij}\right)}{\partial x^k}\right\} = 0 \qquad \text{(VI-34)}$$

$$\nabla^2\left(x^\mu\right) = -\frac{1}{2}\eta^{\mu k}\left\{\frac{\partial\left(h^j_k\right)}{\partial x^j} + \frac{\partial\left(h^i_k\right)}{\partial x^i} - \frac{\partial\left(h\right)}{\partial x^k}\right\} = 0 \qquad \text{(VI-35)}$$

A sufficient, but not necessary, condition for the relationship above is verified, is that the expression between braces is zero.

We can deduce the condition that sets the harmonic coordinates:

$$\frac{\partial\left(h^j_k\right)}{\partial x^j} + \frac{\partial\left(h^i_k\right)}{\partial x^i} - \frac{\partial\left(h\right)}{\partial x^k} = 0 \qquad \text{(VI-36)}$$

The first two terms of the sum above are the same, which leads to a more concise expression of the condition defining the harmonic coordinates relative to the perturbation h^i_k :

$$2\frac{\partial\left(h^i_k\right)}{\partial x^i} - \frac{\partial\left(h\right)}{\partial x^k} = 0 \qquad \text{(VI-37)}$$

III – The wave equation in harmonic coordinates.

The RICCI tensor obtained in the previous chapter, in a framework of linearized, perturbation, is given by the relationship:

$$R_{ij} = \frac{1}{2}\left\{ \frac{\partial^2 h_j^k}{\partial x^i \partial x^k} - \frac{\partial^2 h}{\partial x^i \partial x^j} + \frac{\partial^2 h_i^k}{\partial x^j \partial x^k} - \Box\left(h_{ij}\right) \right\} \tag{VI-38}$$

If we rewrite it in harmonic coordinates, by using the relationship (VI-37), the first 3 terms between braces cancel out, and it remains:

$$R_{ij} = -\frac{1}{2}\Box\left(h_{ij}\right) \tag{VI-39}$$

It follows the form of the scalar curvature in harmonic coordinates:

$$R = \eta^{ij} R_{ij} = -\frac{1}{2}\eta^{ij}\Box\left(h_{ij}\right) = -\frac{1}{2}\Box\left(\eta^{ij} h_{ij}\right) = -\frac{1}{2}\Box(h) \tag{VI-40}$$

One can then write the EINSTEIN equation relative to the perturbation in harmonic coordinates, in its two classical forms:

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} \tag{VI-41}$$

$$R_{ij} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T \right) \tag{VI-42}$$

which leads with the approximation $\eta_{ij} = g_{ij}$ to two equivalent wave equations:

$$\Box\left(h_{ij}\right) - \frac{1}{2}\eta_{ij}\Box\left(h\right) = -\frac{16\pi G}{c^4} T_{ij} \tag{VI-43}$$

$$\Box\left(h_{ij}\right) = -\frac{16\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T \right) \tag{VI-44}$$

It is recognized in this last expression a classical wave equation which is written in cartesian coordinates:

$$\Box\left(h_{ij}\right) = -\frac{\partial^2 h_{ij}}{\partial (ct)^2} + \frac{\partial^2 h_{ij}}{\partial x^2} + \frac{\partial^2 h_{ij}}{\partial y^2} + \frac{\partial^2 h_{ij}}{\partial z^2} = -\frac{16\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T \right) \tag{VI-45}$$

The change of variable:

$$\overline{h}_{ij} = h_{ij} - \frac{1}{2}\eta_{ij} h \tag{VI-46}$$

allows an equivalent formulation of the equation (VI-43), since the Laplacian is going through all the terms on the right of the relationship (VI-46).

It highlights explicitly a propagation equation applied to the modified perturbation $\overline{h}_{ij}$:

$$\Box\left(\overline{h}_{ij}\right) = -\frac{\partial^2 \overline{h}_{ij}}{\partial (ct)^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial x^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial y^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial z^2} = -\frac{16\pi G}{c^4} T_{ij} \qquad \text{(VI-47)}$$

If necessary, we can reverse the relationship (VI-46) in order to express the perturbation h_{ij} as a function of the modified perturbation $\overline{h}_{ij}$. Noting that:

$$\overline{h} = \eta^{ij}\overline{h}_{ij} = \eta^{ij}\left(h_{ij} - \frac{1}{2}\eta_{ij}h\right) = \eta^{ij}h_{ij} - \frac{1}{2}\eta^{ij}\eta_{ij}h \qquad \text{(VI-48)}$$

and by making use of the already explicit result:

$$\eta^{ij}\eta_{ij} = 4 \qquad \text{(VI-49)}$$

We get :

$$\overline{h} = h - 2h = -h \qquad \text{(VI-50)}$$

so, either by introducing this relationship in (VI - 46):

$$h_{ij} = \overline{h}_{ij} - \frac{1}{2}\eta_{ij}\overline{h} \qquad \text{(VI-51)}$$

It remains to derive the condition of harmonic gauge associated with the equation of propagation (VI-47) as a function of the modified perturbation $\overline{h}_{ij}$.

Harmonic coordinates have been defined based on perturbation h_{ij} by the relationship (VI-37) recalled to memory below after exchange of the role of indices i and k:

$$\frac{\partial\left(h_i^k\right)}{\partial x^k} - \frac{1}{2}\frac{\partial (h)}{\partial x^i} = 0 \qquad \text{(VI-52)}$$

One deduce from (VI-46) :

$$\eta^{jk}\overline{h}_{ij} = \eta^{jk}h_{ij} - \frac{1}{2}\eta^{jk}\eta_{ij}h \qquad \text{(VI-53)}$$

And then:

$$\overline{h}_i^k = h_i^k - \frac{1}{2}\eta_i^k h \qquad \text{(VI-54)}$$

Or still :

$$h_i^k = \overline{h}_i^k + \frac{1}{2}\eta_i^k h = \overline{h}_i^k - \frac{1}{2}\eta_i^k \overline{h} \qquad \text{(VI-55)}$$

and by reporting this result in (VI-52):

$$\frac{\partial\left(\bar{h}_i^k\right)}{\partial x^k} - \frac{1}{2}\eta_i^k\frac{\partial\left(\bar{h}\right)}{\partial x^k} + \frac{1}{2}\frac{\partial\left(\bar{h}\right)}{\partial x^i} = 0 \qquad \text{(VI-56)}$$

The mixed metric coefficients $\eta^k{}_i$ are evaluated according to the relationship:

$$\eta_i^k = \eta^{jk}\eta_{ji} = \eta^{0k}\eta_{0i} + \eta^{1k}\eta_{1i} + \eta^{2k}\eta_{2i} + \eta^{3k}\eta_{3i} \qquad \text{(VI-57)}$$

They are not null only for i = k, and then set the value to 1. We can deduce from (VI-56):

$$\frac{\partial\left(\bar{h}_i^k\right)}{\partial x^k} - \frac{1}{2}\frac{\partial\left(\bar{h}\right)}{\partial x^i} + \frac{1}{2}\frac{\partial\left(\bar{h}\right)}{\partial x^i} = 0 \qquad \text{(VI-58)}$$

and as conclusion the relationship defining the harmonic coordinates from the modified perturbation:

$$\frac{\partial\left(\bar{h}_i^k\right)}{\partial x^k} = 0 \qquad \text{(VI-59)}$$

IV – Summary and conclusions.

For a small disturbance h_{ij} of the metric of MINKOWSKY η_{ij}, the wave equation associated with the harmonic coordinates comes in two forms:

$$\square\left(h_{ij}\right) - \frac{1}{2}\eta_{ij}\square\left(h\right) = -\frac{16\pi G}{c^4}T_{ij} \qquad \text{with} \qquad 2\frac{\partial\left(h_k^i\right)}{\partial x^i} - \frac{\partial\left(h\right)}{\partial x^k} = 0 \qquad \text{(VI-60)}$$

$$\square\left(h_{ij}\right) = -\frac{16\pi G}{c^4}\left(T_{ij} - \frac{1}{2}g_{ij}T\right) \qquad \text{with} \qquad 2\frac{\partial\left(h_k^i\right)}{\partial x^i} - \frac{\partial\left(h\right)}{\partial x^k} = 0 \qquad \text{(VI-61)}$$

The introduction of the modified perturbation:

$$\bar{h}_{ij} = h_{ij} - \frac{1}{2}\eta_{ij}h \qquad \text{(VI-62)}$$

allows to rewrite the two wave equations (VI-60, 61) under the following form, associated with harmonic coordinates relative to the modified perturbation:

$$\square\left(\bar{h}_{ij}\right) = -\frac{16\pi G}{c^4}T_{ij} \qquad \text{with} \qquad \frac{\partial\left(\bar{h}_i^k\right)}{\partial x^k} = 0 \qquad \text{(VI-63)}$$

$$\square\left(\bar{h}_{ij}\right) - \frac{1}{2}\eta_{ij}\square\left(\bar{h}\right) = -\frac{16\pi G}{c^4}\left(T_{ij} - \frac{1}{2}g_{ij}T\right) \qquad \text{with} \qquad \frac{\partial\left(\bar{h}_i^k\right)}{\partial x^k} = 0 \qquad \text{(VI-64)}$$

If we place in vacuum, that is to say we cancel the T_{ij} energy momentum tensor, an immediate simplifications appears. Equations (VI-60) and (VI-61) become:

$$\Box\left(h_{ij}\right)-\frac{1}{2}\eta_{ij}\Box\left(h\right)=0 \qquad \text{(VI-65)}$$

$$\Box\left(h_{ij}\right)=0 \qquad \text{(VI-66)}$$

We can deduce that the Laplacian of the trace of the perturbation $\Box(h)$ is necessarily null. This condition does define h in a unique way.

A simplifier choice is h = 0. In this case, the modified perturbation $\bar{h}_{ij}$ given by (VI-62) is equal to the perturbation h_{ij} and we conclude that the four equations (VI-60, 61, 63, 64) is reduced to a single:

$$\Box\left(h_{ij}\right)=0 \qquad \text{(VI-67)}$$

We have to add the gauge conditions that we have imposed:

$$h = 0 \qquad \text{and} \qquad \frac{\partial\left(h^{i}_{k}\right)}{\partial x^{i}} = 0 \qquad \text{(VI-68)}$$

VII - Identification of MAXWELL's equations

We showed in the previous chapter that the linearized EINSTEIN equations reduce to a wave equation:

$$\Box\left(\overline{h}_{ij}\right)=-\frac{\partial^2\overline{h}_{ij}}{\partial(ct)^2}+\frac{\partial^2\overline{h}_{ij}}{\partial x^2}+\frac{\partial^2\overline{h}_{ij}}{\partial y^2}+\frac{\partial^2\overline{h}_{ij}}{\partial z^2}=-\frac{16\pi G}{c^4}T_{ij}=-2\chi T_{ij} \tag{VII-1}$$

This equation must be associated with the condition of harmonic gauge:

$$\frac{\partial\left(\overline{h}_i^k\right)}{\partial x^k}=\frac{\partial\left(\overline{h}_i^0\right)}{\partial x^0}+\frac{\partial\left(\overline{h}_i^1\right)}{\partial x^1}+\frac{\partial\left(\overline{h}_i^2\right)}{\partial x^2}+\frac{\partial\left(\overline{h}_i^3\right)}{\partial x^3}=0 \tag{VII-2}$$

That is, after multiplication by the approached metric tensor:

$$-\frac{\partial\left(\overline{h}_{i0}\right)}{\partial x^0}+\frac{\partial\left(\overline{h}_{i1}\right)}{\partial x^1}+\frac{\partial\left(\overline{h}_{i2}\right)}{\partial x^2}+\frac{\partial\left(\overline{h}_{i3}\right)}{\partial x^3}=0 \tag{VII-3}$$

We can give a slightly more developed writing of the general wave equation (VII-1):

$$\begin{pmatrix}\Box\overline{h}_{00} & \Box\overline{h}_{01} & \Box\overline{h}_{02} & \Box\overline{h}_{03}\\ \Box\overline{h}_{10} & \Box\overline{h}_{11} & \Box\overline{h}_{12} & \Box\overline{h}_{13}\\ \Box\overline{h}_{20} & \Box\overline{h}_{21} & \Box\overline{h}_{22} & \Box\overline{h}_{23}\\ \Box\overline{h}_{30} & \Box\overline{h}_{31} & \Box\overline{h}_{32} & \Box\overline{h}_{33}\end{pmatrix}=-\frac{16\pi G}{c^4}\begin{pmatrix}T_{00} & T_{01} & T_{02} & T_{03}\\ T_{10} & T_{11} & T_{12} & T_{13}\\ T_{20} & T_{21} & T_{22} & T_{23}\\ T_{30} & T_{31} & T_{32} & T_{33}\end{pmatrix} \tag{VII-4}$$

We introduce the momentum-energy tensor of electromagnetic sources that has been proposed in previous chapters:

$$\begin{pmatrix}\Box\overline{h}_{00} & \Box\overline{h}_{01} & \Box\overline{h}_{02} & \Box\overline{h}_{03}\\ \Box\overline{h}_{10} & \Box\overline{h}_{11} & \Box\overline{h}_{12} & \Box\overline{h}_{13}\\ \Box\overline{h}_{20} & \Box\overline{h}_{21} & \Box\overline{h}_{22} & \Box\overline{h}_{23}\\ \Box\overline{h}_{30} & \Box\overline{h}_{31} & \Box\overline{h}_{32} & \Box\overline{h}_{33}\end{pmatrix}=-\frac{16\pi G}{c^4}\lambda\begin{pmatrix}-\rho c^2 & cJ_x & cJ_y & cJ_z\\ cJ_x & -J_xv_x & -J_xv_y & -J_xv_z\\ cJ_y & -J_yv_x & -J_yv_y & -J_yv_z\\ cJ_z & -J_zv_x & -J_zv_y & -J_zv_z\end{pmatrix} \tag{VII-5}$$

The question we ask ourselves is: is it possible to identify MAXWELL's equations in the form of a wave equation relative to electromagnetic potential in equation (VII-5)?

We make the assumption that the metric of the electromagnetic field is described in terms of the tensor $\bar{h}_{ij}$, and we are continuing the identification under this assumption.

The wave equations of potential are recalled to memory:

$$\vec{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} = -\mu_0\rho c^2$$
$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \qquad \text{(VII-4)}$$

That is, in a developed writing:

$$\begin{aligned}
-\frac{\partial^2(\phi/c)}{\partial(ct)^2} + \frac{\partial^2(\phi/c)}{\partial x^2} + \frac{\partial^2(\phi/c)}{\partial y^2} + \frac{\partial^2(\phi/c)}{\partial z^2} &= -\mu_0\rho c \\
-\frac{\partial^2 A_x}{\partial(ct)^2} + \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} &= -\mu_0 J_x \\
-\frac{\partial^2 A_y}{\partial(ct)^2} + \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} &= -\mu_0 J_y \\
-\frac{\partial^2 A_z}{\partial(ct)^2} + \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} &= -\mu_0 J_z
\end{aligned} \qquad \text{(VII-5)}$$

We then faced the following difficulty: there are only 4 components for the electromagnetic potential, while there are 16 terms in the tensor that represents the modified metric.

We conjecture that if MAXWELL's equations should appear in the metric tensor, so it will necessarily be in the dominant terms of this tensor.

We do not know, a priori, the dominant terms of the metric tensor, but we know those of the energy momentum tensor: these are the ones that contain the term c or c^2. If we assume the speed of charges much less than the speed of light, we have approximately:

$$\begin{pmatrix} \Box\bar{h}_{00} & \Box\bar{h}_{01} & \Box\bar{h}_{02} & \Box\bar{h}_{03} \\ \Box\bar{h}_{10} & \Box\bar{h}_{11} & \Box\bar{h}_{12} & \Box\bar{h}_{13} \\ \Box\bar{h}_{20} & \Box\bar{h}_{21} & \Box\bar{h}_{22} & \Box\bar{h}_{23} \\ \Box\bar{h}_{30} & \Box\bar{h}_{31} & \Box\bar{h}_{32} & \Box\bar{h}_{33} \end{pmatrix} = -\frac{16\pi G}{c^4}\lambda \begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & 0 & 0 & 0 \\ cJ_y & 0 & 0 & 0 \\ cJ_z & 0 & 0 & 0 \end{pmatrix} \qquad \text{(VII-6)}$$

We conjecture that the dominant terms of the tensor $\bar{h}_{ij}$ will be similar to the dominant terms of the tensor T_{ij}:

$$\begin{pmatrix} \Box\bar{h}_{00} & \Box\bar{h}_{01} & \Box\bar{h}_{02} & \Box\bar{h}_{03} \\ \Box\bar{h}_{10} & 0 & 0 & 0 \\ \Box\bar{h}_{20} & 0 & 0 & 0 \\ \Box\bar{h}_{30} & 0 & 0 & 0 \end{pmatrix} = -\frac{16\pi G}{c^4}\lambda \begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & 0 & 0 & 0 \\ cJ_y & 0 & 0 & 0 \\ cJ_z & 0 & 0 & 0 \end{pmatrix} \qquad \text{(VII-7)}$$

It should be noted that the relationship (VII-6) above does not involve the nullity of the coefficients $\bar{h}_{ij}$ for i and j different from 0 which is proposed (VII-7). But if we assume that MAXWELL's equations should appear through the metric $\bar{h}_{ij}$, this leads to admit that the dominant terms of this metric are those containing at least once the index 0. It is under this assumption that we consider that terms such as i and j different of 0 can be neglected in a first approximation.

Writing developed of the first line is written explicitly:

$$\begin{aligned}
&-\frac{\partial^2\bar{h}_{00}}{\partial(ct)^2}+\frac{\partial^2\bar{h}_{00}}{\partial x^2}+\frac{\partial^2\bar{h}_{00}}{\partial y^2}+\frac{\partial^2\bar{h}_{00}}{\partial z^2}=-\frac{16\pi G}{c^4}T_{00}=-\frac{16\pi G}{c^4}\left(-\lambda\rho c^2\right)\\
&-\frac{\partial^2\bar{h}_{01}}{\partial(ct)^2}+\frac{\partial^2\bar{h}_{01}}{\partial x^2}+\frac{\partial^2\bar{h}_{01}}{\partial y^2}+\frac{\partial^2\bar{h}_{01}}{\partial z^2}=-\frac{16\pi G}{c^4}T_{01}=-\frac{16\pi G}{c^4}\left(\lambda cJ_x\right)\\
&-\frac{\partial^2\bar{h}_{02}}{\partial(ct)^2}+\frac{\partial^2\bar{h}_{02}}{\partial x^2}+\frac{\partial^2\bar{h}_{02}}{\partial y^2}+\frac{\partial^2\bar{h}_{02}}{\partial z^2}=-\frac{16\pi G}{c^4}T_{02}=-\frac{16\pi G}{c^4}\left(\lambda cJ_y\right)\\
&-\frac{\partial^2\bar{h}_{03}}{\partial(ct)^2}+\frac{\partial^2\bar{h}_{03}}{\partial x^2}+\frac{\partial^2\bar{h}_{03}}{\partial y^2}+\frac{\partial^2\bar{h}_{03}}{\partial z^2}=-\frac{16\pi G}{c^4}T_{03}=-\frac{16\pi G}{c^4}\left(\lambda cJ_z\right)
\end{aligned}\qquad\text{(VII-8)}$$

These relationships must be associated with the condition of harmonic gauge:

$$-\frac{\partial\left(\bar{h}_{i0}\right)}{\partial x^0}+\frac{\partial\left(\bar{h}_{i1}\right)}{\partial x^1}+\frac{\partial\left(\bar{h}_{i2}\right)}{\partial x^2}+\frac{\partial\left(\bar{h}_{i3}\right)}{\partial x^3}=0\qquad\text{(VII-9)}$$

The identification of the equations of potential (VII-5) with equations (VII-8) led to ask:

$$\begin{aligned}
&\bar{h}_{00}=-\frac{\phi}{c}\frac{16\pi G}{c^4}\frac{\left(\lambda\rho c^2\right)}{\mu_0\rho c}=-\frac{\phi}{c}\frac{16\pi G}{\mu_0 c^4}\frac{c}{\sqrt{4\pi\varepsilon_0 G}}=-\frac{\phi}{c}\left(\frac{4}{\lambda c}\right)\\
&\bar{h}_{01}=A_x\left(\frac{4}{\lambda c}\right)=\bar{h}_{10}\\
&\bar{h}_{02}=A_y\left(\frac{4}{\lambda c}\right)=\bar{h}_{20}\\
&\bar{h}_{03}=A_z\left(\frac{4}{\lambda c}\right)=\bar{h}_{30}
\end{aligned}\qquad\text{(VII-10)}$$

We can then express the harmonic gauge condition for i = 0:

$$\begin{aligned}
&-\frac{\partial\left(\bar{h}_{00}\right)}{\partial x^0}+\frac{\partial\left(\bar{h}_{01}\right)}{\partial x^1}+\frac{\partial\left(\bar{h}_{02}\right)}{\partial x^2}+\frac{\partial\left(\bar{h}_{03}\right)}{\partial x^3}=0\\
&\frac{\partial(\phi/c)}{\partial(ct)}+\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_x}{\partial x}=\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\vec{\nabla}\vec{A}=0
\end{aligned}\qquad\text{(VII-11)}$$

It corresponds exactly to the LORENZ gauge.

For i = 1, 2, and 3, the approached expression of the tensor of electromagnetic sources, we have proposed in (VII-6), shows that the condition of harmonic gauge is no more verified accurately, but only in an approximate way. This condition is indeed written:

$$\begin{aligned}
\frac{\partial(\bar{h}_{10})}{\partial x^0} &= \frac{\partial(\bar{h}_{10})}{\partial(ct)} = \frac{1}{c}\frac{\partial(\bar{h}_{10})}{\partial t} = 0 \\
\frac{\partial(\bar{h}_{20})}{\partial x^0} &= \frac{\partial(\bar{h}_{20})}{\partial(ct)} = \frac{1}{c}\frac{\partial(\bar{h}_{20})}{\partial t} = 0 \\
\frac{\partial(\bar{h}_{30})}{\partial x^0} &= \frac{\partial(\bar{h}_{30})}{\partial(ct)} = \frac{1}{c}\frac{\partial(\bar{h}_{30})}{\partial t} = 0
\end{aligned} \qquad \text{(VII-12)}$$

The 1/c coefficient ensures that these terms are small, but does not guarantee that they would be zero. We will examine the consequences of this approximation in a later chapter.

Strictly speaking, in the considered approximation, harmonic gauge condition is verified exactly only for potential vectors which are independent of time, that is to say related to the magnetostatic. The scalar potential is not affected by this limitation.

We designate by magnetostatic gauge the condition of jauge defined by the above three equations. If we look at the simplified energy-momentum tensor associated with this gauge:

$$\lambda\begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & 0 & 0 & 0 \\ cJ_y & 0 & 0 & 0 \\ cJ_z & 0 & 0 & 0 \end{pmatrix} \qquad \text{(VII-13)}$$

We notice that the equation of conservation of energy on lines 1, 2, and 3 is written:

$$\begin{aligned}
\frac{\partial J_x}{\partial t} &= 0 \\
\frac{\partial J_y}{\partial t} &= 0 \\
\frac{\partial J_z}{\partial t} &= 0
\end{aligned} \qquad \text{(VII-14)}$$

In other words, the null terms we have injected into the momentum-energy tensor (VII-13) allow us a rigorous exploitation of this tensor only for direct currents.

We can check that the terms which express the change of the metric are a much less than 1 order of magnitude. One estimates for this:

$$\left(\frac{4}{\lambda c}\right) = \frac{\sqrt{64\pi\varepsilon_0 G}}{c} \approx \frac{8\sqrt{3{,}14 * 8{,}85.10^{-12} * 6{,}67.10^{-11}}}{(3.10^8)} \approx 1{,}14.10^{-18} \qquad \text{(VII-13)}$$

The coefficient of the metric which describes the scalar potential is recalled below:

$$\bar{h}_{00} = -\frac{\phi}{c}\left(\frac{4}{\lambda c}\right) \qquad \text{(VII-14)}$$

This term approaches the value 1 (in absolute value) when the potential ϕ approaches value about 10^{25} Volt. One knows to date no experience involving such high potentials, which justifies the used linear approximations.

At this stage, formal identification of MAXWELL's equations in the EINSTEIN equations is completed, but it is not satisfactory because it does not treat the currents variable in time. We will see how overcome this difficulty in a later chapter.

VIII - Metric of the electromagnetic field in the magnetostatic gauge

The previous chapter showed that the electromagnetic field can be represented by small perturbations of the metric of MINKOWSKY, or in an energy interpretation, by small changes in the volumic energy density of the vacuum.

In the magnetostatic gauge, the representative tensor of these variations has been approximated in an expression of the form:

$$\bar{h}_{ij} = \begin{pmatrix} \bar{h}_{00} & \bar{h}_{01} & \bar{h}_{02} & \bar{h}_{03} \\ \bar{h}_{10} & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ \bar{h}_{20} & \bar{h}_{21} & \bar{h}_{22} & \bar{h}_{23} \\ \bar{h}_{30} & \bar{h}_{31} & \bar{h}_{32} & \bar{h}_{33} \end{pmatrix} = \frac{\sqrt{64\pi\varepsilon_0 G}}{c} \begin{pmatrix} -\phi/c & A_x & A_y & A_z \\ A_x & 0 & 0 & 0 \\ A_y & 0 & 0 & 0 \\ A_z & 0 & 0 & 0 \end{pmatrix} \qquad \text{(VIII-1)}$$

These variations are expressed using the perturbations of the modified metric (topped with a bar), whose relations with unmodified perturbations have been established in the chapter on the gauges:

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} \qquad \text{(VIII-3)}$$

With:

$$\bar{h} = \eta^{ij}\bar{h}_{ij} = \eta^{00}\bar{h}_{00} + \eta^{11}\bar{h}_{11} + \eta^{22}\bar{h}_{22} + \eta^{33}\bar{h}_{33} = -\bar{h}_{00} \qquad \text{(VIII-4)}$$

We deduce the expression of the perturbations of the metric unmodified:

$$h_{ij} = \bar{h}_{ij} + \frac{1}{2}\eta_{ij}\bar{h}_{00} \qquad \text{(VIII-5)}$$

Each of the terms can be included in a matrix expression:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} = \begin{pmatrix} \frac{\bar{h}_{00}}{2} & \bar{h}_{01} & \bar{h}_{02} & \bar{h}_{03} \\ \bar{h}_{10} & \frac{\bar{h}_{00}}{2} & 0 & 0 \\ \bar{h}_{20} & 0 & \frac{\bar{h}_{00}}{2} & 0 \\ \bar{h}_{30} & 0 & 0 & \frac{\bar{h}_{00}}{2} \end{pmatrix} = \frac{\sqrt{64\pi\varepsilon_0 G}}{c} \begin{pmatrix} -\frac{\phi}{2c} & A_x & A_y & A_z \\ A_x & -\frac{\phi}{2c} & 0 & 0 \\ A_y & 0 & -\frac{\phi}{2c} & 0 \\ A_z & 0 & 0 & -\frac{\phi}{2c} \end{pmatrix} \quad \text{(VIII-6)}$$

The full coefficients of the metric (the g_{ij}) are obtained by adding the previous small variations (the h_{ij}) to the coefficients of the MINKOWSKY metric (the η_{ij}):

$$g_{ij} = \eta_{ij} + h_{ij} = \begin{pmatrix} -1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & \left(\frac{4}{\lambda c}\right)A_x & \left(\frac{4}{\lambda c}\right)A_y & \left(\frac{4}{\lambda c}\right)A_z \\ \left(\frac{4}{\lambda c}\right)A_x & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & 0 & 0 \\ \left(\frac{4}{\lambda c}\right)A_y & 0 & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & 0 \\ \left(\frac{4}{\lambda c}\right)A_z & 0 & 0 & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} \end{pmatrix} \quad \lambda = \frac{1}{\sqrt{4\pi\varepsilon_0 G}} \quad \text{(VIII-7)}$$

We can deduce the squared element of space-time according to the relationship:

$$ds^2 = g_{ij}dx^i dx^j \quad \text{(VIII-8)}$$

We get developed writing as a function of the only g_{ij} not null:

$$ds^2 = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 + 2g_{01}dx^0dx^1 + 2g_{02}dx^0dx^2 + 2g_{03}dx^0dx^3 \quad \text{(VIII-9)}$$

And finally, the expression of the metric based on the electromagnetic potential:

$$\begin{aligned} ds^2 &= \left(-1-\frac{2\phi}{\lambda c^2}\right)d(ct)^2 + \left(1-\frac{2\phi}{\lambda c^2}\right)\left(dx^2+dy^2+dz^2\right) \\ &+ 2\left(\frac{4A_x}{\lambda c}\right)d(ct)dx + 2\left(\frac{4A_y}{\lambda c}\right)d(ct)dy + 2\left(\frac{4A_z}{\lambda c}\right)d(ct)dz \end{aligned} \quad \text{(VIII-10)}$$

We conclude that the metric of space-time can be expressed as a function of the electromagnetic potential, in an analogous way to the gravitational phenomena.

IX - Analysis of the relationships obtained in magnetostatic gauge

In the previous chapters, we have been led to cancel some terms, to allow an identification of certain coefficients of the perturbed metric with the electromagnetic potential. We observed that this approximation had broken the exact condition of harmonic gauge. The complexity of the EINSTEIN equations suggest us to examine in detail the consequences of this approximation.

In this chapter, we propose to derive a direct and detailed calculation of the coefficients of the RICCI tensor, in order to evaluate the consequences of the assumptions that we have proposed on each term.

I – Reminder on the working assumptions

For this check, we will use the following form of the EINSTEIN equation:

$$R_{ij} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2}g_{ij}T\right) \tag{IX-1}$$

We define the MINKOWSKY metric by the η_{ij} coefficients:

$$\eta_{ij} = \eta^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{IX-2}$$

This metric experiences small perturbations h_{ij}, and one can put in a matrix representation:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} \quad \text{avec} \quad \left|h_{ij}\right| << 1 \tag{IX-3}$$

We can deduce the matrix representation of the metric associated with the electromagnetic potential:

$$g_{ij} = \eta_{ij} + h_{ij} = \begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \tag{IX-4}$$

In the previous chapters, we have proposed an energy-momentum tensor T_{ij} of the sources of the electromagnetic field. By cancelling the minor terms, we obtained an approximate expression that is recalled below:

$$T_{ij} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} = \frac{1}{\sqrt{4\pi\varepsilon_0 G}} \begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & 0 & 0 & 0 \\ cJ_y & 0 & 0 & 0 \\ cJ_z & 0 & 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} -\rho c^2 & cJ_x & cJ_y & cJ_z \\ cJ_x & 0 & 0 & 0 \\ cJ_y & 0 & 0 & 0 \\ cJ_z & 0 & 0 & 0 \end{pmatrix} \tag{IX-5}$$

We can evaluate an approached contraction of this tensor:

$$T = g^{ij}T_{ij} \approx \eta^{ij}T_{ij} = \eta^{00}T_{00} + \eta^{11}T_{11} + \eta^{22}T_{22} + \eta^{33}T_{33} = \eta^{00}T_{00} = -T_{00} \tag{IX-6}$$

We assumed that the modified metric coefficients are related to the electromagnetic potentials in the following way:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} = \frac{\sqrt{64\pi\varepsilon_0 G}}{c} \begin{pmatrix} -\frac{\phi}{2c} & A_x & A_y & A_z \\ A_x & -\frac{\phi}{2c} & 0 & 0 \\ A_y & 0 & -\frac{\phi}{2c} & 0 \\ A_z & 0 & 0 & -\frac{\phi}{2c} \end{pmatrix} \tag{IX-7}$$

In calculations of terms of the RICCI tensor, we will use a few properties that are apparent in the coefficients of the h_{ij} tensor which is given above:

1) It is a symmetric tensor: $h_{ij} = h_{ji}$.
2) Several terms of this tensor are supposed have value zero, to first approximation:

$$h_{12} = h_{13} = h_{21} = h_{23} = h_{31} = h_{32} = 0 \tag{IX-8}$$

3) The terms of the main diagonal are equal:

$$h_{00} = h_{11} = h_{22} = h_{33} \tag{IX-9}$$

4) Representative potential terms must respect the LORENZ gauge:

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\vec{\nabla}\vec{A}=\frac{\partial(\phi/c)}{\partial(ct)}+\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_x}{\partial x}=-2\frac{\partial h_{00}}{\partial x^0}+\frac{\partial h_{01}}{\partial x^1}+\frac{\partial h_{02}}{\partial x^2}+\frac{\partial h_{03}}{\partial x^3}=0 \qquad \text{(IX-10)}$$

We recall the expression of the multiplicative constant λ:

$$\lambda=\frac{1}{\sqrt{4\pi\varepsilon_0 G}} \qquad \text{(IX-11)}$$

The terms of the modified metric h_{ij} are therefore written using the constant λ:

$$h_{ij}=\begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03}\\ h_{10} & h_{11} & h_{12} & h_{13}\\ h_{20} & h_{21} & h_{22} & h_{23}\\ h_{30} & h_{31} & h_{32} & h_{33}\end{pmatrix}=\frac{4}{\lambda c}\begin{pmatrix} -\frac{\phi}{2c} & A_x & A_y & A_z\\ A_x & -\frac{\phi}{2c} & 0 & 0\\ A_y & 0 & -\frac{\phi}{2c} & 0\\ A_z & 0 & 0 & -\frac{\phi}{2c}\end{pmatrix} \qquad \text{(IX-12)}$$

We deduce the coefficients of the metric associated with electromagnetic potential:

$$g_{ij}=\eta_{ij}+h_{ij}=\begin{pmatrix} -1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & \left(\frac{4}{\lambda c}\right)A_x & \left(\frac{4}{\lambda c}\right)A_y & \left(\frac{4}{\lambda c}\right)A_z\\ \left(\frac{4}{\lambda c}\right)A_x & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & 0 & 0\\ \left(\frac{4}{\lambda c}\right)A_y & 0 & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c} & 0\\ \left(\frac{4}{\lambda c}\right)A_z & 0 & 0 & 1-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c}\end{pmatrix} \qquad \text{(IX-13)}$$

We now have all the elements allowing a direct verification of the EINSTEIN equations.

II – The EINSTEIN equations

We adopt the following representation of the RICCI tensor:

$$R_{ij}=\Gamma^r_{mr}\Gamma^m_{ij}-\Gamma^r_{mj}\Gamma^m_{ir}+\frac{\partial\Gamma^r_{ij}}{\partial x^r}-\frac{\partial\Gamma^r_{ir}}{\partial x^j} \qquad \text{(IX-14)}$$

In a linearized framework, we admit that products of the CHRISTOFFEL coefficients can be neglected as an infinitely small amount of second order. The terms of the RICCI tensor will be evaluated using only the expression involving derivatives of CHRISTOFFEL coefficients:

$$R_{ij} \approx \frac{\partial \Gamma^r_{ij}}{\partial x^r} - \frac{\partial \Gamma^r_{ir}}{\partial x^j} = \left\{ \frac{\partial \Gamma^0_{ij}}{\partial x^0} + \frac{\partial \Gamma^1_{ij}}{\partial x^1} + \frac{\partial \Gamma^2_{ij}}{\partial x^2} + \frac{\partial \Gamma^3_{ij}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{i0}}{\partial x^j} + \frac{\partial \Gamma^1_{i1}}{\partial x^j} + \frac{\partial \Gamma^2_{i2}}{\partial x^j} + \frac{\partial \Gamma^3_{i3}}{\partial x^j} \right\} \tag{IX-15}$$

The general expression of these coefficients is:

$$\Gamma^r_{ij} = \frac{1}{2} g^{rk} \left\{ \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right\} \tag{IX-16}$$

Their writing developed following the dummy variable k is given by:

$$\begin{aligned} \Gamma^r_{ij} &= \frac{1}{2} g^{r0} \left\{ \frac{\partial g_{i0}}{\partial x^j} + \frac{\partial g_{j0}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^0} \right\} + \frac{1}{2} g^{r1} \left\{ \frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1} \right\} \\ &+ \frac{1}{2} g^{r2} \left\{ \frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2} \right\} + \frac{1}{2} g^{r3} \left\{ \frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3} \right\} \end{aligned} \tag{IX-17}$$

The very particular shape of the $g_{ij} = \eta_{ij} + h_{ij}$, with η_{ij} equal to a constant (+ 1 or - 1) and: $|h_{ij}| <<1$ shows that this expression becomes in a linearized framework:

$$\begin{aligned} \Gamma^r_{ij} &\approx \frac{1}{2} \eta^{r0} \left\{ \frac{\partial h_{i0}}{\partial x^j} + \frac{\partial h_{j0}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^0} \right\} + \frac{1}{2} \eta^{r1} \left\{ \frac{\partial h_{i1}}{\partial x^j} + \frac{\partial h_{j1}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^1} \right\} \\ &+ \frac{1}{2} \eta^{r2} \left\{ \frac{\partial h_{i2}}{\partial x^j} + \frac{\partial h_{j2}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^2} \right\} + \frac{1}{2} \eta^{r3} \left\{ \frac{\partial h_{i3}}{\partial x^j} + \frac{\partial h_{j3}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^3} \right\} \end{aligned} \tag{IX-18}$$

Since the only η^{rk} not null are those for which r = k, we obtain the final expression that will be used in the calculations:

$$\Gamma^r_{ij} \approx \frac{1}{2} \eta^{rr} \left\{ \frac{\partial h_{ir}}{\partial x^j} + \frac{\partial h_{jr}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^r} \right\} \tag{IX-19}$$

We get explicitly for each value of r:

$$\begin{aligned} \Gamma^0_{ij} &\approx \frac{1}{2} \eta^{00} \left\{ \frac{\partial h_{i0}}{\partial x^j} + \frac{\partial h_{j0}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^0} \right\} \\ \Gamma^1_{ij} &\approx \frac{1}{2} \eta^{11} \left\{ \frac{\partial h_{i1}}{\partial x^j} + \frac{\partial h_{j1}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^1} \right\} \\ \Gamma^2_{ij} &\approx \frac{1}{2} \eta^{22} \left\{ \frac{\partial h_{i2}}{\partial x^j} + \frac{\partial h_{j2}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^2} \right\} \\ \Gamma^3_{ij} &\approx \frac{1}{2} \eta^{33} \left\{ \frac{\partial h_{i3}}{\partial x^j} + \frac{\partial h_{j3}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^3} \right\} \end{aligned} \tag{IX-20}$$

The check will now consist in injecting these coefficients in the linearized RICCI tensor recalled in (IX-15).

III – The term R_{00}

From the expression of the linearized RICCI tensor (IX-15), we get for i = 0 and j = 0:

$$R_{00} \approx \frac{\partial \Gamma^r_{00}}{\partial x^r} - \frac{\partial \Gamma^r_{0r}}{\partial x^0} = \left\{ \frac{\partial \Gamma^0_{00}}{\partial x^0} + \frac{\partial \Gamma^1_{00}}{\partial x^1} + \frac{\partial \Gamma^2_{00}}{\partial x^2} + \frac{\partial \Gamma^3_{00}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{00}}{\partial x^0} + \frac{\partial \Gamma^1_{01}}{\partial x^0} + \frac{\partial \Gamma^2_{02}}{\partial x^0} + \frac{\partial \Gamma^3_{03}}{\partial x^0} \right\} \quad \text{(IX-21)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{00} are deducted from relations (IX-20), and expressed below based on the only not null h_{ij}:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{00}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{00}}{\partial x^0} \right\} \\
\Gamma^1_{00} &= \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{01}}{\partial x^0} + \frac{\partial h_{01}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^1} \right\} = \frac{1}{2}\left\{ 2\frac{\partial h_{01}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^1} \right\} \\
\Gamma^2_{00} &= \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{02}}{\partial x^0} + \frac{\partial h_{02}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^2} \right\} = \frac{1}{2}\left\{ 2\frac{\partial h_{02}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^2} \right\} \\
\Gamma^3_{00} &= \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{03}}{\partial x^0} + \frac{\partial h_{03}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^3} \right\} = \frac{1}{2}\left\{ 2\frac{\partial h_{03}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^3} \right\}
\end{aligned} \quad \text{(IX-22)}$$

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{00}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{00}}{\partial x^0} \right\} \\
\Gamma^1_{01} &= \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^0} \right\} \\
\Gamma^2_{02} &= \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{02}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^0} - \frac{\partial h_{02}}{\partial x^2} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{22}}{\partial x^0} \right\} \\
\Gamma^3_{03} &= \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{03}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^0} - \frac{\partial h_{03}}{\partial x^3} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{33}}{\partial x^0} \right\}
\end{aligned} \quad \text{(IX-23)}$$

The detailed above CHRISTOFFEL coefficients are now introduced in the expression of R_{00} recalled to memory:

$$R_{00} \approx \left\{ \frac{\partial \Gamma^0_{00}}{\partial x^0} + \frac{\partial \Gamma^1_{00}}{\partial x^1} + \frac{\partial \Gamma^2_{00}}{\partial x^2} + \frac{\partial \Gamma^3_{00}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{00}}{\partial x^0} + \frac{\partial \Gamma^1_{01}}{\partial x^0} + \frac{\partial \Gamma^2_{02}}{\partial x^0} + \frac{\partial \Gamma^3_{03}}{\partial x^0} \right\}$$

$$R_{00} \approx \frac{1}{2}\left\{ 2\frac{\partial^2 h_{01}}{\partial x^1 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^1 \partial x^1} + 2\frac{\partial^2 h_{02}}{\partial x^2 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^2 \partial x^2} + 2\frac{\partial^2 h_{03}}{\partial x^3 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^3 \partial x^3} - \frac{\partial^2 h_{00}}{\partial x^0 \partial x^0} \right\}$$
$$-\frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{22}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{33}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^0 \partial x^0} \right\}$$

(IX-24)

We get after a reorganization of the terms:

$$R_{00} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{\partial x^1 \partial x^1} + \frac{\partial^2 h_{00}}{\partial x^2 \partial x^2} + \frac{\partial^2 h_{00}}{\partial x^3 \partial x^3} - \frac{\partial^2 h_{00}}{\partial x^0 \partial x^0}\right\} + \left\{\frac{\partial^2 h_{01}}{\partial x^1 \partial x^0} + \frac{\partial^2 h_{02}}{\partial x^2 \partial x^0} + \frac{\partial^2 h_{03}}{\partial x^3 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^0 \partial x^0}\right\}$$
$$-\frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{22}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{33}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{00}}{\partial x^0 \partial x^0}\right\}$$
$$R_{00} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{(\partial x^1)^2} + \frac{\partial^2 h_{00}}{(\partial x^2)^2} + \frac{\partial^2 h_{00}}{(\partial x^3)^2} - \frac{\partial^2 h_{00}}{(\partial x^0)^2}\right\} +$$
$$\frac{\partial}{\partial x^0}\left[\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{02}}{\partial x^2} + \frac{\partial h_{03}}{\partial x^3} - \frac{\partial h_{00}}{\partial x^0}\right\} - \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^0} + \frac{\partial h_{22}}{\partial x^0} + \frac{\partial h_{33}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^0}\right\}\right]$$
(IX-25)

Making use of the relationship $h_{00} = h_{11} = h_{22} = h_{33}$, it comes:

$$R_{00} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{(\partial x^1)^2} + \frac{\partial^2 h_{00}}{(\partial x^2)^2} + \frac{\partial^2 h_{00}}{(\partial x^3)^2} - \frac{\partial^2 h_{00}}{(\partial x^0)^2}\right\} + \frac{\partial}{\partial x^0}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{02}}{\partial x^2} + \frac{\partial h_{03}}{\partial x^3} - 2\frac{\partial h_{00}}{\partial x^0}\right\}$$
(IX-26)

The second term on the right of equality is null because it expresses the conservation of energy (IX-10). It remains finally only the d'Alembertian of h_{00}, to a multiplication factor close:

$$R_{00} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{(\partial x^1)^2} + \frac{\partial^2 h_{00}}{(\partial x^2)^2} + \frac{\partial^2 h_{00}}{(\partial x^3)^2} - \frac{\partial^2 h_{00}}{(\partial x^0)^2}\right\} \qquad \text{(IX-27)}$$

It appears that the simplifying assumptions that led to cancel the coefficients h_{12}, h_{13}, h_{21}, h_{23}, h_{31}, h_{32}, are without affecting the determination of the term R_{00}. In other words, the condition of magnetostatic gauge is not necessary for the term R_{00} strictly represents the wave equation relative to the scalar potential.

We must now make the connection with the energy momentum tensor using the EINSTEIN equation (IX-1). With i = 0 and j = 0, we have:

$$R_{00} = \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}g_{00}T\right) \approx \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}\eta_{00}T\right) = \frac{8\pi G}{c^4}\left(T_{00} + \frac{1}{2}T\right) = \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}T_{00}\right)$$
$$R_{00} = \frac{8\pi G}{c^4}\frac{T_{00}}{2} = \frac{8\pi G}{c^4}\left(\frac{-\lambda\rho c^2}{2}\right)$$
(IX-28)

We get so explicitly:

$$R_{00} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{\left(\partial x^1\right)^2}+\frac{\partial^2 h_{00}}{\left(\partial x^2\right)^2}+\frac{\partial^2 h_{00}}{\left(\partial x^3\right)^2}-\frac{\partial^2 h_{00}}{\left(\partial x^0\right)^2}\right\}=\frac{8\pi G}{c^4}\left(\frac{-\lambda\rho c^2}{2}\right)$$

$$R_{00} \approx \left\{\frac{\partial^2 h_{00}}{\left(\partial x^1\right)^2}+\frac{\partial^2 h_{00}}{\left(\partial x^2\right)^2}+\frac{\partial^2 h_{00}}{\left(\partial x^3\right)^2}-\frac{\partial^2 h_{00}}{\left(\partial x^0\right)^2}\right\}=\frac{8\pi G}{c^4}\left(\lambda\rho c^2\right) \qquad \text{with} \quad \lambda=\frac{1}{\sqrt{4\pi\varepsilon_0 G}}, \quad h_{00}=-\left(\frac{4}{\lambda c}\right)\frac{\phi}{2c}=-\frac{2\phi}{\lambda c^2}$$

$$R_{00} \approx \left\{\frac{\partial^2 h_{00}}{\partial x^2}+\frac{\partial^2 h_{00}}{\partial y^2}+\frac{\partial^2 h_{00}}{\partial z^2}-\frac{\partial^2 h_{00}}{\partial (ct)^2}\right\}=\frac{8\pi G}{c^4}\left(\lambda\rho c^2\right)$$

(IX-29)

The substitution of h_{00} as a function of ϕ led to the final expression of the wave equation on the scalar potential ϕ:

$$\left\{\frac{\partial^2(\phi/c)}{\partial x^2}+\frac{\partial^2(\phi/c)}{\partial y^2}+\frac{\partial^2(\phi/c)}{\partial z^2}-\frac{\partial^2(\phi/c)}{\partial (ct)^2}\right\}=-\frac{8\pi G}{c^4}\left(\lambda\rho c^2\right)\frac{\lambda c}{2}$$

$$\left\{\frac{\partial^2(\phi/c)}{\partial x^2}+\frac{\partial^2(\phi/c)}{\partial y^2}+\frac{\partial^2(\phi/c)}{\partial z^2}-\frac{\partial^2(\phi/c)}{\partial (ct)^2}\right\}=-\left(\rho c^2\right)\frac{4\pi G}{c^4}\frac{c}{4\pi\varepsilon_0 G}$$

$$\left\{\frac{\partial^2(\phi/c)}{\partial x^2}+\frac{\partial^2(\phi/c)}{\partial y^2}+\frac{\partial^2(\phi/c)}{\partial z^2}-\frac{\partial^2(\phi/c)}{\partial (ct)^2}\right\}=-\mu_0\rho c \qquad \text{(IX-30)}$$

$$\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\phi}{\partial (ct)^2}\right\}=-\mu_0\rho c^2=-\frac{\rho}{\varepsilon_0}$$

IV – The term R_{01} (and analogous terms R_{02}, R_{03}, R_{10}, R_{20}, R_{30})

From the expression of the linearized RICCI tensor (IX-15), we get for i = 0 and j = 1:

$$R_{01} \approx \frac{\partial\Gamma^r_{01}}{\partial x^r}-\frac{\partial\Gamma^r_{0r}}{\partial x^1}=\left\{\frac{\partial\Gamma^0_{01}}{\partial x^0}+\frac{\partial\Gamma^1_{01}}{\partial x^1}+\frac{\partial\Gamma^2_{01}}{\partial x^2}+\frac{\partial\Gamma^3_{01}}{\partial x^3}\right\}-\left\{\frac{\partial\Gamma^0_{00}}{\partial x^1}+\frac{\partial\Gamma^1_{01}}{\partial x^1}+\frac{\partial\Gamma^2_{02}}{\partial x^1}+\frac{\partial\Gamma^3_{03}}{\partial x^1}\right\} \qquad \text{(IX-31)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{01} are deducted from relationships (IX-20) and listed below:

$$\Gamma^0_{01} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{00}}{\partial x^1}+\frac{\partial h_{10}}{\partial x^0}-\frac{\partial h_{01}}{\partial x^0}\right\}=-\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^1}\right\}$$

$$\Gamma^1_{01} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1}+\frac{\partial h_{11}}{\partial x^0}-\frac{\partial h_{01}}{\partial x^1}\right\}=\frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^0}\right\}$$

$$\Gamma^2_{01} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{02}}{\partial x^1}+\frac{\partial h_{12}}{\partial x^0}-\frac{\partial h_{01}}{\partial x^2}\right\}=\frac{1}{2}\left\{\frac{\partial h_{02}}{\partial x^1}-\frac{\partial h_{01}}{\partial x^2}\right\} \qquad \text{(IX-32)}$$

$$\Gamma^3_{01} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{03}}{\partial x^1}+\frac{\partial h_{13}}{\partial x^0}-\frac{\partial h_{01}}{\partial x^3}\right\}=\frac{1}{2}\left\{\frac{\partial h_{03}}{\partial x^1}-\frac{\partial h_{01}}{\partial x^3}\right\}$$

$$\Gamma_{00}^{0} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{00}}{\partial x^0}+\frac{\partial h_{00}}{\partial x^0}-\frac{\partial h_{00}}{\partial x^0}\right\}=-\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^0}\right\}$$

$$\Gamma_{01}^{1} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1}+\frac{\partial h_{11}}{\partial x^0}-\frac{\partial h_{01}}{\partial x^1}\right\}=\frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^0}\right\}$$

$$\Gamma_{02}^{2} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{02}}{\partial x^2}+\frac{\partial h_{22}}{\partial x^0}-\frac{\partial h_{02}}{\partial x^2}\right\}=\frac{1}{2}\left\{\frac{\partial h_{22}}{\partial x^0}\right\} \qquad \text{(IX-33)}$$

$$\Gamma_{03}^{3} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{03}}{\partial x^3}+\frac{\partial h_{33}}{\partial x^0}-\frac{\partial h_{03}}{\partial x^3}\right\}=\frac{1}{2}\left\{\frac{\partial h_{33}}{\partial x^0}\right\}$$

The detailed above CHRISTOFFEL coefficients are now introduced in the expression of R_{01} recalled to memory:

$$R_{01} \approx \frac{\partial \Gamma_{01}^{r}}{\partial x^r}-\frac{\partial \Gamma_{0r}^{r}}{\partial x^1}=\left\{\frac{\partial \Gamma_{01}^{0}}{\partial x^0}+\frac{\partial \Gamma_{01}^{1}}{\partial x^1}+\frac{\partial \Gamma_{01}^{2}}{\partial x^2}+\frac{\partial \Gamma_{01}^{3}}{\partial x^3}\right\}-\left\{\frac{\partial \Gamma_{00}^{0}}{\partial x^1}+\frac{\partial \Gamma_{01}^{1}}{\partial x^1}+\frac{\partial \Gamma_{02}^{2}}{\partial x^1}+\frac{\partial \Gamma_{03}^{3}}{\partial x^1}\right\}$$

$$R_{01} \approx \frac{1}{2}\left\{-\frac{\partial^2 h_{00}}{\partial x^0\partial x^1}+\frac{\partial^2 h_{11}}{\partial x^1\partial x^0}+\frac{\partial^2 h_{02}}{\partial x^1\partial x^2}-\frac{\partial^2 h_{01}}{\partial x^2\partial x^2}+\frac{\partial^2 h_{03}}{\partial x^1\partial x^3}-\frac{\partial^2 h_{01}}{\partial x^3\partial x^3}\right\} \qquad \text{(IX-34)}$$

$$-\frac{1}{2}\left\{-\frac{\partial^2 h_{00}}{\partial x^0\partial x^1}+\frac{\partial^2 h_{11}}{\partial x^0\partial x^1}+\frac{\partial^2 h_{22}}{\partial x^0\partial x^1}+\frac{\partial^2 h_{33}}{\partial x^0\partial x^1}\right\}$$

We obtained after a reorganization of the terms, and by using the relationship $h_{00} = h_{11} = h_{22} = h_{33}$:

$$R_{01} \approx \frac{1}{2}\left(-\frac{\partial^2 h_{01}}{\partial x^1\partial x^1}-\frac{\partial^2 h_{01}}{\partial x^2\partial x^2}-\frac{\partial^2 h_{01}}{\partial x^3\partial x^3}\right)+\frac{1}{2}\left\{\frac{\partial^2 h_{01}}{\partial x^1\partial x^1}+\frac{\partial^2 h_{02}}{\partial x^1\partial x^2}+\frac{\partial^2 h_{03}}{\partial x^1\partial x^3}-2\frac{\partial^2 h_{00}}{\partial x^0\partial x^1}\right\}$$

$$R_{01} \approx \frac{1}{2}\left(-\frac{\partial^2 h_{01}}{\partial x^1\partial x^1}-\frac{\partial^2 h_{01}}{\partial x^2\partial x^2}-\frac{\partial^2 h_{01}}{\partial x^3\partial x^3}\right)+\frac{1}{2}\frac{\partial}{\partial x^1}\left\{\frac{\partial h_{01}}{\partial x^1}+\frac{\partial h_{02}}{\partial x^2}+\frac{\partial h_{03}}{\partial x^3}-2\frac{\partial h_{00}}{\partial x^0}\right\}$$

(IX-35)

The condition of harmonic gauge cancels the second term right of equality, and it remains:

$$R_{01} \approx \frac{1}{2}\left(-\frac{\partial^2 h_{01}}{\partial x^1\partial x^1}-\frac{\partial^2 h_{01}}{\partial x^2\partial x^2}-\frac{\partial^2 h_{01}}{\partial x^3\partial x^3}\right) \qquad \text{(IX-36)}$$

or still by showing the Laplacian generalized at the time variable:

$$R_{01} \approx \frac{1}{2}\left(\frac{\partial^2 h_{01}}{\partial x^0\partial x^0}-\frac{\partial^2 h_{01}}{\partial x^1\partial x^1}-\frac{\partial^2 h_{01}}{\partial x^2\partial x^2}-\frac{\partial^2 h_{01}}{\partial x^3\partial x^3}\right)-\frac{1}{2}\left\{\frac{\partial^2 h_{01}}{\partial x^0\partial x^0}\right\} \qquad \text{(IX-37)}$$

The condition of harmonic gauge modified by the null terms that we introduced in the metric of the h_{ij}, designated by condition of gauge magnetostatics, is recalled to memory:

$$\frac{\partial\left(\bar{h}_{10}\right)}{\partial x^0} = \frac{\partial\left(h_{10}\right)}{\partial x^0} = 0 \qquad \text{(IX-38)}$$

It turns out that the wave equation relative to the vector potential is rigorously checked only if the condition of magnetostatic gauge is true, i.e. if the time dependence of the potential vector is zero. At this condition, the term R_{01} described exactly the space Laplacian of the component of the potential vector along the x axis.

We can make the link with the energy momentum tensor using the EINSTEIN equation (IX-1). With i = 0 and j = 1, we have:

$$R_{01} = \frac{8\pi G}{c^4}\left(T_{01} - \frac{1}{2}g_{01}T\right) \approx \frac{8\pi G}{c^4}\left(T_{01} - \frac{1}{2}\eta_{01}T\right) = \frac{8\pi G}{c^4}T_{01} = \frac{8\pi G}{c^4}\left(\lambda c J_x\right) \qquad \text{(IX-39)}$$

We get explicitly:

$$R_{01} \approx \frac{1}{2}\left\{\frac{\partial^2 h_{01}}{\left(\partial x^0\right)^2} - \frac{\partial^2 h_{01}}{\left(\partial x^1\right)^2} - \frac{\partial^2 h_{01}}{\left(\partial x^2\right)^2} - \frac{\partial^2 h_{01}}{\left(\partial x^3\right)^2}\right\} = \frac{8\pi G}{c^4}\left(\lambda c J_x\right)$$

$$R_{01} \approx \left\{\frac{\partial^2 h_{01}}{\partial (ct)^2} - \frac{\partial^2 h_{01}}{\partial x^2} - \frac{\partial^2 h_{01}}{\partial y^2} - \frac{\partial^2 h_{01}}{\partial z^2}\right\} = \frac{16\pi G}{c^4}\left(\lambda c J_x\right)$$

with

$$\lambda = \frac{1}{\sqrt{4\pi\varepsilon_0 G}} \qquad h_{01} = \left(\frac{4}{\lambda c}\right)A_x \qquad \text{(IX-40)}$$

Substitution of h_{01} as a function of A_x leads to the final expression of the wave equation on the component along x of the vector potential:

$$\left\{-\frac{\partial^2 A_x}{\partial (ct)^2} + \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right\} = -\frac{16\pi G}{c^4}\left(\lambda c J_x\right)\frac{\lambda c}{4}$$

$$\left\{-\frac{\partial^2 A_x}{\partial (ct)^2} + \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right\} = -\frac{4\pi G}{c^2}\lambda^2 J_x = -\frac{4\pi G}{c^2}\frac{1}{4\pi\varepsilon_0 G}J_x = -\frac{J_x}{c^2\varepsilon_0} = -\mu_0 J_x \qquad \text{(IX-41)}$$

V – The term R_{11} (and analogous terms R_{22}, R_{33})

From the expression of the linearized RICCI tensor (IX-15), we obtain with i = 1 and j = 1:

$$R_{11} \approx \frac{\partial \Gamma^r_{11}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^1} = \left\{\frac{\partial \Gamma^0_{11}}{\partial x^0} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{11}}{\partial x^2} + \frac{\partial \Gamma^3_{11}}{\partial x^3}\right\} - \left\{\frac{\partial \Gamma^0_{10}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^1} + \frac{\partial \Gamma^3_{13}}{\partial x^1}\right\} \qquad \text{(IX-42)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{11} are deducted from relationships (IX-20) and listed below:

$$\Gamma^0_{11} = \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{10}}{\partial x^1} + \frac{\partial h_{10}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^0}\right\} = -\frac{1}{2}\left\{2\frac{\partial h_{10}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^0}\right\}$$

$$\Gamma^1_{11} = \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^1}\right\}$$

$$\Gamma^2_{11} = \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{12}}{\partial x^1} + \frac{\partial h_{12}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^2}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^2}\right\} \qquad \text{(IX-43)}$$

$$\Gamma^3_{11} = \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{13}}{\partial x^1} + \frac{\partial h_{13}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^3}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^3}\right\}$$

$$\Gamma^0_{10} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^1} - \frac{\partial h_{10}}{\partial x^0}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^1}\right\}$$

$$\Gamma^1_{11} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^1}\right\}$$

$$\Gamma^2_{12} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2}\right\} = \frac{1}{2}\left\{\frac{\partial h_{22}}{\partial x^1}\right\} \qquad \text{(IX-44)}$$

$$\Gamma^3_{13} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{13}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^1} - \frac{\partial h_{13}}{\partial x^3}\right\} = \frac{1}{2}\left\{\frac{\partial h_{33}}{\partial x^1}\right\}$$

The detailed above CHRISTOFFEL coefficients are now introduced in the expression of R_{11} recalled to memory:

$$R_{11} \approx \frac{\partial \Gamma^r_{11}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^1} = \left\{\frac{\partial \Gamma^0_{11}}{\partial x^0} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{11}}{\partial x^2} + \frac{\partial \Gamma^3_{11}}{\partial x^3}\right\} - \left\{\frac{\partial \Gamma^0_{10}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^1} + \frac{\partial \Gamma^3_{13}}{\partial x^1}\right\}$$

$$R_{11} \approx -\frac{1}{2}\left\{2\frac{\partial^2 h_{10}}{\partial x^1 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0}\right\} + \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^1 \partial x^1}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^2 \partial x^2}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^3 \partial x^3}\right\} \qquad \text{(IX-45)}$$

$$+\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{\partial x^1 \partial x^1}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^1 \partial x^1}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{22}}{\partial x^1 \partial x^1}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{33}}{\partial x^1 \partial x^1}\right\}$$

Making use of the relationship $h_{00} = h_{11} = h_{22} = h_{33}$, we get:

$$R_{11} \approx \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{11}}{\partial x^3 \partial x^3}\right\} - \left\{\frac{\partial^2 h_{10}}{\partial x^1 \partial x^0}\right\} \qquad \text{(IX-46)}$$

As in the previous paragraph, the term R_{11} describes rigorously the wave equation relative to the potential scalar only when the condition of magnetostatic gauge is verified.

VI – The term R_{12} (and the analogous terms R_{13}, R_{21}, R_{23}, R_{31}, R_{32})

From the expression of the linearized RICCI tensor (IX-15), we obtain with i = 1 and j = 2:

$$R_{12} \approx \frac{\partial \Gamma^r_{12}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^2} = \left\{ \frac{\partial \Gamma^0_{12}}{\partial x^0} + \frac{\partial \Gamma^1_{12}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{12}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{10}}{\partial x^2} + \frac{\partial \Gamma^1_{11}}{\partial x^2} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{13}}{\partial x^2} \right\} \qquad \text{(IX-47)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{12} are deducted from relationships (IX-20) and listed below:

$$\begin{aligned}
\Gamma^0_{12} &= \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{10}}{\partial x^2} + \frac{\partial h_{20}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{10}}{\partial x^2} + \frac{\partial h_{20}}{\partial x^1} \right\} \\
\Gamma^1_{12} &= \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{11}}{\partial x^2} + \frac{\partial h_{21}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^2} \right\} \\
\Gamma^2_{12} &= \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{22}}{\partial x^1} \right\} \\
\Gamma^3_{12} &= \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{13}}{\partial x^2} + \frac{\partial h_{23}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^3} \right\} = 0
\end{aligned} \qquad \text{(IX-48)}$$

$$\begin{aligned}
\Gamma^0_{10} &\approx \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^1} - \frac{\partial h_{10}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{00}}{\partial x^1} \right\} \\
\Gamma^1_{11} &\approx \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^1} \right\} \\
\Gamma^2_{12} &\approx \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{22}}{\partial x^1} \right\} \\
\Gamma^3_{13} &\approx \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{13}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^1} - \frac{\partial h_{13}}{\partial x^3} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{33}}{\partial x^1} \right\}
\end{aligned} \qquad \text{(IX-49)}$$

The detailed above CHRISTOFFEL coefficients are now introduced in the expression of R_{12} recalled to memory:

$$\begin{aligned}
R_{12} &\approx \frac{\partial \Gamma^r_{12}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^2} = \left\{ \frac{\partial \Gamma^0_{12}}{\partial x^0} + \frac{\partial \Gamma^1_{12}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{12}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{10}}{\partial x^2} + \frac{\partial \Gamma^1_{11}}{\partial x^2} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{13}}{\partial x^2} \right\} \\
R_{12} &\approx -\frac{1}{2}\left\{ \frac{\partial^2 h_{10}}{\partial x^0 \partial x^2} + \frac{\partial^2 h_{20}}{\partial x^0 \partial x^1} \right\} + \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^1 \partial x^2} \right\} + \frac{1}{2}\left\{ \frac{\partial^2 h_{22}}{\partial x^1 \partial x^2} \right\} \\
&+ \frac{1}{2}\left\{ \frac{\partial^2 h_{00}}{\partial x^1 \partial x^2} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^1 \partial x^2} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{22}}{\partial x^1 \partial x^2} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{33}}{\partial x^1 \partial x^2} \right\}
\end{aligned} \qquad \text{(IX-50)}$$

Making use of the relationship $h_{00} = h_{11} = h_{22} = h_{33}$, we get finally:

$$R_{12} \approx -\frac{1}{2}\left\{ \frac{\partial^2 h_{10}}{\partial x^0 \partial x^2} + \frac{\partial^2 h_{20}}{\partial x^0 \partial x^1} \right\} \qquad \text{(IX-51)}$$

It appears that the term R_{12} of the RICCI tensor is exactly zero only in the condition of magnetostatic gauge, i.e. when the time derivatives of the potential vector is null.

IV – Conclusion

Linearized gravity equations provide a framework in which we tried to bring up MAXWELL's equations.

When trying an energy approach assuming a momentum-energy tensor for a moving charge perfectly analogous to the momentum-energy tensor that one knows for a moving mass, we face several challenges.

MAXWELL's equations indicate us that densities of charges are at the origin of the electromagnetic field. These densities form a four-vector, and have therefore 4 components, while a symmetric momentum-energy tensor has 10. So, we have assumed that certain components of the momentum-energy tensor can be neglected in a first approximation.

When we set up this hypothesis, we make the observation that the harmonic gauge condition, which led to the wave equation from the EINSTEIN linearized equations, is no more rigorously checked.

So we must admit that the simplified framework in which we hoped to see MAXWELL's equations does not allow a correct identification. So we must continue the work of this chapter by keeping all terms in the momentum-energy tensor: this will be the subject of the next chapter.

X – general case

The previous chapter showed us that the simplifications proposed, by canceling some terms of the energy-momentum tensor and of the perturbations of the metric tensor, do not allow to find the equations of the potentials on first line and first column of the tensor h_{ij}.

We start again the study considering the momentum-energy tensor of the radiation sources that we have proposed in (IV-6), without neglecting any term:

$$T_{ij} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} = \lambda\rho \begin{pmatrix} -c^2 & cv_x & cv_y & cv_z \\ cv_x & -v_x v_x & -v_x v_y & -v_x v_z \\ cv_y & -v_y v_x & -v_y v_y & -v_y v_z \\ cv_z & -v_z v_x & -v_z v_y & -v_z v_z \end{pmatrix} \qquad \text{(X-1)}$$

In a linearized metric, the product of this tensor with the metric tensor gives the relationship:

$$\begin{aligned} T &= g^{ij}T_{ij} \approx \eta^{ij}T_{ij} = \eta^{00}T_{00} + \eta^{11}T_{11} + \eta^{22}T_{22} + \eta^{33}T_{33} = -T_{00} + T_{11} + T_{22} + T_{33} \\ T &= \lambda\rho\left(c^2 - v_x^2 - v_y^2 - v_z^2\right) = \lambda\rho\left(c^2 - v^2\right) \end{aligned} \qquad \text{(X-2)}$$

We infer the tensor which is the second member of the equations of gravitation, always in a linearized metric where we have about $g_{ij} \sim \eta_{ij}$:

$$T_{ij} - \frac{1}{2}g_{ij}T = \lambda\rho \begin{pmatrix} -\frac{1}{2}\left(c^2 + v^2\right) & cv_x & cv_y & cv_z \\ cv_x & -v_x v_x - \frac{1}{2}\left(c^2 - v^2\right) & -v_x v_y & -v_x v_z \\ cv_y & -v_y v_x & -v_y v_y - \frac{1}{2}\left(c^2 - v^2\right) & -v_y v_z \\ cv_z & -v_z v_x & -v_z v_y & -v_z v_z - \frac{1}{2}\left(c^2 - v^2\right) \end{pmatrix}$$

(X-3)

In a first order approximation, we neglect the terms in v^2 before c^2 terms in the main diagonal:

$$T_{ij} - \frac{1}{2} g_{ij} T \approx \lambda\rho \begin{pmatrix} -\frac{c^2}{2} & cv_x & cv_y & cv_z \\ cv_x & -\frac{c^2}{2} & -v_x v_y & -v_x v_z \\ cv_y & -v_y v_x & -\frac{c^2}{2} & -v_y v_z \\ cv_z & -v_z v_x & -v_z v_y & -\frac{c^2}{2} \end{pmatrix} \tag{X-4}$$

We do a similar work on the tensor of perturbations. Starting with the tensor of the modified perturbations:

$$\bar{h}_{ij} = \begin{pmatrix} \bar{h}_{00} & \bar{h}_{01} & \bar{h}_{02} & \bar{h}_{03} \\ \bar{h}_{10} & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ \bar{h}_{20} & \bar{h}_{21} & \bar{h}_{22} & \bar{h}_{23} \\ \bar{h}_{30} & \bar{h}_{31} & \bar{h}_{32} & \bar{h}_{33} \end{pmatrix} = \left(\frac{4}{\lambda c}\right) \begin{pmatrix} -\phi/c & A_x & A_y & A_z \\ A_x & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ A_y & \bar{h}_{21} & \bar{h}_{22} & \bar{h}_{23} \\ A_z & \bar{h}_{31} & \bar{h}_{32} & \bar{h}_{33} \end{pmatrix} \tag{X-5}$$

We infer the tensor of the unmodified perturbations by the relationship:

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2} \eta_{ij} \bar{h} \tag{X-6}$$

With:

$$\bar{h} = \eta^{ij} \bar{h}_{ij} = \eta^{00} \bar{h}_{00} + \eta^{11} \bar{h}_{11} + \eta^{22} \bar{h}_{22} + \eta^{33} \bar{h}_{33} = -\bar{h}_{00} + \bar{h}_{11} + \bar{h}_{22} + \bar{h}_{33} \tag{X-7}$$

The treatment is similar to that which has been made above to the momentum-energy tensor, and we get:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} \approx \left(\frac{4}{\lambda c}\right) \begin{pmatrix} -\frac{\phi}{2c} & A_x & A_y & A_z \\ A_x & -\frac{\phi}{2c} & h_{12} & h_{13} \\ A_y & h_{21} & -\frac{\phi}{2c} & h_{23} \\ A_z & h_{31} & h_{32} & -\frac{\phi}{2c} \end{pmatrix} \tag{X-8}$$

Terms $h_{12} = h_{21}$, $h_{13} = h_{31}$, and $h_{23} = h_{32}$ remain temporarily undetermined.

We will commonly use subsequently, equality of all terms of the main diagonal: $h_{00} = h_{11} = h_{22} = h_{33}$

As previously, we want to check directly the impact of the the second-order approximation that we did in (X-4) on the EINSTEIN equations recalled to memory:

$$R_{ij} = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2} g_{ij} T\right) \qquad \text{(X-9)}$$

In a linearized metric, using established above partial results, these equations take the following form:

$$\left(\frac{4}{\lambda c}\right)\begin{pmatrix} \Box h_{00} & \Box h_{01} & \Box h_{02} & \Box h_{03} \\ \Box h_{10} & \Box h_{11} & \Box h_{12} & \Box h_{13} \\ \Box h_{20} & \Box h_{21} & \Box h_{22} & \Box h_{23} \\ \Box h_{30} & \Box h_{31} & \Box h_{32} & \Box h_{33} \end{pmatrix} = \lambda\rho \begin{pmatrix} -\frac{c^2}{2} & cv_x & cv_y & cv_z \\ cv_x & -\frac{c^2}{2} & -v_x v_y & -v_x v_z \\ cv_y & -v_y v_x & -\frac{c^2}{2} & -v_y v_z \\ cv_z & -v_z v_x & -v_z v_y & -\frac{c^2}{2} \end{pmatrix} \qquad \text{(X-10)}$$

Constraints supported by the h_{ij} are essentially those of the harmonic gauge. We are developing a few additional elements on this gauge in the next paragraph.

I – Back on the constraint of harmonic gauge

When we look for solutions of the wave equation:

$$\Box\left(\overline{h}_{ij}\right) = -\frac{\partial^2 \overline{h}_{ij}}{\partial (ct)^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial x^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial y^2} + \frac{\partial^2 \overline{h}_{ij}}{\partial z^2} = -\frac{16\pi G}{c^4} T_{ij} = -2\chi T_{ij} \qquad \text{(X-11)}$$

We have two constraints to meet so that these solutions correspond to solutions of the equations of gravitation: the first one deals with conservation of energy of the energy-momentum tensor, the second one deals with the harmonic gauge which reduces each component of the RICCI tensor to a d'Alembertian.

To better understand how these two constraints interact, we propose to detail the relationship between these two constraints. On the first line of the tensor twice covariant (i = 0), the following relationships must be true:

$$\begin{aligned} -\frac{\partial T_{00}}{\partial x^0} + \frac{\partial T_{01}}{\partial x^1} + \frac{\partial T_{02}}{\partial x^2} + \frac{\partial^2 T_{03}}{\partial x^3} &= 0 \\ -\frac{\partial \overline{h}_{00}}{\partial x^0} + \frac{\partial \overline{h}_{01}}{\partial x^1} + \frac{\partial \overline{h}_{02}}{\partial x^2} + \frac{\partial^2 \overline{h}_{03}}{\partial x^3} &= 0 \end{aligned} \qquad \text{(X-12)}$$

But we also know that these quantities are related by the wave equation:

$$
\begin{aligned}
&-\frac{\partial^2\bar{h}_{00}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^3\right)^2}=-2\chi T_{00}\\
&-\frac{\partial^2\bar{h}_{01}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^3\right)^2}=-2\chi T_{01}\\
&-\frac{\partial^2\bar{h}_{02}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^3\right)^2}=-2\chi T_{02}\\
&-\frac{\partial^2\bar{h}_{03}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^3\right)^2}=-2\chi T_{03}
\end{aligned}
\qquad \text{(X-13)}
$$

When we write the relationship of conservation of energy on terms T_{0j}, this implies the following relation on terms $\bar{h}_{0j}$:

$$
\begin{aligned}
&-\frac{\partial}{\partial x^0}\left\{-\frac{\partial^2\bar{h}_{00}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{00}}{\partial\left(x^3\right)^2}\right\}+\frac{\partial}{\partial x^1}\left\{-\frac{\partial^2\bar{h}_{01}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{01}}{\partial\left(x^3\right)^2}\right\}\\
&+\frac{\partial}{\partial x^2}\left\{-\frac{\partial^2\bar{h}_{02}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{02}}{\partial\left(x^3\right)^2}\right\}+\frac{\partial}{\partial x^3}\left\{-\frac{\partial^2\bar{h}_{03}}{\partial\left(x^0\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^1\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^2\right)^2}+\frac{\partial^2\bar{h}_{03}}{\partial\left(x^3\right)^2}\right\}=0
\end{aligned}
$$

(X-14)

We get after reorganization of the terms:

$$
\begin{aligned}
&-\frac{\partial^2}{\left(\partial x^0\right)^2}\left\{-\frac{\partial\bar{h}_{00}}{\partial x^0}+\frac{\partial\bar{h}_{01}}{\partial x^1}+\frac{\partial\bar{h}_{02}}{\partial x^2}+\frac{\partial\bar{h}_{03}}{\partial x^3}\right\}+\frac{\partial^2}{\left(\partial x^1\right)^2}\left\{-\frac{\partial\bar{h}_{00}}{\partial x^0}+\frac{\partial\bar{h}_{01}}{\partial x^1}+\frac{\partial\bar{h}_{02}}{\partial x^2}+\frac{\partial\bar{h}_{03}}{\partial x^3}\right\}\\
&+\frac{\partial^2}{\left(\partial x^2\right)^2}\left\{-\frac{\partial\bar{h}_{00}}{\partial x^0}+\frac{\partial\bar{h}_{01}}{\partial x^1}+\frac{\partial\bar{h}_{02}}{\partial x^2}+\frac{\partial\bar{h}_{03}}{\partial x^3}\right\}+\frac{\partial^2}{\left(\partial x^3\right)^2}\left\{-\frac{\partial\bar{h}_{00}}{\partial x^0}+\frac{\partial\bar{h}_{01}}{\partial x^1}+\frac{\partial\bar{h}_{02}}{\partial x^2}+\frac{\partial\bar{h}_{03}}{\partial x^3}\right\}=0
\end{aligned}
$$

(X-15)

The other lines are treated similarly. It is concluded that there is compatibility between the two constraints and that they can therefore be met simultaneously.

It remains to write explicitly, in the general case, the constraint of harmonic gauge for twice covariant perturbation of the metric tensor h_{ij}. This constraint has been already detailed for the mixed tensor in the chapter on the gauges:

$$
\begin{aligned}
&\frac{\partial h^i_k}{\partial x^i}-\frac{1}{2}\frac{\partial(h)}{\partial x^k}=0\\
&\frac{\partial h^0_k}{\partial x^0}+\frac{\partial h^1_k}{\partial x^1}+\frac{\partial h^2_k}{\partial x^2}+\frac{\partial h^3_k}{\partial x^3}-\frac{1}{2}\frac{\partial\left(-h_{00}+h_{11}+h_{22}+h_{33}\right)}{\partial x^k}=0\\
&-\frac{\partial h_{0k}}{\partial x^0}+\frac{\partial h_{1k}}{\partial x^1}+\frac{\partial h_{2k}}{\partial x^2}+\frac{\partial h_{3k}}{\partial x^3}-\frac{1}{2}\frac{\partial\left(-h_{00}+h_{11}+h_{22}+h_{33}\right)}{\partial x^k}=0
\end{aligned}
\qquad \text{(X-16)}
$$

We get

for $k = 0$:

$$-\frac{\partial h_{00}}{\partial x^0}+\frac{\partial h_{10}}{\partial x^1}+\frac{\partial h_{20}}{\partial x^2}+\frac{\partial h_{30}}{\partial x^3}-\frac{1}{2}\frac{\partial\left(-h_{00}+h_{11}+h_{22}+h_{33}\right)}{\partial x^0}=0 \qquad \text{(X-17)}$$

or still with the condition $h_{00} = h_{11} = h_{22} = h_{33}$:

$$-2\frac{\partial h_{00}}{\partial x^0}+\frac{\partial h_{10}}{\partial x^1}+\frac{\partial h_{20}}{\partial x^2}+\frac{\partial h_{30}}{\partial x^3}=0 \qquad \text{(X-18)}$$

for k = 1 :

$$-\frac{\partial h_{01}}{\partial x^0}+\frac{\partial h_{11}}{\partial x^1}+\frac{\partial h_{21}}{\partial x^2}+\frac{\partial h_{31}}{\partial x^3}-\frac{1}{2}\frac{\partial\left(-h_{00}+h_{11}+h_{22}+h_{33}\right)}{\partial x^1}=0 \qquad \text{(X-19)}$$

or still with the condition $h_{00} = h_{11} = h_{22} = h_{33}$:

$$-\frac{\partial h_{01}}{\partial x^0}+\frac{\partial h_{21}}{\partial x^2}+\frac{\partial h_{31}}{\partial x^3}=0 \qquad \text{(X-20)}$$

The relationship is analogous for k =2 and k = 3:

$$\begin{aligned}&-\frac{\partial h_{02}}{\partial x^0}+\frac{\partial h_{12}}{\partial x^1}+\frac{\partial h_{32}}{\partial x^3}=0\\&-\frac{\partial h_{03}}{\partial x^0}+\frac{\partial h_{13}}{\partial x^1}+\frac{\partial h_{23}}{\partial x^2}=0\end{aligned} \qquad \text{(X-21)}$$

We have all the elements allowing a direct verification of the equations of gravitation in the general case.

II – The term R_{00}

New terms introduced in the tensor of perturbations ($h_{12} = h_{21}$, $h_{13} = h_{31}$, et $h_{23} = h_{32}$) don't make any changes compared to the previous chapter:

$$R_{00}\approx\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{\left(\partial x^0\right)^2}-\frac{\partial^2 h_{00}}{\left(\partial x^1\right)^2}-\frac{\partial^2 h_{00}}{\left(\partial x^2\right)^2}-\frac{\partial^2 h_{00}}{\left(\partial x^3\right)^2}\right\} \qquad \text{(X-22)}$$

III – The Term R_{01} (and analogous terms R_{02}, R_{03}, R_{10}, R_{20}, R_{30})

The linearized expression of the term R_{01} is recalled below:

$$R_{01}\approx\frac{\partial\Gamma^r_{01}}{\partial x^r}-\frac{\partial\Gamma^r_{0r}}{\partial x^1}=\left\{\frac{\partial\Gamma^0_{01}}{\partial x^0}+\frac{\partial\Gamma^1_{01}}{\partial x^1}+\frac{\partial\Gamma^2_{01}}{\partial x^2}+\frac{\partial\Gamma^3_{01}}{\partial x^3}\right\}-\left\{\frac{\partial\Gamma^0_{00}}{\partial x^1}+\frac{\partial\Gamma^1_{01}}{\partial x^1}+\frac{\partial\Gamma^2_{02}}{\partial x^1}+\frac{\partial\Gamma^3_{03}}{\partial x^1}\right\} \qquad \text{(X-23)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{01} are listed below:

$$\Gamma^0_{01} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{00}}{\partial x^1} + \frac{\partial h_{10}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^0}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^1}\right\}$$

$$\Gamma^1_{01} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^0}\right\}$$

$$\Gamma^2_{01} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{02}}{\partial x^1} + \frac{\partial h_{12}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^2}\right\} = \frac{1}{2}\left\{\frac{\partial h_{02}}{\partial x^1} + \frac{\partial h_{12}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^2}\right\} \qquad \text{(X-24)}$$

$$\Gamma^3_{01} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{03}}{\partial x^1} + \frac{\partial h_{13}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^3}\right\} = \frac{1}{2}\left\{\frac{\partial h_{03}}{\partial x^1} + \frac{\partial h_{13}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^3}\right\}$$

$$\Gamma^0_{00} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{00}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^0}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^0}\right\}$$

$$\Gamma^1_{01} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^0}\right\}$$

$$\Gamma^2_{02} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{02}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^0} - \frac{\partial h_{02}}{\partial x^2}\right\} = \frac{1}{2}\left\{\frac{\partial h_{22}}{\partial x^0}\right\} \qquad \text{(X-25)}$$

$$\Gamma^3_{03} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{03}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^0} - \frac{\partial h_{03}}{\partial x^3}\right\} = \frac{1}{2}\left\{\frac{\partial h_{33}}{\partial x^0}\right\}$$

The coefficients of CHRISTOFFEL detailed above are introduced in the expression of R_{01} recalled to memory:

$$R_{01} \approx \frac{\partial \Gamma^r_{01}}{\partial x^r} - \frac{\partial \Gamma^r_{0r}}{\partial x^1} = \left\{\frac{\partial \Gamma^0_{01}}{\partial x^0} + \frac{\partial \Gamma^1_{01}}{\partial x^1} + \frac{\partial \Gamma^2_{01}}{\partial x^2} + \frac{\partial \Gamma^3_{01}}{\partial x^3}\right\} - \left\{\frac{\partial \Gamma^0_{00}}{\partial x^1} + \frac{\partial \Gamma^1_{01}}{\partial x^1} + \frac{\partial \Gamma^2_{02}}{\partial x^1} + \frac{\partial \Gamma^3_{03}}{\partial x^1}\right\}$$

$$R_{01} \approx \frac{1}{2}\left\{-\frac{\partial^2 h_{00}}{\partial x^0 \partial x^1} + \frac{\partial^2 h_{11}}{\partial x^1 \partial x^0} + \frac{\partial^2 h_{02}}{\partial x^1 \partial x^2} + \frac{\partial^2 h_{12}}{\partial x^0 \partial x^2} - \frac{\partial^2 h_{01}}{\partial x^2 \partial x^2} + \frac{\partial^2 h_{03}}{\partial x^1 \partial x^3} + \frac{\partial^2 h_{13}}{\partial x^0 \partial x^3} - \frac{\partial^2 h_{01}}{\partial x^3 \partial x^3}\right\}$$
$$-\frac{1}{2}\left\{-\frac{\partial^2 h_{00}}{\partial x^0 \partial x^1} + \frac{\partial^2 h_{11}}{\partial x^0 \partial x^1} + \frac{\partial^2 h_{22}}{\partial x^0 \partial x^1} + \frac{\partial^2 h_{33}}{\partial x^0 \partial x^1}\right\}$$

(X-26)

We get after a reorganization of the terms, and by using the relationship $h_{00} = h_{11} = h_{22} = h_{33}$:

$$R_{01} \approx \frac{1}{2}\left(\frac{\partial^2 h_{01}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{01}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{01}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{01}}{\partial x^3 \partial x^3}\right) + \frac{1}{2}\left\{\frac{\partial^2 h_{01}}{\partial x^1 \partial x^1} + \frac{\partial^2 h_{02}}{\partial x^1 \partial x^2} + \frac{\partial^2 h_{03}}{\partial x^1 \partial x^3} - 2\frac{\partial^2 h_{00}}{\partial x^0 \partial x^1}\right\}$$
$$+\frac{1}{2}\left\{-\frac{\partial^2 h_{01}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{12}}{\partial x^0 \partial x^2} + \frac{\partial^2 h_{13}}{\partial x^0 \partial x^3}\right\}$$

$$R_{01} \approx \frac{1}{2}\left(\frac{\partial^2 h_{01}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{01}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{01}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{01}}{\partial x^3 \partial x^3}\right) + \frac{1}{2}\frac{\partial}{\partial x^1}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{02}}{\partial x^2} + \frac{\partial h_{03}}{\partial x^3} - 2\frac{\partial h_{00}}{\partial x^0}\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial x^0}\left\{-\frac{\partial h_{01}}{\partial x^0} + \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{13}}{\partial x^3}\right\}$$

(X-27)

Harmonic gauge conditions cancel out the second and third terms to the right of equality, and it remains:

$$R_{01} \approx \frac{1}{2}\left(\frac{\partial^2 h_{01}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{01}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{01}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{01}}{\partial x^3 \partial x^3} \right) \qquad \text{(X-28)}$$

IV – The term R_{11} (and analogous terms R_{22}, R_{33})

The linearized expression of the term R_{11} is recalled below:

$$R_{11} \approx \frac{\partial \Gamma^r_{11}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^1} = \left\{ \frac{\partial \Gamma^0_{11}}{\partial x^0} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{11}}{\partial x^2} + \frac{\partial \Gamma^3_{11}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{10}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^1} + \frac{\partial \Gamma^3_{13}}{\partial x^1} \right\} \qquad \text{(X-29)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{11} are listed below:

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{10}}{\partial x^1} + \frac{\partial h_{10}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ 2\frac{\partial h_{10}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^0} \right\} \\
\Gamma^1_{11} &= \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^1} \right\} \\
\Gamma^2_{11} &= \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{12}}{\partial x^1} + \frac{\partial h_{12}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^2} \right\} = \frac{1}{2}\left\{ 2\frac{\partial h_{12}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^2} \right\} \\
\Gamma^3_{11} &= \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{13}}{\partial x^1} + \frac{\partial h_{13}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^3} \right\} = \frac{1}{2}\left\{ 2\frac{\partial h_{13}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^3} \right\}
\end{aligned} \qquad \text{(X-30)}$$

$$\begin{aligned}
\Gamma^0_{10} &\approx \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^1} - \frac{\partial h_{10}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{00}}{\partial x^1} \right\} \\
\Gamma^1_{11} &\approx \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^1} \right\} \\
\Gamma^2_{12} &\approx \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{22}}{\partial x^1} \right\} \\
\Gamma^3_{13} &\approx \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{13}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^1} - \frac{\partial h_{13}}{\partial x^3} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{33}}{\partial x^1} \right\}
\end{aligned} \qquad \text{(X-31)}$$

The coefficients of CHRISTOFFEL detailed above are introduced in the expression of R_{11} recalled to memory:

$$R_{11} \approx \frac{\partial \Gamma^r_{11}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^1} = \left\{ \frac{\partial \Gamma^0_{11}}{\partial x^0} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{11}}{\partial x^2} + \frac{\partial \Gamma^3_{11}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{10}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^1} + \frac{\partial \Gamma^3_{13}}{\partial x^1} \right\}$$

$$R_{11} \approx -\frac{1}{2}\left\{ 2\frac{\partial^2 h_{10}}{\partial x^1 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} \right\} + \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} \right\} + \frac{1}{2}\left\{ 2\frac{\partial^2 h_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 h_{11}}{\partial x^2 \partial x^2} \right\} + \frac{1}{2}\left\{ 2\frac{\partial^2 h_{13}}{\partial x^1 \partial x^3} - \frac{\partial^2 h_{11}}{\partial x^3 \partial x^3} \right\}$$

$$+\frac{1}{2}\left\{ \frac{\partial^2 h_{00}}{\partial x^1 \partial x^1} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{22}}{\partial x^1 \partial x^1} \right\} - \frac{1}{2}\left\{ \frac{\partial^2 h_{33}}{\partial x^1 \partial x^1} \right\} \qquad \text{(X-32)}$$

Making use of the relationship $h_{00} = h_{11} = h_{22} = h_{33}$, we get:

$$R_{11} \approx \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{11}}{\partial x^3 \partial x^3} \right\} + \left\{ -\frac{\partial^2 h_{10}}{\partial x^1 \partial x^0} + \frac{\partial^2 h_{12}}{\partial x^1 \partial x^2} + \frac{\partial^2 h_{13}}{\partial x^1 \partial x^3} \right\}$$

$$R_{11} \approx \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{11}}{\partial x^3 \partial x^3} \right\} + \frac{\partial}{\partial x^1}\left\{ -\frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{13}}{\partial x^3} \right\} \qquad \text{(X-33)}$$

The condition of harmonic gauge cancels the second term right of equality, and it remains:

$$R_{11} \approx \frac{1}{2}\left\{ \frac{\partial^2 h_{11}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{11}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{11}}{\partial x^3 \partial x^3} \right\} \qquad \text{(X-34)}$$

V – The term R_{12} (and analogous R_{13}, R_{21}, R_{23}, R_{31}, R_{32})

The linearized expression of the term R_{12} is recalled below:

$$R_{12} \approx \frac{\partial \Gamma^r_{12}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^2} = \left\{ \frac{\partial \Gamma^0_{12}}{\partial x^0} + \frac{\partial \Gamma^1_{12}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{12}}{\partial x^3} \right\} - \left\{ \frac{\partial \Gamma^0_{10}}{\partial x^2} + \frac{\partial \Gamma^1_{11}}{\partial x^2} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{13}}{\partial x^2} \right\} \qquad \text{(X-35)}$$

CHRISTOFFEL coefficients necessary for the development of the term R_{12} are listed below:

$$\Gamma^0_{12} = \frac{1}{2}\eta^{00}\left\{ \frac{\partial h_{10}}{\partial x^2} + \frac{\partial h_{20}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^0} \right\} = -\frac{1}{2}\left\{ \frac{\partial h_{10}}{\partial x^2} + \frac{\partial h_{20}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^0} \right\}$$

$$\Gamma^1_{12} = \frac{1}{2}\eta^{11}\left\{ \frac{\partial h_{11}}{\partial x^2} + \frac{\partial h_{21}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^1} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{11}}{\partial x^2} \right\}$$

$$\Gamma^2_{12} = \frac{1}{2}\eta^{22}\left\{ \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{22}}{\partial x^1} \right\} \qquad \text{(X-36)}$$

$$\Gamma^3_{12} = \frac{1}{2}\eta^{33}\left\{ \frac{\partial h_{13}}{\partial x^2} + \frac{\partial h_{23}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^3} \right\} = \frac{1}{2}\left\{ \frac{\partial h_{13}}{\partial x^2} + \frac{\partial h_{23}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^3} \right\}$$

$$\Gamma^0_{10} \approx \frac{1}{2}\eta^{00}\left\{\frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{00}}{\partial x^1} - \frac{\partial h_{10}}{\partial x^0}\right\} = -\frac{1}{2}\left\{\frac{\partial h_{00}}{\partial x^1}\right\}$$
$$\Gamma^1_{11} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{11}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^1} - \frac{\partial h_{11}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{11}}{\partial x^1}\right\}$$
$$\Gamma^2_{12} \approx \frac{1}{2}\eta^{22}\left\{\frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{22}}{\partial x^1} - \frac{\partial h_{12}}{\partial x^2}\right\} = \frac{1}{2}\left\{\frac{\partial h_{22}}{\partial x^1}\right\} \tag{X-37}$$
$$\Gamma^3_{13} \approx \frac{1}{2}\eta^{33}\left\{\frac{\partial h_{13}}{\partial x^3} + \frac{\partial h_{33}}{\partial x^1} - \frac{\partial h_{13}}{\partial x^3}\right\} = \frac{1}{2}\left\{\frac{\partial h_{33}}{\partial x^1}\right\}$$

The coefficients of CHRISTOFFEL detailed above are introduced in the expression of R_{12} recalled to memory:

$$R_{12} \approx \frac{\partial \Gamma^r_{12}}{\partial x^r} - \frac{\partial \Gamma^r_{1r}}{\partial x^2} = \left\{\frac{\partial \Gamma^0_{12}}{\partial x^0} + \frac{\partial \Gamma^1_{12}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{12}}{\partial x^3}\right\} - \left\{\frac{\partial \Gamma^0_{10}}{\partial x^2} + \frac{\partial \Gamma^1_{11}}{\partial x^2} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{13}}{\partial x^2}\right\}$$
$$R_{12} \approx -\frac{1}{2}\left\{\frac{\partial^2 h_{10}}{\partial x^0 \partial x^2} + \frac{\partial^2 h_{20}}{\partial x^0 \partial x^1} - \frac{\partial^2 h_{12}}{\partial x^0 \partial x^0}\right\} + \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^1 \partial x^2}\right\} + \frac{1}{2}\left\{\frac{\partial^2 h_{22}}{\partial x^1 \partial x^2}\right\}$$
$$+\frac{1}{2}\left\{\frac{\partial^2 h_{13}}{\partial x^2 \partial x^3} + \frac{\partial^2 h_{23}}{\partial x^1 \partial x^3} - \frac{\partial^2 h_{12}}{\partial x^3 \partial x^3}\right\} \tag{X-38}$$
$$+\frac{1}{2}\left\{\frac{\partial^2 h_{00}}{\partial x^1 \partial x^2}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{11}}{\partial x^1 \partial x^2}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{22}}{\partial x^1 \partial x^2}\right\} - \frac{1}{2}\left\{\frac{\partial^2 h_{33}}{\partial x^1 \partial x^2}\right\}$$

Making use of the relationship $h_{00} = h_{11} = h_{22} = h_{33}$, we get finally:

$$R_{12} \approx \frac{1}{2}\left\{\frac{\partial^2 h_{12}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{12}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{12}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{12}}{\partial x^3 \partial x^3}\right\} + \frac{1}{2}\left\{-\frac{\partial^2 h_{10}}{\partial x^0 \partial x^2} + \frac{\partial^2 h_{12}}{\partial x^2 \partial x^2} + \frac{\partial^2 h_{13}}{\partial x^2 \partial x^3}\right\}$$
$$+\frac{1}{2}\left\{-\frac{\partial^2 h_{20}}{\partial x^0 \partial x^1} + \frac{\partial^2 h_{21}}{\partial x^1 \partial x^1} + \frac{\partial^2 h_{23}}{\partial x^1 \partial x^3}\right\}$$
$$R_{12} \approx \frac{1}{2}\left\{\frac{\partial^2 h_{12}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{12}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{12}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{12}}{\partial x^3 \partial x^3}\right\} + \frac{1}{2}\frac{\partial}{\partial x^2}\left\{-\frac{\partial h_{10}}{\partial x^0} + \frac{\partial h_{12}}{\partial x^2} + \frac{\partial h_{13}}{\partial x^3}\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial x^1}\left\{-\frac{\partial h_{20}}{\partial x^0} + \frac{\partial h_{21}}{\partial x^1} + \frac{\partial h_{23}}{\partial x^3}\right\} \tag{X-39}$$

Harmonic gauge conditions cancel out the second and third terms to the right of equality, and it remains:

$$R_{12} \approx \frac{1}{2}\left\{\frac{\partial^2 h_{12}}{\partial x^0 \partial x^0} - \frac{\partial^2 h_{12}}{\partial x^1 \partial x^1} - \frac{\partial^2 h_{12}}{\partial x^2 \partial x^2} - \frac{\partial^2 h_{12}}{\partial x^3 \partial x^3}\right\} \tag{X-40}$$

VI – Conclusion

Compared to the potential vector A_x, A_y, A_z, defined in electromagnetism, the terms $h_{12} = h_{21}$, $h_{13} = h_{31}$, et $h_{23} = h_{32}$ that we have to consider in the tensor of perturbations h_{ij} should check the conditions of harmonic gauge:

$$\frac{\partial h_{12}}{\partial x^2}+\frac{\partial h_{13}}{\partial x^3}=\frac{\partial h_{10}}{\partial x^0}=\left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)\frac{\partial A_x}{\partial(ct)}$$
$$\frac{\partial h_{21}}{\partial x^1}+\frac{\partial h_{23}}{\partial x^3}=\frac{\partial h_{20}}{\partial x^0}=\left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)\frac{\partial A_y}{\partial(ct)} \qquad \text{(X-41)}$$
$$\frac{\partial h_{31}}{\partial x^1}+\frac{\partial h_{32}}{\partial x^3}=\frac{\partial h_{30}}{\partial x^0}=\left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)\frac{\partial A_z}{\partial(ct)}$$

In respect of these constraints, the introduction of an energy-momentum tensor of the sources of the electromagnetic field in the linearized EINSTEIN equations allows to find the equations of the electromagnetic potentials on the first line and the the first column of the RICCI tensor.

Because of potential equations are equivalent to MAXWELL's equations as long as you satisfy the LORENZ gauge, this implies that MAXWELL's equations can be considered to be included in the linearized equations of gravitation.

This similar treatment of electromagnetism and gravitation may only appear if we consider all of the terms of the energy momentum tensor and the modified metric perturbation tensor. We can however, as has been done in this chapter, neglect in the main diagonal terms squared. This approximation, considered as a second order one, keeps all the constraints of jauge and linearization. In particular, it does not impact the relativistic aspect of the equations of potential.

The element of space-time squared is obtained using now all terms of the perturbation of the metric tensor:

$$\begin{aligned} ds^2 &= g_{ij}dx^i dx^j = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \\ &+ 2g_{01}dx^0dx^1 + 2g_{02}dx^0dx^2 + 2g_{03}dx^0dx^3 + 2g_{12}dx^1dx^2 + 2g_{13}dx^1dx^3 + 2g_{23}dx^2dx^3 \end{aligned} \qquad \text{(X-42)}$$

We get in terms of potential:

$$\begin{aligned} ds^2 &= \left(-1-\frac{2\phi}{\lambda c^2}\right)d(ct)^2 + \left(1-\frac{2\phi}{\lambda c^2}\right)\left(dx^2+dy^2+dz^2\right) \\ &+2\left(\frac{4A_x}{\lambda c}\right)d(ct)dx + 2\left(\frac{4A_y}{\lambda c}\right)d(ct)dy + 2\left(\frac{4A_z}{\lambda c}\right)d(ct)dz \\ &+ 2h_{12}dxdy + 2h_{13}dxdz + 2h_{23}dydz \end{aligned} \qquad \text{(X-43)}$$

Terms $h_{12} = h_{21}$, $h_{13} = h_{31}$, et $h_{23} = h_{32}$ both check the harmonic gauge conditions recalled in (X-41) and the general wave equations relative to each term of the tensor h_{ij}. The compatibility between these constraints has been shown in paragraph I) above.

XI - The electrostatic energy of a point charge

I – Relations with the PLANCK units

The approach of the previous chapters led us to propose a momentum-energy tensor for the sources of the electromagnetic field:

$$T_{ij} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} = \lambda\rho \begin{pmatrix} -c^2 & cv_x & cv_y & cv_z \\ cv_x & -v_x v_x & -v_x v_y & -v_x v_z \\ cv_y & -v_y v_x & -v_y v_y & -v_y v_z \\ cv_z & -v_z v_x & -v_z v_y & -v_z v_z \end{pmatrix} \qquad \text{(XI-1)}$$

In this expression, v_x, v_y, v_z, represents the velocity components of the charges following every direction of space. For the static charges, this tensor takes the simplified expression:

$$T_{ij} = \lambda\rho \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi\varepsilon_0 G}} \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(XI-2)}$$

The total volumic density of electromagnetic energy is given by the absolute value of the term T_{00}. By integrating this density on the whole space, we obtain the total charge of the particle that we put equal to q. The purely electromagnetic energy W_q associated with the particle is deducted immediately:

$$W_q = \frac{q}{\sqrt{4\pi\varepsilon_0 G}} c^2 \qquad \text{(XI-3)}$$

We wonder about this relationship which we did not find evidence in the scientific literature, and are looking for a few items that may confirm or deny its interest.

An element of consistency appears in the potential associated with the charge q to provide it with its electromagnetic energy:

If we put:

$$W_q = \frac{q}{\sqrt{4\pi\varepsilon_0 G}} c^2 = qV_p \qquad \text{(XI-4)}$$

The potential V_p which is highlighted is not a potential any: it corresponds exactly to the potential described by PLANCK units, or PLANCK potential.

$$V_p = \frac{c^2}{\sqrt{4\pi\varepsilon_0 G}} = 1{,}0432 \quad 10^{27} \text{ Volt} \qquad \text{(XI-5)}$$

If we assimilate this potential to the vacuum potential, the relationship (XI-4) is consistent with the classical physics: the electromagnetic energy of a charge placed in the vacuum obeys the same relationship when it is placed in any potential.

Specifically, if a charge q is placed in a potential V_0 compared to the vacuum, its total electromagnetic energy will be equal to:

$$W_0 = q\,(V_p + V_0) \qquad \text{(XI-6)}$$

If this charge moves up to a point of potential V_1 compared to the vacuum, its total energy becomes:

$$W_1 = q\,(V_p + V_1) \qquad \text{(XI-7)}$$

The energy exchanged during this movement is written:

$$\Delta W = W_1 - W_0 = q\,(V_1 - V_0) \qquad \text{(XI-8)}$$

According to the classical result of electrodynamics.

In the approach developed in this document, potential variations are described by variations of the tensor h_{ij}.

$$\begin{aligned}
h_{00} &= -\left(\frac{2\sqrt{4\pi\varepsilon_0 G}}{c^2}\right)\phi \\
h_{01} = h_{10} &= \left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)A_x \\
h_{02} = h_{20} &= \left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)A_y \\
h_{03} = h_{30} &= \left(\frac{4\sqrt{4\pi\varepsilon_0 G}}{c}\right)A_z
\end{aligned} \qquad \text{(XI-9)}$$

These potential changes are directly related to changes in the volumic densities of electromagnetic energy of vacuum, in accordance with the EINSTEIN equations, interpreted in their energy aspect.

For an electron of charge e, we can give an other expression of electromagnetic energy based on PLANCK units.

The fine structure constant, designated by α, is given by the expression:

$$\alpha = \left(\frac{e}{q_p}\right)^2 = \frac{e^2}{4\pi\varepsilon_0 c\hbar} = \frac{1}{137.046} \qquad \text{(XI-20)}$$

where q_p is the PLANCK charge.

The total electromagnetic energy of the electron W_e is given by relationship (XI-4) with $q = e$:

$$W_e = \frac{e}{\sqrt{4\pi\varepsilon_0 G}} c^2 \qquad \text{(XI-21)}$$

We can express this energy in terms of the fine structure constant:

$$W_e = \frac{e}{\sqrt{4\pi\varepsilon_0 G}} c^2 = \frac{e}{\sqrt{4\pi\varepsilon_0 c\hbar}} c^2 \sqrt{\frac{c\hbar}{G}} = \sqrt{\alpha} c^2 \sqrt{\frac{c\hbar}{G}} = \sqrt{\alpha} E_p \qquad \text{(XI-22)}$$

where E_p is the PLANCK energy.

We can also look to a dimension, sometimes improperly called "dimension of the electron". This dimension is obtained by assuming that the electromagnetic energy of the electron W_e is distributed from a distribution with spherical symmetry of radius a, and it is governed by the electrostatic relationship until the 'radius' a of the electron. The total electrostatic energy is calculated by integrating the volumic density of electrostatic energy from the radius r = a to the radius r infinite:

$$W_e = \frac{\varepsilon_0}{2}\int E^2 dV = \frac{\varepsilon_0}{2} 4\pi \int E_r^2 r^2 dr = \frac{\varepsilon_0}{2} 4\pi \left(\frac{e}{4\pi\varepsilon_0}\right)^2 \int_{r=a}^{\infty} \frac{dr}{r^2} = \frac{e^2}{8\pi\varepsilon_0 a} \qquad \text{(XI-23)}$$

Equality with the expression deduced of the term of the electromagnetic energy-momentum tensor T_{00} provides the relationship:

$$W_e = \left(\frac{e}{\sqrt{4\pi G\varepsilon_0}}\right) c^2 = \frac{e^2}{8\pi\varepsilon_0 a} \qquad \text{(XI-24)}$$

We can deduce a "diameter" d = 2a of the electron in the form:

$$d = 2a = \frac{e}{\sqrt{4\pi\varepsilon_0 c\hbar}} \sqrt{\frac{G\hbar}{c^3}} = \sqrt{\alpha} L_p \qquad \text{(XI-25)}$$

where L_p is the PLANCK length:

$$L_p = \sqrt{\frac{G\hbar}{c^3}} \quad \text{(XI-26)}$$

The time taken by light to "cross" the electron is related to the PLANCK time by the fine structure constant:

$$t = \frac{d}{c} = \sqrt{\alpha}\,\frac{l_p}{c} = \sqrt{\alpha}\,t_p \quad \text{(XI-27)}$$

We can also look at the relationship between the electromagnetic energy and the energy of mass, at the PLANCK scale.

The mass energy W_m associated with mass m is given by the EINSTEIN relation:

$$W_m = mc^2 \quad \text{(XI-28)}$$

The electromagnetic energy W_q associated with a charge q is given by the term T_{00} given by the electromagnetic momentum-energy tensor:

$$W_q = \frac{q}{\sqrt{4\pi\varepsilon_0 G}}c^2 \quad \text{(XI-29)}$$

We can deduce that the mass m which contains the same amount of energy as the charge q is given by the relationship:

$$m = \frac{q}{\sqrt{4\pi\varepsilon_0 G}} = \lambda q \quad \text{(XI-30)}$$

These relations highlight a simple analogy between the expressions of the mass energy W_m and electromagnetic energy W_q of a charge, to the PLANCK scale (remembered that E_p is the PLANCK energy):

$$W_q = \frac{q}{q_p}E_p \qquad W_m = \frac{m}{m_p}E_p \quad \text{(XI-31)}$$

If the vacuum is made, as suggested by DIRAC, by a sea of electrons located on the lowest level of energy and respecting the PAULI exclusion principle, the relations above shall be interpreted in a simple way: the vacuum properties that are underlying PLANK units are directly related to the properties of the electron by the fine structure constant α.

II) Back on the force notion

The concept of force is related to the notion of energy by a simple relationship: energy = force multiplied by displacement. However, considering two motionless point masses or two motionless point charges, the notion of force is present, while no exchange of energy is put into play.

We can interpret this phenomenon by adopting a purely energy point of view.

Let us consider two systems that generate each a certain distribution of energy in space. If it is admitted, by a basic law of physics, that this energy has a natural tendency to become more homogeneous, or said otherwise, to evolve to a state where it would be overall more stable, then there is a force between these two systems.

This change of perspective seems insignificant, but it illuminates physics issue from the vacuum energy in a different way: energy is no longer the consequence of a force that moved, it's the force that is the consequence of a certain distribution of energy seeking to evolve in space.

If we adopt this point of view, then similar distributions of energy should lead to similar forces.

Then consider a point mass m and a point charge q that have the same energy, distributed in space with a same spherical symmetry. They should check the relationship:

$$mc^2 = \frac{q}{\sqrt{4\pi\varepsilon_0 G}} c^2 \qquad \text{(XI-32)}$$

Or still :

$$m = \frac{q}{\sqrt{4\pi\varepsilon_0 G}} = \lambda q \qquad \text{(XI-32)}$$

The energy reasoning suggested above tells us then that the force between two masses F_m must be identical to the force between two charges F_q, when they are in the same space conditions:

$$F_m = G\frac{m^2}{r^2} \qquad F_q = \frac{q^2}{4\pi\varepsilon_0 r^2} \qquad \text{(XI-33)}$$

For m = λq, we get :

$$F_m = G\frac{m^2}{r^2} = G\frac{(\lambda q)^2}{r^2} = G\lambda^2\frac{q^2}{r^2} = \frac{G}{4\pi\varepsilon_0 G}\frac{q^2}{r^2} = \frac{q^2}{4\pi\varepsilon_0 r^2} = F_q \qquad \text{(XI-34)}$$

It appears that the total electromagnetic energy associated with a charge, deducted from the momentum-energy tensor that is proposed in this paper, seems consistent with fundamental physics laws that governs the forces between charges and forces between masses.

XII - Motion of a charged particle

I – Heuristic approach of the equation of motion of a charged particle

The theory of general relativity provides a new concept to predict the path of a massive object in a gravitational field. This trajectory is no longer determined by driving forces on the moving mass, but by the curvature of space-time. This curvature of space-time "guide" the test mass following a geodesic equation:

$$\frac{d^2x^r}{d\tau^2} + \Gamma^r_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \qquad \text{(XII-1)}$$

In this relationship, τ is the proper time, and the CHRISTOFFEL coefficients are given by the relationship:

$$\Gamma^r_{ij} = \frac{1}{2}g^{rk}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \qquad \text{(XII-2)}$$

The curvature of space-time appears in the derivatives of the metric terms (the g_{ij}). Since the gravitation potential is present throughout these terms, we can deduce that curvature represented by the coefficients of CHRISTOFFEL is imposed by the gravitation potential.

When trying to understand how this relationship joined NEWTON's law, it is required to do several approximations.

One replaces the proper time τ by the absolute Newtonian time t:

$$\frac{d^2x^r}{dt^2} + \Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad \text{(XII-3)}$$

We multiply this equality by the test mass m:

$$m\frac{d^2x^r}{dt^2} + m\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad \text{(XII-4)}$$

One holds only indices r = 1, r = 2, r = 3, which correspond to the components of forces according to x, y, and z:

$$
\begin{aligned}
m\frac{d^2x^1}{dt^2} &= -m\Gamma^1_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\\
m\frac{d^2x^2}{dt^2} &= -m\Gamma^2_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\\
m\frac{d^2x^3}{dt^2} &= -m\Gamma^3_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}
\end{aligned}
\qquad \text{(XII-5)}
$$

We are then in the presence of NEWTON's law, in which the force exerted on the test mass is expressed in two ways. The first (left of equality) is identified as the force of inertia, while the second (right of equality) is identified as the external force.

In the case of a charged particle in an electromagnetic field, the driving force depends only on the charge and the nature of the field, which suggests a suitable modification of the term right of equality.

CHRISTOFFEL coefficients reflect the curve generated by the presence of the electromagnetic field in which the charge is moving. In addition, the energy present in this term relates to the electromagnetic energy of the charge q. These observations suggest to change the equation of motion of the particle in the following way:

$$
(m)\frac{d^2x^r}{d\tau^2} + (\lambda q)\Gamma^r_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \qquad \text{(XII-6)}
$$

The heuristic approach that led to this formulation does not constitute a rigorous justification, and it should therefore be considered with caution.

It turns out that a rigorous justification is not simple to get. We are in possession of a representative tensor of electromagnetic potentials, and so in possession of the CHRISTOFFEL coefficients relative to this tensor. But the energy relation above is more demanding: it assumes that we rigorously know this tensor when the electromagnetic field is perturbed by the test charge. This requirement can be linked to that which is necessary for the derivative of the gravitational waves.

We will nevertheless propose a draft verification by trying to identify the LORENTZ force in the electromagnetic energy term. It should be assessed at their fair value items that will follow: this draft check does not constitute proof of the validity of the above (XII-6) relationship, but just some elements for thought in agreement with the proposed conjecture.

In order to make appear the LORENTZ force, we replace the proper time τ by the newtonian time t, and we write the relationship (XII-6) in the form:

$$
m\frac{d^2x^r}{dt^2} = -(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \qquad \text{(XII-7)}
$$

The left term of equality represents the force of inertia, while the right term must represent the LORENTZ force for components such as r = 1, r = 2, and r = 3.

Development in relation to index i, allows to get:

$$-(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = -(\lambda q)\left\{\Gamma^r_{0j}\frac{dx^0}{dt}\frac{dx^j}{dt} + \Gamma^r_{1j}\frac{dx^1}{dt}\frac{dx^j}{dt} + \Gamma^r_{2j}\frac{dx^2}{dt}\frac{dx^j}{dt} + \Gamma^r_{3j}\frac{dx^3}{dt}\frac{dx^j}{dt}\right\}$$
(XII-8)

We can deduce the expression developed in relation to the j index:

$$-(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = -(\lambda q)\left\{\begin{array}{l}\Gamma^r_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} + \Gamma^r_{01}\frac{dx^0}{dt}\frac{dx^1}{dt} + \Gamma^r_{02}\frac{dx^0}{dt}\frac{dx^2}{dt} + \Gamma^r_{03}\frac{dx^0}{dt}\frac{dx^3}{dt}\\ + \Gamma^r_{10}\frac{dx^1}{dt}\frac{dx^0}{dt} + \Gamma^r_{11}\frac{dx^1}{dt}\frac{dx^1}{dt} + \Gamma^r_{12}\frac{dx^1}{dt}\frac{dx^2}{dt} + \Gamma^r_{13}\frac{dx^1}{dt}\frac{dx^3}{dt}\\ + \Gamma^r_{20}\frac{dx^2}{dt}\frac{dx^0}{dt} + \Gamma^r_{21}\frac{dx^2}{dt}\frac{dx^1}{dt} + \Gamma^r_{22}\frac{dx^2}{dt}\frac{dx^2}{dt} + \Gamma^r_{23}\frac{dx^2}{dt}\frac{dx^3}{dt}\\ + \Gamma^r_{30}\frac{dx^3}{dt}\frac{dx^0}{dt} + \Gamma^r_{31}\frac{dx^3}{dt}\frac{dx^1}{dt} + \Gamma^r_{32}\frac{dx^3}{dt}\frac{dx^2}{dt} + \Gamma^r_{33}\frac{dx^3}{dt}\frac{dx^3}{dt}\end{array}\right\}$$
(XII-9)

In this expression, the terms do not have the same order of magnitude: it is apparent by puting $x^0 = ct$ and then, $dx^0/dt = c$:

$$-(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = -(\lambda q)\left\{\begin{array}{l}\Gamma^r_{00}c^2 + \Gamma^r_{01}c\frac{dx^1}{dt} + \Gamma^r_{02}c\frac{dx^2}{dt} + \Gamma^r_{03}c\frac{dx^3}{dt}\\ + \Gamma^r_{10}c\frac{dx^1}{dt} + \Gamma^r_{11}\frac{dx^1}{dt}\frac{dx^1}{dt} + \Gamma^r_{12}\frac{dx^1}{dt}\frac{dx^2}{dt} + \Gamma^r_{13}\frac{dx^1}{dt}\frac{dx^3}{dt}\\ + \Gamma^r_{20}c\frac{dx^2}{dt} + \Gamma^r_{21}\frac{dx^2}{dt}\frac{dx^1}{dt} + \Gamma^r_{22}\frac{dx^2}{dt}\frac{dx^2}{dt} + \Gamma^r_{23}\frac{dx^2}{dt}\frac{dx^3}{dt}\\ + \Gamma^r_{30}c\frac{dx^3}{dt} + \Gamma^r_{31}\frac{dx^3}{dt}\frac{dx^1}{dt} + \Gamma^r_{32}\frac{dx^3}{dt}\frac{dx^2}{dt} + \Gamma^r_{33}\frac{dx^3}{dt}\frac{dx^3}{dt}\end{array}\right\}$$
(XII-10)

Assuming the velocity components small in front of the speed of light, we keep only the dominant terms:

$$-(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \approx -(\lambda q)\left\{\Gamma^r_{00}c^2 + \left(\Gamma^r_{01} + \Gamma^r_{10}\right)c\frac{dx^1}{dt} + \left(\Gamma^r_{02} + \Gamma^r_{20}\right)c\frac{dx^2}{dt} + \left(\Gamma^r_{03} + \Gamma^r_{30}\right)c\frac{dx^3}{dt}\right\}$$
(XII-11)

The symmetry of the CHRISTOFFEL coefficients allows finally to get the following expression:

$$-(\lambda q)\Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \approx -(\lambda q)\left\{\Gamma^r_{00}c^2 + 2\Gamma^r_{01}c\frac{dx^1}{dt} + 2\Gamma^r_{02}c\frac{dx^2}{dt} + 2\Gamma^r_{03}c\frac{dx^3}{dt}\right\} \qquad \text{(XII-12)}$$

<u>II – Motion in a scalar potential</u>

We place the particle in a scalar potential ϕ, defined by the h_{ij} representing the MINKOWSKY metric perturbations:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} = \left(\frac{4}{\lambda c}\right) \begin{pmatrix} -\frac{\phi}{2c} & 0 & 0 & 0 \\ 0 & -\frac{\phi}{2c} & 0 & 0 \\ 0 & 0 & -\frac{\phi}{2c} & 0 \\ 0 & 0 & 0 & -\frac{\phi}{2c} \end{pmatrix} \tag{XII-13}$$

This scalar potential must check the condition of harmonic gauge, therefore we will admit that it is independent of time.

We assume that the charged test particle does not interfere, to first order, with this metric that contains the dominant terms of the tensor h_{ij}.

We deduce the coefficients of CHRISTOFFEL useful for the description of the LORENTZ force. For the component force depending on x which corresponds to r = 1, we get:

$$\begin{aligned}
\Gamma^1_{00} &\approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^0} + \frac{\partial h_{01}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^1}\right\} = \frac{1}{2}\left\{-\frac{\partial h_{00}}{\partial x^1}\right\} \\
\Gamma^1_{01} &\approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^1}\right\} = 0 \\
\Gamma^1_{02} &\approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^2} + \frac{\partial h_{21}}{\partial x^0} - \frac{\partial h_{02}}{\partial x^1}\right\} = 0 \\
\Gamma^1_{03} &\approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^3} + \frac{\partial h_{31}}{\partial x^0} - \frac{\partial h_{03}}{\partial x^1}\right\} = 0
\end{aligned} \tag{XII-14}$$

We infer the strength component relative to x:

$$\begin{aligned}
F_x &\approx -(\lambda q)\Gamma^1_{00}c^2 = -\frac{1}{2}(\lambda q)\left\{-\frac{\partial h_{00}}{\partial x^1}\right\}c^2 = \frac{1}{2}(\lambda q)\left\{\frac{\partial h_{00}}{\partial x}\right\}c^2 = \frac{1}{2}(\lambda q)\left\{\left(\frac{4}{\lambda c}\right)\frac{\partial}{\partial x}\left(\frac{-\phi}{2c}\right)\right\}c^2 \\
F_x &\approx -(\lambda q)\Gamma^1_{00}c^2 = q\left\{-\frac{\partial \phi}{\partial x}\right\} = qE_x
\end{aligned} \tag{XII-15}$$

In this expression, E_x represents the component of electric field deducted from the variation of the scalar potential relative to x, according to the relationship:

$$E_x = -\frac{\partial \phi}{\partial x} \tag{XII-16}$$

We get in a similar way for other components:

$$F_y \approx -(\lambda q)\Gamma^2_{00}c^2 = q\left\{-\frac{\partial\phi}{\partial y}\right\} = qE_y$$

$$F_z \approx -(\lambda q)\Gamma^3_{00}c^2 = q\left\{-\frac{\partial\phi}{\partial z}\right\} = qE_z \qquad \text{(XII-17)}$$

In the simplifying assumptions that we have set out, it appears that the conjecture suggested by an energy analysis, and amending the geodesic equation following the relationship (XII-7), led exactly to the terms that describe the LORENTZ force.

III – Motion in a vector potential

We place the particle in a potential vector of components A_x, A_y, A_z, defined by the h_{ij} representing the corresponding changes in the metric of MINKOWSKY:

$$h_{ij} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix} = \left(\frac{4}{\lambda c}\right)\begin{pmatrix} 0 & A_x & A_y & A_z \\ A_x & 0 & 0 & 0 \\ A_y & 0 & 0 & 0 \\ A_z & 0 & 0 & 0 \end{pmatrix} \qquad \text{(XII-18)}$$

This vector potential must check the condition of harmonic gauge, therefore we will admit that it is independent of time.

We assume, as in the previous paragraph that the charged particle does not interfere, to first order, with this metric, but we must keep in mind that the h_{ij} representing the vector potential are an order of magnitude less than those who represent the scalar potential. In other words, we went down to an order of magnitude compared to the scalar potential, and it is not excluded that the presence of the charge has effects which are more significant on this metric.

We evaluate the coefficients of CHRISTOFFEL useful for the description of the LORENTZ force. For the force component depending on x which corresponds to r = 1, we get:

$$\Gamma^1_{00} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^0} + \frac{\partial h_{01}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^1}\right\} = 0$$

$$\Gamma^1_{01} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^1} + \frac{\partial h_{11}}{\partial x^0} - \frac{\partial h_{01}}{\partial x^1}\right\} = 0$$

$$\Gamma^1_{02} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^2} + \frac{\partial h_{21}}{\partial x^0} - \frac{\partial h_{02}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{01}}{\partial x^2} - \frac{\partial h_{02}}{\partial x^1}\right\} \qquad \text{(XII-19)}$$

$$\Gamma^1_{03} \approx \frac{1}{2}\eta^{11}\left\{\frac{\partial h_{01}}{\partial x^3} + \frac{\partial h_{31}}{\partial x^0} - \frac{\partial h_{03}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{\partial h_{01}}{\partial x^3} - \frac{\partial h_{03}}{\partial x^1}\right\}$$

We infer the strength component relative to x:

$$F_x = -(\lambda q)\left\{\Gamma^r_{00}c^2 + 2\Gamma^r_{01}c\frac{dx^1}{dt} + 2\Gamma^r_{02}c\frac{dx^2}{dt} + 2\Gamma^r_{03}c\frac{dx^3}{dt}\right\}$$

$$F_x = -(\lambda q)\left\{\left(\frac{\partial h_{01}}{\partial x^2} - \frac{\partial h_{02}}{\partial x^1}\right)c\frac{dx^2}{dt} + \left(\frac{\partial h_{01}}{\partial x^3} - \frac{\partial h_{03}}{\partial x^1}\right)c\frac{dx^3}{dt}\right\} \qquad \text{(XII-20)}$$

$$F_x = -(\lambda q)\left\{\left(\frac{\partial h_{01}}{\partial y} - \frac{\partial h_{02}}{\partial x}\right)c\frac{dy}{dt} + \left(\frac{\partial h_{01}}{\partial z} - \frac{\partial h_{03}}{\partial x}\right)c\frac{dz}{dt}\right\}$$

We get after substitution of the expressions of potential based on the h_{ij}:

$$F_x = -(\lambda q)\left(\frac{4}{\lambda c}\right)\left\{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)c\frac{dy}{dt} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)c\frac{dz}{dt}\right\}$$

$$F_x = -4q\left\{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\frac{dy}{dt} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\frac{dz}{dt}\right\} \qquad \text{(XII-21)}$$

The expression of the electric and magnetic fields as a function of potential is recalled to memory:

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \overrightarrow{Rot}(\vec{A}) = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} \qquad \vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -\overrightarrow{Grad}(\phi) - \frac{\partial \vec{A}}{\partial t} = \begin{pmatrix} -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \\ -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} \\ -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \end{pmatrix}$$

(XII-22)

One gets by identifying:

$$F_x = 4q\left\{B_z\frac{dy}{dt} - B_y\frac{dz}{dt}\right\} \qquad \text{(XII-23)}$$

The components along y and z generate similar relationships:

$$F_y = 4q\left\{B_x\frac{dz}{dt} - B_z\frac{dx}{dt}\right\}$$

$$F_z = 4q\left\{B_y\frac{dx}{dt} - B_x\frac{dy}{dt}\right\} \qquad \text{(XII-24)}$$

It appears that the resulting expression covers exactly the force of LORENTZ, but to a factor 4 close.

We may naturally suspect the metric modifications by the test charge that are not taken into account in the reasoning, but we should provide some convincing evidence that have not been established yet.

These results, although encouraging, show that complementary comprehension work must be done before we can reverse or confirm the hypothesis proposed in a change in the physical sense of the geodesic equation.

XIII - Energy expression of MAXWELL equations

We have the elements to formulate an energetic expression of MAXWELL equations. We hope to see the emergence of a formulation whose equations of dimensions will be similar to those of the equations of gravitation.

The term "analog" here means that the terms that represent the sources of electromagnetic or gravitational fields will have the dimension of a volumic density of energy.

Terms that represent the evolutionary relationships between fields or potentials, and space-time (usually partial derivatives), will be expressed as the product of two terms, each having the dimension of the square root of a volumic density energy.

I – A look back at the energy expression of gravitational equations

We recall, first of all, how it is possible to bring an energetic aspect to the equations of EINSTEIN. Starting from the usual formulation:

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} = \chi T_{ij} \qquad \text{(XIII-1)}$$

This equality is divided by the EINSTEIN constant:

$$\frac{R_{ij}}{\chi} - \frac{R}{2\chi} g_{ij} = T_{ij} \qquad \text{(XIII-2)}$$

We get a relationship in which each of the main terms has the dimension of a volumic density of energy.

The RICCI tensor is then expressed according to CHRISTOFFEL coefficients:

$$\frac{R_{ij}}{\chi} = \frac{\Gamma^k_{mk}\Gamma^m_{ij}}{\chi} - \frac{\Gamma^k_{mj}\Gamma^m_{ik}}{\chi} + \frac{\partial \Gamma^k_{ij}}{\chi \partial x^k} - \frac{\partial \Gamma^k_{ik}}{\chi \partial x^j} \qquad \text{(XIII-3)}$$

In the search for a physical dimension of each of the terms, one proceeds empirically, breaking down this tensor in the form:

$$\frac{R_{ij}}{\chi} = \frac{\Gamma^k_{mk}}{\sqrt{\chi}}\frac{\Gamma^m_{ij}}{\sqrt{\chi}} - \frac{\Gamma^k_{mj}}{\sqrt{\chi}}\frac{\Gamma^m_{ik}}{\sqrt{\chi}} + \frac{\partial \frac{\Gamma^k_{ij}}{\sqrt{\chi}}}{\partial\left(\sqrt{\chi}x^k\right)} - \frac{\partial \frac{\Gamma^k_{ik}}{\sqrt{\chi}}}{\partial\left(\sqrt{\chi}x^j\right)} \qquad \sqrt{\chi} = \frac{\sqrt{8\pi G}}{c^2} \qquad \text{(XIII-4)}$$

This expression reveals two categories of terms:

$$\frac{\Gamma^k_{ij}}{\sqrt{\chi}} \qquad \text{(XIII-5)}$$

$$\frac{\partial}{\partial\left(\sqrt{\chi}x^k\right)} \qquad \text{(XIII-6)}$$

Each of these terms has the dimension of the square root of a volumic density of energy.

The expression (XIII-6) interprets any spatial derivative (or temporal multiplied by c) as a quantity that is homogeneous to the square root of a volumic density of energy. We will now use this term in the search for energy formulations.

II – Energy expression of MAXWELL equations

In vacuum, and in their temporal description, MAXWELL equations may be written:

$$\begin{aligned}
&\vec{\nabla}\Lambda\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}\\
&\vec{\nabla}\Lambda\vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\\
&\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}\\
&\vec{\nabla}.\vec{H} = 0
\end{aligned} \qquad \text{(XIII-7)}$$

The third equation allows a direct treatment in relation to the energy relationships established in this brief. We start from its developed writing:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0} \qquad \text{(XIII-8)}$$

We multiply the two members of this equation by the factor that makes the right term homogeneous to a volumic density of energy:

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\frac{\varepsilon_0 c^2}{\sqrt{4\pi\varepsilon_0 G}} = \left(\frac{\rho}{\varepsilon_0}\right)\frac{\varepsilon_0 c^2}{\sqrt{4\pi\varepsilon_0 G}} \qquad \text{(XIII-9)}$$

We introduce the energy expression of the spatial derivative (XIII-6) deduced from EINSTEIN's equations:

$$\left(\frac{\partial\left(\sqrt{\chi}E_x\right)}{\partial\left(\sqrt{\chi}x\right)}+\frac{\partial\left(\sqrt{\chi}E_y\right)}{\partial\left(\sqrt{\chi}y\right)}+\frac{\partial\left(\sqrt{\chi}E_z\right)}{\partial\left(\sqrt{\chi}z\right)}\right)\frac{\varepsilon_0 c^2}{\sqrt{4\pi\varepsilon_0 G}}=\frac{\rho c^2}{\sqrt{4\pi\varepsilon_0 G}} \qquad \text{(XIII-10)}$$

Or again :

$$\frac{\partial\left(\sqrt{2\varepsilon_0}E_x\right)}{\partial\left(\sqrt{\chi}x\right)}+\frac{\partial\left(\sqrt{2\varepsilon_0}E_y\right)}{\partial\left(\sqrt{\chi}y\right)}+\frac{\partial\left(\sqrt{2\varepsilon_0}E_z\right)}{\partial\left(\sqrt{\chi}z\right)}=\frac{\rho c^2}{\sqrt{4\pi\varepsilon_0 G}} \qquad \text{(XIII-11)}$$

The right term represents the volumic density of energy associated with the charge q, while the terms of the left are made up by the association of two quantities:

$\sqrt{2\varepsilon_0}E_x$ represents a quantity directly proportional to the square root of the electromagnetic energy density associated with the component E_x.

$\frac{\partial}{\partial\left(\sqrt{\chi}x\right)}$ represents a spatial derivative that is homogeneous at the square root of a volumic density of energy.

Similar work on other equations allows us to propose an energy formulation of MAXWELL equations in the form:

$$\begin{aligned}
&\frac{\vec{\nabla}}{\sqrt{\chi}}\Lambda\left(\sqrt{2\varepsilon_0}\vec{E}\right)=-\frac{\partial\left(\sqrt{2\mu_0}\vec{H}\right)}{\partial\left(\sqrt{\chi}ct\right)}\\
&\frac{\vec{\nabla}}{\sqrt{\chi}}\Lambda\left(\sqrt{2\mu_0}\vec{H}\right)=\frac{\vec{J}c}{\sqrt{4\pi\varepsilon_0 G}}+\frac{\partial\left(\sqrt{2\varepsilon_0}\vec{E}\right)}{\partial\left(\sqrt{\chi}ct\right)}\\
&\frac{\vec{\nabla}}{\sqrt{\chi}}.\left(\sqrt{2\varepsilon_0}\vec{E}\right)=\frac{\rho c^2}{\sqrt{4\pi\varepsilon_0 G}}\\
&\frac{\vec{\nabla}}{\sqrt{\chi}}.\left(\sqrt{2\mu_0}\vec{H}\right)=0
\end{aligned} \qquad \text{(XIII-12)}$$

It is clear that such a formulation does not bring anything more with regard to the mathematical treatment of these equations.

On the other hand, from the point of view of physics, this writing highlights the fundamental role of the square root of the density of energy associated with each component of the electromagnetic field. It highlights a possible interpretation of electromagnetism from variations in the electromagnetic energy density of the vacuum, consistent with the work of previous chapters.

XIV - Conclusion

The concept of energy is a fundamental concept in physics. Combining it with the principle of least action, we can describe the whole of the fundamental laws that govern both the infinitely small and the macroscopic world. Combining it with an additional assumption that assumes that all energy is extracted from the energy of the vacuum, we get a coherent description of the whole of physics.

To underline this consistency, it is useful to highlight the energy aspects of the major laws of physics.

About quantum mechanics, it appears that many properties can find an energy interpretation (Cf An energy and determinist approach of quantum mechanics).

About gravitation, chapter III shows that the EINSTEIN equations can be analyzed as energy equations by a simple change of point of view.

About electromagnetism, it seems that this field of physics can be interpreted analogously to gravitation, if we are able to define a momentum-energy tensor of sources of electromagnetic field. Such a tensor has been proposed, and some reflection elements have been developed around this idea.

In an energy approach, the framework of relativity refers to the vacuum energy that is distributed uniformly throughout the space. This even distribution is associated with a flat space-time. So we can associate a non-uniform distribution of the vacuum energy to a curved space-time.

When we introduce a mass or a charge in a flat space-time, we change its curvature, and so we also changes the distribution of the vacuum energy that will be no more homogenous as in the case of special relativity.

If we place a test particle in non-homogeneous space, this particle and its surrounding own energy will evolve according to the principle of least action. The notion of force appears naturally as a consequence of energy densities that are different in two points in space, which allows to reach the point of view of the classical mechanics without effort.

Energy point of view leads us to assume that electromagnetism and gravitation are governed by the same equations. Several reflections are advanced in this sense in this document.

This similar treatment of electromagnetic and gravitational energy is not without raising some questions. The first of these appears to be the following: why are there no magnetic effect in the phenomena of gravity since they are very easily observable for electromagnetic phenomena?

We can try to bring an early response in the following way.

For charges in motion at speeds much lower than the speed of light, magnetic phenomena are an order of magnitude less than the electric phenomena in a report equals c. In other words, we must fetch the 7th or 8th decimal place in the electric phenomena to observe the change brought by the magnetic effects.

If these effects are nevertheless noticeable for currents that circulate and generate a magnetic field, it is because these currents are electrically neutral: the electrostatic effects are null (to each electron that circulates in the wire, there is a proton, which compensates for its electrostatic field), and it remains only effects due to the movement of the charges in the wire. Although weak, these effects can be highlighted easily because they are practically the only ones which exist.

The problem with the gravitation, is that there is no negative masses. The gravitational effect similar to the electrostatic effect can never be cancelled. It follows that magnetic effects would hardly discernible, because drowned in the "gravitostatic" effects

If this similar behavior of the electromagnetic and gravitational energy was confirmed, it would induce consequences on our vision of MAXWELL's equations, whose a more complete formulation is given by the EINSTEIN equations, associated with a momentum-energy tensor of electromagnetic sources. It was showed that the MINKOWSKY metric variations induced by the presence of electromagnetic potential were extremely low compared to 1 ($\sim 10^{-18}$), which allows to understand why this metric remains particularly well suited to the study of electromagnetism.

It remains nevertheless that on very long distances, or very long time, low variations may appear, compared to electromagnetism of MAXWELL. In particular, it remains a work of understanding for interpretation of terms ($h_{12} = h_{21}$, $h_{13} = h_{31}$, and $h_{23} = h_{32}$) which appeared in the perturbations tensor relative to electromagnetic potential.

In conclusion, the work presented in this paper developed analogies of behavior between the laws of electromagnetism and gravitation, when they are examined through the energy prism of the EINSTEIN equations. If we add the energy aspect of the DIRAC equations, these are large parts of physics that seem to find consistency around the concept of energy.

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