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## A study on topogenic integer additive set-labeled graphs

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**Abstract.** Let  $\mathbb{N}_0$  denote the set of all non-negative integers and  $\mathcal{P}(\mathbb{N}_0)$  be its power set. An integer additive set-labeling (IASL) of a graph  $G$  is an injective function  $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$  such that the induced function  $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$  is defined by  $f^+(uv) = f(u) + f(v)$ , where  $f(u) + f(v)$  is the sumset of  $f(u)$  and  $f(v)$ . The IASL  $f$  is said to be an integer additive set-indexer (IASI) if the associated function  $f^+$  is also injective. If  $f^+(uv) = k \forall uv \in E(G)$ , then  $f$  is said to be a  $k$ -uniform integer additive set-indexer. In this paper, we study the admissibility of a particular type of integer additive set-labeling, called topogenic IASL, by certain graphs.

**Keywords:** Integer additive set-labeled graphs; Topological integer additive set-labeled graphs; Topogenic integer additive set-labeled graphs; Topogenic integer additive set-graceful graphs.

**Mathematics Subject Classification 2010:** 05C78.

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## 1 Introduction

For all terms and definitions, other than newly defined or specifically mentioned in this paper, we refer to [3], [9] and [16] and for the concepts in topology, we further refer to [7] and [10]. Unless mentioned otherwise, the graphs considered in this paper are simple, finite, non-trivial and connected.

Let  $A$  and  $B$  be two non-empty sets. The *sumset* of  $A$  and  $B$  is denoted by  $A + B$  and is defined by  $A + B = \{a + b : a \in A, b \in B\}$ . If  $X = A + B$ , then  $A$  and  $B$  are said to be the *summands* of  $X$ . For any set  $A$ ,  $A = \{0\} + A$ . In this context, we say that  $\{0\}$  and  $A$  are the *trivial summands* of the set  $A$  and  $A$  is said to be the *trivial sumset* of

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$\{0\}$  and  $A$  itself. In this paper, by a sumset, we mean a non-trivial sumset unless stated otherwise.

Using the concepts of sumsets of two sets, the following notion has been introduced.

Let  $\mathbb{N}_0$  denote the set of all non-negative integers and  $\mathcal{P}(\mathbb{N}_0)$  be its power set. An *integer additive set-labeling* (IASL) of a graph  $G$  is an injective function  $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$  such that the induced function  $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$  is defined by  $f^+(uv) = f(u) + f(v)$ , where  $f(u) + f(v)$  is the sumset of  $f(u)$  and  $f(v)$ . A graph that admits an IASL is called an *integer additive set-labeled graph* (IASL-graph). An IASL  $f$  of a given graph  $G$  is said to be an *integer additive set-indexer* (IASI) if the associated function  $f^+$  is also injective (see [4, 5]).

An IASL (or an IASI)  $f$  of a graph  $G$  is said to be a *k-uniform IASL* (or a *k-uniform IASI*) if  $f^+(uv) = k \forall uv \in E(G)$ . The cardinality of the set-label of an element (a vertex or an edge) of a graph  $G$  is said to be the *set-indexing number* of that element (see [5, 11]).

Since the set-label of every edge of  $G$  is the sumset of the set-labels of its end vertices, it can be noted that no vertex of an IASL-graph  $G$  can have the empty set as its set-label. If any of the given two sets is countably infinite, then their sumset is also a countably infinite set. Hence, all sets we consider in this paper are non-empty finite sets of non-negative integers.

The notions of certain types of integer additive set-labelings are given below.

**Definition 1.1.** [13] An IASL  $f$  of a graph  $G$ , with respect to a ground set  $X \subset \mathbb{N}_0$ , is said to be a *topological IASL* (TIASL) of  $G$  if  $\mathcal{T} = f(V(G)) \cup \emptyset$  is a topology on  $X$ . A graph which admits a topological integer additive set-labeling is called a *topological integer additive set-labeled graph* (TIASL-Graph).

**Definition 1.2.** [14] Let  $f$  be an IASI of a graph  $G$  with respect to a ground set  $X$ . Then,  $f$  is said to be an *integer additive set-graceful labeling* (IASGL) of  $G$  if  $f^+(E) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$ . A graph which admits an integer additive set-graceful labeling is called an *integer additive set-graceful graph* (IASG-Graph).

**Definition 1.3.** [15] Let  $f$  be an IASI of a graph  $G$  with respect to a ground set  $X$ . Then,  $f$  is said to be an *integer additive set-sequential labeling* (IASSL) of  $G$  if  $f^*(G) = \mathcal{P}(X) - \{\emptyset\}$ . A graph which admits an integer additive set-sequential labeling is called an *integer additive set-sequential graph* (IASS-Graph).

Certain characteristics and structural properties of TIASL-graphs have been studied in [13]. The characteristics and properties of integer additive set-graceful graphs and integer additive set-sequential graphs have been discussed in [14] and [15] respectively.

In this paper, we introduce a new type of IASL, called a topogenic IASL and initiate a study about topogenic IASLs and the graphs which admit topogenic IASLs.

## 2 Topogenic IASL-graphs

Motivated from the studies made in [1], [2] and [6], we define a sumset analogue of the topogenic set-indexers of graphs as follows.

**Definition 2.1.** Let  $X$  be a finite non-empty set of non-negative integers. An integer additive set-indexer  $f$  of a given graph  $G$ , defined by  $f : V(G) \rightarrow \mathcal{P}(X)$ , is said to be a *topogenic integer additive set-labeling* of  $G$ , with respect to the ground set  $X$ , if  $\mathcal{T} = f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\}$  is a topology on  $X$ . A graph  $G$  that admits a topogenic IASL is said to be a topogenic IASL-graph.

Let us now define a new function  $f^*$  on the given graph  $G$  as follows.

$$f^*(G) = \begin{cases} f(x) & \text{if } x \in V(G) \\ f^+(x) & \text{if } x \in E(G) \end{cases}$$

. Therefore,  $f^*(G) = f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\}$ . Hence,  $f$  is a topogenic IASL of  $G$  if  $f^*(G) \cup \{\emptyset\}$  is a topology on the ground set  $X$ .

For a topogenic IASL  $f$  defined on a given graph  $G$ , the topology  $f(V) \cup f^+(E) \cup \{\emptyset\}$  of the ground set  $X$  is denoted by  $\mathcal{T}_f$ .

**Example 2.1.** For a non-empty finite set  $X$  of non-negative integers, the trivial graph  $K_1$  admits a topogenic IASL with respect to the indiscrete topology of  $X$ .

**Example 2.2.** Let  $X$  be a two point set of non-negative integers. Let  $\mathcal{T}$  be a topology on  $X$ , which is neither the indiscrete topology nor the discrete topology on  $X$ . Then,  $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$  where  $a \in X$ . Then, the graph  $K_2$  admits a topogenic IASL with respect to the set  $X$  if and only if  $a = 0$ .

The following results establish the relations of topogenic IASLs of given graphs with the IASGL and IASSL of those graphs.

**Proposition 2.2.** Every IASGL of a graph  $G$  is a topogenic IASL of  $G$ .

*Proof.* Let  $f$  be an IASGL of a given graph  $G$ . Then,  $f^+(E) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$ . Since  $f$  be an IASGL of  $G$ ,  $\{0\}$  must be a set-label of vertex of  $G$ . (see [14]). Therefore,  $f^*(G) = \mathcal{P}(X) - \{\emptyset\}$ . Clearly,  $f^*(G) \cup \{\emptyset\}$  is the discrete topology on  $X$ . Hence,  $f$  is a topogenic IASL of  $G$ .  $\square$

**Proposition 2.3.** Every IASSL of a graph  $G$  is a topogenic IASL of  $G$ .

*Proof.* Let  $f$  be an IASSL of a given graph  $G$ . Then,  $f^*(G) = \mathcal{P}(X) - \{\emptyset\}$  and we have  $f^*(G) \cup \{\emptyset\}$  is the discrete topology on  $X$ . Hence,  $f$  is a topogenic IASL of  $G$ .  $\square$

A significant observation on the topogenic IASL-graphs is given below.

**Observation 2.4.** If  $f : V(G) \rightarrow \mathcal{P}(X)$  is a topogenic IASL of  $G$ , then it is possible for a vertex  $v \in V(G)$  and an edge  $e \in E(G)$ , that  $f(v) = f^+(e)$ .

In observation 2.4, we observed that there may exist a vertex and an edge in  $G$  having the same set-label. In the following theorem, we discuss the minimum number of vertices and edges in a graph  $G$  which have the same set-labels.

**Lemma 2.5.** Let  $f$  be a topogenic IASL defined on a given graph  $G$  with respect to a topology  $\mathcal{T}$  of a ground set  $X$ . Then, there are at least  $\rho'$  elements in  $\mathcal{T}$  common to  $f(V)$  and  $f^+(E)$ , where  $\rho'$  is the number of elements in  $\mathcal{T}$  which are neither the summands nor the sumsets of some other sets in  $\mathcal{T}$ .

*Proof.* Let  $\mathcal{B}$  be the subset of  $\mathcal{T}$ , consisting the elements which are neither the summands nor the sumsets of any other sets in  $\mathcal{T}$ . Let  $|\mathcal{B}| = \rho'$ . Since no element in  $\mathcal{B}$  is a non-trivial sumset of elements of  $\mathcal{T}$ , all elements of  $\mathcal{B}$  must be the set-label of some vertices of  $G$ . Let  $V'$  be the set of these vertices with set-labels from  $\mathcal{B}$ . Since, these elements are not non-trivial summands of other sets in  $\mathcal{T}$ , they can be adjacent only to the vertex having the set-label  $\{0\}$ , if it exists. Since  $G$  is connected graph, then  $\{0\}$  must be the set-label of a vertex, say  $v$  of  $G$ . Then, the subgraph of  $G$  induced by  $V' \cup \{v\}$  is a star graph and the edges in this graph have the same set-label of their end vertices other than  $v$ . Therefore,  $|f(V) \cap f^+(E)| \geq |\mathcal{B}| = \rho'$ .  $\square$

The following lemma is an immediate consequence of Lemma 2.5.

**Lemma 2.6.** Let  $X$  be a non-empty finite ground set. If  $X$  is not a sumset of two subsets of  $X$ , then a graph  $G$  admits a topogenic IASL  $f$  if

1.  $0 \in X$  and  $\{0\} \in f(V)$ .
2.  $G$  has at least one pendant vertex.

*Proof.* Let  $f$  is a topogenic IASL of a graph  $G$  with respect to the non-empty finite ground set  $X$ . If  $X$  is not the sumset of its subsets, then it has only trivial summands. Since  $X \in f(V)$  and  $G$  has no trivial vertices, the vertex  $v$  having the set-label  $X$  must be adjacent to some other vertex of  $G$ . But, since  $X$  has only trivial summands,  $v$  can be adjacent to another vertex  $u$  if  $u$  has the set-label  $\{0\}$ . Therefore,  $\{0\} \in f(V)$  and hence  $0 \in X$ . More over,  $v$  can be adjacent to only one vertex of  $G$ . Hence,  $v$  is a pendant vertex of  $G$ .  $\square$

In view of the above result, we have

**Theorem 2.7.** Let  $X$  be non-empty finite ground set and let  $\mathcal{T}$  be a topology on  $X$  such that  $X$  is not a sumset of any elements in  $\mathcal{T}$ . If a graph  $G$  admits a topogenic IASL  $f$  with respect to  $\mathcal{T}$ , then  $\{0\} \in \mathcal{T}$  and  $G$  has at least one pendant vertex.

*Proof.* Given that  $\mathcal{T}$  is a topology on  $X$  such that  $X$  is not a sumset of any elements of  $\mathcal{T}$ . Then,  $X$  can not be the set-label of any edge of  $G$  and  $X$  is the set-label of a vertex, say  $v$ , of  $G$ . More over,  $v$  can be adjacent to another vertex, say  $u$ , of  $G$  if  $u$  has the set-label  $\{0\}$ .  $\square$

In view of the above result, we can find a necessary condition for a topogenic IASL-graph to have pendant vertices.

**Theorem 2.8.** Let  $f$  be a topogenic IASL of a graph  $G$  with respect to a topology  $\mathcal{T}$  of the ground set  $X$ . If  $\mathcal{T}$ , consists of  $\rho'$  sets which are neither the sumsets nor the summands of some other sets in  $\mathcal{T}$ , then  $G$  should necessarily have at least  $\rho'$  pendant vertices.

In view of the above results, we note an important observation as given below.

**Observation 2.9.** If  $f : V(G) \rightarrow \mathcal{P}(X)$  is a topogenic IASL of a graph  $G$  with respect to a ground set  $X$ , then the set  $X$  is the set-label of an edge or a pendant vertex of  $G$ .

The following theorem is an immediate consequence of the above results.

**Theorem 2.10.** Let  $f$  be a topogenic IASL defined on a graph  $G$  with respect to a non-empty finite set  $X$ . Then, every set in  $\mathcal{T}_f$  is either a summand of some sets in  $\mathcal{T}_f$  or the sumset of two sets in  $\mathcal{T}_f$ .

Figure 1 illustrates a graph which admits a topogenic IASL with respect to the ground set  $X = \{0, 1, 2, 3\}$ .

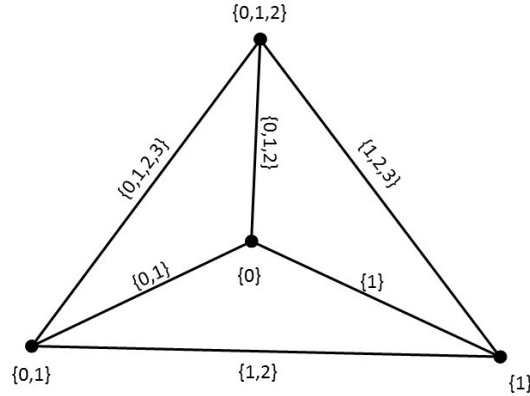


Figure 1: A topogenic IASL-graph without pendant vertices

**Definition 2.11.** Let  $X$  be a non-empty finite ground set and let  $\mathcal{T}$  be a topology on  $X$ . Then, a graph  $G$  is said to be a *topogenic graphical realisation* of  $\mathcal{T}$  if there exists a topogenic IASL  $f$  on  $G$  such that  $f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\} = \mathcal{T}$ .

The following theorem establishes the existence of topogenic graphical realisation for a given topology of the ground set  $X$ .

**Theorem 2.12.** Let  $X$  be a ground set which contains the element 0 and  $\mathcal{T}$  be any topology on  $X$  which contains the set  $\{0\}$ . Then, there exists a topogenic graphical realisation for the topology  $\mathcal{T}$ .

*Proof.* Let  $\mathcal{T}$  be a topology on the ground set  $X$  containing the set  $\{0\}$ . What we have to do is to construct a graph  $G$  such that  $f(V) \cup f^+(E) = \mathcal{T} - \{\emptyset\}$ . Plot a vertex, say  $v_1$  and label this vertex by the set  $\{0\}$ . Let  $\mathcal{A}$  be the subset of  $\mathcal{T}$  which contains the sets in  $\mathcal{T}$  which are not the sumsets of any other sets in  $\mathcal{T}$ . Take  $|\mathcal{A}|$  new vertices and join all these vertices to the vertex  $v_1$ . Now, draw edges between the vertices, the sumset of whose set-labels are in  $\mathcal{T}$ . The graph thus obtained is a topogenic graphical realisation of  $\mathcal{T}$ .  $\square$

Figure 2 illustrates the existence of a topogenic graphical realisation of a topology  $\mathcal{T} = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}, \{0, 1, 2, 3\}, \{0, 1, 2, 3, 4\}\}$  of the ground set  $X = \{0, 1, 2, 3, 4\}$ .

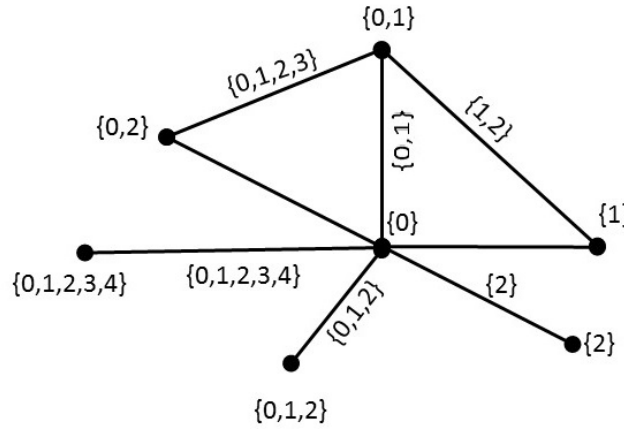


Figure 2: A topogenic graphical realisation of Topology of  $X$ .

A necessary and sufficient condition for the existence of a topogenic graphical realisation for a given topology of the ground set  $X$ .

**Theorem 2.13.** Let  $X$  be a non-empty finite ground set and let  $\mathcal{T}$  be a topology on  $X$ . Then, a graph  $G$  is a topogenic graphical realisation of  $\mathcal{T}$  if and only if every non-empty set of  $\mathcal{T}$  is either a summand of some other elements of  $\mathcal{T}$  or the sumset of two elements of  $\mathcal{T}$ .

*Proof.* Given that  $\mathcal{T}$  be a topology on the ground set  $X$ . First assume that  $\mathcal{T}$  has a topogenic graphical realisation, say  $G$ . Then, there exists an IASI  $f$  for  $G$  such that  $f(V) \cup f^+(E) = \mathcal{T} - \{\emptyset\}$ . That is, every non-empty set in  $X$  is either a set-label of a vertex or an edge of  $G$ . Therefore, every non-empty set of  $\mathcal{T}$  is either a summand of some other elements of  $\mathcal{T}$  or the sumset of two elements of  $\mathcal{T}$ .

Conversely, assume that every non-empty set of  $\mathcal{T}$  is either a summand of some other elements of  $\mathcal{T}$  or the sumset of two elements of  $\mathcal{T}$ . The sets in  $\mathcal{T}$ , which are summands of some other sets of  $\mathcal{T}$ , but not sumsets of any sets in  $\mathcal{T}$ , must be the set-labels of

vertices of  $G$ . All other sets in  $\mathcal{T}$  must be the set-labels of other vertices or edges of  $G$ . Therefore,  $f(V) \cup f^+(E) = \mathcal{T} - \{\emptyset\}$  and hence  $f$  is a topogenic IASL of  $G$ .  $\square$

The following theorem is on the number of elements of a graph which admits a topogenic IASL with respect to a given ground set  $X$ .

**Theorem 2.14.** Let  $G$  be a topogenic IASL-graph with respect to a topology  $\mathcal{T}$  of a given ground set  $X$  and let  $\mathcal{T}$  contains  $\rho$  sets which are not the sumsets of any other sets in  $\mathcal{T}$ . Then,  $|V(G)| \geq \rho$  and  $|E(G)| \geq \tau - \rho$ , where  $\tau = |\mathcal{T}|$ .

*Proof.* Let  $f$  be a topological IASL defined on  $G$  with respect to a ground set  $X$  and  $\mathcal{T}$  be the topology of  $X$  such that  $f(V) \cup f^+(E) = \mathcal{T} - \{\emptyset\}$ , where  $|\mathcal{T}| = \tau$ . Let  $\mathcal{A}$  be the subset of  $\mathcal{T}$  whose elements are not the sumset of any elements in  $\mathcal{T}$ . Let  $|\mathcal{A}| = \rho$ . Then, no edge of  $G$  can have any element of  $\mathcal{T}$  as their set-labels and hence all elements of  $\mathcal{A}$  must be the set-labels of the vertices of  $G$ . Then,  $|V(G)| = |\mathcal{A}| = \rho$  and the number of edges in  $G$  is  $|\mathcal{T}| - |\mathcal{A}| = \tau - \rho$ . Therefore, an arbitrary topogenic IASL-graph  $G$ ,  $|V(G)| \geq \rho$  and  $|E(G)| \geq \tau - \rho$ .  $\square$

### 3 Conclusion

In this paper, we have introduced the notion of a topogenic IASL-graphs and discussed certain structural properties of topogenic IASL-graphs. Some of the following problems which we have identified in this area are given below.

**Problem 1.** Determine the minimum cardinality of a ground set  $X$  with respect to which a given graph  $G$  admits a topogenic IASL.

**Problem 2.** For a given ground set  $X$ , Find the number of topologies on  $X$  with respect to which a given graph  $G$  admits a topogenic IASL.

**Problem 3.** Characterise the graph classes according to whether they admit topogenic IASLs or not.

**Problem 4.** Verify the existence of a topogenic graphical realisation for an arbitrary topology of a ground set  $X$ .

More properties and characteristics of this type IASL, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain other types IASLs are still open. Studies about those IASLs which assign sets having specific properties, to the elements of a given graph are also noteworthy. All these facts highlight a wide scope for further studies in this area.

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