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| Davide Giraud. Limit theorems for U-statistics of Bernoulli data. 2019. hal-02097836

HAL Id: hal-02097836

<https://hal.science/hal-02097836v1>

Preprint submitted on 12 Apr 2019

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LIMIT THEOREMS FOR U-STATISTICS OF BERNOULLI DATA

DAVIDE GIRAUDO

ABSTRACT. In this paper, we consider U-statistics whose data is a strictly stationary sequence which can be expressed as a functional of an i.i.d. one. We establish a strong law of large numbers, a bounded law of the iterated logarithms and a central limit theorem under a dependence condition. The main ingredients for the proof are an approximation by U-statistics whose data is a functional of ℓ i.i.d. random variables and an analogue of the Hoeffding's decomposition for U-statistics of this type.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Context.** Let $(X_j)_{j \geq 1}$ be a strictly stationary sequence, in the sense that the vectors $(X_i)_{i=1}^n$ and $(X_{i+k})_{i=1}^n$ have the same distribution for all n and $k \geq 1$. The U-statistic of kernel $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and data $(X_j)_{j \geq 1}$ is defined as

$$U_n := \sum_{1 \leq i < j \leq n} h(X_i, X_j), \quad n \geq 2. \quad (1.1)$$

The study of the asymptotic behavior of the sequence $(U_n)_{n \geq 2}$ properly normalized is a question of interest in probability theory and the applications. We will be interested in the following three limit theorems.

(1) Law of large numbers: let $1 \leq p < 2$; the following convergence holds

$$\frac{1}{n^{1+1/p}} (U_n - \mathbb{E}[U_n]) \rightarrow 0 \text{ a. s.}; \quad (1.2)$$

(2) Bounded law of the iterated logarithms: the random variable

$$\sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |U_n - \mathbb{E}[U_n]| \quad (1.3)$$

is almost surely finite, where $L: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by $L(x) = \max\{\ln x, 1\}$ and $\text{LL}(x) := L \circ L(x)$.

(3) Central limit theorem: there exists a $\sigma > 0$ such that

$$\frac{1}{n^{3/2}} (U_n - \mathbb{E}[U_n]) \rightarrow \sigma N \text{ in distribution}, \quad (1.4)$$

where N is a standard normal random variable.

Usually, the conditions for guaranting this kind of limit theorems are on the dependence of the sequence $(X_j)_{j \geq 1}$ and also on the kernel h , for example by requiring some integrability conditions on $h(X_1, X_2)$. We will first review a few results for the case where the data $(X_j)_{j \geq 1}$ is i.i.d. and the kernel h is symmetric.

(1) If $1 \leq p < 2$ and $h(X_1, X_2) \in L^p$, then (1.2) holds [11].

Date: April 12, 2019.

Key words and phrases. Law of large numbers; law of the iterated logarithms; central limit theorem; U-statistics; functionals of i.i.d.

- (2) If $h(X_1, X_2) \in \mathbb{L}^2$, then the random variable defined by (1.3) is almost surely finite. Moreover, for all $1 < p < 2$, according to Theorem 2.5 in [5], the following inequality holds:

$$\left\| \sup_{n \geq 1} \frac{1}{\sqrt{nL(L(n))}} \left| U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})] \right| \right\|_p \leq C_p \|h(X_1, X_2)\|_2. \quad (1.5)$$

- (3) The convergence (1.4) has been established in [13].

The extension of these results to the case of stationary dependent data is a challenging problem. The law of large numbers has been established under a β -mixing assumption in [3, 10]. In the case $p = 1$, the independence assumption can be replaced by a 4-dependence assumption, that is, for all 4-uple of distinct integers (i_1, i_2, i_3, i_4) , the collection of random variables $(X_{i_k})_{k=1}^4$ is independent. If the sequence $(X_j)_{j \geq 1}$ is identically distributed, then the law of large numbers hold (Theorem 1 in [4]). The question of an equivalent of the ergodic theorem when the data is any strictly stationary sequence was considered in [1]. The question of the law of the iterated logarithms was also addressed in [9] where the data is allowed to be α -mixing or a functional of a β -mixing sequence. A central limit theorem has been established under a β -mixing assumption in [2] and under an α -mixing condition in [8].

In this paper, we will be interested in establishing the law of large number, the law of the iterated logarithms and the central limit theorem when the data can be expressed as a functional of an i.i.d. sequence. In a similar context on the data but for weighted U-statistics, the central limit theorem was investigated in [14].

Let us precise the context. Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be an independent identically distributed sequence with values in \mathbb{R}^k , $k \geq 1$. Given measurable functions $h: \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ and $f: (\mathbb{R}^k)^{\mathbb{Z}} \rightarrow \mathbb{R}$, we are interested in the asymptotic behavior of the U-statistic of order two defined by

$$U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) = \sum_{1 \leq i < j \leq n} h(f((\varepsilon_{i-k})_{k \in \mathbb{Z}}), f((\varepsilon_{j-k})_{k \in \mathbb{Z}})), \quad (1.6)$$

that is, letting $X_j := f((\varepsilon_{j-k})_{k \in \mathbb{Z}})$, U_n is a U-statistic of kernel h and the data is the strictly stationary sequence $(X_j)_{j \geq 1}$. More precisely, we are interested in conditions involving the kernel h and the sequence $(f((\varepsilon_{j-k})_{k \in \mathbb{Z}}))_{j \geq 1}$ which guarantee the previously mentioned limit theorems.

In all the paper, the kernel h is supposed to be symmetric in the sense that $h(x, y) = h(y, x)$ for all $x, y \in \mathbb{R}$.

The paper is organized as follows. In Subsection 1.2, we will introduce a measure of dependence of a U-statistic whose data is a functional of an i.i.d. sequence. In Subsection 1.3, we formulate an analogue of the Hoeffding decomposition for such U-statistics. Subsections 1.4, 1.5 and 1.6 are devoted respectively to the statements of the law of large numbers, the bounded law of the iterated logarithms and the central limit theorems for U-statistics of Bernoulli data. In Subsection 1.7, we give examples of kernels h for which the measure of dependence can be estimated only with the help of the dependence of the data. Section 2 is devoted to the proofs of the previously mentioned results.

1.2. Measure of dependence. The extension of the results of the i.i.d. case requires a measure of dependence. Let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence, $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ a measurable function and $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. In order to deal with the dependence which comes into play in $U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})$, we need the following notations. Denote $X_j := f((\varepsilon_{j-i})_{i \in \mathbb{Z}})$ and define the random vectors

$$V_{j,\ell} := (\varepsilon_u)_{u=j-\ell}^{j+\ell}. \quad (1.7)$$

The random variable $\mathbb{E}[X_j | V_{j,\ell}]$ can be written as a function of $V_{j,\ell}$ and by stationarity and Lemma A.6, the involved function does not depend on j . Therefore, we write

$$\mathbb{E}[X_j | V_{j,\ell}] = f_\ell(V_{j,\ell}). \quad (1.8)$$

We then define for $p \geq 1$ and $\ell \geq 1$ the coefficient of dependence

$$\theta_{\ell,p} := \sup_{j \geq 0} \|h(f_\ell(V_{0,\ell}), f_\ell(V_{j,\ell})) - h(f_{\ell-1}(V_{0,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))\|_p \quad (1.9)$$

and for $\ell = 0$, $\theta_{0,p} = \sup_{j \geq 0} \|h(f_0(V_{j,0}), f_0(V_{0,0}))\|_p$.

In particular, finiteness of $\sum_{\ell \geq 1} \theta_{\ell,p}$ allows to write

$$\begin{aligned} U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) &= \sum_{1 \leq i < j \leq n} \sum_{\ell \geq 1} h(f_\ell(V_{i,0}), f_\ell(V_{j,0})) \\ &\quad + \sum_{1 \leq i < j \leq n} h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) - h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1})), \end{aligned} \quad (1.10)$$

where the convergence takes place in \mathbb{L}^p and almost surely. The interest of the decomposition (1.10) is that it reduces the treatment of the original U-statistic to that of U-statistics whose data is a strictly stationary sequence which is a functional of $2\ell + 1$ independent identically distributed random variables. Nevertheless, this task requires some work in order to be reduced to U-statistics of independent data.

1.3. A generalized Hoeffding's decomposition. A useful tool to establish limit theorems for U-statistics with i.i.d. data is the Hoeffding's decomposition [13]. Let $h: \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a symmetric measurable function and $(X_j)_{j \in \mathbb{Z}}$ be an i.i.d. sequence. We write decompose h in the following way:

$$h(x, y) = \theta + h_1(x) + h_1(y) + h_2(x, y), \quad (1.11)$$

where $\theta = \mathbb{E}[h(X_1, X_2)]$,

$$h_1(x) = \mathbb{E}[h(X_1, x)] - \theta \text{ and} \quad (1.12)$$

$$h_2(x, y) = h(x, y) - h_1(x) - h_1(y) - \theta. \quad (1.13)$$

In this way, the following equality holds

$$\sum_{1 \leq i < j \leq n} h(X_i, X_j) = \binom{n}{2} \theta + n \sum_{i=1}^n h_1(X_i) + \sum_{1 \leq i < j \leq n} h_2(X_i, X_j). \quad (1.14)$$

The part involving h_2 can be treated by martingale techniques, since the sequence $(\sum_{i=1}^{j-1} h_2(X_i, X_j))_{j \geq 1}$ is a martingale differences sequence with respect to the filtration $(\sigma(X_u, u \leq j))$ and the terms $\sum_{i=1}^{j-1} h_2(X_i, X_j)$ can be treated thanks to a reverse martingale differences property.

We would like to extend this to the setting mentioned in 1.1, that is, $X_j := f((\varepsilon_{j-k})_{k \in \mathbb{Z}})$, where $(\varepsilon_u)_{u \in \mathbb{Z}}$ is an i.i.d. sequence. We introduce the following notation

$$U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}}) = \sum_{1 \leq i < j \leq n} h(\varepsilon_i, \varepsilon_j), \quad (1.15)$$

where $(\varepsilon_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence of random variables with values in \mathbb{R}^k and $h: \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a measurable function.

One naturally expects the decomposition to be more complicated as in the independent case. Let us point out the major differences and common points. Like in the independent case, the decomposition of the centered U-statistic involves a stationary sequence and a degenerated U-statistic. But in the context of Bernoulli data, one get a series involving stationary sequences and a series of degenerated U-statistics. An other difference is that we also have remainder terms which are not directly associated to the involved stationary sequences or degenerated U-statistic. The origin of these terms will be explained during the proof.

Proposition 1.1. *Let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence of random variables, $(\varepsilon'_u)_{u \in \mathbb{Z}}$ an independent copy of $(\varepsilon_u)_{u \in \mathbb{Z}}$ and $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be a measurable function. Let $V_{k,\ell} := (\varepsilon_u)_{u=k-\ell}^{k+\ell}$ and $V'_{k,\ell} := (\varepsilon'_u)_{u=k-\ell}^{k+\ell}$. Let $f_\ell: \mathbb{R}^{2\ell+1} \rightarrow \mathbb{R}$ be a function such that*

$$f_\ell(V_{k,\ell}) = \mathbb{E} \left[f \left((\varepsilon_{k-u})_{u \in \mathbb{Z}} \right) \mid V_{k,\ell} \right] \text{ a.s.} \quad (1.16)$$

Let $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric measurable function. Assume that the following convergence holds almost surely for all $1 \leq i < j$:

$$\lim_{\ell \rightarrow +\infty} h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) = h \left(f \left((\varepsilon_{i-u})_{u \in \mathbb{Z}} \right), f \left((\varepsilon_{j-u})_{u \in \mathbb{Z}} \right) \right). \quad (1.17)$$

Then the following equality holds:

$$\begin{aligned} U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})] &= n \sum_{k=1}^n \mathbb{E} [h(f_0(V_{k,0}), f_0(V'_{k,0})) \mid V_{k,0}] + U_n^{\text{ind}}(h^{(0)}, (\varepsilon_i)_i) + \\ &\sum_{\ell \geq 1} (4\ell + 1) \left\lfloor \frac{n}{4\ell + 1} \right\rfloor \sum_{k=1}^{(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1} \mathbb{E} [h(f_\ell(V_{k,\ell}), f_\ell(V'_{0,\ell})) \mid V_{k,\ell}] - \mathbb{E} [h(f_{\ell-1}(V_{k,\ell-1}), f_{\ell-1}(V'_{0,\ell-1})) \mid V_{k,\ell-1}] \\ &+ \sum_{\ell \geq 1} \sum_{a, b \in [4\ell+2]} U_{\lfloor \frac{n}{2\ell} \rfloor}^{\text{ind}} \left(h_{a,b}^{(\ell)}, (\varepsilon_i^{a,b}) \right) + R_{n,1,1} + R_{n,1,2} + \sum_{k=2}^6 R_{n,k}, \end{aligned} \quad (1.18)$$

where for each $\ell \geq 1$ and all $a, b \in [4\ell + 2]$, the U -statistic $U_{\lfloor \frac{n}{2\ell} \rfloor}^{\text{ind}} \left(h_{a,b}^{(\ell)}, (\varepsilon_i^{a,b}) \right)$ has independent data and is degenerated, and the remainder terms are defined as

$$R_{n,1,1} := \sum_{\ell \geq 1} \sum_{j=(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{i=(4\ell+2) \lfloor \frac{j-1}{4\ell+2} \rfloor + 1}^{j-1} H_{i,j}^{(\ell)} \quad (1.19)$$

$$R_{n,1,2} := \sum_{\ell \geq 1} \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} H_{(4\ell+2)k+a,j}^{(\ell)}; \quad (1.20)$$

$$R_{n,2} := \sum_{\ell \geq 1} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a, b \in [4\ell+2] \\ a < b}} H_{u(4\ell+2)+a, u(4\ell+2)+b}^{(\ell)} \quad (1.21)$$

$$R_{n,3} := \sum_{\ell \geq 1} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a, b \in [4\ell+2] \\ 0 \leq a-b \leq (2\ell+1)-1}} \left(H_{a, v(4\ell+2)+b}^{(\ell)} + H_{b, v(4\ell+2)+a}^{(\ell)} \right) \quad (1.22)$$

$$R_{n,4} = \sum_{\ell \geq 1} \sum_{\substack{a, b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \left(H_{a, (v+1)(4\ell+2)+b}^{(\ell)} + H_{(4\ell+2)+b, v(4\ell+2)+a}^{(\ell)} \right) \quad (1.23)$$

$$R_{n,5} = \sum_{\ell \geq 1} \sum_{\substack{a, b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 2} \left(H_{u(4\ell+2)+a, (u+1)(4\ell+2)+b}^{(\ell)} - H_{u(4\ell+2)+a, 2m(2\ell+1)+b}^{(\ell)} \right) \quad (1.24)$$

$$R_{n,6} = \sum_{\ell \geq 1} \sum_{\substack{a, b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \left(H_{b, v(4\ell+2)+a}^{(\ell)} - H_{v(4\ell+2)+b, v(4\ell+2)+a}^{(\ell)} \right), \quad (1.25)$$

with

$$\begin{aligned} H_{i,j}^{(\ell)} &:= h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) - \mathbb{E}[h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell}))] \\ &\quad - (h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1})) - \mathbb{E}[h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))]). \end{aligned} \quad (1.26)$$

1.4. Law of large numbers. We first present a result on the Marcinkiewicz law of large numbers for U-statistics of i.i.d. data. In order to extend it to the context of functional of i.i.d., we need a control on a maximal function.

Proposition 1.2 (Marcinkiewicz-Zygmund law of large numbers). *Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence with values in \mathbb{R}^d and let $h: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. For $1 \leq p < 2$, define*

$$M_p := \sup_{n \geq 1} \frac{1}{n^{2/p}} \left| U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}}) \right|. \quad (1.27)$$

Assume that $\mathbb{E}[|h(\varepsilon_0, \varepsilon_1)|^p]$ is finite and that for all $x \in \mathbb{R}^d$, $\mathbb{E}[h(\varepsilon_0, x)] = 0$. Then the following statements hold:

- (1) the sequence $(n^{-2/p} |U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}})|)_{n \geq 1}$ converges to zero almost surely ;
- (2) for any positive x ,

$$x^p \mathbb{P}\{M_p > x\} \leq \kappa_p \mathbb{E}[|h(\varepsilon_0, \varepsilon_1)|^p], \quad (1.28)$$

where κ_p is bigger than 1 and depends only on p .

Corollary 1.3. *Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence with values in \mathbb{R}^d and let $h: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. For $1 \leq p < 2$, define*

$$M_p := \sup_{n \geq 1} \frac{1}{n^{1+1/p}} \left| U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}}) \right|. \quad (1.29)$$

Assume that $\mathbb{E}[|h(\varepsilon_0, \varepsilon_1)|^p]$ is finite and $\mathbb{E}[h(\varepsilon_0, \varepsilon_1)] = 0$. Then the following statements hold:

- (1) the sequence $(n^{-1-1/p} |U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}})|)_{n \geq 1}$ converges to zero almost surely ;
- (2) for any positive x ,

$$x^p \mathbb{P}\{M_p > x\} \leq \kappa_p \mathbb{E}[|h(\varepsilon_0, \varepsilon_1)|^p], \quad (1.30)$$

where κ_p is bigger than 1 and depends only on p .

The previous Proposition gives a control in terms of the weak- L^p -semi-norm, defined as $\|X\|_{p,w} := \sup_{x>0} (x^p \mathbb{P}\{|X| > x\})^{1/p}$, $p > 1$. However, the triangular inequality may fail, which is not convenient in view of the decomposition obtained in Proposition 1.1. For this reason, we introduce the weak- L^p -norm, defined as

$$\|X\|_{p,\infty} := \sup_{A: \mathbb{P}(A) > 0} \mathbb{P}(A)^{1/p-1} \mathbb{E}[|X| \mathbf{1}_A]. \quad (1.31)$$

Notice that for all fixed $p > 1$, there exists constants c_p and C_p such that for all random variable X , $c_p \|X\|_{p,w} \leq \|X\|_{p,\infty} \leq C_p \|X\|_{p,w}$. Moreover, the observation that for any sequence of non-negative random variables $(Y_n)_{n \geq 1}$,

$$\left\| \sup_{n \geq 1} Y_n \right\|_{p,w}^p \leq \sum_{n \geq 1} \|Y_n\|_{p,w}^p \quad (1.32)$$

will be useful in the sequel.

We are now in position to present our first result, which gives a control on the maximal function of a U-statistic whose data is a functional of an i.i.d. sequence.

Theorem 1.4. *Let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence, $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions. Let $p \in [1, 2)$. Then*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} \left| U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}})] \right| \right\|_{p, \infty} \leq c_p \left(\theta_{0,p} + \sum_{\ell \geq 1} \ell^{1-1/p} \theta_{\ell,p} \right). \quad (1.33)$$

The second result of this Subsection is a Marcinkiewicz law of large numbers.

Theorem 1.5 (Marcinkiewicz law of large numbers). *Let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence, $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions. Let $p \in [1, 2)$. Suppose that*

$$\sum_{\ell \geq 0} \ell^{1-1/p} \theta_{\ell,p} < +\infty. \quad (1.34)$$

Then the following almost sure convergence holds:

$$\frac{1}{n^{1+1/p}} \left| U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}})] \right| \rightarrow 0. \quad (1.35)$$

1.5. Bounded law of the iterated logarithms. Let us present our next result concerning the bounded law of the iterated logarithms. Like in the independent case, one can control the moments of order $p \in (1, 2)$ of the maximal function of U-statistic with the normalisation $n^{3/2} \text{LL}(n)$.

Theorem 1.6. *Let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence, $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions. Let $p \in [1, 2)$. For all $1 \leq p < 2$, the following inequality holds:*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} \left| U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}})] \right| \right\|_p \leq c_p \left(\theta_{0,2} + \sum_{\ell \geq 1} \ell^{1/2} \theta_{\ell,2} \right), \quad (1.36)$$

where c_p depends only on p .

1.6. Central limit theorem. Let us present our result concerning the central limit theorem. We will make essentially three assumptions on the dependence of our U-statistic. The first and second one involve the coefficients $\delta_{\cdot,2}$ and $\theta_{\ell,1}$ respectively and the third condition is imposed is summability of the family of covariances of a strictly stationary sequence which comes from the generalized Hoeffding decomposition of Proposition 1.1.

Theorem 1.7. *Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function, $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be a measurable function and $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. sequence. Let*

$$X_j := f((\varepsilon_{j-k})_{k \in \mathbb{Z}}), \quad U_n := \sum_{1 \leq i < j \leq n} h(X_i, X_j). \quad (1.37)$$

Suppose that

$$\sum_{\ell \geq 0} \ell^{1/2} \theta_{\ell,2} < +\infty; \quad \sum_{\ell \geq 0} \ell^2 \theta_{\ell,1} < +\infty \quad (1.38)$$

and that

$$\sum_{k \in \mathbb{Z}} |\text{Cov}(Y_0, Y_k)| < +\infty, \quad (1.39)$$

where the random variable Y_k is defined by the following \mathbb{L}^2 -convergence

$$Y_k = \lim_{\ell \rightarrow +\infty} \mathbb{E} [h(f_\ell(V_{k,\ell}), f_\ell(V'_{0,\ell})) \mid V_{k,\ell}] - \mathbb{E} [h(f_\ell(V_{k,\ell}), f_\ell(V'_{0,\ell}))] \quad (1.40)$$

and $V_{k,\ell} = (\varepsilon_u)_{u=k-\ell}^{k+\ell}$, $V'_{k,\ell} = (\varepsilon'_u)_{u=k-\ell}^{k+\ell}$, where $(\varepsilon'_u)_{u \in \mathbb{Z}}$ is an independent copy of $(\varepsilon_u)_{u \in \mathbb{Z}}$.

Then the following convergence in distribution holds:

$$\frac{1}{n^{3/2}} (U_n - \mathbb{E}[U_n]) \rightarrow N(0, \sigma^2), \quad (1.41)$$

where

$$\sigma^2 := \sum_{k \in \mathbb{Z}} \text{Cov}(Y_0, Y_k). \quad (1.42)$$

Remark 1.8. It is not clear to us whether the condition (1.38) guarantees (1.39). Nevertheless, conditions of the type

$$\sum_{j \geq 0} \|\mathbb{E}[Y_j | \sigma(\varepsilon_u, u \leq 0)] - \mathbb{E}[Y_j | \sigma(\varepsilon_u, u \leq -1)]\|_2 < \infty \quad (1.43)$$

guarantee (1.39) and are not too hard to check for the choices of kernel involved in Subsection 1.7.

1.7. Applications. In this Subsection, we give examples of kernels h for which the measure of dependence defined by (1.9) can be estimated.

(1) Uniformly continuous kernel

Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function. We assume that there exists a non-negative function $\omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is increasing, satisfies $\omega(0) = 0$ and for each $x, x', y, y' \in \mathbb{R}$,

$$|h(x, y) - h(x', y')| \leq \omega(|x - x'|) + \omega(|y - y'|). \quad (1.44)$$

Then the following inequality holds:

$$\begin{aligned} & \|h(f_\ell(V_{0,\ell}), f_\ell(V_{j,\ell})) - h(f_{\ell-1}(V_{0,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))\|_p \\ & \leq \|\omega(|f_\ell(V_{0,\ell}) - f_{\ell-1}(V_{0,\ell-1})|)\|_p + \|\omega(|f_\ell(V_{j,\ell}) - f_{\ell-1}(V_{j,\ell-1})|)\|_p. \end{aligned} \quad (1.45)$$

Since $(V_{j,\ell}, V_{j,\ell-1})$ has the same distribution as $(V_{0,\ell}, V_{0,\ell-1})$, the two terms of the right hand side of (1.45) are equal hence

$$\theta_{\ell,p} \leq 2 \|\omega(|f_\ell(V_{0,\ell}) - f_{\ell-1}(V_{0,\ell-1})|)\|_p. \quad (1.46)$$

In particular, if h is α -Hölder continuous for some $\alpha \in (0, 1)$, we can choose $\omega: t \mapsto ct^\alpha$ for some constant c and in this case, the estimate (1.46) becomes

$$\theta_{\ell,p} \leq 2c \|f_\ell(V_{0,\ell}) - f_{\ell-1}(V_{0,\ell-1})\|_{p\alpha}^\alpha, \quad (1.47)$$

which can be rewritten as

$$\theta_{\ell,p} \leq 2c \|\mathbb{E}[X_0 | V_{0,\ell}] - \mathbb{E}[X_0 | V_{0,\ell-1}]\|_{p\alpha}^\alpha. \quad (1.48)$$

(2) Variance estimation. Consider the kernel $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $h(x, y) := (x - y)^2 / 2$. The associated U-statistic is (after normalization) the classical variance estimator. For this choice of kernel, one can estimate the measure of dependence defined by (1.9). One need to control the \mathbb{L}^p -norm of the difference of the square of two random variables Y and Z . Since

$$\|Y^2 - Z^2\|_p^p = \mathbb{E}[|Y - Z|^p |Y + Z|^p] \leq \mathbb{E}[|Y - Z|^{2p}]^{1/2} \mathbb{E}[|Y + Z|^{2p}]^{1/2} \quad (1.49)$$

we derive that

$$\|Y^2 - Z^2\|_p \leq \|Y - Z\|_{2p} (\|Y\|_{2p} + \|Z\|_{2p}). \quad (1.50)$$

We use this for a fixed $\ell \geq 1$ to $Y = f_\ell(V_{0,\ell}) - f_\ell(V_{j,\ell})$ and $Z = f_{\ell-1}(V_{0,\ell-1}) - f_{\ell-1}(V_{j,\ell-1})$. Accounting the following bounds (which are a consequence of stationarity):

$$\|Y\|_{2p} + \|Z\|_{2p} \leq 2 \|f_\ell(V_{0,\ell})\|_{2p} + 2 \|f_{\ell-1}(V_{0,\ell-1})\|_{2p} \leq 4 \|X_0\|_{2p}, \quad (1.51)$$

we get that

$$\theta_{\ell,p} \leq 2 \|X_0\|_{2p} \|\mathbb{E}[X_0 | V_{0,\ell}] - \mathbb{E}[X_0 | V_{0,\ell-1}]\|_p. \quad (1.52)$$

2. PROOFS

2.1. Proof of the generalised Hoeffding's decomposition. Let us explain the idea of proof of Proposition 1.1. First, we write $U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})$ as a series (index by ℓ) of U-statistics. For the term of index ℓ , the data of the corresponding U-statistic is a function of $2\ell + 1$ i.i.d. random variables. We then decompose this a sum of $4(2\ell + 1)$ U-statistics of independent data.

We divide the proof in two steps:

- (1) First we treat the case where $X_j = f((\varepsilon_u)_{u=j-\ell}^{j+\ell})$ and $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a symmetric function.
- (2) Then we write the mentioned U-statistic a sum of U-statistics tractable with the work of step 1, plus a remainder term.

Step 1: we decompose a U-statistic whose data is a function of $2\ell + 1$ i.i.d. random variables as a sum of $4\ell + 2$ U-statistics of i.i.d. data plus remainder terms.

Lemma 2.1. *Let $\ell \geq 1$ be an integer, $h: \mathbb{R}^{2\ell+1} \times \mathbb{R}^{2\ell+1} \rightarrow \mathbb{R}$ be a measurable function, let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. real-valued sequence and $(\varepsilon'_u)_{u \in \mathbb{Z}}$ an independent copy of $(\varepsilon_u)_{u \in \mathbb{Z}}$. Define $U_n(h, (\varepsilon_u)_{u \in \mathbb{Z}}) := \sum_{1 \leq i < j \leq n} h(V_i, V_j) = \sum_{1 \leq i < j \leq n} H_{i,j}$, where $V_j = (\varepsilon_u)_{u=j-\ell}^{j+\ell}$ and $H_{i,j} := h(V_i, V_j)$.*

Then

$$U_n(h, (\varepsilon_u)_{u \in \mathbb{Z}}) = \sum_{a,b \in [4\ell+2]} U_{\lfloor \frac{n}{4\ell+2} \rfloor}^{\text{ind}}(h_{a,b}, (\varepsilon_i^{a,b})) + \sum_{k=1}^6 R_{n,k}, \quad (2.1)$$

where

- the function $h_{a,b}: \mathbb{R}^{4\ell+2} \times \mathbb{R}^{4\ell+2} \rightarrow \mathbb{R}$ is defined for $0 \leq a - b \leq 2\ell$ by

$$h_{a,b}((x_i)_{i=1}^{4\ell+2}, (y_j)_{j=1}^{4\ell+2}) := h((x_{i+a-b})_{i=1}^{2\ell+1}, (y_j)_{j=1}^{2\ell+1}) + h((y_{j+a-b})_{j=1}^{2\ell+1}, (x_i)_{i=1}^{2\ell+1}) \quad (2.2)$$

and for $(2\ell + 1) \leq a - b \leq (4\ell + 2) - 1$ by

$$h_{a,b}((x_i)_{i=1}^{4\ell+2}, (y_j)_{j=1}^{4\ell+2}) := h((x_{i+a-b-(2\ell+1)})_{i=1}^{2\ell+1}, (y_j)_{j=1}^{2\ell+1}) + h((y_{j+a-b-(2\ell+1)})_{j=1}^{2\ell+1}, (x_i)_{i=1}^{2\ell+1}), \quad (2.3)$$

- the random vectors $\varepsilon_u^{a,b}$ are defined by

$$\varepsilon_u^{a,b} := (\varepsilon_j)_{j=u(4\ell+2)+b-\ell}^{(u+1)(4\ell+2)+b-\ell-1}, \quad 0 \leq a - b \leq 2\ell; \quad (2.4)$$

$$\varepsilon_u^{a,b} := (\varepsilon_j)_{j=u(4\ell+2)+b+\ell+1}^{(u+1)(4\ell+2)+b+\ell+1}, \quad (2\ell + 1) \leq a - b \leq (4\ell + 2) - 1; \quad (2.5)$$

- the remainder terms are defined (with the convention that $\sum_{u=1}^{-k} = 0, k \leq 0$) by

$$R_{n,1} = \sum_{1 \leq i < j \leq n} H_{i,j} - \sum_{1 \leq i < j \leq (4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor} H_{i,j}; \quad (2.6)$$

$$R_{n,2} := \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a,b \in [4\ell+2] \\ a < b}} H_{u(4\ell+2)+a, u(4\ell+2)+b}; \quad (2.7)$$

$$R_{n,3} = \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a,b \in [4\ell+2] \\ 0 \leq a-b \leq (2\ell+1)-1}} (H_{a, v(4\ell+2)+b} + H_{b, v(4\ell+2)+a}); \quad (2.8)$$

$$R_{n,4} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} (H_{a, (v+1)(4\ell+2)+b} + H_{(4\ell+2)+b, v(4\ell+2)+a}); \quad (2.9)$$

$$R_{n,5} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 2} (H_{u(4\ell+2)+a, (u+1)(4\ell+2)+b} - H_{u(4\ell+2)+a, 2m(2\ell+1)+b}); \quad (2.10)$$

$$R_{n,6} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} (H_{b, v(4\ell+2)+a} - H_{v(4\ell+2)+b, v(4\ell+2)+a}). \quad (2.11)$$

Proof. Before going into the details of the proof, let us explain the general idea, which will also explain the origin of the remainder terms. First, it turns out that it would be more convenient that n is a multiple of $4\ell + 2$. However, it has no reason to be the case and we should also take into account the difference between U_n and $U_{n'}$ where n' is a multiple of $4\ell + 2$ close to n . The control of the difference of these two terms is precisely $R_{n,1}$. Now that we are reduced to the case where n is a multiple of $4\ell + 2$, we look that the terms $H_{i,j}$ and the remainder of i and j by the Euclidian division by $4\ell + 2$. If these remainder are closed enough, then we can write the corresponding sum as a U-statistic whose data are independent vectors of length $4\ell + 2$. If the remainders are too far way, we have to add and subtract a term to be reduced to the previous case, and this leads to the definition of $R_{n,k}$, $3 \leq k \leq 6$. If the quotient in the Euclidian division by $4\ell + 1$ of i and j are the same, then the corresponding sum is $R_{n,2}$.

From the equality $U_n(h_2, f, (\varepsilon_i)_{i \in \mathbb{Z}}) = \sum_{1 \leq i < j \leq n} H_{i,j}$ we get by (2.6) that

$$U_n(h_2, (\varepsilon_i)_{i \in \mathbb{Z}}) = \sum_{1 \leq i < j \leq (4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor} H_{i,j} + R_{n,1}. \quad (2.12)$$

For simplicity, let us denote by I the set $[4\ell + 2]$ and $m := \lfloor \frac{n}{4\ell+2} \rfloor$. With these notations and in view of (2.7), the following equality takes place

$$\begin{aligned} & \sum_{1 \leq i < j \leq (4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor} H_{i,j} \\ &= \sum_{0 \leq u < v \leq m-1} \sum_{\substack{a,b \in I \\ 0 \leq a-b \leq (2\ell+1)-1}} (H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, v(4\ell+2)+a}) \\ &+ \sum_{0 \leq u < v \leq m-1} \sum_{\substack{a,b \in I \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} (H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, v(4\ell+2)+a}) + R_{n,2}. \end{aligned} \quad (2.13)$$

Let us treat the first term. For $a, b \in I$ such that $0 \leq a - b \leq 2\ell$, in view of the definitions (2.2) and (2.4) and the symmetry of h_2 , the following equality holds

$$h_{a,b}(\varepsilon_u^{a,b}, \varepsilon_v^{a,b}) = H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, 2v(4\ell+2)+a} \quad (2.14)$$

hence

$$\begin{aligned} & \sum_{0 \leq u < v \leq m-1} \sum_{\substack{a,b \in I \\ 0 \leq a-b \leq (2\ell+1)-1}} (H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, v(4\ell+2)+a}) \\ &= U_m^{\text{ind}}(h_{a,b}, (\varepsilon_u^{a,b})_{u \in \mathbb{Z}}) + R_{n,3}. \end{aligned} \quad (2.15)$$

Let us treat the second term in the right hand side of (2.13). Adding and subtracting the terms $H_{u(4\ell+2)+a, (v+1)(4\ell+2)+b}$ and $H_{(u+1)(4\ell+2)+b, v(4\ell+2)+a}$ gives, after having rewritten the corresponding sums as double sums and exploited a telescoping of the inside sum,

$$\sum_{0 \leq u < v \leq m-1} \left(H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, v(4\ell+2)+a} \right) = S + R_{n,5} + R_{n,6}, \quad (2.16)$$

where

$$S := \sum_{0 \leq u < v \leq m-1} \left(H_{u(4\ell+2)+a, (v+1)(4\ell+2)+b} + H_{2(u+1)(2\ell+1)+b, v(4\ell+2)+a} \right) \quad (2.17)$$

We express S as a U-statistic of independent data. Noticing that $0 \leq a - (b + (2\ell + 1)) \leq (2\ell + 1) - 1$, we are in a similar situation as in the case $0 \leq a - b \leq (2\ell + 1) - 1$, with b replaced by $b + (2\ell + 1)$. Therefore, in view of (2.5), (2.3) and (2.9), we obtain that

$$S = U_{m-1}^{\text{ind}} \left(h_{a,b}, (\varepsilon_u^{a,b})_{u \in \mathbb{Z}} \right) + R_{n,4}. \quad (2.18)$$

Collecting these terms gives

$$\begin{aligned} & \sum_{\substack{a,b \in I \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{0 \leq u < v \leq m-1} \left(H_{u(4\ell+2)+a, v(4\ell+2)+b} + H_{u(4\ell+2)+b, v(4\ell+2)+a} \right) \\ &= \sum_{\substack{a,b \in I \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} U_{m-1}^{\text{ind}} \left(h_{a,b}, (\varepsilon_u^{a,b})_{u \in \mathbb{Z}} \right) + R_{n,4} + R_{n,5} + R_{n,6}. \end{aligned} \quad (2.19)$$

We end the proof of Lemma 2.1 by combining the equalities (2.13), (2.15) and (2.19). \square

Now, we can apply the Hoeffding decomposition to each U-statistic of i.i.d. data and rewrite the remainder term in a more tractable form.

Lemma 2.2. *Let $\ell \geq 1$ be an integer, $h: \mathbb{R}^{2\ell+1} \times \mathbb{R}^{2\ell+1} \rightarrow \mathbb{R}$ be a measurable function, let $(\varepsilon_u)_{u \in \mathbb{Z}}$ be an i.i.d. real-valued sequence and $(\varepsilon'_u)_{u \in \mathbb{Z}}$ an independent copy of $(\varepsilon_u)_{u \in \mathbb{Z}}$. Define $U_n(h, (\varepsilon_u)_{u \in \mathbb{Z}}) := \sum_{1 \leq i < j \leq n} h(V_i, V_j) - \mathbb{E}[h(V_i, V_j)] = \sum_{1 \leq i < j \leq n} H_{i,j}$, where $V_j = (\varepsilon_u)_{u=j-\ell}^{j+\ell}$, $V'_j = (\varepsilon'_u)_{u=j-\ell}^{j+\ell}$ and $H_{i,j} := h(V_i, V_j) - \mathbb{E}[V_i, V_j]$.*

Then

$$\begin{aligned} U_n(h, (\varepsilon_u)_{u \in \mathbb{Z}}) &= (4\ell + 2) \left[\frac{n}{4\ell + 2} \right] \sum_{k=1}^{(4\ell+2) \left(\lfloor \frac{n}{4\ell+2} \rfloor \right) + 1} \left(\mathbb{E}[h(V_k, V'_0) \mid V_k] - \mathbb{E}[h(V_k, V'_0)] \right) \\ &\quad + \sum_{a,b \in [4\ell+2]} U_{\lfloor \frac{n}{4\ell+2} \rfloor}^{\text{ind}} \left(h_{a,b}^{(2)}, (\varepsilon_i^{a,b}) \right) + \sum_{k=1}^6 R_{n,k}, \end{aligned} \quad (2.20)$$

where the function $h_{a,b}: \mathbb{R}^{4\ell+2} \times \mathbb{R}^{4\ell+2} \rightarrow \mathbb{R}$ is defined

$$\begin{aligned} h_{a,b}^{(2)} &:= h_{a,b} \left((x_i)_{i=1}^{4\ell+2}, (y_j)_{j=1}^{4\ell+2} \right) \\ &\quad - h_{a,b} \left((x_i)_{i=1}^{4\ell+2}, \varepsilon_0^{a,b} \right) - h_{a,b} \left((y_j)_{j=1}^{4\ell+2}, \varepsilon_0^{a,b} \right) + \mathbb{E} \left[h_{a,b} \left(\varepsilon_0^{a,b}, \varepsilon_0'^{a,b} \right) \right] \end{aligned} \quad (2.21)$$

and

- the function $h_{a,b}: \mathbb{R}^{4\ell+2} \times \mathbb{R}^{4\ell+2} \rightarrow \mathbb{R}$ is defined for $0 \leq a - b \leq 2\ell$ by

$$h_{a,b} \left((x_i)_{i=1}^{4\ell+2}, (y_j)_{j=1}^{4\ell+2} \right) := h \left((x_{i+a-b})_{i=1}^{2\ell+1}, (y_j)_{j=1}^{2\ell+1} \right) + h \left((y_{j+a-b})_{j=1}^{2\ell+1}, (x_i)_{i=1}^{2\ell+1} \right) \quad (2.22)$$

and for $(2\ell + 1) \leq a - b \leq (4\ell + 2) - 1$ by

$$\begin{aligned} h_{a,b} \left((x_i)_{i=1}^{4\ell+2}, (y_j)_{j=1}^{4\ell+2} \right) \\ := h \left((x_{i+a-b-(2\ell+1)})_{i=1}^{2\ell+1}, (y_j)_{j=1}^{2\ell+1} \right) + h \left((y_{j+a-b-(2\ell+1)})_{j=1}^{2\ell+1}, (x_i)_{i=1}^{2\ell+1} \right), \end{aligned} \quad (2.23)$$

- the random vectors $\varepsilon_u^{a,b}$ are defined by

$$\varepsilon_u^{a,b} := (\varepsilon_j)_{j=u(4\ell+2)+b-\ell}^{(u+1)(4\ell+2)+b-\ell-1}, 0 \leq a-b \leq 2\ell; \quad (2.24)$$

$$\varepsilon_u^{a,b} := (\varepsilon_j)_{j=u(4\ell+2)+b+\ell+1}^{(u+1)(4\ell+2)+b+\ell}, (2\ell+1) \leq a-b \leq (4\ell+2)-1; \quad (2.25)$$

- the remainder terms are defined (with the convention that $\sum_{u=1}^{-k} = 0, k \leq 0$) by

$$R_{n,1,1} := \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{i=(4\ell+2)\lfloor \frac{j-1}{4\ell+2} \rfloor + 1}^{j-1} H_{i,j} \quad (2.26)$$

$$R_{n,1,2} := \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} H_{(4\ell+2)k+a,j}; \quad (2.27)$$

$$R_{n,2} := \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a,b \in [4\ell+2] \\ a < b}} H_{u(4\ell+2)+a, u(4\ell+2)+b} \quad (2.28)$$

$$R_{n,3} := \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \sum_{\substack{a,b \in [4\ell+2] \\ 0 \leq a-b \leq (2\ell+1)-1}} (H_{a, v(4\ell+2)+b} + H_{b, v(4\ell+2)+a}) \quad (2.29)$$

$$R_{n,4} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} (H_{a, (v+1)(4\ell+2)+b} + H_{(4\ell+2)+b, v(4\ell+2)+a}) \quad (2.30)$$

$$R_{n,5} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 2} (H_{u(4\ell+2)+a, (u+1)(4\ell+2)+b} - H_{u(4\ell+2)+a, 2m(2\ell+1)+b}) \quad (2.31)$$

$$R_{n,6} = \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} (H_{b, v(4\ell+2)+a} - H_{v(4\ell+2)+b, v(4\ell+2)+a}), \quad (2.32)$$

To sum up, the centered U-statistic whose data is a function of $2\ell+1$ i.i.d. random variables can be decomposed as a partial sum of a strictly stationary sequence, a sum of degenerated U-statistics plus a remainder term.

Proof of Proposition 1.1. By the assumption (1.17), the following equality holds almost surely:

$$U_n(h, f, (\varepsilon_u)_{u \in \mathbb{Z}}) = U_n(h, f_0, (\varepsilon_u)_{u \in \mathbb{Z}}) + \sum_{\ell=1}^{+\infty} (U_n(h, f_\ell, (\varepsilon_u)_{u \in \mathbb{Z}}) - U_n(h, f_{\ell-1}, (\varepsilon_u)_{u \in \mathbb{Z}})). \quad (2.33)$$

Since $U_n(h, f_0, (\varepsilon_u)_{u \in \mathbb{Z}})$ is a U-statistic of independent identically distributed data, it can be treated by the classical Hoeffding's decomposition written in (1.14).

To proceed, we need to decompose for a fixed $\ell \geq 1$ the term $U_n(h, f_\ell, (\varepsilon_u)_{u \in \mathbb{Z}}) - U_n(h, f_{\ell-1}, (\varepsilon_u)_{u \in \mathbb{Z}})$. To this aim, we apply Lemma 2.2 in the following setting: the function h is replaced by $\tilde{h}: \mathbb{R}^{2\ell+1} \times \mathbb{R}^{2\ell+1} \rightarrow \mathbb{R}$, which is defined by

$$\begin{aligned} \tilde{h}((x_i)_{i=1}^{2\ell+1}, (y_j)_{j=1}^{2\ell+1}) \\ = h(f_\ell((x_i)_{i=1}^{2\ell+1}), f_\ell((y_j)_{j=1}^{2\ell+1})) - h(f_{\ell-1}((x_i)_{i=2}^{2\ell}), f_{\ell-1}((y_j)_{j=2}^{2\ell})). \end{aligned} \quad (2.34)$$

In this way,

$$U_n(h, f_\ell, (\varepsilon_u)_{u \in \mathbb{Z}}) - U_n(h, f_{\ell-1}, (\varepsilon_u)_{u \in \mathbb{Z}}) = \sum_{1 \leq i < j \leq n} \tilde{h}((\varepsilon_{i-u})_{u=-\ell}^\ell, (\varepsilon_{j-v})_{v=-\ell}^\ell) \quad (2.35)$$

and with the notation $V_{i,\ell} = (\varepsilon_u)_{u=i-\ell}^{i+\ell}$, Lemma 2.1 gives

$$\begin{aligned} & U_n(h, f_\ell, (\varepsilon_u)_{u \in \mathbb{Z}}) - U_n(h, f_{\ell-1}, (\varepsilon_u)_{u \in \mathbb{Z}}) - \mathbb{E} [U_n(h, f_\ell, (\varepsilon_u)_{u \in \mathbb{Z}}) - U_n(h, f_{\ell-1}, (\varepsilon_u)_{u \in \mathbb{Z}})] \\ &= (4\ell + 2) \left\lfloor \frac{n}{4\ell + 2} \right\rfloor \sum_{k=1}^{(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1} \left(\mathbb{E} [\tilde{h}(V_{k,\ell}, V'_{0,\ell}) \mid V_{k,\ell}] - \mathbb{E} [\tilde{h}(V_{k,\ell}, V'_{0,\ell}) \mid V_{k,\ell}] \right) \\ & \quad + \sum_{a,b \in [4\ell+2]} U_{\lfloor \frac{n}{4\ell+2} \rfloor}^{\text{ind}} \left(\tilde{h}_{a,b}^{(2)}, (\varepsilon_i^{a,b}) \right) + \sum_{k=1}^6 R_{n,k}, \end{aligned} \quad (2.36)$$

Observe that

$$\mathbb{E} [\tilde{h}(V_{k,\ell}, V'_{0,\ell}) \mid V_{k,\ell}] = \mathbb{E} [h(f_\ell(V_{k,\ell}), f_\ell(V'_{0,\ell})) \mid V_{k,\ell}] - \mathbb{E} [h(f_{\ell-1}(V_{k,\ell-1}), f_{\ell-1}(V'_{0,\ell-1})) \mid V_{k,\ell}] \quad (2.37)$$

and using Lemma A.5, this equality becomes

$$\mathbb{E} [\tilde{h}(V_{k,\ell}, V'_{0,\ell}) \mid V_{k,\ell}] = \mathbb{E} [h(f_\ell(V_{k,\ell}), f_\ell(V'_{0,\ell})) \mid V_{k,\ell}] - \mathbb{E} [h(f_{\ell-1}(V_{k,\ell-1}), f_{\ell-1}(V'_{0,\ell-1})) \mid V_{k,\ell-1}] \quad (2.38)$$

Moreover,

$$U_{\lfloor \frac{n}{4\ell+2} \rfloor}^{\text{ind}} \left(\tilde{h}_{a,b}^{(2)}, (\varepsilon_i^{a,b}) \right) = \sum_{1 \leq u < v \leq \lfloor \frac{n}{4\ell+2} \rfloor} (Y_{\ell,u,v}^{a,b} + Y_{\ell,u,v}^{b,a} - Y_{\ell-1,u,v}^{a,b} - Y_{\ell-1,u,v}^{b,a}) \quad (2.39)$$

where

$$\begin{aligned} Y_{\ell,u,v}^{a,b} &= h(f_\ell(V_{(4\ell+2)u+a,\ell}), f_\ell(V_{(4\ell+2)v+b,\ell})) - \mathbb{E} [h(f_\ell(V_{(4\ell+2)u+a,\ell}), f_\ell(V'_{0,\ell})) \mid V_{(4\ell+2)u+a,\ell}] \\ & \quad - \mathbb{E} [h(f_\ell(V_{(4\ell+2)v+b,\ell}), f_\ell(V'_{0,\ell})) \mid V_{(4\ell+2)v+b,\ell}] + \mathbb{E} [h(f_\ell(V_{0,\ell}), f_\ell(V'_{0,\ell}))]. \end{aligned} \quad (2.40)$$

We conclude by collecting all the terms. \square

2.2. Proof of the results of Subsection 1.4.

Proof of Proposition 1.2. It will be more convenient to work with dyadics, since the martingale property will be useful to handle the maximums. First observe that $M_p \leq 2^{1/p} M'_p$, where

$$M'_p := \sup_{N \geq 1} 2^{-2N/p} \max_{2 \leq n \leq 2^N} |U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E} [U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}})]|. \quad (2.41)$$

For a fixed integer n , consider the event

$$A_N := \left\{ 2^{-2N/p} \max_{2 \leq n \leq 2^N} |U_n^{\text{ind}}(h, (\varepsilon_i)_{i \in \mathbb{Z}})| > 2 \right\}. \quad (2.42)$$

It suffices to prove that there exists a constant c_p (depending only on p) such that

$$\sum_{N=1}^{+\infty} \mathbb{P}(A_N) \leq c_p \mathbb{E} [|h(\varepsilon_0, \varepsilon_1)|^p]. \quad (2.43)$$

Indeed, item 1 follows from an application of (2.43) to h/ε for a positive ε and the Borel-Cantelli lemma. In order to prove item 2, we notice that $\mathbb{P}\{M'_p > 2\} \leq \sum_{N=1}^{+\infty} \mathbb{P}(A_N)$ and we apply (2.43) to h/x for each positive x . Consequently, we focus on establishing a satisfactory bound for $\mathbb{P}(A_N)$.

Define for $j \geq 2$ the random variable $D_j := \sum_{i=1}^{j-1} h(\varepsilon_i, \varepsilon_j)$. Let \mathcal{F}_j denote the σ -algebra generated by the random variables $\varepsilon_k, 1 \leq k \leq j$. Define

$$D'_j := D_j \mathbf{1} \{|D_j| \leq 2^{2N/p}\} - \mathbb{E} [D_j \mathbf{1} \{|D_j| \leq 2^{2N/p}\} | \mathcal{F}_{j-1}] \quad \text{and} \quad (2.44)$$

$$D''_j := D_j \mathbf{1} \{|D_j| > 2^{2N/p}\} - \mathbb{E} [D_j \mathbf{1} \{|D_j| > 2^{2N/p}\} | \mathcal{F}_{j-1}]. \quad (2.45)$$

Since $\mathbb{E} [D_j | \mathcal{F}_{j-1}] = 0$, it follows that $D_j = D'_j + D''_j$ hence $A_N \subset A'_N \cup A''_N$, where

$$A'_N := \left\{ 2^{-4N/p} \max_{2 \leq k \leq 2^N} \left| \sum_{j=2}^k D'_j \right| > 1 \right\} \quad \text{and} \quad (2.46)$$

$$A''_N := \left\{ 2^{-4n/p} \max_{2 \leq k \leq 2^N} \left| \sum_{j=2}^k D''_j \right| > 1 \right\}. \quad (2.47)$$

Let us bound $p'_N := \mathbb{P}(A'_N)$. Markov's inequality entails

$$p'_N \leq 2^{-4N/p} \mathbb{E} \left[\max_{2 \leq k \leq 2^N} \left| \sum_{j=2}^k D'_j \right|^2 \right] \quad (2.48)$$

and since $(D'_j)_{j \geq 2}$ is a martingale differences sequence, we obtain by Doob's inequality and orthogonality of increments that

$$p'_N \leq 2^{1-4N/p} \sum_{j=2}^{2^N} \mathbb{E} [D_j'^2] \leq 2^{2-4N/p} \sum_{j=2}^{2^N} \mathbb{E} [D_j^2 \mathbf{1} \{|D_j| \leq 2^{N/p}\}]. \quad (2.49)$$

Now, we use

$$\mathbb{E} [D_j^2 \mathbf{1} \{|D_j| \leq 2^{2N/p}\}] = 2 \int_0^{2^{2N/p}} t \mathbb{P} \{t < |D_j| \leq 2^{2N/p}\} dt \quad (2.50)$$

$$= 2 \int_0^{2^{2N/p}} t \mathbb{P} \{|D_j| > t\} dt - 2^{2N/p} \mathbb{P} \{|D_j| > 2^{2N/p}\} \quad (2.51)$$

$$\leq 2 \int_0^{2^{2N/p}} t \mathbb{P} \{|D_j| > t\} dt \quad (2.52)$$

and after the substitution $s = 2^{-2N/p}t$, we get

$$\mathbb{E} [D_j^2 \mathbf{1} \{|D_j| \leq 2^{2N/p}\}] \leq 2^{1+4N/p} \int_0^1 s \mathbb{P} \{|D_j| > 2^{2N/p}s\} ds. \quad (2.53)$$

We thus obtained the estimate

$$p'_N \leq 8 \sum_{j=2}^{2^N} \int_0^1 s \mathbb{P} \{|D_j| > 2^{2N/p}s\} ds. \quad (2.54)$$

In order to bound $p''_N := \mathbb{P}(A''_N)$, we start by Markov's inequality to get

$$p''_N \leq 2^{-2N/p} \sum_{j=2}^{2^N} \mathbb{E} [|D_j''|] \leq 2^{1-2N/p} \sum_{j=2}^{2^N} \mathbb{E} [D_j \mathbf{1} \{|D_j| > 2^{2N/p}\}]. \quad (2.55)$$

Since there exists a constant C_p depending only on p such that

$$\begin{aligned} \mathbb{E} [|D_j| \mathbf{1} \{ |D_j| > 2^{2N/p} \}] &= 2^{2N/p} \mathbb{P} \{ |D_j| > 2^{2N/p} \} + \int_{2^{2N/p}}^{+\infty} \mathbb{P} \{ |D_j| > t \} dt \\ &\leq C_p \int_{2^{2N/p-1}}^{+\infty} \mathbb{P} \{ |D_j| > t \} dt, \end{aligned} \quad (2.56)$$

we derive after the substitution $t = 2^{2N/p} s$ that

$$p''_N \leq C_p 2 \sum_{j=2}^{2^N} \int_{1/2}^{+\infty} \mathbb{P} \{ |D_j| > 2^{2N/p} s \} ds. \quad (2.57)$$

The combination of (2.54) with (2.57) yields

$$\mathbb{P} (A_N) \leq C_p \sum_{j=2}^{2^N} \int_0^{+\infty} \min \{ 1, s \} \mathbb{P} \{ |D_j| > 2^{2N/p} s \} ds. \quad (2.58)$$

We are thus reduced to control the tail of D_j , which will be done by using Proposition A.1. Our particular setting permits some simplification of the involved terms.

We first observe that D_j has the same distribution as $\sum_{i=1}^{j-1} h(\varepsilon_0, \varepsilon_i)$ (since the vectors $(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j)$ and $(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_0)$ are identically distributed). Define $d_i := h(\varepsilon_0, \varepsilon_i)$. Since $\mathbb{E}[h(\varepsilon_0, x)] = 0$ for all $x \in \mathbb{R}^d$, the sequence $(d_i)_{i=1}^d$ is a martingale differences sequence for the filtration $(\mathcal{G}_i)_{i=1}^n$ where \mathcal{G}_i is the σ -algebra generated by ε_k , $0 \leq k \leq i$. We apply Proposition A.1 to $x = 2^{2n/p} s$ for a fixed positive s and $q = 2p$. let $i \in \{1, \dots, j-1\}$. By Lemma A.5 applied to $Y = d_i$, $\mathcal{F} = \sigma(\varepsilon_0)$ and $\mathcal{G} = \sigma(\varepsilon_1, \dots, \varepsilon_{i-1})$, we have

$$\mathbb{E} [|d_i|^p | \mathcal{G}_{i-1}] = \mathbb{E} [|h(\varepsilon_0, \varepsilon_j)|^p | \sigma(\varepsilon_0)]. \quad (2.59)$$

Using Lemma A.6 with $Y = \varepsilon_j$, $Z = \varepsilon_0$ and $f = h$, we derive that

$$\mathbb{E} [|d_i|^p | \mathcal{G}_{i-1}] = \mathbb{E} [|h(\varepsilon_0, \varepsilon_1)|^p | \sigma(\varepsilon_0)]. \quad (2.60)$$

Using this equality combined with the fact that the random variables d_i , $1 \leq i \leq j-1$ have the same distribution as d_1 , one gets

$$\begin{aligned} \mathbb{P} \{ |D_j| > 2^{2N/p} s \} &\leq c_1 (j-1) \int_0^1 \mathbb{P} \{ |d_1| > x 2^{2N/p} s u c_2 \} u^{q-1} du \\ &\quad + c_1 \int_0^1 \mathbb{P} \left\{ (j-1)^{1/p} (\mathbb{E} [|h(\varepsilon_0, \varepsilon_1)|^p | \sigma(\varepsilon_0)])^{1/p} > 2^{2N/p} s u c_2 \right\} u^{q-1} du. \end{aligned} \quad (2.61)$$

In view of (2.58), we derive that

$$\begin{aligned} \mathbb{P} (A_N) &\leq c_1 2^{2N} \int_0^{+\infty} \int_0^1 \mathbb{P} \{ |d_1| > x 2^{2N/p} s u c_2 \} u^{2p-1} du \min \{ 1, s \} ds \\ &\quad + c_1 2^N \int_0^{+\infty} \int_0^1 \mathbb{P} \left\{ (\mathbb{E} [|h(\varepsilon_0, \varepsilon_1)|^p | \sigma(\varepsilon_0)])^{1/p} > 2^{N/p} s u c_2 \right\} u^{2p-1} du \min \{ 1, s \} ds. \end{aligned} \quad (2.62)$$

Summing over N , we get (2.43) in view of the inequality

$$\sum_{N \geq 1} 2^{2N} \mathbb{P} (Y > 2^{2N/p}) \leq 2 \mathbb{E} [Y^p] \quad (2.63)$$

for a non-negative random variable Y and the convergence of the integrals $\int_0^1 u^{p-1} du$ and $\int_0^{+\infty} \min \{ 1, s \} s^{-p} ds$. This ends the proof of Proposition 1.2. \square

2.2.1. *Treatment of $R_{n,1,1}$ and $R_{n,1,2}$.* Recall that

$$H_{i,j}^{(\ell)} := h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) - \mathbb{E}[h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell}))] \\ - (h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1})) - \mathbb{E}[h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))]). \quad (2.64)$$

and

$$R_{n,1,1} := \sum_{\ell \geq 1} Y_{n,\ell}; \quad Y_{n,\ell} := \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{i=(4\ell+2)\lfloor \frac{j-1}{4\ell+2} \rfloor + 1}^{j-1} H_{i,j}^{(\ell)} \quad (2.65)$$

$$R_{n,1,2} := \sum_{\ell \geq 1} Z_{n,\ell}; \quad Z_{n,\ell} := \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} H_{(4\ell+2)k+a,j}^{(\ell)}. \quad (2.66)$$

For a fix $\ell \geq 1$, we evaluate the contribution of $Y_{n,\ell}$ and $Z_{n,\ell}$.

Lemma 2.3. *Let $\ell \geq 1$. The following inequalities hold:*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Y_{n,\ell}| \right\|_{p,\infty} \leq c_p \ell^{1-1/p} \theta_{\ell,p}; \quad (2.67)$$

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Z_{n,\ell}| \right\|_{p,\infty} \leq c_p \ell^{1-1/p} \theta_{\ell,p}, \quad (2.68)$$

where c_p depends only on p .

Proof. First observe that $Y_{n,\ell}$ is a sum of at most $(4\ell+2)^2$ random variables whose weak- \mathbb{L}^p -norm does not exceed $\theta_{\ell,p}$ hence by cutting the supremum where n is between two consecutive multiples of $4\ell+2$ gives

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Y_{n,\ell}| \right\|_{p,\infty} \leq \left(\sum_{n \geq 1} \left\| \frac{1}{n^{1+1/p}} |Y_{n,\ell}| \right\|_{p,\infty}^p \right)^{1/p} \quad (2.69)$$

$$\leq (4\ell+2)^{2-1/p} \left(\sum_{n \geq 1} n^{-1-p} \right)^{1/p} \theta_{\ell,p}, \quad (2.70)$$

In order to treat $Z_{n,\ell}$, we decompose it as $Z'_{n,\ell} + Z''_{n,\ell}$, where

$$Z'_{n,\ell} = \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \quad (2.71)$$

$$Z''_{n,\ell} = \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right]. \quad (2.72)$$

Using (1.32), it follows that

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Z'_{n,\ell}| \right\|_{p,\infty} \leq \left(\sum_{n \geq 1} \left\| \frac{1}{n^{1+1/p}} |Z'_{n,\ell}| \right\|_{p,\infty}^p \right)^{1/p} \quad (2.73)$$

$$\leq \left(\sum_{N \geq 1} N^{-p-1} (4\ell+2)^{-p-1} \left(\sum_{n=(4\ell+2)N}^{(4\ell+2)(N+1)-1} \left\| Z'_{n,\ell} \right\|_{p,\infty} \right)^p \right)^{1/p} \quad (2.74)$$

and for all n such that $(4\ell + 2)N \leq n \leq (4\ell + 2)(N + 1) - 1$,

$$\|Z'_{n,\ell}\|_{p,\infty} \leq \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \left\| \sum_{k=0}^{\lfloor \frac{n}{4\ell+2} \rfloor - 1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \quad (2.75)$$

$$\leq \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)N+1}^{(4\ell+2)(N+1)} \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \quad (2.76)$$

hence

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Z_{n,\ell}| \right\|_{p,\infty} \leq \left(\sum_{N \geq 1} N^{-p-1} (4\ell + 2)^{-p-1} \left(\sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)N+1}^{(4\ell+2)(N+1)} \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \right)^p \right)^{1/p}. \quad (2.77)$$

For all fixed j , we notice using Lemma A.5 that $\left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right)_{0 \leq k \leq N-1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_k)_{0 \leq k \leq N-1}$ where

$$\mathcal{F}_k := \sigma(V_{j,\ell}) \vee \sigma(V_{(4\ell+2)i+a,j}, i \leq k)$$

hence by Burkholder's inequality,

$$\begin{aligned} & \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty}^p \\ & \leq c_p \sum_{k=0}^{N-1} \left\| H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right\|_{p,\infty}^p \\ & \leq 2^p c_p \sum_{k=0}^{N-1} \left\| H_{(4\ell+2)k+a,j}^{(\ell)} \right\|_p^p \leq 2^p c_p N \theta_{\ell,p}^p \end{aligned} \quad (2.78)$$

and plugging this estimate into (2.77) gives

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |Z'_{n,\ell}| \right\|_{p,\infty} \leq C_p \left(\sum_{N \geq 1} N^{-p-1} (4\ell + 2)^{-p-1} (4\ell + 2)^{2p} (N \theta_{\ell,p}^p)^p \right)^{1/p} \leq C'_p \theta_{\ell,p} \ell^{1-1/p}. \quad (2.79)$$

In order to treat the contribution of $Z''_{n,\ell}$, we observe that $\mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right]$ is independent of k hence

$$Z''_{n,\ell} = \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \left\lfloor \frac{n}{4\ell+2} \right\rfloor \mathbb{E} \left[H_{a,j}^{(\ell)} \mid V_{j,\ell} \right]. \quad (2.80)$$

Consequently, the control of the contribution of $\sup_{n \geq 1} n^{-1-1/p} |Z''_{n,\ell}|$ can be done thanks to Proposition A.2.

This ends the proof of Lemma 2.3. \square

2.2.2. *Treatment of terms of the form $\sum_u H_{a,(4\ell+2)u+b}^{(\ell)}$ and $\sum_u H_{(4\ell+2)u+a,(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + b}^{(\ell)}$.*

Lemma 2.4. *For all $\ell \geq 1$ and all $a, b \in [4\ell + 2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{a,(4\ell+2)u+b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell + 2)^{-1-1/p} \theta_{\ell,p}. \quad (2.81)$$

Lemma 2.5. *For all $\ell \geq 1$ and all $a, b \in [4\ell + 2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{(4\ell+2)u+a,(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell + 2)^{-1-1/p} \theta_{\ell,p}. \quad (2.82)$$

These two lemmas are the consequence of the following observations.

- (1) We first assume that $0 \leq a - b \leq 2\ell$; if not we add and subtract $H_{a,(4\ell+2)(u+1)+b}^{(\ell)}$ and use telescoping to treat instead $\sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{a,(4\ell+2)(u+1)+b}^{(\ell)}$ (where we will apply the previous case to $\tilde{a} = a$ and $\tilde{b} = b + 2\ell + 1$). A similar method can be used to treat the sum involved in Lemma 2.5.
- (2) The supremum involved in the statement can be restricted to the integers n with are a multiple of $4\ell + 2$.
- (3) For all integer N and all sequence of random variables $W_{i,j}$, $i, j \geq 1$, we write

$$S_n := \sum_{1 \leq i < j \leq n} W_{i,j}, \quad S'_n := \sum_{0 \leq i < j \leq n} W_{i,j}. \quad (2.83)$$

With the choice $W_{i,j} := H_{(4\ell+2)i+a,(4\ell+2)j+b}^{(\ell)}$, the following equality holds:

$$\sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{a,(4\ell+2)u+b}^{(\ell)} = S'_N - S_N, \quad (2.84)$$

when $n = (4\ell + 2)N$. Since S'_N and S_N can be both expressed as U-statistics of i.i.d. data, we can use Corollary 1.3 to treat these terms.

Lemma 2.5 can be done in a similar way: we express this time the involved sum as $S_N - S_{N-1}$.

2.2.3. *Treatment of terms of the form $\sum_u H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)}$.*

Lemma 2.6. *For all $\ell \geq 1$ and all $a, b \in [4\ell + 2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell + 2)^{-1-1/p} \theta_{\ell,p}. \quad (2.85)$$

This follows from the fact that $\left(H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)} \right)_{u \geq 1}$ forms a two-dependent sequence.

2.2.4. *Proof of Theorem 1.4 and Theorem 1.5.* Theorem 1.4 is a consequence of the combination of Proposition 1.1, the estimates of Lemmas 2.3, 2.4, 2.6 and 2.5.

In order to prove Theorem 1.5, we start from the decomposition

$$U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E} \left[U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) \right] = A_{n,L} + B_{n,L}, \quad (2.86)$$

where for a fixed L , $A_{n,L}$ is the sum for the indexes ℓ smaller or equal to L (viewing the terms associated to $V_{k,0}$ as the corresponding ones for $\ell = 0$) and $B_{n,L}$ the remaining term. We have to prove that for each positive ε ,

$$\lim_{N \rightarrow +\infty} \mathbb{P} \left\{ \sup_{n \geq N} \frac{1}{n^{1+1/p}} |U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E}[U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})]| > 2\varepsilon \right\} = 0. \quad (2.87)$$

Using (2.86) and the fact that $A_{n,L}$ consists of sums of terms which can be treated by Lemmas 2.3, 2.4, 2.6 and 2.5, we derive that for all fixed L ,

$$\limsup_{N \rightarrow +\infty} \mathbb{P} \left\{ \sup_{n \geq N} \frac{1}{n^{1+1/p}} |U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E}[U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}})]| > 2\varepsilon \right\} \leq \limsup_{N \rightarrow +\infty} \mathbb{P} \left\{ \sup_{n \geq N} \frac{1}{n^{1+1/p}} |B_{n,L}| > \varepsilon \right\}. \quad (2.88)$$

Bounding the latter probability by $\varepsilon^{-p} \left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |B_{n,L}| \right\|_{p,w}^p$ and using Lemmas 2.3, 2.4, 2.6 and 2.5, we can see that for some constant C depending only on p ,

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1+1/p}} |B_{n,L}| \right\|_{p,w} \leq C \sum_{\ell \geq L} \ell^2 \theta_{\ell,p}, \quad (2.89)$$

which can be made arbitrarily small.

2.3. Proof of Theorem 1.6. Like for the results on the strong law of larger numbers, we have to control the contribution of the extra-terms in the decomposition obtained in Proposition 1.1.

2.3.1. *Treatment of $R_{n,1,1}$ and $R_{n,1,2}$.* Recall that

$$\begin{aligned} H_{i,j}^{(\ell)} &:= h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) - \mathbb{E}[h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell}))] \\ &\quad - (h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1})) - \mathbb{E}[h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))]). \end{aligned} \quad (2.90)$$

and

$$R_{n,1,1} := \sum_{\ell \geq 1} Y_{n,\ell}; \quad Y_{n,\ell} := \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{i=(4\ell+2)\lfloor \frac{j-1}{4\ell+2} \rfloor + 1}^{j-1} H_{i,j}^{(\ell)} \quad (2.91)$$

$$R_{n,1,2} := \sum_{\ell \geq 1} Z_{n,\ell}; \quad Z_{n,\ell} := \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} H_{(4\ell+2)k+a,j}^{(\ell)}. \quad (2.92)$$

For a fix $\ell \geq 1$, we evaluate the contribution of $Y_{n,\ell}$ and $Z_{n,\ell}$.

Lemma 2.7. *Let $\ell \geq 1$. The following inequalities hold.*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Y_{n,\ell}| \right\|_{p,\infty} \leq c_p \ell^{1-1/p} \theta_{\ell,p} \quad (2.93)$$

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Z_{n,\ell}| \right\|_{p,\infty} \leq c_p \ell^2 \theta_{\ell,p}. \quad (2.94)$$

Proof. First observe that $Y_{n,\ell}$ is a sum of at most $(4\ell+2)^2$ random variables whose weak- \mathbb{L}^p -norm does not exceed $\theta_{\ell,p}$ hence by cutting the supremum where n is between two consecutive multiples of

$4\ell + 2$ gives

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Y_{n,\ell}| \right\|_{p,\infty} \leq \left(\sum_{n \geq 1} \left\| \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Y_{n,\ell}| \right\|_{p,\infty}^p \right)^{1/p} \quad (2.95)$$

$$\leq c(4\ell + 2)^{1-1/p} \left(\sum_{n \geq 1} n^{-3p/2} \right)^{1/p} \theta_{\ell,p}. \quad (2.96)$$

In order to treat $Z_{n,\ell}$, we decompose it as $Z'_{n,\ell} + Z''_{n,\ell}$, where

$$Z'_{n,\ell} = \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \quad (2.97)$$

$$Z''_{n,\ell} = \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right]. \quad (2.98)$$

We first use (1.32) to get

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Z'_{n,\ell}| \right\|_{p,\infty} \leq \left(\sum_{n \geq 3} \left\| \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Z'_{n,\ell}| \right\|_{p,\infty}^p \right)^{1/p} \quad (2.99)$$

$$\leq \left(\sum_{N \geq 1} N^{-3p/2} (4\ell + 2)^{-3p/2} \left(\sum_{n=(4\ell+2)N}^{(4\ell+2)(N+1)-1} \|Z'_{n,\ell}\|_{p,\infty} \right)^p \right)^{1/p} \quad (2.100)$$

and for all n such that $(4\ell + 2)N \leq n \leq (4\ell + 2)(N + 1) - 1$,

$$\|Z'_{n,\ell}\|_{p,\infty} \leq \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \left\| \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor - 1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \quad (2.101)$$

$$\leq \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)N+1}^{(4\ell+2)(N+1)} \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \quad (2.102)$$

hence

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Z'_{n,\ell}| \right\|_{p,\infty}^p \leq \sum_{N \geq 1} N^{-p-1} (4\ell + 2)^{-3p/2} \left(\sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)N+1}^{(4\ell+2)(N+1)} \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty} \right)^p. \quad (2.103)$$

For all fixed j , we notice using Lemma A.5 that $\left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right)_{0 \leq k \leq N-1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_k)_{0 \leq k \leq N-1}$ where

$$\mathcal{F}_k := \sigma(V_{j,\ell}) \vee \sigma(V_{(4\ell+2)i+a,j}, i \leq k)$$

hence

$$\begin{aligned}
& \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_{p,\infty}^p \\
& \leq \left\| \sum_{k=0}^{N-1} \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_2^p \\
& \leq \left(\sum_{k=0}^{N-1} \left\| \left(H_{(4\ell+2)k+a,j}^{(\ell)} - \mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right] \right) \right\|_2 \right)^{p/2} \leq c_p N^{p/2} \theta_{\ell,2}^p \quad (2.104)
\end{aligned}$$

and plugging this estimate into (2.103) gives

$$\begin{aligned}
\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} |Z'_{n,\ell}| \right\|_{p,\infty} & \leq C_p \left(\sum_{N \geq 1} N^{-1-p/2} (4\ell+2)^{-3p/2+2} (4\ell+2)^{2p} N \theta_{\ell,p}^p \right)^{1/p} \\
& \leq C'_p \theta_{\ell,2} \ell^{1/2}. \quad (2.105)
\end{aligned}$$

We control the contribution of $Z''_{n,\ell}$ by noticing that $\mathbb{E} \left[H_{(4\ell+2)k+a,j}^{(\ell)} \mid V_{j,\ell} \right]$ is actually independent on k .

This ends the proof of Lemma 2.7. \square

2.3.2. *Treatment of terms of the form $\sum_u H_{a,(4\ell+2)u+b}^{(\ell)}$ and $\sum_u H_{(4\ell+2)u+a,(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + b}^{(\ell)}$.*

Lemma 2.8. *For all $\ell \geq 1$ and all $a, b \in [4\ell+2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{a,(4\ell+2)u+b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell+2)^{-3/2} \theta_{\ell,p}. \quad (2.106)$$

Lemma 2.9. *For all $\ell \geq 1$ and all $a, b \in [4\ell+2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2} \sqrt{\text{LL}(n)}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{(4\ell+2)u+a,(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell+2)^{-1-1/p} \theta_{\ell,p}. \quad (2.107)$$

The proof follows exactly the same idea as the proof of Lemmas 2.4 and 2.5, where the use of Corollary 1.3 is replaced by that of Proposition A.4.

2.3.3. *Treatment of terms of the form $\sum_u H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)}$.*

Lemma 2.10. *For all $\ell \geq 1$ and all $a, b \in [4\ell+2]$, the following inequality holds*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{3/2}} \left\| \sum_{u=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)} \right\| \right\|_{p,\infty} \leq c_p (4\ell+2)^{-3/2} \theta_{\ell,p}. \quad (2.108)$$

This follows from the fact that $\left(H_{(4\ell+2)u+a,(4\ell+2)u+b}^{(\ell)} \right)_{u \geq 1}$ forms a two-dependent sequence.

2.4. **Proof of Theorem 1.7.** Let us explain the idea of the proof. The convergence is essentially due to the partial sums of strictly stationary sequence $(Y_k)_{k \geq 1}$, where

$$Y_k = Y_{k,0} + \sum_{\ell \geq 1} Y_{k,\ell} - Y_{k,\ell-1} \quad (2.109)$$

and

$$Y_{k,\ell} := \mathbb{E} \left[h \left(f_\ell (V_{k,\ell}), f_\ell (V'_{0,\ell}) \right) \mid V_{k,\ell} \right] - \mathbb{E} \left[h \left(f_\ell (V_{k,\ell}), f_\ell (V'_{0,\ell}) \right) \right]. \quad (2.110)$$

We can establish the convergence of $(n^{-1/2} \sum_{k=1}^n Y_k)_{n \geq 1}$ by showing the convergence of $(n^{-1/2} \sum_{k=1}^n Y_{k,L})_{n \geq 1}$ for a fixed L and by controlling the remainder. We have to prove that all the terms in the decomposition obtained in Proposition 1.1 converge to zero in probability. To sum up, we start by writing

$$\begin{aligned} \frac{1}{n^{3/2}} U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) - \mathbb{E} \left[U_n(h, f, (\varepsilon_i)_{i \in \mathbb{Z}}) \right] &= \frac{1}{\sqrt{n}} \sum_{k=1}^n Y_k + U_n^{\text{ind}}(h^{(0)}, (\varepsilon_i)_i) + \\ &+ \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{a,b \in [4\ell+2]} U_{\lfloor \frac{n}{2\ell} \rfloor}^{\text{ind}} \left(h_{a,b}^{(\ell)}, (\varepsilon_i^{a,b}) \right) + R_{n,1,1} + R_{n,1,2} + \sum_{k=2}^7 R_{n,k}, \end{aligned} \quad (2.111)$$

where for each $\ell \geq 1$ and all $a, b \in [4\ell+2]$, the U-statistic $U_{\lfloor \frac{n}{2\ell} \rfloor}^{\text{ind}} \left(h_{a,b}^{(\ell)}, (\varepsilon_i^{a,b}) \right)$ has independent data and is degenerated, and the remainder terms are defined as

$$R_{n,1,1} := \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor+1}^n \sum_{i=(4\ell+2)\lfloor \frac{j-1}{4\ell+2} \rfloor+1}^{j-1} H_{i,j}^{(\ell)} \quad (2.112)$$

$$R_{n,1,2} := \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{a=1}^{4\ell+2} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor+1}^n \sum_{k=0}^{\lfloor \frac{j-1}{4\ell+2} \rfloor-1} H_{(4\ell+2)k+a,j}^{(\ell)}; \quad (2.113)$$

$$R_{n,2} := \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor-1} \sum_{\substack{a,b \in [4\ell+2] \\ a < b}} H_{u(4\ell+2)+a, u(4\ell+2)+b}^{(\ell)} \quad (2.114)$$

$$R_{n,3} := \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor-1} \sum_{\substack{a,b \in [4\ell+2] \\ 0 \leq a-b \leq (2\ell+1)-1}} \left(H_{a, v(4\ell+2)+b}^{(\ell)} + H_{b, v(4\ell+2)+a}^{(\ell)} \right) \quad (2.115)$$

$$R_{n,4} = \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor-1} \left(H_{a, (v+1)(4\ell+2)+b}^{(\ell)} + H_{(4\ell+2)+b, v(4\ell+2)+a}^{(\ell)} \right) \quad (2.116)$$

$$R_{n,5} = \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{u=0}^{\lfloor \frac{n}{4\ell+2} \rfloor-2} \left(H_{u(4\ell+2)+a, (u+1)(4\ell+2)+b}^{(\ell)} - H_{u(4\ell+2)+a, 2m(2\ell+1)+b}^{(\ell)} \right) \quad (2.117)$$

$$R_{n,6} = \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{\substack{a,b \in [4\ell+2] \\ (2\ell+1) \leq a-b \leq (4\ell+2)-1}} \sum_{v=1}^{\lfloor \frac{n}{4\ell+2} \rfloor-1} \left(H_{b, v(4\ell+2)+a}^{(\ell)} - H_{v(4\ell+2)+b, v(4\ell+2)+a}^{(\ell)} \right), \quad (2.118)$$

$$R_{n,7} := \frac{1}{n^{3/2}} \sum_{\ell \geq 1} R_{n,7}^{(\ell)},$$

$$R_{n,7}^{(\ell)} := (4\ell + 1) \left[\frac{n}{4\ell + 1} \right]^{(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1} \sum_{k=1}^{(4\ell+2) \lfloor \frac{n}{4\ell+2} \rfloor + 1} (Y_{k,\ell} - Y_{k,\ell-1}) - n \sum_{k=1}^n (Y_{k,\ell} - Y_{k,\ell-1}) \quad (2.119)$$

with

$$H_{i,j}^{(\ell)} := h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell})) - \mathbb{E}[h(f_\ell(V_{i,\ell}), f_\ell(V_{j,\ell}))] \\ - (h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1})) - \mathbb{E}[h(f_{\ell-1}(V_{i,\ell-1}), f_{\ell-1}(V_{j,\ell-1}))]). \quad (2.120)$$

We then follow the steps :

- (1) we show that $(n^{-1/2} \sum_{k=1}^n Y_k)_{n \geq 1}$ converges to a centered normal distribution with variance
- (2) We show the convergence in probability to zero of all the terms $R_{n,1,1}, R_{n,1,2}, R_{n,k}, 2 \leq k \leq 7$.

2.4.1. *Convergence of $(n^{-1/2} \sum_{k=1}^n Y_k)_{n \geq 1}$.* We use Theorem 4.2 in [6], which states the following. For $L, n \in \mathbb{N}$, $Z_n, Z_{n,L}, W_L$ and Z are real-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that

- (1) for all $L \in \mathbb{N}$, $Z_{n,L} \rightarrow W_L$ in distribution as $n \rightarrow \infty$;
- (2) $W_L \rightarrow Z$ in distribution as $L \rightarrow \infty$, and
- (3) for each $\varepsilon > 0$, $\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{|Z_{n,L} - Z_n| > \varepsilon\} = 0$.

Then $Z_n \rightarrow Z$ in distribution as $n \rightarrow \infty$.

We will apply the result in the following setting:

$$Z_{n,L} := \frac{1}{\sqrt{n}} \sum_{k=1}^n Y_{k,L}, \quad (2.121)$$

W_L a centered normal variable with variance $\sigma_L^2 := \sum_{k=-L}^L \text{Cov}(Y_{0,L}, Y_{k,L})$ and W a centered normal variable with variance $\sigma^2 = \sum_{k \in \mathbb{Z}} \text{Cov}(Y_0, Y_k)$.

The first item follows from the central limit theorem for $(2L+1)$ -dependent random variables; the second one from the convergence of $(\sigma_L^2)_{L \geq 1}$ to σ^2 , which can be seen by writing $Y_{k,L} - Y_k = \sum_{\ell \geq L} (Y_{k,\ell} - Y_{k,\ell+1})$ and using Cauchy-Schwarz inequality:

$$\left| \sum_{k=-L}^L \text{Cov}(Y_{0,L}, Y_{k,L}) - \text{Cov}(Y_0, Y_k) \right| \leq \sum_{k=-L}^L |\text{Cov}(Y_{0,L}, Y_{k,L} - Y_k)| + \sum_{k=-L}^L |\text{Cov}(Y_{0,L} - Y_0, Y_k)| \\ \leq 2L \|Y_{0,L}\|_2 \sum_{\ell \geq L} \theta_{\ell,2} + 2L \|Y_k\|_2 \sum_{\ell \geq L} \theta_{\ell,2} \quad (2.122)$$

and the quantities $\|Y_{0,L}\|_2$ and $\|Y_k\|_2$ are bounded independently of L and k .

For the third item, we start from Chebychev's inequality:

$$\mathbb{P}\{|Z_{n,L} - Z_n| > \varepsilon\} \leq \varepsilon^{-2} \|Z_{n,L} - Z_n\|_2^2 \quad (2.123)$$

and

$$\|Z_{n,L} - Z_n\|_2 \leq \sum_{\ell \geq L} \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n (Y_{k,\ell} - Y_{k,\ell-1}) \right\|_2 \quad (2.124)$$

and $\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n (Y_{k,\ell} - Y_{k,\ell-1}) \right\|_2$ does not exceed a constant times $\ell^{1/2} \theta_{\ell,2}$ hence

$$\limsup_{n \rightarrow +\infty} \mathbb{P}\{|Z_{n,L} - Z_n| > \varepsilon\} \leq \varepsilon^{-2} \left(\sum_{\ell \geq L} \ell^{1/2} \theta_{\ell,2} \right)^2. \quad (2.125)$$

2.4.2. *Convergence in probability of $R_{n,1,1}$, $R_{n,1,2}$ and $R_{n,k}$, $2 \leq k \leq 6$ to 0.* Observe that

$$\mathbb{E} [|R_{n,1,1}|] \leq \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{j=(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n \sum_{i=(4\ell+2)\lfloor \frac{j-1}{4\ell+2} \rfloor + 1}^{j-1} \left\| H_{i,j}^{(\ell)} \right\|_1, \quad (2.126)$$

that $\left\| H_{i,j}^{(\ell)} \right\|_1 \leq \theta_{\ell,1}$ and that the number of terms in the two inner sums is of order ℓ^2 hence

$$\mathbb{E} [|R_{n,1,1}|] \leq \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \ell^2 \theta_{\ell,1}. \quad (2.127)$$

We use the same method for the terms $R_{n,1,2}$ and $R_{n,k}$, $2 \leq k \leq 6$.

2.4.3. *Treatment of $R_{n,7}$.* First we rewrite $R_{n,7}$ as a double sum, namely, as

$$R_{n,7} = \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \sum_{k \geq 1} (Y_{k,\ell} - Y_{k,\ell-1}) c_{n,k,\ell} \quad (2.128)$$

where

$$c_{n,k,\ell} = (4\ell + 2) \left\lfloor \frac{n}{4\ell + 2} \right\rfloor \left[k \leq (4\ell + 2) \left\lfloor \frac{n}{4\ell + 2} \right\rfloor + 1 \right] - n [k \leq n], \quad (2.129)$$

with the notation $[P] = 1$ if the assertion P holds and 0 otherwise. Write n as $(4\ell + 2)N + r$, where $0 \leq r \leq 4\ell + 1$. Then

$$c_{n,k,\ell} = -q [k \leq N(4\ell + 2) + 1] - n [N(4\ell + 2) \leq k \leq n], \quad (2.130)$$

hence

$$|R_{n,7}| \leq \frac{1}{n^{3/2}} \sum_{\ell \geq 1} \ell \left| \sum_{k=1}^{(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1} (Y_{k,\ell} - Y_{k,\ell-1}) \right| + \frac{1}{n^{1/2}} \sum_{\ell \geq 1} \left| \sum_{(4\ell+2)\lfloor \frac{n}{4\ell+2} \rfloor + 1}^n (Y_{k,\ell} - Y_{k,\ell-1}) \right| \quad (2.131)$$

and taking the expectation give that

$$\|R_{n,7}\|_1 \leq 2n^{-1/2} \sum_{\ell \geq 1} \ell \theta_{\ell,1}. \quad (2.132)$$

A. APPENDIX

In this appendix, we collect some fact about partial sums of martingales or functional of independent sequences that we will need in the proof.

The first is a probability inequatlity for martingales; Proposition A.2 and A.3 give a control of the maximal function involved in the strong law of large numbers, respectively for martingales and functionals of a fixed number of i.i.d. random variables. We end the Appendix by two lemmas on conditional expectation.

Proposition A.1 (Theorem 1.3 in [12]). *Let $1 < p < 2$ and $q > p$. Then there exists constants c_1 and c_2 depending only on $\{p, q\}$ such that if $(d_i)_{i=1}^n$ is a martingale differences sequence with respect to a filtration $(\mathcal{F}_i)_{i=1}^n$, then for each integer n and each positive x ,*

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n d_i \right| > x \right\} \leq c_1 \sum_{i=1}^n \int_0^1 \mathbb{P} \{ |d_i| > xuc_2 \} u^{q-1} du + c_1 \int_0^1 \mathbb{P} \left\{ \left(\sum_{i=1}^n \mathbb{E} [|d_i|^p | \mathcal{F}_{i-1}] \right)^{1/p} > xuc_2 \right\} u^{q-1} du. \quad (A.1)$$

The next Proposition gives a control of the maximal function involved in the strong law of large numbers. A control on the r -th moment for $r < p$ was obtained in the proof of Theorem 4.1 in [17]. The control on the weak- \mathbb{L}^p -moment was explicitly established in [7], p. 324, under a stationarity assumption, but the proof work for martingale with identically distributed increments.

Proposition A.2. *Let $(d_j)_{j \geq 1}$ be an identically distributed martingale differences sequence with respect to the filtration $(\mathcal{F}_j)_{j \geq 0}$. Then for all $1 < p < 2$,*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1/p}} \left| \sum_{j=1}^n d_j \right| \right\|_{p, \infty} \leq c_p \|d_1\|_p. \quad (\text{A.2})$$

Proposition A.3. *Let $\ell \geq 1$ and $(X_j)_{j \geq 1}$ be a sequence such that $X_j = f((\varepsilon_u)_{u=j-\ell}^{j+\ell})$, where $(\varepsilon_u)_{u \in \mathbb{Z}}$ is i.i.d. and X_1 is centered. Then for all $1 < p < 2$,*

$$\left\| \sup_{n \geq 1} \frac{1}{n^{1/p}} \left| \sum_{j=1}^n X_j \right| \right\|_{p, \infty} \leq c_p \ell^{1-1/p} \|X_1\|_p. \quad (\text{A.3})$$

Proof. We first notice that

$$\sup_{n \geq 1} \frac{1}{n^{1/p}} \left| \sum_{j=1}^n X_j \right| \leq \sum_{a \in [4\ell+2]} \sup_{n \geq 1} \frac{1}{n^{1/p}} \left| \sum_{k=1}^{\lfloor \frac{n}{4\ell+2} \rfloor} X_{(4\ell+2)k+a} \right| \quad (\text{A.4})$$

and we apply Proposition A.2 to the sequences $(X_{(4\ell+2)k+a})_{k \geq 1}$ for all fixed $a \in [4\ell+2]$. \square

The next Proposition give a control of the weak- \mathbb{L}^p -norm of the maximum function involved in the bounded law of the iterated logarithms for partial sum of stationary sequences. When the involved sequence is i.i.d. and centered, this reduces to Théorème 1 in [15]. This can be extended to the context of the functional of $2\ell+1$ i.i.d. random variables by the same method as in the proof of Proposition A.4.

Proposition A.4. *Let $\ell \geq 1$ and $(X_j)_{j \geq 1}$ be a sequence such that $X_j = f((\varepsilon_u)_{u=j-\ell}^{j+\ell})$, where $(\varepsilon_u)_{u \in \mathbb{Z}}$ is i.i.d. and X_1 is centered. Then for all $1 < p < 2$,*

$$\left\| \sup_{n \geq 1} \frac{1}{\sqrt{n \text{LL}(n)}} \left| \sum_{j=1}^n X_j \right| \right\|_{p, \infty} \leq c_p \ell^{1/2} \|X_1\|_2. \quad (\text{A.5})$$

Lemma A.5 (Proposition 2 p. 1693 in [16]). *Let Y be a real-valued random variable and let \mathcal{F} and \mathcal{G} be two sub- σ -algebra such that \mathcal{G} is independent of the σ -algebra generated by Y and \mathcal{F} . Then*

$$\mathbb{E}[Y \mid \mathcal{F} \vee \mathcal{G}] = \mathbb{E}[Y \mid \mathcal{F}]. \quad (\text{A.6})$$

Finally, the next lemma is well-known.

Lemma A.6. *Let Y and Z be two independent random variables with values in \mathbb{R}^d . Let $f: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. Then*

$$\mathbb{E}[f(Y, Z) \mid \sigma(Z)] = g(Z), \quad (\text{A.7})$$

where $g: \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by $g(z) = \mathbb{E}[f(Y, z)]$.

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