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RATIONAL REAL ALGEBRAIC MODELS OF COMPACT DIFFERENTIAL SURFACES WITH CIRCLE ACTIONS

ADRIEN DUBOULOZ AND CHARLIE PETITJEAN

Dedicated to Lucy Moser-Jauslin on her 60th birthday

ABSTRACT. We give an algebro-geometric classification of smooth real affine algebraic surfaces endowed with an effective action of the real algebraic circle group $\mathbb{S}^1$ up to equivariant isomorphisms. As an application, we show that every compact differentiable surface endowed with an action of the circle S^1 admits a unique smooth rational real quasi-projective model up to $\mathbb{S}^1$ -equivariant birational diffeomorphism.

INTRODUCTION

A description of normal complex affine surfaces admitting an effective action of the complex multiplicative group $\mathbb{G}_{m,\mathbb{C}} = (\mathbb{C}^*, \times)$ was given by Flenner and Zaidenberg [9] in terms of their graded coordinate rings. Generalizing earlier constructions due to Dolgachev-Pinkham-Demazure [7, 13, 6], they described these graded rings as rings of sections of divisors with rational coefficients on suitable one-dimensional rational quotients of the given action. This type of presentation, which is nowadays called the *DPD-presentation* of normal affine surfaces with $\mathbb{G}_{m,\mathbb{C}}$ -actions, was generalized vastly by Altmann and Hausen [1] to give presentations of normal complex affine varieties of any dimension endowed with effective actions of tori $\mathbb{G}_{m,\mathbb{C}}^r$ in terms of so-called polyhedral Weil divisors on suitable rational quotients obtained as limits of GIT quotients.

For normal affine varieties over arbitrary base fields k endowed with effective actions of non necessarily split tori, that is, commutative k -groups schemes G whose base extension to a separable closure k^s of k is isomorphic to split tori $\mathbb{G}_{m,k^s}^r$, only partial extensions of the Altmann-Hausen formalism are known so far. Besides the toric case considered by several authors, a first step was made by Langlois [10] who obtained a generalization to affine varieties endowed with effective actions of complexity one. In another direction, Liendo and the first author [8] recently extended the Altmann-Hausen framework to describe normal real affine varieties endowed with an effective action of the 1-dimensional non-split real torus, the circle

$$\mathbb{S}^1 = \operatorname{Spec}(\mathbb{R}[u, v]/(u^2 + v^2 - 1)) \cong \operatorname{SO}_2(\mathbb{R}).$$

The common approach in these generalizations is based on the understanding of the interplay between the algebro-combinatorial data in an Altmann-Hausen presentation of the variety with split torus action obtained by base extension to a separable closure k^s of the base field k and Galois descent with respect to the Galois group $\operatorname{Gal}(k^s/k)$. In the real case, this amounts to describe normal real affine varieties as normal complex affine varieties endowed with an anti-regular involution σ , called a *real structure*. The results in [8] then essentially consist of a description of $\mathbb{S}^1$ -actions on normal real affine varieties in the language of [1] extended to complex affine varieties with real structures.

The goal of this article is to give a survey of this description and some applications in the special case of normal real affine surfaces, formulated in the language of DPD-presentations of Flenner and Zaidenberg. The first main result, Theorem 14, describes the one-to-one correspondence between normal real affine surfaces with effective $\mathbb{S}^1$ -actions and certain pairs (D, h) called *real DPD-pairs* on smooth real affine curves C , consisting of a Weil $\mathbb{Q}$ -divisor D on the complexification of C and rational function h on C . We also

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characterize which such pairs correspond to smooth real affine surfaces. A second main result, Theorem 20, consists of a classification of real $\mathbb{S}^1$ -orbits on a smooth real affine surface in relation to the structure of the real fibers of the quotient morphism for the $\mathbb{S}^1$ -action.

To give an illustration of the flavor of these results, consider the smooth complex affine surface

$$S = \{xy^2 = 1 - z^2\} \subset \text{Spec}(\mathbb{C}[x^{\pm 1}, y, z]).$$

The group $\mathbb{G}_{m,\mathbb{C}}$ acts effectively on S by $t \cdot (x, y, z) = (t^2x, t^{-1}y, z)$ and the categorical quotient for this action is the projection $\pi = \text{pr}_z : S \rightarrow \mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z])$. All fibers of π consist of a unique $\mathbb{G}_{m,\mathbb{C}}$ -orbit, isomorphic to $\mathbb{G}_{m,\mathbb{C}}$ acting on itself by translations, except for $\pi^{-1}(-1)$ and $\pi^{-1}(1)$ which are isomorphic to $\mathbb{G}_{m,\mathbb{C}}$ on which $\mathbb{G}_{m,\mathbb{C}}$ acts with stabilizer equal to the group μ_2 of complex square roots of unity. The composition of the involution $t \mapsto t^{-1}$ of $\mathbb{G}_{m,\mathbb{C}}$ with the complex conjugation defines a real structure σ_0 on $\mathbb{G}_{m,\mathbb{C}}$ for which the pair $(\mathbb{G}_{m,\mathbb{C}}, \sigma_0)$ describes a real algebraic group isomorphic to $\mathbb{S}^1$. The composition of the involution $(x, y, z) \mapsto (x^{-1}, xy, z)$ of S with the complex conjugation defines a real structure σ on S , making the pair (S, σ) into a smooth real affine surface. The $\mathbb{G}_{m,\mathbb{C}}$ -action on S is compatible with these two real structures and defines a real action of $\mathbb{S}^1$ on the real affine surface (S, σ) . The quotient morphism π can in turn be interpreted as a real morphism $\pi : (S, \sigma) \rightarrow \mathbb{A}_{\mathbb{R}}^1 = \text{Spec}(\mathbb{R}[z])$ which is the categorical quotient of (S, σ) for the $\mathbb{S}^1$ -action in the category of real algebraic varieties. A real DPD-pair (D, h) describing the $\mathbb{S}^1$ -action on (S, σ) is then given for instance by the Weil $\mathbb{Q}$ -divisor $D = \frac{1}{2}\{-1\} + \frac{1}{2}\{1\}$ on $\mathbb{A}_{\mathbb{C}}^1$ and the rational function $h = 1 - z^2$ on $\mathbb{A}_{\mathbb{R}}^1$. The fibers of $\pi : (S, \sigma) \rightarrow \mathbb{A}_{\mathbb{R}}^1$ over real points c of $\mathbb{A}_{\mathbb{R}}^1$ all consist of a single $\mathbb{S}^1$ -orbit which is isomorphic to $\mathbb{S}^1$ -acting on itself by translations if $c \in]-1, 1[$, to $\mathbb{S}^1$ acting on itself with stabilizer μ_2 if $c \in \{-1, 1\}$, and to the real affine curve without real point $\{u^2 + v^2 = -1\}$ otherwise.

The set of real points of the above surface (S, σ) endowed with the induced Euclidean topology is diffeomorphic to the Klein bottle K (see subsection 3.1.4). Furthermore, the $\mathbb{S}^1$ -action on (S, σ) induces a differentiable action of the real circle S^1 on K which coincides with the standard S^1 -action on K viewed as the S^1 -equivariant connected sum $\mathbb{R}P^2 \#_{S^1} \mathbb{R}P^2$ of two copies of the projective plane $\mathbb{R}P^2$. In other words, (S, σ) endowed with its $\mathbb{S}^1$ -action is an equivariant *real affine algebraic model* of the Klein bottle K endowed with its differentiable S^1 -action. By a result of Comessatti [5], a compact connected differential manifold of dimension 2 without boundary admits a projective rational real algebraic model if and only if it is non-orientable or diffeomorphic to the sphere S^2 or the torus $T = S^1 \times S^1$. It was established later on by Biswas and Huisman [3] that such a projective model is unique up to so-called *birational diffeomorphisms*, that is, diffeomorphisms induced by birational maps defined at every real point and admitting inverses of the same type. As an application of the real DPD-presentation formalism, we establish the following uniqueness property of rational models of compact differentiable surfaces with S^1 -actions among all smooth rational quasi-projective real algebraic surfaces with $\mathbb{S}^1$ -actions.

Theorem 1. *A connected compact real differential manifold of dimension 2 without boundary endowed with an effective differentiable S^1 -action admits a smooth rational quasi-projective real algebraic model with $\mathbb{S}^1$ -action, unique up to $\mathbb{S}^1$ -equivariant birational diffeomorphism.*

The article is organized as follows. In the first section we review the equivalence of categories between quasi-projective real varieties and quasi-projective complex varieties equipped with real structures and recall the interpretation of $\mathbb{S}^1$ -actions on such varieties as forms of $\mathbb{G}_{m,\mathbb{C}}$ -actions on complex varieties with real structures. We also describe a correspondence between $\mathbb{S}^1$ -torsors and certain pairs consisting of an invertible sheaf and a rational function on their base space which to our knowledge did not appear before in the literature. The second section is devoted to the presentation of smooth real affine surfaces with $\mathbb{S}^1$ -action in terms of real DPD-pairs and to the study of their real $\mathbb{S}^1$ -orbits. In the third section, we first present different constructions of rational projective and affine real algebraic models of compact differential surfaces with $\mathbb{S}^1$ -actions and then proceed to the proof of Theorem 1.

1. PRELIMINARIES

In what follows the term algebraic variety over field k refers to a geometrically integral scheme X of finite type over k . In the sequel, k will be equal to either $\mathbb{R}$ or $\mathbb{C}$, and we will say that X is a real, respectively complex, algebraic variety.

1.1. Real and complex quasi-projective algebraic varieties. Every complex algebraic variety V can be viewed as a scheme over the field $\mathbb{R}$ of real numbers via the composition of its structure morphism $p : V \rightarrow \text{Spec}(\mathbb{C})$ with the étale double cover $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$ induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[i]/(i^2 + 1)$. The Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mu_2$ acts on $\text{Spec}(\mathbb{C})$ by the usual complex conjugation $z \mapsto \bar{z}$.

Definition 2. A *real structure* on a complex algebraic variety V consists of an involution of $\mathbb{R}$ -scheme σ of V which lifts the complex conjugation, so that we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ p \downarrow & & \downarrow p \\ \text{Spec}(\mathbb{C}) & \xrightarrow{z \mapsto \bar{z}} & \text{Spec}(\mathbb{C}) \\ & \searrow & \swarrow \\ & \text{Spec}(\mathbb{R}) & \end{array}$$

When $V = \text{Spec}(A)$ is affine, a real structure σ is equivalently determined by its comomorphism $\sigma^* : A \rightarrow A$, which is an involution of A viewed as an $\mathbb{R}$ -algebra.

A *real morphism* (resp. *real rational map*) between complex algebraic varieties with real structures (V', σ') and (V, σ) is a morphism (resp. a rational map) of complex algebraic varieties $f : V' \rightarrow V$ such that $\sigma \circ f = f \circ \sigma'$ as morphisms of $\mathbb{R}$ -schemes.

For every real algebraic variety X , the *complexification*

$$X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C} := X \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C})$$

of X comes equipped with a canonical real structure σ_X given by the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ by complex conjugation on the second factor. Conversely, if a complex variety $p : V \rightarrow \text{Spec}(\mathbb{C})$ is equipped with a real structure σ and covered by σ -invariant affine open subsets -so for instance if V is quasi-projective-, then the quotient $\pi : V \rightarrow V/\langle \sigma \rangle$ exists in the category of schemes and the structure morphism $p : V \rightarrow \text{Spec}(\mathbb{C})$ descends to a morphism $V/\langle \sigma \rangle \rightarrow \text{Spec}(\mathbb{R}) = \text{Spec}(\mathbb{C})/\langle z \mapsto \bar{z} \rangle$ making $V/\langle \sigma \rangle$ into a real algebraic variety X such that $V \cong X_{\mathbb{C}}$. In the case where $V = \text{Spec}(A)$ is affine, the algebraic variety $X = V/\langle \sigma \rangle$ is affine, equal to the spectrum of the ring A^{σ^*} of σ^* -invariant elements of A . This correspondence extends to a well-known equivalence of categories:

Lemma 3. [4] *The functor $X \mapsto (X_{\mathbb{C}}, \sigma_X)$ is an equivalence between the category of quasi-projective real algebraic varieties and the category of pairs (V, σ) consisting of a complex quasi-projective variety V endowed with a real structure σ .*

In what follows, we will switch freely from one point of view to the other, by viewing a quasi-projective real algebraic variety X either as a geometrically integral $\mathbb{R}$ -scheme of finite type or as a pair (V, σ) consisting of a quasi-projective complex algebraic variety V endowed with a real structure σ . A *real form* of a given real algebraic variety (V, σ) is a real algebraic variety (V', σ') such that the complex varieties V and V' are isomorphic. A *real closed subscheme* Z of a real algebraic variety (V, σ) is a σ -invariant closed subscheme of V , endowed with the induced real structure $\sigma|_Z$.

The set $V(\mathbb{C})$ of complex points of a smooth complex algebraic variety V can be endowed with a natural structure of real smooth manifold locally inherited from that on $\mathbb{A}_{\mathbb{C}}^n(\mathbb{C}) \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ [14, Lemme 1 and Proposition 2]. Every morphism of smooth complex algebraic varieties $f : V' \rightarrow V$ induces a differentiable map $f(\mathbb{C}) : V(\mathbb{C}) \rightarrow V(\mathbb{C})$ which is a diffeomorphism when f is an isomorphism. Similarly, a real structure

σ on V induces a differentiable involution of $V(\mathbb{C})$, whose set of fixed points $V(\mathbb{C})^\sigma$, called the *real locus* of (V, σ) , is a smooth differential real manifold. The real algebraic variety (V, σ) is then said to be an *algebraic model* of this differential manifold.

Definition 4. A *birational diffeomorphism* $\varphi : (V', \sigma') \dashrightarrow (V, \sigma)$ between smooth real algebraic varieties with non empty real loci is a real birational map whose restriction to the real locus $V'(\mathbb{C})^{\sigma'}$ of (V', σ') is a diffeomorphism onto the real locus $V(\mathbb{C})^\sigma$ of (V, σ) , and which admits a rational inverse of the same type.

Example 5. Let $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ be the complexification of the smooth affine curve Q_1 in $\mathbb{A}_{\mathbb{R}}^2 = \text{Spec}(\mathbb{R}[u, v])$ defined by the equation $u^2 + v^2 = 1$. The stereographic projection from the real point $N = (0, 1)$ of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ induces an everywhere defined birational diffeomorphism

$$\pi_N : (Q_{1,\mathbb{C}} \setminus \{N\}, \sigma_{Q_1}|_{Q_{1,\mathbb{C}} \setminus \{N\}}) \rightarrow (\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z]), \sigma_{\mathbb{A}_{\mathbb{C}}^1}), \quad (u, v) \mapsto \frac{u}{1-v}$$

with image equal to $\mathbb{A}_{\mathbb{C}}^1 \setminus \{\pm i\}$. Its inverse is given by

$$z \mapsto (u, v) = \left(\frac{2z}{z^2 + 1}, \frac{z^2 - 1}{z^2 + 1} \right).$$

1.2. Circle actions as real forms of hyperbolic $\mathbb{G}_m$ -actions.

Definition 6. The *circle* $\mathbb{S}^1$ is the nontrivial real form $(\mathbb{G}_{m,\mathbb{C}}, \sigma_0)$ of $(\mathbb{G}_{m,\mathbb{C}}, \sigma_{\mathbb{G}_{m,\mathbb{R}}})$ whose real structure is the composition of the involution $t \mapsto t^{-1}$ of $\mathbb{G}_{m,\mathbb{C}} = \text{Spec}(\mathbb{C}[t^{\pm 1}])$ with the complex conjugation. It is a real algebraic group isomorphic to the group

$$SO_2(\mathbb{R}) = \text{Spec}(\mathbb{C}[t^{\pm 1}]^{\sigma_0^*}) \cong \text{Spec}(\mathbb{R}[u, v]/(u^2 + v^2 - 1)),$$

with group law given by $(u, v) \cdot (u', v') = (uu' - vv', uv' + u'v)$.

An action of $\mathbb{S}^1$ on a real algebraic variety (V, σ) is a real action of $(\mathbb{G}_{m,\mathbb{C}}, \sigma_0)$ on (V, σ) , that is, an action $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \rightarrow V$ of $\mathbb{G}_{m,\mathbb{C}}$ on V for which the following diagram commutes

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{C}} \times V & \xrightarrow{\mu} & V \\ \sigma_0 \times \sigma \downarrow & & \downarrow \sigma \\ \mathbb{G}_{m,\mathbb{C}} \times V & \xrightarrow{\mu} & V \end{array}$$

Let $\pi : (V, \sigma) \rightarrow (C, \tau)$ be an affine morphism between real algebraic varieties and let $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \rightarrow V$ be an $\mathbb{S}^1$ -action on (V, σ) by real (C, τ) -automorphisms. Putting $\mathcal{A} = \pi_* \mathcal{O}_V$, μ is uniquely determined by its associated $\mathcal{O}_C$ -algebra co-action homomorphism

$$\mu^* : \mathcal{A} \rightarrow \mathcal{A} \otimes_{\mathcal{O}_C} \mathcal{O}_C[t^{\pm 1}].$$

The latter determines a $\mathbb{Z}$ -grading of $\mathcal{A}$ by its $\mathcal{O}_C$ -submodules

$$\mathcal{A}_m = \{f \in \mathcal{A}, \mu^* f = f \otimes t^m\}, \quad m \in \mathbb{Z},$$

of semi-invariants germs of sections of weight m .

The action μ is said to be *effective* if the set $\{m \in \mathbb{Z}, \mathcal{A}_m \neq \{0\}\}$ is not contained in a proper subgroup of $\mathbb{Z}$, and *hyperbolic* if there exists $m < 0$ and $m' > 0$ such that $\mathcal{A}_m$ and $\mathcal{A}_{m'}$ are non zero. The following lemma is an extension in the relative affine setting of [8, Lemma 1.7].

Lemma 7. Let $\pi : (V, \sigma) \rightarrow (C, \tau)$ be an affine morphism between real algebraic varieties and let $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \rightarrow V$ be an effective $\mathbb{S}^1$ -action on (V, σ) by (C, τ) -automorphisms. Let $\mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m$ be the corresponding decomposition of the quasi-coherent $\mathcal{O}_C$ -algebra $\mathcal{A} = \pi_* \mathcal{O}_V$ into semi-invariants $\mathcal{O}_C$ -submodules. Then the following hold:

- 1) The action μ is hyperbolic and $\sigma^* \mathcal{A}_m = \tau_* \mathcal{A}_{-m}$ for every $m \in \mathbb{Z}$.
- 2) The $\mathcal{O}_C$ -module $\mathcal{A}_0$ is a quasi-coherent $\mathcal{O}_C$ -subalgebra of finite type of $\mathcal{A}$. Furthermore, the restriction $\sigma^* : \mathcal{A}_0 \rightarrow \tau_* \mathcal{A}_0$ of σ^* is the comorphism of a real structure τ_0 on $\text{Spec}_C(\mathcal{A}_0)$ for which the induced morphism $\pi_0 : (\text{Spec}_C(\mathcal{A}_0), \tau_0) \rightarrow (C, \tau)$ is a real morphism.

Proof. Since $\pi : (V, \sigma) \rightarrow (C, \tau)$ is an affine morphism, σ is equivalently determined by its comorphism $\sigma^* : \mathcal{A} \rightarrow \tau_*\mathcal{A}$. Since μ is a non trivial action, there exists a nonzero element $m \in \mathbb{Z}$ such that $\mathcal{A}_m \neq \{0\}$. The commutativity of the diagram in Definition 6 implies that for a local section f of $\mathcal{A}_m$,

$$\mu^*(\sigma^*(f)) = (\sigma^* \otimes \sigma_0^*)(f \otimes t^m) = \sigma^*(f) \otimes t^{-m}$$

hence that $\sigma^*(f) \in \tau_*\mathcal{A}_{-m}$. Thus $\sigma^*\mathcal{A}_m \subseteq \tau_*\mathcal{A}_{-m}$, and since $(\tau_*\sigma^*) \circ \sigma^* = \text{id}_{\mathcal{A}}$, it follows that the equality $\sigma^*\mathcal{A}_m = \tau_*\mathcal{A}_{-m}$ holds. This shows that the action μ is hyperbolic and that $\sigma^*\mathcal{A}_0 = \tau_*\mathcal{A}_0$. The fact that $\mathcal{A}_0$ is a quasi-coherent $\mathcal{O}_C$ -algebra of finite type is well-known [11, Theorem 1.1], and the fact that $\sigma^*|_{\mathcal{A}_0}$ is the comorphism of a real structure τ_0 on $\text{Spec}_C(\mathcal{A}_0)$ making $\pi_0 : (\text{Spec}_C(\mathcal{A}_0), \tau_0) \rightarrow (C, \tau)$ into a real morphism is a straightforward consequence of the definitions. $\square$

Definition 8. In the setting of Lemma 7, the real affine morphism $(V, \sigma) \rightarrow (\text{Spec}_C(\mathcal{A}_0), \tau_0)$ is called the real (categorical) quotient morphism of the $\mathbb{S}^1$ -action on (V, σ) .

1.3. Principal homogeneous $\mathbb{S}^1$ -bundles. Recall that a $\mathbb{G}_{m, \mathbb{C}}$ -torsor over a complex algebraic variety C is a C -scheme $\rho : P \rightarrow C$ endowed with an action $\mu : \mathbb{G}_{m, \mathbb{C}} \times P \rightarrow P$ of $\mathbb{G}_{m, \mathbb{C}}$ by C -scheme automorphisms, such that P is Zariski locally isomorphic over C to $C \times \mathbb{G}_{m, \mathbb{C}}$ on which $\mathbb{G}_{m, \mathbb{C}}$ acts by translations on the second factor.

Definition 9. An $\mathbb{S}^1$ -torsor (also called a principal homogeneous $\mathbb{S}^1$ -bundle) over a real algebraic variety (C, τ) is a real algebraic variety $\rho : (P, \sigma) \rightarrow (C, \tau)$ endowed with an $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times P \rightarrow P$ for which $\rho : P \rightarrow C$ is a $\mathbb{G}_{m, \mathbb{C}}$ -torsor.

Recall that isomorphism classes of $\mathbb{G}_{m, \mathbb{C}}$ -torsors $\rho : P \rightarrow C$ over C are in one-to-one correspondence with elements of the Picard group $\text{Pic}(C) \cong H^1(C, \mathcal{O}_C^*)$ of C . More explicitly, for every such P , there exists an invertible $\mathcal{O}_C$ -submodule $\mathcal{L}$ of the sheaf of rational functions $\mathcal{K}_C$ of C and an isomorphism of $\mathbb{Z}$ -graded algebras

$$\rho_*\mathcal{O}_P \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m},$$

where for $m < 0$, $\mathcal{L}^{\otimes m}$ denotes the $-m$ -th tensor power of the dual $\mathcal{L}^\vee$ of $\mathcal{L}$. Furthermore, two invertible $\mathcal{O}_C$ -submodules of $\mathcal{K}_C$ determine isomorphic $\mathbb{G}_{m, \mathbb{C}}$ -torsors if and only if they are isomorphic. For $\mathbb{S}^1$ -torsors, we have the following counterpart:

Lemma 10. For every $\mathbb{S}^1$ -torsor $\rho : (P, \sigma) \rightarrow (C, \tau)$ there exists a pair $(\mathcal{L}, h)$ consisting of an invertible $\mathcal{O}_C$ -submodule $\mathcal{L} \subset \mathcal{K}_C$ and a nonzero real rational function h on (C, τ) such that $\rho_*\mathcal{O}_P = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m}$ and $\mathcal{L} \otimes \tau^*\mathcal{L} = h^{-1}\mathcal{O}_C$ as $\mathcal{O}_C$ -submodules of $\mathcal{K}_C$.

Furthermore, two such pairs $(\mathcal{L}_1, h_1)$ and $(\mathcal{L}_2, h_2)$ determine isomorphic $\mathbb{S}^1$ -torsors if and only if there exists a rational function $f \in \Gamma(C, \mathcal{K}_C^*)$ such that $\mathcal{L}_1^\vee \otimes \mathcal{L}_2 = f^{-1}\mathcal{O}_C$ and $h_2 = (f\tau^*f)h_1$.

Proof. Let $\mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m$ be the decomposition of $\mathcal{A} = \rho_*\mathcal{O}_P$ into $\mathcal{O}_C$ -submodules of semi-invariants with respect to the action μ and let $\mathcal{L}$ be an invertible $\mathcal{O}_C$ -submodule of $\mathcal{K}_C$ for which we have an isomorphism of graded $\mathcal{O}_C$ -algebras

$$\Psi : \mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m \xrightarrow{\cong} \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m}.$$

By Lemma 7 1), we have $\sigma^*\mathcal{A}_m = \tau_*\mathcal{A}_{-m}$ for every $m \in \mathbb{Z}$. It follows that for every $m \in \mathbb{Z}$, the composition

$$\varphi_m : \tau_*\Psi \circ \sigma^* \circ \Psi^{-1} : \mathcal{L}^{\otimes m} \rightarrow \tau_*\mathcal{L}^{\otimes -m}$$

is an isomorphism of $\mathcal{O}_C$ -modules such that $\varphi_0 = \tau^* : \mathcal{O}_C = \mathcal{L}^{\otimes 0} \rightarrow \tau_*\mathcal{L}^{\otimes -0} = \tau_*\mathcal{O}_C$ and

$$\varphi_{m+m'} = \varphi_m \otimes \varphi_{m'} : \mathcal{L}^{\otimes(m+m')} = \mathcal{L}^{\otimes m} \otimes \mathcal{L}^{\otimes m'} \rightarrow \tau_*\mathcal{L}^{\otimes(-m-m')} = \tau_*\mathcal{L}^{\otimes(-m)} \otimes \tau_*\mathcal{L}^{\otimes(-m')}$$

for every $m, m' \in \mathbb{Z}$. Furthermore, since $\tau_*\sigma^* \circ \sigma^* = \text{id}_{\mathcal{A}}$ and $\tau_*^2 = \text{id}_{\mathcal{O}_C}$, we have

$$\tau_*\varphi_{-m} \circ \varphi_m = \text{id}_{\mathcal{L}^{\otimes m}} \quad \text{and} \quad \tau_*\varphi_m \circ \varphi_{-m} = \text{id}_{\mathcal{L}^{\otimes(-m)}}$$

for every $m \in \mathbb{Z}$. For $m = 1$ and $m' = -1$, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{L}^\vee & \xrightarrow{\varphi_1 \otimes \varphi_{-1}} & \tau_* \mathcal{L}^\vee \otimes \tau_* \mathcal{L} \\ \text{ev} \downarrow & & \downarrow \tau_* \text{ev} \\ \mathcal{O}_C & \xrightarrow{\tau^*} & \tau_* \mathcal{O}_C, \end{array}$$

where $\text{ev} : \mathcal{L} \otimes \mathcal{L}^\vee \xrightarrow{\cong} \mathcal{O}_C$ is the canonical homomorphism $f \otimes f' \mapsto f'(f)$, implies that

$$\varphi_{-1} = ({}^t\varphi_1)^{-1} : \mathcal{L}^\vee \rightarrow \tau_* \mathcal{L}.$$

It follows that $\varphi_m = \varphi_1^{\otimes m}$ for $m \geq 1$ and that $\varphi_m = ({}^t\varphi_1)^{\otimes (-m)}$ for $m \leq -1$. The collection $(\varphi_m)_{m \in \mathbb{Z}}$ is thus uniquely determined by $\varphi_0 = \tau^*$ and an isomorphism $\varphi_1 : \mathcal{L} \rightarrow \tau_* \mathcal{L}^\vee$ satisfying the identity $\tau_* ({}^t\varphi_1)^{-1} \circ \varphi_1 = \text{id}_{\mathcal{L}}$, hence equivalently by an isomorphism $\psi = \tau^* \varphi_1 : \tau^* \mathcal{L} \rightarrow \mathcal{L}^\vee$ such that $({}^t\psi^{-1}) \circ \tau^* \psi = \text{id}_{\mathcal{L}}$. An isomorphism $\psi : \tau^* \mathcal{L} \xrightarrow{\cong} \mathcal{L}^\vee$ is in turn equivalently determined by an isomorphism $\mathcal{O}_C \xrightarrow{\cong} \mathcal{L} \otimes \tau^* \mathcal{L}$, that is, by a rational function $h \in \Gamma(C, \mathcal{K}_C^*)$ such that $\mathcal{L} \otimes \tau^* \mathcal{L} = h^{-1} \mathcal{O}_C$ as $\mathcal{O}_C$ -submodules of $\mathcal{K}_C$. The condition $({}^t\psi^{-1}) \circ \tau^* \psi = \text{id}_{\mathcal{L}}$ then amounts to the property that $h^{-1}(\tau^* h) = 1$, i.e. that h is a real rational function on (C, τ) .

Two invertible $\mathcal{O}_C$ -submodules $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{K}_C$ define equivariantly isomorphic $\mathbb{G}_{m, \mathbb{C}}$ -torsors $\rho_1 : P_1 \rightarrow C$ and $\rho_2 : P_2 \rightarrow C$ if and only if there exists an isomorphism $\alpha : \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$. When $\mathcal{L}_1$ and $\mathcal{L}_2$ come with respective isomorphisms $\psi_1 : \tau^* \mathcal{L}_1 \rightarrow \mathcal{L}_1^\vee$ and $\psi_2 : \tau^* \mathcal{L}_2 \rightarrow \mathcal{L}_2^\vee$ corresponding to $\mathbb{S}^1$ -actions on (P_1, σ_1) and (P_2, σ_2) , the condition that a given isomorphism $\alpha : \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$ induces an $\mathbb{S}^1$ -equivariant isomorphism between (P_1, σ_1) and (P_2, σ_2) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \tau^* \mathcal{L}_1 & \xrightarrow{\psi_1} & \mathcal{L}_1^\vee \\ \tau^* \alpha \downarrow & & \downarrow {}^t\alpha^{-1} \\ \tau^* \mathcal{L}_2 & \xrightarrow{\psi_2} & \mathcal{L}_2^\vee. \end{array}$$

The isomorphism α is uniquely determined by a rational function $f \in \Gamma(C, \mathcal{K}_C^*)$ such that $\mathcal{L}_1^\vee \otimes \mathcal{L}_2 = f^{-1} \mathcal{O}_C$ as $\mathcal{O}_C$ -submodules of $\mathcal{K}_C$. By definition of h_1 and h_2 as the unique nonzero real rational functions on (C, τ) such that $\mathcal{L}_i \otimes \tau^* \mathcal{L}_i = h_i^{-1} \mathcal{O}_C$, $i = 1, 2$, the commutativity of the above diagram is equivalent to the equality $h_2 = (f \tau^* f) h_1$. $\square$

Example 11. By Hilbert's Theorem 90, every $\mathbb{G}_{m, \mathbb{C}}$ -torsor over $\text{Spec}(\mathbb{C})$ is isomorphic to the trivial one, that is, to $\mathbb{G}_{m, \mathbb{C}}$ acting on itself by translations. In contrast, there exists precisely two non-isomorphic $\mathbb{S}^1$ -torsors over $\text{Spec}(\mathbb{R}) = (\text{Spec}(\mathbb{C}), \sigma_{\text{Spec}(\mathbb{R})})$:

1) The trivial one given by $\mathbb{S}^1 = (\mathbb{G}_{m, \mathbb{C}} = \text{Spec}(\mathbb{C}[t^{\pm 1}], \sigma_0)$ acting on itself by translations. A corresponding pair is $(\mathcal{L}, h) = (\mathbb{C}, 1)$,

2) A nontrivial one $\hat{\mathbb{S}}^1 = (\text{Spec}(\mathbb{C}[u^{\pm 1}], \hat{\sigma}_0)$ whose real structure $\hat{\sigma}_0$ is the composition of the involution $u \mapsto -u^{-1}$ with the complex conjugation, endowed with the $\mathbb{S}^1$ -action given by $t \cdot u = tu$. A corresponding pair is $(\mathcal{L}, h) = (\mathbb{C}, -1)$.

Note that the real locus of $\mathbb{S}^1$ is isomorphic to the real circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ whereas the real locus of $\hat{\mathbb{S}}^1$ is empty.

2. CIRCLE ACTIONS ON SMOOTH REAL AFFINE SURFACES

In this section, we first review the correspondence between normal real affine surfaces (S, σ) with effective $\mathbb{S}^1$ -actions and suitable pairs consisting of a Weil $\mathbb{Q}$ -divisor and a rational function on smooth real affine curves (C, τ) , which we call DPD-pairs. We characterize smooth affine surfaces in terms of properties of their corresponding pairs. We then describe the structure of exceptional orbits of $\mathbb{S}^1$ -actions on smooth surfaces (S, σ) in relation to degenerate fibers of their quotient morphisms.

2.1. Real DPD-presentations of smooth real affine surfaces with $\mathbb{S}^1$ -actions. Recall that a Weil $\mathbb{Q}$ -divisor on a smooth real affine curve $(C = \text{Spec}(A_0), \tau)$ is an element of the abelian group consisting of formal sums

$$D = \sum_{c \in C} D(c)\{c\} \in \mathbb{Q} \otimes_{\mathbb{Z}} \text{Div}(C)$$

such that $D(c) \in \mathbb{Q}$ is equal to zero for all but finitely many points $c \in C$. The *support* of D is the finite set of points $c \in C$ such that $D(c) \neq 0$. The group of Weil $\mathbb{Q}$ -divisors is partially ordered by the relation

$$(D \leq D' \Leftrightarrow D(c) \leq D'(c) \quad \forall c \in C).$$

Every nonzero rational function f on C determines a Weil $\mathbb{Q}$ -divisor $\text{div}(f) = \sum_{c \in C} (\text{ord}_c f)\{c\}$ with integral coefficients. For every Weil $\mathbb{Q}$ -divisor D on C , we denote by $\Gamma(C, \mathcal{O}_C(D))$ the A_0 -submodule of the field of fractions $\text{Frac}(A_0)$ of A_0 generated by nonzero rational functions f on C such that $\text{div}(f) + D \geq 0$.

Given an automorphism α of C as a scheme over $\mathbb{R}$ or $\mathbb{C}$, the pull-back of $D = \sum_{c \in C} D(c)\{c\}$ by α is the Weil $\mathbb{Q}$ -divisor

$$\alpha^* D = \sum_{c \in C} D(c)\{\alpha^{-1}(c)\} = \sum_{c \in C} D(\alpha(c))\{c\}.$$

Definition 12. A *real DPD-pair* on a smooth real affine curve (C, τ) is a pair (D, h) consisting of a Weil $\mathbb{Q}$ -divisor D on C and a nonzero real rational function h on (C, τ) satisfying $D + \tau^* D \leq \text{div}(h)$.

We say that two rational numbers $r_i = p_i/q_i$, $i = 1, 2$, where $\gcd(p_i, q_i) = 1$, form a *regular pair* if $|p_1 q_2 - p_2 q_1| = 1$.

Definition 13. A real DPD-pair (D, h) on a smooth real affine curve (C, τ) is said to be *regular* if for every $c \in C$ such that $D(c) + D(\tau(c)) < \text{ord}_c(h)$ the rational numbers $D(c)$ and $D(\tau(c)) - \text{ord}_c(h)$ form a regular pair.

Given a smooth real affine curve (C, τ) , a pair $(\mathcal{L}, h)$ consisting of an invertible $\mathcal{O}_C$ -submodule $\mathcal{L} \subset \mathcal{K}_C$ and a real rational function h on (C, τ) such that $\mathcal{L} \otimes \tau^* \mathcal{L} = h^{-1} \mathcal{O}_C$ as $\mathcal{O}_C$ -submodules of $\mathcal{K}_C$ determines a Cartier divisor D on C such that $D + \tau^* D = \text{div}(h)$, hence a regular real DPD-pair (D, h) on (C, τ) . By Lemma 10, every smooth real affine surface (S, σ) endowed with the structure of an $\mathbb{S}^1$ -torsor over (C, τ) is determined by such a regular real DPD-pair (D, h) . More generally, we have the following:

Theorem 14. Every normal real affine surface (S, σ) with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$ is determined by a smooth real affine curve (C, τ) and a real DPD-pair (D, h) on it. Furthermore, the following hold:

1) Two DPD-pairs (D_1, h_1) and (D_2, h_2) on the same curve (C, τ) determine $\mathbb{S}^1$ -equivariantly isomorphic real affine surfaces if and only if there exists a real automorphism ψ of (C, τ) and a rational function f on C such that

$$\psi^* D_2 = D_1 + \text{div}(f) \quad \text{and} \quad \psi^* h_2 = (f \tau^* f) h_1.$$

2) The normal real affine surface (S, σ) determined by a real DPD-pair (D, h) is smooth if and only if the pair is regular.

The correspondence between normal real affine surface (S, σ) with an effective $\mathbb{S}^1$ -actions and real DPD-pairs on smooth real affine curves (C, τ) was established in [8, Proposition 3.2] as a particular case of a general correspondence between $\mathbb{S}^1$ -actions on normal real affine varieties and suitable pairs (D, h) on certain normal real semi-projective varieties [8, Corollary 2.16], whose proof uses the formalism of polyhedral divisors due to Altmann and Hausen [1]. Since this correspondence provides an explicit method to determine the data (S, σ) and (C, τ) , (D, h) from each others, we will review it in detail using the DPD-formalism of [9] in the next subsections.

Proof of Theorem 14. Assertion 1) follows from Corollary 2.16 in [8]. Note that if (D_2, h_2) is regular real DPD-pair then for every real automorphism ψ of (C, τ) and every rational function f on C , the real DPD-pair

$$(D_1, h_1) = (\psi^* D_2 - \text{div}(f), (f \tau^* f)^{-1} \psi^* h_2)$$

is regular due to the fact that $\operatorname{div}(f)$ and $\operatorname{div}((f\tau^*f)^{-1}\psi^*h_2)$ are integral Weil divisors on C .

To prove 2), let (S, σ) be the normal real affine surface with $\mathbb{S}^1$ -action determined by real DPD-pair (D, h) on a smooth real affine curve (C, τ) as in § 2.1.1 below. Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be its real quotient morphism and let $D_+ = D$ and $D_- = \tau^*D - \operatorname{div}(h)$. By [9, Theorem 4.15], the singular locus of S is contained in the fibers of the quotient morphism $\pi : S \rightarrow C$ over the points $c \in C$ such that $D_+(c) + D_-(c) < 0$. Furthermore, for such a point c , S is smooth at every point of $\pi^{-1}(c)$ if and only if the rational numbers $D_+(c)$ and $D_-(c)$ form a regular pair. $\square$

2.1.1. From real DPD-pairs to normal real affine surfaces with effective $\mathbb{S}^1$ -actions. Given a real DPD-pair (D, h) on a smooth real affine curve $(C = \operatorname{Spec}(A_0), \tau)$, we set $D_+ = D$ and $D_- = \tau^*D_+ - \operatorname{div}(h)$. The condition $D + \tau^*D \leq \operatorname{div}(h)$ implies that $D_+ + D_- \leq 0$, so that for every $m' \leq 0 \leq m$, the product

$$\Gamma(C, \mathcal{O}_C(-m'D_-)) \cdot \Gamma(C, \mathcal{O}_C(mD_+))$$

in $\operatorname{Frac}(A_0)$ is contained either in $\Gamma(C, \mathcal{O}_C(-(m' + m)D_-))$ if $m' + m \leq 0$ or in $\Gamma(C, \mathcal{O}_C((m' + m)D_+))$ if $m' + m \geq 0$. It follows that the graded A_0 -module

$$A_0[D_-, D_+] = \bigoplus_{m < 0} \Gamma(C, \mathcal{O}_C(-mD_-)) \oplus \Gamma(C, \mathcal{O}_C) \oplus \bigoplus_{m > 0} \Gamma(C, \mathcal{O}_C(mD_+))$$

is a graded A_0 -algebra for the multiplication law given by component wise multiplication in $\operatorname{Frac}(A_0)$. By [9, § 4.2], $A_0[D_+, D_-, h]$ is finitely generated over $\mathbb{C}$ and normal. The grading then corresponds to an effective hyperbolic $\mathbb{G}_{m, \mathbb{C}}$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$ on the normal complex affine surface $S = \operatorname{Spec}(A_0[D_-, D_+])$. The ring of invariants for this action is equal to A_0 and the morphism $\pi : S \rightarrow C = \operatorname{Spec}(A_0)$ induced by the inclusion $A_0 \subset A_0[D_-, D_+]$ is the categorical quotient morphism. Since $D_- = \tau^*D_+ - \operatorname{div}(h)$, for every $m \geq 1$, the homomorphism

$$\tau_m^* : \Gamma(C, \mathcal{O}_C(mD_+)) \rightarrow \Gamma(C, \mathcal{O}_C(mD_-)), f \mapsto h^m \tau^* f$$

is an isomorphism with inverse

$$\tau_{-m}^* : \Gamma(C, \mathcal{O}_C(-mD_-)) \rightarrow \Gamma(C, \mathcal{O}_C(-mD_+)), f \mapsto h^m \tau^* f.$$

Letting $\tau_0 = \tau$, these isomorphisms collect into an automorphism $\sigma^* = \bigoplus_{m \in \mathbb{Z}} \tau_m^*$ of $A_0[D_-, D_+]$ which is the comorphism of a real structure σ on S for which we have $\sigma \circ \mu = \mu \circ (\sigma_0 \times \sigma)$. It follows that (S, σ) is a normal real affine surface and that $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$ is an effective $\mathbb{S}^1$ -action on (S, σ) in the sense of Definition 6.

Example 15. Let $(C = \operatorname{Spec}(A_0), \tau)$ be a smooth real affine curve with a real point c whose defining ideal is principal, generated by a real regular function h on (C, τ) . Let D be the trivial divisor 0. Then (D, h) is a real DPD-pair on (C, τ) for which we have $D_+ = D = 0$ and $D_- = \tau^*D - \operatorname{div}(h) = -\{c\}$. It follows that $\Gamma(C, \mathcal{O}_C(mD_+)) = A_0$ and that $\Gamma(C, \mathcal{O}_C(mD_-)) = h^m A_0$ for every $m \geq 0$. The corresponding homomorphism

$$\tau_m^* : \Gamma(C, \mathcal{O}_C(mD_+)) = A_0 \rightarrow h^m A_0 = \Gamma(C, \mathcal{O}_C(mD_-))$$

is the multiplication by h^m . The algebra $A_0[D_+, D_-]$ is generated by the homogeneous elements

$$x = 1 \in \Gamma(C, \mathcal{O}_C(D_+)) = \Gamma(C, \mathcal{O}_C) \quad \text{and} \quad y = h \in \Gamma(C, \mathcal{O}_C(D_-)) = \Gamma(C, \mathcal{O}_C(-\{c\}))$$

of degree 1 and -1 respectively. These satisfy the obvious homogeneous relation $xy = h$, and we have

$$A_0[D_+, D_-] \cong A_0[x, y]/(xy - h).$$

The corresponding $\mathbb{G}_{m, \mathbb{C}}$ -action μ on $S = \operatorname{Spec}(A_0[D_+, D_-])$ is given by $t \cdot (x, y) = (tx, t^{-1}y)$ and the real structure σ for which μ becomes an $\mathbb{S}^1$ -action on (S, σ) is the lift of τ defined by $\sigma^*x = y$ and $\sigma^*y = x$.

2.1.2. From normal real affine surfaces with effective $\mathbb{S}^1$ -actions to real DPD-pairs. Given a normal real affine surface (S, σ) with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$, it follows from Lemma 7 that the coordinate ring A of S decomposes as the direct sum $A = \bigoplus_{m \in \mathbb{Z}} A_m$ of semi-invariants sub-spaces such that $\sigma^*(A_m) = A_{-m}$ for every $m \in \mathbb{Z}$. The curve $C = \text{Spec}(A_0)$ is the categorical quotient of the $\mathbb{G}_{m, \mathbb{C}}$ -action on S . The restriction of σ^* to A_0 induces a real structure τ on C . Let $s \in \text{Frac}(A)$ be any semi-invariant rational function of weight 1 and let $h = s\sigma^*s \in \text{Frac}(A)$. Since σ^*s is a semi-invariant rational function of weight -1 , h is a σ^* -invariant rational function of weight 0, hence a τ^* -invariant element of $\text{Frac}(A_0)$. For every $m \in \mathbb{Z}$, $s^{-m}A_m$ is a locally free A_0 -submodule of $\text{Frac}(A_0)$. By [9, § 4.2], there exists Weil $\mathbb{Q}$ -divisors D_+ and D_- on C satisfying $D_+ + D_- \leq 0$ such that for every $m \geq 0$ we have

$$s^{-m}A_m = \Gamma(C, \mathcal{O}_C(mD_+)) \quad \text{and} \quad s^mA_{-m} = \Gamma(C, \mathcal{O}_C(mD_-))$$

as A_0 -submodules of $\text{Frac}(A_0)$. Since by Lemma 7 and the definition of h , we have

$$\tau^*(s^{-m} \cdot A_m) = h^{-m}(s^m \cdot A_{-m}) \quad \forall m \in \mathbb{Z},$$

it follows that $D_- = \tau^*D_+ - \text{div}(h)$. So setting $D = D_+$, the pair (D, h) is a real DPD- pair on the smooth real affine curve (C, τ) . By construction, $S \cong \text{Spec}(A_0[D_-, D_+])$ and the real structure σ on S coincides with that constructed from (D, h) in the previous subsection.

Example 16. Let $(C = \text{Spec}(R), \tau)$ be a smooth real affine curve with a real point c whose defining ideal is principal, generated by some real regular function f on (C, τ) . Let $A = R[x^{\pm 1}, y]/(xy^2 - f)$ and let $S = \text{Spec}(A)$. The morphism $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$, $(t, (x, y)) \mapsto (t^2x, t^{-1}y)$ defines an $\mathbb{G}_{m, \mathbb{C}}$ -action on S by C -automorphisms, which becomes an $\mathbb{S}^1$ -action by (C, τ) -automorphisms when S is endowed with the unique real structure σ lifting τ such $\sigma^*x = x^{-1}$ and $\sigma^*y = xy$. The ring of $\mathbb{G}_{m, \mathbb{C}}$ -invariant A_0 is equal to $R[xy^2]/(xy^2 - f) \cong R$. Choosing $s = y^{-1}$ as semi-invariant rational function of weight 1, we have $h = y^{-1}\sigma^*(y^{-1}) = x^{-1}y^{-2} = f^{-1} \in \text{Frac}(R)$. The decomposition of A into subspaces of semi-invariants functions is then given for every $m \geq 0$ by

$$\begin{aligned} s^{-m}A_m &= s^{-m}R \cdot (xy)^m = R \cdot (xy^2)^m = f^mR = \Gamma(C, \mathcal{O}_C(mD_+)), \\ s^{2m+1}A_{-2m-1} &= s^{2m+1}R \cdot (x^{-m}y) = R \cdot (xy^2)^{-m} = f^{-m}R = \Gamma(C, \mathcal{O}_C((2m+1)D_-)), \\ s^{2m}A_{-2m} &= s^{2m}R \cdot x^{-m} = R \cdot (x^{-m}y^{-2m}) = f^{-m}R = \Gamma(C, \mathcal{O}_C(2mD_-)). \end{aligned}$$

It follows that $D_+ = -r\{c\}$ for some rational number $r \in]0, 1]$ and that

$$D_- = \tau^*(D_+) - \text{div}(h) = r\{c\} - \text{div}(f^{-1}) = (1-r)\{c\}.$$

Since $f^{-m}R = \Gamma(C, \mathcal{O}_C((2m+1)D_-)) = \Gamma(C, \mathcal{O}_C(2mD_-))$ for every $m \geq 0$, it follows that

$$\frac{m}{2m+1} \leq (1-r) < \frac{m+1}{2m+1} \quad \text{and} \quad \frac{1}{2} \leq (1-r) < \frac{m+1}{2m}$$

for every $m \geq 0$. Thus $(1-r) = \frac{1}{2}$ and a real DPD-pair on (C, τ) corresponding to (S, σ) endowed with the $\mathbb{S}^1$ -action μ is $(D, h) = (-\frac{1}{2}\{c\}, f^{-1})$.

2.2. Real fibers of the quotient morphism: principal and exceptional orbits. Let (S, σ) be a smooth real affine surface with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$, and let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be its real quotient morphism. Recall that $\pi : S \rightarrow C = \text{Spec}(\Gamma(S, \mathcal{O}_S)^{\mathbb{G}_{m, \mathbb{C}}})$ is surjective and that each fiber of π contains a unique closed $\mathbb{G}_{m, \mathbb{C}}$ -orbit Z and is the union of all $\mathbb{G}_{m, \mathbb{C}}$ -orbits in S containing Z in their closure. In the complex case, [9, Theorem 18] provides a description of the structure of the fibers of π in terms of a pair of Weil $\mathbb{Q}$ -divisors D_+ and D_- on C for which $\Gamma(S, \mathcal{O}_S) = A_0[D_-, D_+]$ (see § 2.1.2). In this subsection, we specialize this description to fibers of π over points in the real locus of (C, τ) . We begin with the following example which illustrates different possibilities for such fibers.

Example 17. Let $S_\varepsilon \subset \mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ be the smooth complex affine surface with equation

$$xy = z^2 + \varepsilon, \quad \text{where } \varepsilon = \pm 1,$$

endowed with the real structure σ defined as the composition of the involution $(x, y, z) \mapsto (y, x, z)$ with the complex conjugation. The effective $\mathbb{G}_{m, \mathbb{C}}$ -action μ on S_ε given by $t \cdot (x, y, z) = (tx, t^{-1}y, z)$ defines an $\mathbb{S}^1$ -action on (S_ε, σ) whose real quotient morphism coincides with the projection

$$\pi = \text{pr}_z : (S_\varepsilon, \sigma) \rightarrow (C, \tau) = (\text{Spec}(\mathbb{C}[z], \sigma_{\mathbb{A}_{\mathbb{R}}^1}).$$

A corresponding real DPD-pair is for instance $(D, h) = (0, z^2 + \varepsilon)$ where 0 denotes the trivial Weil divisor.

The morphism $(x, y, z) \mapsto (-x, -y, -z)$ defines a fixed point free real action of $\mathbb{Z}_2$ on (S_ε, σ) commuting with the $\mathbb{S}^1$ -action. The quotient surface $\bar{S}_\varepsilon = S_\varepsilon / \mathbb{Z}_2$ is smooth and σ descends to a real structure $\bar{\sigma}$ on it. The morphism π descends to a real morphism $\bar{\pi} : (\bar{S}_\varepsilon, \bar{\sigma}) \rightarrow (\bar{C}, \bar{\tau}) = (\text{Spec}(\mathbb{C}[z^2]), \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ which coincides with the real quotient morphism of the induced $\mathbb{S}^1$ -action on $\bar{S}_\varepsilon$.

1) If $\varepsilon = 1$, then since $z^2 + 1 = (z - i)(z + i) = f\tau^*f$, it follows from Theorem 14 1) and Lemma 10 that $\pi : (S_1, \sigma) \rightarrow (C, \tau)$ restricts to the trivial $\mathbb{S}^1$ -torsor over the principal real affine open subset $(C_h = \text{Spec}(\mathbb{C}[z]_{z^2+1}), \tau|_{C_h})$ of C . In particular, for every real point c of (C, τ) , $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is isomorphic to $\mathbb{S}^1$ on which $\mathbb{S}^1$ -acts by translations.

Since the real point $0 \in (C, \tau)$ is a fixed point of the $\mathbb{Z}_2$ -action on C , the fiber of $\bar{\pi} : (\bar{S}_1, \bar{\sigma}) \rightarrow (\bar{C}, \bar{\tau})$ over the real point $0 \in (\bar{C}, \bar{\tau})$ has multiplicity two. When endowed with its reduced structure, it is isomorphic to the quotient of $\text{Spec}(\mathbb{C}[x, y]/(xy - 1))$ by the involution $(x, y) \mapsto (-x, -y)$, hence to $\mathbb{A}_{\mathbb{R}}^1 = \text{Spec}(\mathbb{C}[w^{\pm 1}])$, where $w = x^2$. The real structure is given by the composition of the involution $w \mapsto w^{-1}$ with the complex conjugation, and the group $\mathbb{G}_{m, \mathbb{C}}$ acts on it by $t \cdot w = t^2w$. So $(\bar{\pi}^{-1}(0)_{\text{red}}, \bar{\sigma}|_{\bar{\pi}^{-1}(0)_{\text{red}}})$ is isomorphic to $\mathbb{S}^1$ on which $\mathbb{S}^1$ acts with stabilizer μ_2 .

2) If $\varepsilon = -1$, then, in contrast with the previous case, there is no rational function $f \in \mathbb{C}(z)$ such that $z^2 - 1 = f\tau^*f$. Consequently, there is no real open subset of (C, τ) over which $\pi : (S_{-1}, \sigma) \rightarrow (C, \tau)$ restricts to the trivial $\mathbb{S}^1$ -torsor. For a real point c of (C, τ) , $h = z^2 - 1$ takes negative value at c if $c \in]-1, 1[$ and positive value if $c \in]-\infty, -1[\cup]1, +\infty[$. The fiber $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is thus isomorphic to the nontrivial $\mathbb{S}^1$ -torsor $\hat{\mathbb{S}}^1$ of Example 11 in the first case, and to the trivial $\mathbb{S}^1$ -torsor $\mathbb{S}^1$ in the second case.

The fiber of π over the point ± 1 is isomorphic to $\text{Spec}(\mathbb{C}[x, y]/(xy))$ and thus consists of two affine lines $\bar{O}^+ = \text{Spec}(\mathbb{C}[x])$ and $\bar{O}^- = \text{Spec}(\mathbb{C}[y])$ exchanged by the real structure σ , intersecting at the real point $p = (0, 0, 0)$ of (S_{-1}, σ) . The curves $O^\pm = \bar{O}^\pm \setminus \{p\} \cong \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ endowed with the induced $\mathbb{G}_{m, \mathbb{C}}$ -actions are trivial $\mathbb{G}_{m, \mathbb{C}}$ -torsors and p is an $\mathbb{S}^1$ -fixed point.

As in the previous case, since the real point $0 \in (C, \tau)$ is a fixed point of the $\mathbb{Z}_2$ -action on C , the fiber of $\bar{\pi} : (\bar{S}_{-1}, \bar{\sigma}) \rightarrow (\bar{C}, \bar{\tau})$ over the real point $0 \in (\bar{C}, \bar{\tau})$ has multiplicity two. When endowed with its reduced structure, it is isomorphic to the quotient of $\text{Spec}(\mathbb{C}[x, y]/(xy + 1))$ by the involution $(x, y) \mapsto (-x, -y)$ hence to $\mathbb{A}_{\mathbb{R}}^1 = \text{Spec}(\mathbb{C}[w^{\pm 1}])$, where $w = x^2$. The induced real structure is the composition of the involution $w \mapsto w^{-1}$ with the complex conjugation. The induced $\mathbb{G}_{m, \mathbb{C}}$ -action is given by $t \cdot w = t^2w$. Thus $(\bar{\pi}^{-1}(0)_{\text{red}}, \bar{\sigma}|_{\bar{\pi}^{-1}(0)_{\text{red}}})$ is isomorphic to $\mathbb{S}^1$ on which $\mathbb{S}^1$ -acts with stabilizer μ_2 .

Lemma 18. *Let (S, σ) be a smooth real affine surface with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$ determined by a regular real DPD-pair (D, h) on a smooth real affine curve (C, τ) , and let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be the corresponding real quotient morphism. Then for every real point c of (C, τ) there exists a principal real affine open neighborhood $(U, \tau|_U)$ of c and a regular real DPD-pair (D', h') on $(U, \tau|_U)$ with the following properties:*

- 1) $D'|_{U \setminus \{c\}}$ is the trivial divisor, $D'(c) \in [0, 1[$ and $h' \in \Gamma(U, \mathcal{O}_U) \cap \Gamma(U \setminus \{c\}, \mathcal{O}_{U \setminus \{c\}}^*)$.
- 2) The surface $(\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$ is $\mathbb{S}^1$ -equivariantly isomorphic to that determined by the real DPD-pair (D', h') on $(U, \tau|_U)$.

In particular, $\pi|_{\pi^{-1}(U \setminus \{c\})} : (\pi^{-1}(U \setminus \{c\}), \sigma|_{\pi^{-1}(U \setminus \{c\})}) \rightarrow (U \setminus \{c\}, \tau|_{U \setminus \{c\}})$ is an $\mathbb{S}^1$ -torsor.

Proof. Recall that by the construction described in § 2.1.1, we have

$$\Gamma(S, \mathcal{O}_S) = A_0[D_+, D] = \bigoplus_{m < 0} \Gamma(C, \mathcal{O}_C(-mD_-)) \oplus \Gamma(C, \mathcal{O}_C) \oplus \bigoplus_{m > 0} \Gamma(C, \mathcal{O}_C(mD_+))$$

where $A_0 = \Gamma(C, \mathcal{O}_C)$, $D_+ = D$ and $D_- = \tau^*D - \text{div}(h)$. Let $U = C_g$ be a real principal affine open neighborhood of c for some real regular function g on (C, τ) , and let $(S|_U, \sigma|_U) = (\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$ be endowed with the induced $\mathbb{S}^1$ -action. The graded coordinate ring of $S|_U$ is isomorphic to the homogeneous localization

$$\Gamma(S, \mathcal{O}_S)_{(g)} \cong \bigoplus_{m < 0} \Gamma(U, \mathcal{O}_C(-mD_-)) \oplus \Gamma(U, \mathcal{O}_U) \oplus \bigoplus_{m > 0} \Gamma(U, \mathcal{O}_C(mD_+))$$

of $\Gamma(S, \mathcal{O}_S)$ with respect to $g \in \Gamma(C, \mathcal{O}_C)$. It follows that $(S|_U, \sigma|_U)$ is $\mathbb{S}^1$ -equivariantly isomorphic to the real affine surface determined by the real DPD-pair $(D|_U, h|_U)$ on the smooth real affine curve $(U, \tau|_U)$. For a small enough such real affine neighborhood U of c , we have $D(c') = 0$ for every $c' \in U \setminus \{c\}$ and $h \in \Gamma(U \setminus \{c\}, \mathcal{O}_{U \setminus \{c\}}^*)$. In particular, $D|_{U \setminus \{c\}}$ is a principal Cartier divisor such that $D|_{U \setminus \{c\}} + \tau|_{U \setminus \{c\}}^* D|_{U \setminus \{c\}} = \text{div}(h|_{U \setminus \{c\}})$, which implies by Lemma 10 that $\pi : (S|_{U \setminus \{c\}}, \sigma|_{U \setminus \{c\}}) \rightarrow (U \setminus \{c\}, \tau|_{U \setminus \{c\}})$ is an $\mathbb{S}^1$ -torsor. Shrinking U further if necessary, we can ensure in addition that $c = \text{div}(f)$ for some real regular function on $(U, \tau|_U)$. Letting $\delta = \lfloor D(c) \rfloor$ be the round-down of the rational number $D(c)$, it follows from Theorem 14 1) that $(S|_U, \sigma|_U)$ is $\mathbb{S}^1$ -equivariantly isomorphic to the surface determined by the regular real DPD-pair

$$(D', h') = (D|_U - \text{div}(f^\delta), (f^{-\delta} \tau^* f^{-\delta})h),$$

on $(U, \tau|_U)$. By construction, we have $D' = (D(c) - \delta)\{c\}$ where $D(c) - \delta \in [0, 1[$ and $h' \in \Gamma(U \setminus \{c\}, \mathcal{O}_{U \setminus \{c\}}^*)$. Since (D', h') is a real DPD-pair,

$$\text{ord}_c(h') \geq D'(c) + \tau^*(D')(c) = 2D'(c) \geq 0,$$

which implies that $h' \in \Gamma(U, \mathcal{O}_U) \cap \Gamma(U \setminus \{c\}, \mathcal{O}_{U \setminus \{c\}}^*)$. $\square$

Definition 19. Let (S, σ) be a smooth real affine surface with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$. A $\mathbb{G}_{m, \mathbb{C}}$ -orbit Z is called *principal* if Z endowed with the $\mathbb{G}_{m, \mathbb{C}}$ -action induced by μ is the trivial $\mathbb{G}_{m, \mathbb{C}}$ -torsor. It is called *exceptional* otherwise. If Z is in addition irreducible and σ -invariant, we say that $(Z, \sigma|_Z)$ is a principal $\mathbb{S}^1$ -orbit if Z is a principal $\mathbb{G}_{m, \mathbb{C}}$ -orbit, and an exceptional $\mathbb{S}^1$ -orbit otherwise.

Theorem 20. Let (S, σ) be a smooth real affine surface with an effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m, \mathbb{C}} \times S \rightarrow S$ determined by a regular real DPD-pair (D, h) on a smooth real affine curve (C, τ) . Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be the corresponding real quotient morphism and let c be a real point of (C, τ) . Then exactly one of the following three possibilities occurs:

- a) $D(c) \in \mathbb{Z}$ and $\text{ord}_c(h) = 2D(c)$. In this case, there exists a real affine open neighborhood $(U, \tau|_U)$ of c such that $\pi|_{\pi^{-1}(U)} : (\pi^{-1}(U), \sigma|_{\pi^{-1}(U)}) \rightarrow (U, \tau|_U)$ is an $\mathbb{S}^1$ -torsor. The fiber $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is an $\mathbb{S}^1$ -torsor over $(c, \sigma_{\text{Spec}(\mathbb{R})})$ which is either isomorphic to $\mathbb{S}^1$ if $\pi^{-1}(c)$ contains a real point of (S, σ) , or to the nontrivial $\mathbb{S}^1$ -torsor $\hat{\mathbb{S}}^1$ of Example 11 otherwise.
- b) $D(c) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $\text{ord}_c(h) = 2D(c)$. In this case, $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is a multiple fiber of multiplicity 2, whose reduction is an exceptional $\mathbb{S}^1$ -orbit, isomorphic to $\mathbb{S}^1$ on which $\mathbb{S}^1$ acts with stabilizer μ_2 .
- c) $D(c) \in \mathbb{Z}$ and $\text{ord}_c(h) = 2D(c) + 1$. In this case, the fiber $\pi^{-1}(c)$ is reduced, consisting of the closures of two principal $\mathbb{G}_{m, \mathbb{C}}$ orbits O^+ and O^- exchanged by the real structure σ , whose closures $\overline{O}^\pm$ in S are affine lines intersecting transversally at an $\mathbb{S}^1$ -fixed real point p of (S, σ) .

Furthermore, in cases b) and c), for every real affine open neighborhood $(U, \tau|_U)$ of $c \in C$, the restriction

$$\pi|_{\pi^{-1}(U \setminus \{c\})} : (\pi^{-1}(U \setminus \{c\}), \sigma|_{\pi^{-1}(U \setminus \{c\})}) \rightarrow (U \setminus \{c\}, \tau|_{U \setminus \{c\}})$$

is a nontrivial $\mathbb{S}^1$ -torsor.

Proof. By Lemma 18, there exists a real affine open neighborhood $(U, \tau|_U)$ of c such that $c = \text{div}(f)$ for some real regular function f on $(U, \tau|_U)$ and such that $(S|_U, \sigma|_U)$ is $\mathbb{S}^1$ -equivariantly isomorphic over

$(U, \tau|_U)$ to the real affine surface determined by a regular real DPD-pair (D', h') such that $D' = D'(c)c$ where $D'(c) \in [0, 1[$ and $h' \in \Gamma(U, \mathcal{O}_U) \cap \Gamma(U \setminus \{c\}, \mathcal{O}_{U \setminus \{c\}}^*)$. Since $D'(c) + \tau^* D'(c) \leq \text{ord}_c(h')$ by definition of a real DPD-pair, this leads to the following dichotomy:

I) If $2D'(c) = \text{ord}_c(h')$ then since $D'(c) \in [0, 1[$ and $\text{ord}_c(h')$ is an integer, we have either $D'(c) = 0$ and $\text{ord}_c(h') = 0$ or $D'(c) = \frac{1}{2}$ and $\text{ord}_c(h') = 1$. In first case, D' is the trivial divisor, and so is $\tau^* D'$. Furthermore, since h' does not vanish on U , $\pi|_{\pi^{-1}(U)} : (S|_U, \sigma|_U) = (\pi^{-1}(U), \sigma|_{\pi^{-1}(U)}) \rightarrow (U, \tau|_U)$ is an $\mathbb{S}^1$ -torsor by Lemma 10. By Theorem 14 1) and Example 11, $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is isomorphic either to $\mathbb{S}^1$ if $h_c(c) \in \mathbb{R}_{>0}$, or to $\hat{\mathbb{S}}^1$ otherwise. This yields case a).

In the second case, we have $D' = \frac{1}{2}\{c\}$ and it follows from [9, Theorem 18 (a)] that $\pi^{-1}(c) = 2Z$ where Z is an exceptional $\mathbb{G}_{m, \mathbb{C}}$ -orbit isomorphic to a punctured affine line on which $\mathbb{G}_{m, \mathbb{C}}$ acts with stabilizer μ_2 . The real curve $(Z, \sigma|_Z)$ endowed with the restriction of μ is thus isomorphic either to $\mathbb{S}^1$ if it contains a real point or to $\hat{\mathbb{S}}^1$ otherwise, on which $\mathbb{S}^1$ acts with stabilizer μ_2 . We claim that the case where $(Z, \sigma|_Z)$ is isomorphic to $\hat{\mathbb{S}}^1$ does not occur. Indeed, the real structure $\tau|_U$ on U lifts in a unique way to a real structure $\tilde{\tau}$ on $\tilde{U} = \text{Spec}(\Gamma(U, \mathcal{O}_U)[X]/(X^2 - f))$ such that $\tilde{\tau}^* X = X$, for which the morphism $\psi : (\tilde{U}, \tilde{\tau}) \rightarrow (U, \tau|_U)$ the induced morphism $\psi : (\tilde{U}, \tilde{\tau}) \rightarrow (U, \tau|_U)$ is a real double cover totally ramified over c and étale elsewhere. The normalization over the fiber product $S \times_U \tilde{U}$ is a smooth real affine surface $(\tilde{S}, \tilde{\sigma})$ and the action of the Galois group $\mathbb{Z}_2$ of the cover $\psi : (\tilde{U}, \tilde{\tau}) \rightarrow (U, \tau|_U)$ lifts to a free real $\mathbb{Z}_2$ -action on $(\tilde{S}, \tilde{\sigma})$ for which we have $(S|_U, \sigma|_U) \cong (\tilde{S}, \tilde{\sigma})/\mathbb{Z}_2$, the quotient morphism $\Psi : (\tilde{S}, \tilde{\sigma}) \rightarrow (S|_U, \sigma|_U) \cong (\tilde{S}, \tilde{\sigma})/\mathbb{Z}_2$ being étale. The $\mathbb{S}^1$ -action μ on S lifts to an effective $\mathbb{S}^1$ -action $\tilde{\mu}$ on $(\tilde{S}, \tilde{\sigma})$, whose real quotient morphism $\tilde{\pi} : (\tilde{S}, \tilde{\sigma}) \rightarrow (\tilde{U}, \tilde{\tau})$ is equal to the composition of the normalization morphism $\nu : \tilde{S} \rightarrow S \times_U \tilde{U}$ with the projection $\text{pr}_{\tilde{U}}$. Letting $\tilde{c} = \psi^{-1}(c)$, $\tilde{\pi}^{-1}(\tilde{c})$ is reduced and $(\tilde{\pi}^{-1}(\tilde{c}), \tilde{\sigma}|_{\tilde{\pi}^{-1}(\tilde{c})})$ is an $\mathbb{S}^1$ -torsor over $(\tilde{c}, \sigma_{\text{Spec}(\mathbb{R})})$. By Example 11, it is isomorphic to $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} = \text{Spec}(\mathbb{C}[u^{\pm 1}])$ endowed with a real structure given as the composition of the complex conjugation either with the involution $u \mapsto u^{-1}$ or with the involution $u \mapsto -u^{-1}$. Furthermore, the $\mathbb{Z}_2$ -action on $\tilde{S}$ restricts to a $\mathbb{G}_{m, \mathbb{C}}$ -equivariant free $\mathbb{Z}_2$ -action on $\tilde{\pi}^{-1}(\tilde{c}) \cong \text{Spec}(\mathbb{C}[u^{\pm 1}])$ compatible with the real structure $\tilde{\sigma}|_{\tilde{\pi}^{-1}(\tilde{c})}$. The latter is thus necessarily given by $u \mapsto -u$, and the quotient morphism Ψ restricts on $(\tilde{\pi}^{-1}(\tilde{c}), \tilde{\sigma}|_{\tilde{\pi}^{-1}(\tilde{c})})$ to an étale double cover $(\tilde{\pi}^{-1}(c), \tilde{\sigma}|_{\tilde{\pi}^{-1}(\tilde{c})}) \rightarrow (Z, \sigma|_Z)$. We conclude that $Z \cong \text{Spec}(\mathbb{C}[w^{\pm 1}])$, where $w = u^2$ and that $\sigma|_Z$ is the real structure given as the composition of the involution $w \mapsto w^{-1}$ with the complex conjugation, which shows that $(Z, \sigma|_Z)$ is isomorphic to $\mathbb{S}^1$. Finally, since $\text{ord}_c(h') = 1$, there cannot exist any rational function on U such that $h' = g\tau^*g$. It follows that for every real affine open neighborhood $(V, \tau|_V)$ of c contained in $(U, \tau|_U)$, the restriction of π over $V \setminus \{c\}$ is a nontrivial $\mathbb{S}^1$ -torsor. This yields case b).

II) Otherwise, if $2D'(c) - \text{ord}_c(h) < 0$, then since by hypothesis (D, h) whence (D', h') is a regular real DPD-pair, the rational numbers $D'(c)$ and $D'(\tau(c)) - \text{ord}_c(h') = D'(c) - \text{ord}_c(h')$ form a regular pair. Since $D'(c) \in [0, 1[$ and $\text{ord}_c(h') > 2D'(c)$ is an integer, the only possibility is that $D'(c) = 0$ and $\text{ord}_c(h') = 1$. By [9, Theorem 18 (b)], the fiber $\pi^{-1}(c)$ is then reduced consisting of the closure of two principal $\mathbb{G}_{m, \mathbb{C}}$ -orbits O^+ and O^- whose closures $\overline{O}^{\pm}$ in S are affine lines intersecting transversally at a $\mathbb{G}_{m, \mathbb{C}}$ -fixed point $p \in \pi^{-1}(c)$. The defining ideals of $\overline{O}^+$ and $\overline{O}^-$ in the graded coordinate ring $\Gamma(S|_U, \mathcal{O}_S) \otimes_{\Gamma(U, \mathcal{O}_U)} (\Gamma(U, \mathcal{O}_U)/f)$ of the scheme theoretic fiber $\pi^{-1}(c)$ are the positive and negative part respectively. Since by Lemma 7, σ^* exchanges the positive and negative parts of the grading of $\Gamma(S|_U, \mathcal{O}_S)$, it follows that σ exchanges $\overline{O}^+$ and $\overline{O}^-$, hence that p is a σ -invariant point. As in the previous case, the fact that $\text{ord}_c(h') = 1$ implies that for every real affine open neighborhood $(V, \tau|_V)$ of c contained in $(U, \tau|_U)$, the restriction of π over $V \setminus \{c\}$ is a nontrivial $\mathbb{S}^1$ -torsor. This yields case c). $\square$

Since the only proper algebraic subgroups of $\mathbb{G}_{m, \mathbb{C}}$ are cyclic groups, the exceptional orbits of a $\mathbb{G}_{m, \mathbb{C}}$ -action are either $\mathbb{G}_{m, \mathbb{C}}$ -fixed points or closed curves isomorphic to the punctured affine line $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ on which $\mathbb{G}_{m, \mathbb{C}}$ acts with stabilizer isomorphic to a cyclic group μ_m of order $m \geq 2$. While there exist smooth complex affine surfaces S endowed with hyperbolic $\mathbb{G}_{m, \mathbb{C}}$ -actions admitting 1-dimensional exceptional orbits with stabilizers μ_m for every $m \geq 2$, for instance $(\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}) \times \mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[x^{\pm 1}, y])$ endowed with the

$\mathbb{G}_{m,\mathbb{C}}$ -action $t \cdot (x, y) = (t^m x, t^{-1}y)$, the possible types of exceptional $\mathbb{S}^1$ -orbits on smooth real affine surfaces with effective $\mathbb{S}^1$ -actions are much more restricted:

Corollary 21. *The exceptional $\mathbb{S}^1$ -orbits on a smooth real affine surface (S, σ) with an effective $\mathbb{S}^1$ -action are either real $\mathbb{S}^1$ -fixed points or closed curves isomorphic to $\mathbb{S}^1$ on which $\mathbb{S}^1$ acts with stabilizer μ_2 .*

Proof. Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be the real quotient morphism for the given $\mathbb{S}^1$ -action. The image by π of a real exceptional $\mathbb{S}^1$ -orbit $(Z, \sigma|_Z)$ is a τ -invariant proper closed subset of C , which is irreducible since Z is irreducible. So $\pi(Z)$ is a real point c of (C, τ) . Since Z is an exceptional $\mathbb{G}_{m,\mathbb{C}}$ -orbit, the assertion follows from Theorem 20, cases b) and c). $\square$

The following proposition records the possible structures of fibers of the real quotient morphism $\pi : (S, \sigma) \rightarrow (C, \tau)$ over pairs of non-real complex points q and $\tau(q)$ of C . Its proof, which is similar to that of Theorem 20, is left to the reader.

Proposition 22. *Let (S, σ) be the smooth real affine surface with effective $\mathbb{S}^1$ -action $\mu : \mathbb{G}_{m,\mathbb{C}} \times S \rightarrow S$ determined by a regular real DPD-pair (D, h) on a smooth real affine curve (C, τ) . Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be its real quotient morphism, and let q and $\tau(q)$ be a pair of non-real complex points of C exchanged by the real structure τ . Then exactly one of the following possibilities occurs:*

- (1) $D(q) + D(\tau(q)) = \text{ord}_q(h) = \text{ord}_{\tau(q)}(h)$ and:
 - a) Either $D(q)$ and $D(\tau(q))$ both belong to $\mathbb{Z}$ and then $\pi^{-1}(q)$ and $\pi^{-1}(\tau(q))$ are principal $\mathbb{G}_{m,\mathbb{C}}$ -orbits. Furthermore, there exists a real affine open neighborhood $(U, \tau|_U)$ of $q \cup \tau(q)$ such that $\pi|_{\pi^{-1}(U)} : (\pi^{-1}(U), \sigma|_{\pi^{-1}(U)}) \rightarrow (U, \tau|_U)$ is an $\mathbb{S}^1$ -torsor.
 - b) Or $D(q)$ and $D(\tau(q))$ both belong to $\mathbb{Q} \setminus \mathbb{Z}$ and then $\pi^{-1}(q)$ and $\pi^{-1}(\tau(q)) = \sigma(\pi^{-1}(q))$ are 1-dimensional exceptional $\mathbb{G}_{m,\mathbb{C}}$ -orbits of multiplicity $m \geq 2$ on which $\mathbb{G}_{m,\mathbb{C}}$ acts with stabilizer μ_m .
- (2) $D(q) + D(\tau(q)) < \text{ord}_q(h) = \text{ord}_{\tau(q)}(h)$. Then $\pi^{-1}(q)_{\text{red}} = \overline{O}_q^+ \cup \overline{O}_q^-$, where O_q^+ and O_q^- are 1-dimensional $\mathbb{G}_{m,\mathbb{C}}$ -orbits whose closures $\overline{O}_q^\pm$ in S are affine lines intersecting transversally at a $\mathbb{G}_{m,\mathbb{C}}$ -fixed point p . Furthermore, the fiber $\pi^{-1}(\tau(q))_{\text{red}} = \sigma(\pi^{-1}(q)_{\text{red}})$ is equal to

$$\pi^{-1}(\tau(q))_{\text{red}} = \overline{O}_{\tau(q)}^+ \cup \overline{O}_{\tau(q)}^- = \sigma(\overline{O}_q^-) \cup \sigma(\overline{O}_q^+).$$

Example 23. Let (S_ε, σ) , where $\varepsilon = \pm 1$, be the smooth real affine surface with equation $xy = \varepsilon(z^2 + 1)$ in $\mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ endowed with the real structure given by the composition of the involution $(x, y, z) \mapsto (y, x, z)$ with the complex conjugation. The $\mathbb{G}_{m,\mathbb{C}}$ -action μ on S given by $t \cdot (x, y, z) = (tx, t^{-1}y, z)$ defines a real action of $\mathbb{S}^1$ on (S_ε, σ) . The categorical quotient for the $\mathbb{G}_{m,\mathbb{C}}$ -action is the affine line $C = \text{Spec}(A_0)$, where $A_0 = \mathbb{C}[xy, z]/(xy - \varepsilon(z^2 + 1)) \cong \mathbb{C}[z]$, and the quotient morphism $\pi : S_\varepsilon \rightarrow C$ is a real morphism for the real structures σ and $\tau = \sigma_{\mathbb{A}_{\mathbb{C}}^1}$ on S_ε and C respectively. The decomposition of the coordinate ring A_ε of S_ε into semi-invariant subspaces is given by

$$\begin{aligned} A_\varepsilon &= \bigoplus_{m \in \mathbb{Z}} A_{\varepsilon, m} = \bigoplus_{m < 0} A_0 \cdot y^{-m} \oplus A_0 \oplus \bigoplus_{m > 0} A_0 \cdot x^m \\ &= \bigoplus_{m < 0} A_0 \cdot (xy)^{-m} x^m \oplus A_0 \oplus \bigoplus_{m > 0} A_0 \cdot x^m \\ &= \bigoplus_{m < 0} A_0 \cdot (\varepsilon(z^2 + 1))^{-m} s^m \oplus A_0 \oplus \bigoplus_{m > 0} A_0 \cdot s^m \end{aligned}$$

where $s = x$. A corresponding real DPD-pair on (C, τ) is thus given by $(D, h) = (0, x\sigma^*x) = (0, \varepsilon(z^2 + 1))$. Noting that $1 + z^2 = (1 + iz)(1 - iz) = f\tau^*f$, we deduce from Theorem 14 1) that (S_ε, σ) is also given by the real DPD-pair $(D', h') = (D - \text{div}(f), \varepsilon) = (1 \cdot \{i\}, \varepsilon)$.

It then follows from Lemma 10 and Example 11 that the restriction of $\pi : (S_\varepsilon, \sigma) \rightarrow (C, \tau)$ over the real affine open subset $U = C \setminus \{\pm i\}$ is either the trivial $\mathbb{S}^1$ -torsor $(U, \tau|_U) \times \mathbb{S}^1$ if $\varepsilon = 1$, or the $\mathbb{S}^1$ -torsor $(U, \tau|_U) \times \hat{\mathbb{S}}^1$ if $\varepsilon = -1$. On the other hand, $\pi^{-1}(\{\pm i\}) \cong \text{Spec}(\mathbb{C}[x, y]/(xy))$ is the union of two copies

$\overline{O}_{\pm i}^+ = \{x = z \mp i = 0\}$ and $\overline{O}_{\pm i}^- = \{y = z \mp i = 0\}$ of the complex affine line intersecting at the point $\{x = y = z \mp i = 0\}$, and since $\sigma^*x = y$, we have $\sigma(\overline{O}_{\pm i}^\pm) = \overline{O}_{\mp i}^\mp$.

3. RATIONAL REAL ALGEBRAIC MODELS OF COMPACT DIFFERENTIAL SURFACES WITH CIRCLE ACTIONS

This section is devoted to the proof of Theorem 1. We first construct explicit rational projective and affine real algebraic models of compact real manifolds of dimension 2 without boundary endowed with effective differentiable S^1 -actions. Then we show that each rational quasi-projective real algebraic model of such a manifold is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to one of these models.

3.1. Rational affine models with compact real loci. It is a classical result (see e.g. [2, I.3.a]) that a compact connected real manifold of dimension 2 without boundary endowed with an effective differentiable S^1 -action is equivariantly diffeomorphic to one of the following manifolds: the torus $T = S^1 \times S^1$, the sphere S^2 , the projective plane $\mathbb{RP}^2$ and the Klein bottle K . We now describe smooth rational projective and affine real algebraic models with $\mathbb{S}^1$ -actions of these compact differential surfaces.

3.1.1. Equivariant rational models of the torus. The group S^1 acts on the torus $T = S^1 \times S^1$ by translations on the second factor. All the orbits are principal, and the orbit space is equal to S^1 .

A rational projective model of T is the complexification $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1})$ of $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ on which $\mathbb{S}^1$ acts on the second factor via the projective representation induced by the representation $\mathbb{S}^1 \rightarrow SO_2(\mathbb{R})$ in Definition 6. The action of $\mathbb{S}^1$ on $(\mathbb{P}_{\mathbb{C}}^1 = \text{Proj}(\mathbb{C}[u, v]), \sigma_{\mathbb{P}_{\mathbb{R}}^1})$ induced by the representation $\mathbb{S}^1 \rightarrow SO_2(\mathbb{R})$ has a pair of non-real fixed points $[1 : i]$ and $[1 : -i]$ exchanged by the real structure $\sigma_{\mathbb{P}_{\mathbb{R}}^1}$. Their complement is isomorphic to the trivial $\mathbb{S}^1$ -torsor. The affine open subset

$$S_1 = (\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1}) \setminus \{[1 : \pm i] \times \mathbb{P}_{\mathbb{C}}^1 \cup \mathbb{P}_{\mathbb{C}}^1 \times [1 : \pm i]\}$$

is $\sigma_{\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1}$ -invariant and $\mathbb{S}^1$ -invariant. Furthermore, letting σ_1 be the restriction of $\sigma_{\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1}$ to S_1 , the inclusion $(S_1, \sigma_1) \hookrightarrow (\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1})$ is an $\mathbb{S}^1$ -equivariant birational diffeomorphism. It follows that (S_1, σ_1) is a rational affine model of T , equivariantly isomorphic to the product $(Q_{1, \mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ of $\mathbb{S}^1$ with the complexification of the smooth affine quadric curve $Q_1 \subset \text{Spec}(\mathbb{R}[u, v])$ with equation $u^2 + v^2 = 1$, on which $\mathbb{S}^1$ acts by translations on the second factor.

The projection

$$\pi_1 = \text{pr}_{Q_{1, \mathbb{C}}} : (S_1, \sigma_1) = (Q_{1, \mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1 \rightarrow (C_1, \tau_1) = (Q_{1, \mathbb{C}}, \sigma_{Q_1})$$

is the trivial $\mathbb{S}^1$ -torsor. A corresponding real DPD-pair on (C_1, τ_1) is $(D_1, h_1) = (0, 1)$, where 0 denotes the trivial Weil divisor. The image by π_1 of the real locus of (S_1, σ_1) is equal to the real locus S^1 of (C_1, τ_1) .

3.1.2. Equivariant rational models of the sphere. The group S^1 acts on the unit sphere S^2 in $\mathbb{R}^3$ by rotations around a fixed axis. All the orbits are principal, except for the two fixed points where the axis meets the sphere, and the orbit space is a closed interval, each of its ends corresponding to a non-principal orbit.

A rational projective model is given by the complexification of the smooth quadric

$$Q = \{u^2 + v^2 + z^2 - w^2 = 0\} \subset \mathbb{P}_{\mathbb{R}}^3 = \text{Proj}_{\mathbb{R}}(\mathbb{R}[u, v, z, w])$$

endowed with the restriction of the $\mathbb{S}^1$ -action on $(\mathbb{P}_{\mathbb{C}}^3, \sigma_{\mathbb{P}_{\mathbb{R}}^3})$ defined by the projective representation induced by the direct sum of the representation $\mathbb{S}^1 \rightarrow SO_2(\mathbb{R})$ with the trivial 2-dimensional representation.

The $\mathbb{S}^1$ -invariant real hyperplane section $H = \{w = 0\}$ of $(Q_{\mathbb{C}}, \sigma_Q)$ has empty real locus. Its complement is $\mathbb{S}^1$ -equivariantly isomorphic to the complexification $(S_2, \sigma_2) = (\mathbb{S}_{\mathbb{C}}^2, \sigma_{\mathbb{S}^2})$ of the smooth affine quadric $\mathbb{S}^2$ in $\text{Spec}(\mathbb{R}[u, v, z])$ defined by the equation $u^2 + v^2 + z^2 = 1$, on which $\mathbb{S}^1$ acts by the restriction of the direct sum of the representation $\mathbb{S}^1 \rightarrow SO_2(\mathbb{R})$ with the trivial 1-dimensional representation. By construction, the inclusion $(S_2, \sigma_2) \hookrightarrow (Q_{\mathbb{C}}, \sigma_Q)$ is an $\mathbb{S}^1$ -equivariant birational diffeomorphism.

The real quotient morphism of the $\mathbb{S}^1$ -action on (S_2, σ_2) is the projection

$$\pi_2 = \text{pr}_z : (S_2, \sigma_2) \rightarrow (C_2, \tau_2) = (\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z]), \sigma_{\mathbb{A}_{\mathbb{R}}^1})$$

and a corresponding real DPD-pair on (C_2, τ_2) is $(D_2, h_2) = (0, 1 - z^2)$, where 0 denotes the trivial Weil divisor. The image by π_2 of the real locus of (S_2, σ_2) is the segment $[-1, 1]$ of the real locus $\mathbb{R}$ of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$. The restriction of π_2 over the principal real affine open subset $(\mathbb{A}_{\mathbb{C}}^1 \setminus \{\pm 1\}, \sigma_{\mathbb{A}_{\mathbb{R}}^1}|_{\mathbb{A}_{\mathbb{C}}^1 \setminus \{\pm 1\}})$ is a nontrivial $\mathbb{S}^1$ -torsor. The fibers of π_2 over the real points ± 1 of (C_2, τ_2) are of type c) in Theorem 20. Their respective real loci consist of a unique point $p_{\pm} = (0, 0, \pm 1)$, which is an $\mathbb{S}^1$ -fixed point.

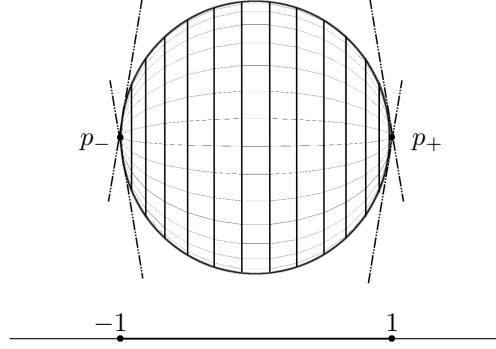


FIGURE 1. Projection of the real locus S^2 of (S_2, σ_2) onto the interval $[-1, 1]$. The dashed lines represent the closures $\overline{O}^{\pm}$ of the two principal $\mathbb{G}_{m, \mathbb{C}}$ -orbits in $\pi_2^{-1}(\pm 1)$ exchanged by σ_2 and intersecting at the unique $\mathbb{S}^1$ -fixed point $p_{\pm} \subset \pi_2^{-1}(\pm 1)$.

3.1.3. Equivariant rational models of the projective plane. The group S^1 acts on $\mathbb{RP}^2$, viewed as the projective compactification of $\mathbb{R}^2$ by adding a “line at infinity” $\mathbb{RP}^1 \cong S^1$, by the extension of the linear action of $S^1 = SO(2)$ on $\mathbb{R}^2$ to an action on $\mathbb{RP}^2$ leaving the line at infinity invariant. All the orbits are principal, except for two of them: one is a fixed point corresponding to the origin of $\mathbb{R}^2$ and the other is the line at infinity, equivariantly isomorphic to S^1 on which S^1 act with stabilizer μ_2 . An S^1 -invariant tubular neighborhood of this second non principal orbit is isomorphic to the quotient $(S^1 \times \mathbb{R})/(z, u) \sim (-z, -u)$, that is, to an open Moebius band B , endowed with the induced S^1 -action. The orbit space of this S^1 -action on $\mathbb{RP}^2$ is a closed interval, each of its ends corresponding to a non-principal orbit.

A rational projective model is the complexification $(\mathbb{P}_{\mathbb{C}}^2 = \text{Proj}(\mathbb{C}[u, v, z]), \sigma_{\mathbb{P}_{\mathbb{R}}^2})$ of $\mathbb{P}_{\mathbb{R}}^2$ endowed with the $\mathbb{S}^1$ -action defined by the projective representation induced by the direct sum of the representation $\mathbb{S}^1 \rightarrow SO_2(\mathbb{R})$ with the trivial 1-dimensional representation. The smooth conic Δ in $\mathbb{P}_{\mathbb{C}}^2$ with equation $u^2 + v^2 + z^2 = 0$ is $\sigma_{\mathbb{P}_{\mathbb{R}}^2}$ -invariant and $\mathbb{S}^1$ -invariant and has empty real locus. Its complement $S_3 = \mathbb{P}_{\mathbb{C}}^2 \setminus \Delta$ endowed with the restriction σ_3 of $\sigma_{\mathbb{P}_{\mathbb{R}}^2}$ and the induced $\mathbb{S}^1$ -action is thus an affine model of $\mathbb{RP}^2$. By construction, the inclusion $(S_3, \sigma_3) \hookrightarrow (\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{R}}^2})$ is an $\mathbb{S}^1$ -equivariant birational diffeomorphism.

Another rational projective model of $\mathbb{RP}^2$ with $\mathbb{S}^1$ -action is obtained by blowing-up the smooth quadric

$$Q_{\mathbb{C}} = \{u^2 + v^2 + z^2 - w^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^3,$$

endowed with the real structure σ_Q and the $\mathbb{S}^1$ -action defined in subsection 3.1.2, at the real $\mathbb{S}^1$ -fixed point $p_- = [0 : 0 : -1 : 1]$. Letting $\alpha : (\tilde{Q}_{\mathbb{C}}, \sigma_{\tilde{Q}}) \rightarrow (Q_{\mathbb{C}}, \sigma_Q)$ be the blow-up morphism, with exceptional divisor $E_- \cong (\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$, the $\mathbb{S}^1$ -action on $(Q_{\mathbb{C}}, \sigma_Q)$ lifts to an action on $(\tilde{Q}_{\mathbb{C}}, \sigma_{\tilde{Q}})$ for which the real locus of $(\tilde{Q}_{\mathbb{C}}, \sigma_{\tilde{Q}})$ endowed with the induced S^1 -action is equivariantly diffeomorphic to the S^1 -equivariant connected sum $\mathbb{RP}^2 \#_{S^1} S^2 \simeq \mathbb{RP}^2$ endowed with the S^1 -action defined above. The proper transforms in $(\tilde{Q}_{\mathbb{C}}, \sigma_{\tilde{Q}})$ of the curves $\ell_- = \{u + iv = z + w = 0\}$ and $\bar{\ell}_- = \{u - iv = z + w = 0\}$ in $(Q_{\mathbb{C}}, \sigma_Q)$ are a pair of non-real disjoint smooth rational curves with self-intersection -1 , exchanged by the real structure $\sigma_{\tilde{Q}}$. Their

union is an $\mathbb{S}^1$ -invariant real closed subset F_- of $(\tilde{Q}_\mathbb{C}, \sigma_{\tilde{Q}})$ and the contraction of F_- is an $\mathbb{S}^1$ -equivariant birational diffeomorphism $\alpha' : (\tilde{Q}_\mathbb{C}, \sigma_{\tilde{Q}}) \rightarrow (\mathbb{P}_\mathbb{C}^2, \sigma_{\mathbb{P}_\mathbb{C}^2})$ which maps the proper transform $\tilde{H} \subset \tilde{Q}_\mathbb{C}$ of the curve $H = \{w = 0\} \subset Q_\mathbb{C}$ onto the smooth conic $\Delta \subset \mathbb{P}_\mathbb{C}^2$ (see Figure 2).

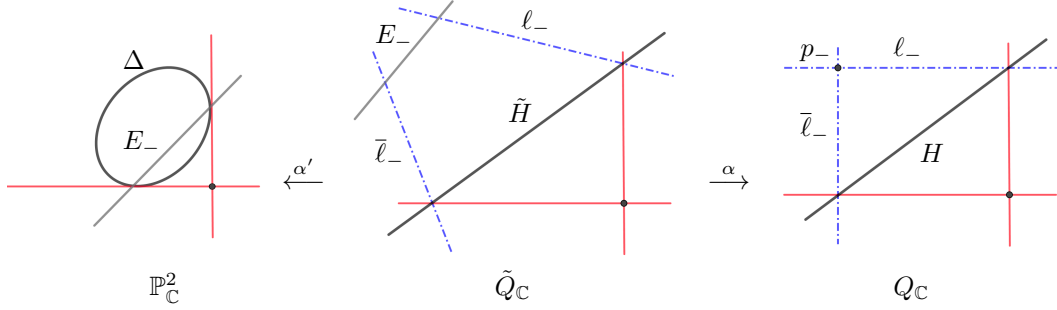


FIGURE 2. A real birational map between $(\mathbb{P}_\mathbb{C}^2, \sigma_{\mathbb{P}_\mathbb{C}^2})$ and $(Q_\mathbb{C}, \sigma_Q)$.

We obtain a diagram

$$(S_3, \sigma_3) \cong (\mathbb{P}_\mathbb{C}^2 \setminus \Delta, \sigma_{\mathbb{P}_\mathbb{C}^2}) \xleftarrow{\alpha'} (\tilde{Q}_\mathbb{C} \setminus (\tilde{H} \cup F_-), \sigma_{\tilde{Q}}) \xrightarrow{\alpha} (Q_\mathbb{C} \setminus H, \sigma_Q) \cong (S_2, \sigma_2)$$

in which the left hand side induced morphism α' is an $\mathbb{S}^1$ -equivariant real isomorphism. The right hand side morphism α realizes $(\tilde{Q}_\mathbb{C} \setminus (\tilde{H} \cup F_-), \sigma_{\tilde{Q}})$ as the $\mathbb{S}^1$ -equivariant real affine modification of (S_2, σ_2) obtained by blowing-up p_- and removing the proper transforms of the closures $\overline{O}^\pm$ of the two principal $\mathbb{G}_{m,\mathbb{C}}$ -orbits in $\pi_2^{-1}(-1)$ exchanged by σ_2 and intersecting at p_- (see Figure 1).

A real DPD-presentation of (S_3, σ_3) can be determined as follows. The smooth real quadric $(Q_\mathbb{C}, \sigma_Q)$ is isomorphic to the Galois double cover of $(\mathbb{P}_\mathbb{C}^2, \sigma_{\mathbb{P}_\mathbb{C}^2})$ branched along the real conic Δ . The commutative diagram

$$\begin{array}{ccc} S_2 = Q_\mathbb{C} \setminus \{w = 0\} & \longrightarrow & Q_\mathbb{C} \\ \downarrow & & \downarrow [u:v:z:w] \mapsto [u:v:z] \\ S_3 = \mathbb{P}_\mathbb{C}^2 \setminus \Delta & \longrightarrow & \mathbb{P}_\mathbb{C}^2 \end{array}$$

then identifies S_3 with the quotient of $S_2 \cong \{u^2 + v^2 + z^2 = 1\} \subset \mathbb{A}_\mathbb{C}^3$ by the antipodal involution $(u, v, z) \mapsto (-u, -v, -z)$. This involution commutes with the real structure σ_2 on S_2 and the quotient morphism $(S_2, \sigma_2) \rightarrow (S_3, \sigma_3) \cong (S_2, \sigma_2)/\mathbb{Z}_2$ is a real morphism, which is equivariant for the $\mathbb{S}^1$ -actions on (S_2, σ_2) and (S_3, σ_3) . The real quotient morphism $\pi_2 : (S_2, \sigma_2) \rightarrow (C_2, \tau_2)$ thus descends to a real morphism

$$\pi_3 : (S_3, \sigma_3) \cong (S_2, \sigma_2)/\mathbb{Z}_2 \rightarrow (C_3, \tau_3) = (C_2, \tau_2)/\mathbb{Z}_2 \cong (\mathbb{A}_\mathbb{C}^1 = \text{Spec}(\mathbb{C}[Z], \sigma_{\mathbb{A}_\mathbb{C}^1}),$$

where $Z = 2z^2 - 1$, which is the real quotient morphism of the induced $\mathbb{S}^1$ -action on (S_3, σ_3) . With this choice of coordinate, a direct calculation shows that a real DPD-pair on (C_3, τ_3) corresponding to (S_3, σ_3) is $(D_3, h_3) = (\frac{1}{2}\{-1\}, 1 - Z^2)$.

The image by π_3 of the real locus $\mathbb{R}P^2$ of (S_3, σ_3) is the segment $[-1, 1]$ of the real locus $\mathbb{R}$ of (C_3, τ_3) . The restriction of π_3 over the principal real affine open subset $(\mathbb{A}_\mathbb{C}^1 \setminus \{\pm 1\}, \sigma_{\mathbb{A}_\mathbb{C}^1}|_{\mathbb{A}_\mathbb{C}^1 \setminus \{\pm 1\}})$ of (C_3, τ_3) is a nontrivial $\mathbb{S}^1$ -torsor. The fibers of π_3 over the real points -1 and 1 of (C_3, τ_3) are respectively of type b) and c) in Theorem 20. Their real loci consist respectively of a copy of S^1 on which S^1 acts with stabilizer μ_2 and a unique point p , which is a fixed point of the induced S^1 -action on the real locus of (S_3, σ_3) .

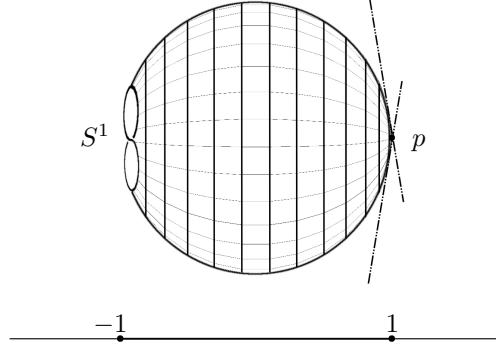


FIGURE 3. Projection of the real locus $\mathbb{RP}^2$ of $(S_3, \sigma_3) = (\mathbb{S}_{\mathbb{C}}^2, \sigma_{\mathbb{S}^2})/\mathbb{Z}_2$ onto the interval $[-1, 1]$. The dashed lines represent the closures $\overline{O}^{\pm}$ of the two principal $\mathbb{G}_{m, \mathbb{C}}$ -orbits in $\pi_3^{-1}(1)$ exchanged by σ_3 and intersecting at the unique real S^1 -fixed point $p \subset \pi_3^{-1}(1)$.

3.1.4. *Equivariant rational models of the Klein bottle.* The Klein bottle K with its S^1 -action is the S^1 -equivariant connected sum $\mathbb{RP}^2 \#_{S^1} \mathbb{RP}^2$ of two copies of $\mathbb{RP}^2$ endowed with the S^1 -action defined in the previous subsection. Namely, $\mathbb{RP}^2 \#_{S^1} \mathbb{RP}^2$ is obtained by removing on each copy of $\mathbb{RP}^2$ an S^1 -invariant open disc containing the unique S^1 -fixed point and gluing together the resulting boundary circles in an S^1 -equivariant way. Equivalently, $\mathbb{RP}^2 \#_{S^1} \mathbb{RP}^2$ is obtained by the S^1 -equivariant gluing of two closed Moebius bands

$$\overline{B} = (S^1 \times [-1, 1]) / (z, u) \sim (-z, -u)$$

with the S^1 -action as in the previous subsection along their boundary circles. The resulting S^1 -action on K has two non principal orbits isomorphic to S^1 with stabilizer μ_2 , and with S^1 -invariant tubular neighborhoods diffeomorphic to open Moebius bands. The orbit space is again closed interval, each of its ends corresponding to a non-principal orbit.

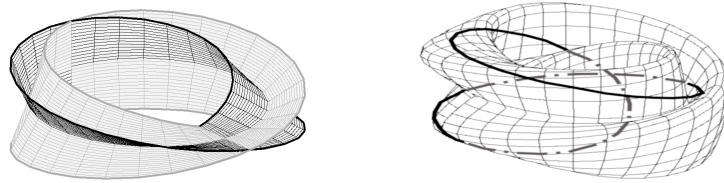


FIGURE 4. Two Moebius bands glued along their boundaries.

A projective rational model of K is obtained as follows: let $(\mathbb{P}_{\mathbb{C}}^2 = \text{Proj}(\mathbb{C}[u, v, z]), \sigma_{\mathbb{P}_{\mathbb{C}}^2})$ be endowed with the S^1 -action defined by the projective representation induced by the direct sum of the representation $S^1 \rightarrow SO_2(\mathbb{R})$ with the trivial 1-dimensional representation. The blow-up of $(\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{C}}^2})$ at the real S^1 -fixed point $[0 : 0 : 1]$ is the real Hirzebruch surface $(\mathbb{F}_{1, \mathbb{C}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}_{\mathbb{R}}^1} \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{R}}^1}(-1)), \sigma_{\mathbb{F}_1})$ in which the exceptional divisor $E \cong (\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$ is the section with self-intersection -1 of the $\mathbb{P}^1$ -bundle structure. The S^1 -action on $(\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{C}}^2})$ lifts to an action on $(\mathbb{F}_{1, \mathbb{C}}, \sigma_{\mathbb{F}_1})$ and the real locus of $(\mathbb{F}_{1, \mathbb{C}}, \sigma_{\mathbb{F}_1})$ endowed with the induced S^1 -action is diffeomorphic to $K \simeq \mathbb{RP}^2 \#_{S^1} \mathbb{RP}^2$ endowed with the S^1 -action defined above.

An affine model of K is in turn obtained from $(\mathbb{F}_{1, \mathbb{C}}, \sigma_{\mathbb{F}_1})$ by removing the union of the proper transform of the conic $\Delta = \{u^2 + v^2 + z^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^2$ and of the proper transforms of the pair of non-real lines

$$\ell = \{u + iv = 0\} \quad \text{and} \quad \sigma_{\mathbb{P}_{\mathbb{R}}^2}(\ell) = \{u - iv = 0\}$$

of $(\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{R}}^2})$ passing through $[0 : 0 : 1]$. Indeed, $\Delta \cup \ell \cup \sigma_{\mathbb{P}_{\mathbb{R}}^2}(\ell)$ is a real $\mathbb{S}^1$ -invariant closed subset of $(\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{R}}^2})$ with $[0 : 0 : 1]$ as a unique real point, whose proper transform B in $(\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ is an ample $\mathbb{S}^1$ -invariant curve with empty real locus. So $(S_4, \sigma_4) = (\mathbb{F}_{1,\mathbb{C}} \setminus B, \sigma_{\mathbb{F}_1}|_{\mathbb{F}_{1,\mathbb{C}} \setminus B})$ endowed with the induced $\mathbb{S}^1$ -action is a real affine surface whose real locus is diffeomorphic to K . By construction, the inclusion $(S_4, \sigma_4) \hookrightarrow (\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ is an $\mathbb{S}^1$ -equivariant birational diffeomorphism.

A alternative projective model of K is obtained from the quadric $(Q_{\mathbb{C}}, \sigma_Q)$ of subsection 3.1.2 by blowing-up the two real $\mathbb{S}^1$ -fixed points $p_{\pm} = [0 : 0 : \pm 1 : 1]$ with respective exceptional divisors $E_{\pm} \cong (\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$. Letting $\beta : (\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}}) \rightarrow (Q_{\mathbb{C}}, \sigma_Q)$ be the real blow-up morphism, the $\mathbb{S}^1$ -action on $(Q_{\mathbb{C}}, \sigma_Q)$ lifts to an action on $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ for which the real locus of $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ endowed with the induced $\mathbb{S}^1$ -action is equivariantly diffeomorphic to the connected sum $\mathbb{RP}^2 \#_{S^1} S^2 \#_{S^1} \mathbb{RP}^2 \simeq \mathbb{RP}^2 \#_{S^1} \mathbb{RP}^2$, hence to the Klein bottle K endowed with the $\mathbb{S}^1$ -action defined above. As in the previous subsection, the proper transforms in $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ of the curves $\ell_- = \{u + iv = z + w = 0\}$ and $\bar{\ell}_- = \{u - iv = z + w = 0\}$ in $(Q_{\mathbb{C}}, \sigma_Q)$ are a pair of non-real disjoint smooth rational curves with self-intersection -1 , exchanged by the real structure $\sigma_{\hat{Q}}$. Their union is an $\mathbb{S}^1$ -invariant real closed subset F_- of $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ and the contraction of F_- is an $\mathbb{S}^1$ -equivariant birational diffeomorphism $\beta' : (\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}}) \rightarrow (\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ which maps the proper transform $\hat{H} \subset \hat{Q}_{\mathbb{C}}$ of the curve $H = \{w = 0\} \subset Q_{\mathbb{C}}$ onto the proper transform in $(\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ of the conic $\Delta = \{u^2 + v^2 + z^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^2$. The proper transforms in $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ of the curves $\ell_+ = \{u + iv = z - w = 0\}$ and $\bar{\ell}_+ = \{u - iv = z - w = 0\}$ in $(Q_{\mathbb{C}}, \sigma_Q)$ are also a pair of non-real disjoint smooth rational curves with self-intersection -1 , exchanged by the real structure $\sigma_{\hat{Q}}$. Their union is an $\mathbb{S}^1$ -invariant real closed subset F_+ of $(\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}})$ whose image by $\beta' : (\hat{Q}_{\mathbb{C}}, \sigma_{\hat{Q}}) \rightarrow (\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ is equal to the union of the proper transforms in $(\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ of the lines ℓ and $\sigma_{\mathbb{P}_{\mathbb{R}}^2}(\ell)$ of $(\mathbb{P}_{\mathbb{C}}^2, \sigma_{\mathbb{P}_{\mathbb{R}}^2})$ (see Figure 5).

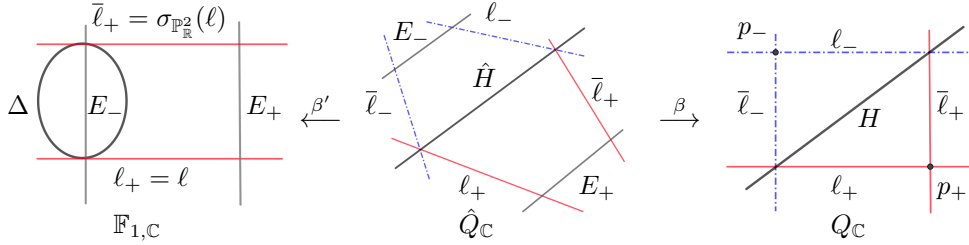


FIGURE 5. A real birational map between $(\mathbb{F}_{1,\mathbb{C}}, \sigma_{\mathbb{F}_1})$ and $(Q_{\mathbb{C}}, \sigma_Q)$.

We obtain a diagram

$$(S_4, \sigma_4) = (\mathbb{F}_{1,\mathbb{C}} \setminus B, \sigma_{\mathbb{F}_1}) \xleftarrow{\beta'} (\hat{Q}_{\mathbb{C}} \setminus (\hat{H} \cup F_- \cup F_+), \sigma_{\hat{Q}}) \xrightarrow{\beta} (Q_{\mathbb{C}} \setminus H, \sigma_Q) \cong (S_2, \sigma_2)$$

in which the left hand side induced morphism β' is an $\mathbb{S}^1$ -equivariant real isomorphism. The right hand side morphism β realizes the real affine surface $(\hat{Q}_{\mathbb{C}} \setminus (\hat{H} \cup F_- \cup F_+), \sigma_{\hat{Q}})$ as the $\mathbb{S}^1$ -equivariant real affine modification of (S_2, σ_2) obtained by blowing-up p_- and p_+ and removing the proper transforms of the closures $\overline{O}_{\pm 1}^{\pm}$ of the two principal $\mathbb{G}_{m,\mathbb{C}}$ -orbits in $\pi_2^{-1}(\pm 1)$ exchanged by σ_2 and intersecting at $p_{\pm}$ (see Figure 1).

The $\mathbb{S}^1$ -equivariant affine modification $\beta : (S_4, \sigma_4) \rightarrow (S_2, \sigma_2)$ can be made explicit as follows. Let (Q_2, σ'_2) be the smooth surface in $\text{Spec}(\mathbb{C}[x, y, z])$ with equation $xy = 1 - z^2$, endowed with the real structure defined as the composition of the involution $(x, y, z) \mapsto (y, x, z)$ with the complex conjugation. The isomorphism

$$Q_2 \rightarrow S_2 \subset \text{Spec}(\mathbb{C}[u, v, z]), \quad (x, y, z) \mapsto \left(\frac{x+y}{2}, \frac{x-y}{2i}, z \right)$$

induces a real isomorphism $(Q_2, \sigma'_2) \cong (S_2, \sigma_2)$ which is equivariant for the $\mathbb{S}^1$ -action on (Q_2, σ'_2) given by the hyperbolic $\mathbb{G}_{m, \mathbb{C}}$ -action $\mu(t, (x, y, z)) = (tx, t^{-1}y, z)$. Via this isomorphism, the $\mathbb{S}^1$ -equivariant real affine modification of (S_2, σ_2) described geometrically above coincides with the affine modification of (Q_2, σ'_2) with center at the real $\mathbb{S}^1$ -invariant closed subscheme with defining ideal $I = (x, y)^2$ and with real $\mathbb{S}^1$ -invariant principal divisor $\text{div}(xy)$. It follows that (S_4, σ_4) is $\mathbb{S}^1$ -equivariantly isomorphic to the complex affine surface in $\text{Spec}(\mathbb{C}[x^{\pm 1}, y, z])$ defined by the equation $xy^2 = 1 - z^2$, endowed with the real structure given by the composition of the involution $(x, y, z) \mapsto (x^{-1}, xy, z)$ with the complex conjugation, equipped with the $\mathbb{S}^1$ -action given by the hyperbolic $\mathbb{G}_{m, \mathbb{C}}$ -action $\mu(t, (x, y, z)) = (t^2x, t^{-1}y, z)$.

We deduce from this description that the real quotient morphism of (S_4, σ_4) is the projection

$$\pi_4 = \text{pr}_z : (S_4, \sigma_4) \rightarrow (C_4, \tau_4) = (\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z], \sigma_{\mathbb{A}_{\mathbb{R}}^1})$$

and that a real DPD-pair on (C_4, τ_4) corresponding to (S_4, σ_4) is $(D_4, h_4) = (\frac{1}{2}\{-1\} + \frac{1}{2}\{1\}, 1 - z^2)$. The image by π_4 of the real locus of (S_4, σ_4) is the segment $[-1, 1]$ of the real locus $\mathbb{R}$ of (C_4, τ_4) . The restriction of π_4 over the principal real affine open subset $(\mathbb{A}_{\mathbb{C}}^1 \setminus \{\pm 1\}, \sigma_{\mathbb{A}_{\mathbb{R}}^1}|_{\mathbb{A}_{\mathbb{C}}^1 \setminus \{\pm 1\}})$ of (C_4, τ_4) is a nontrivial $\mathbb{S}^1$ -torsor. The fibers of π_4 over the real points ± 1 of (C_4, τ_4) are of type b) in Theorem 20. Their real loci consist of a copy of S^1 on which S^1 acts with stabilizer μ_2 .

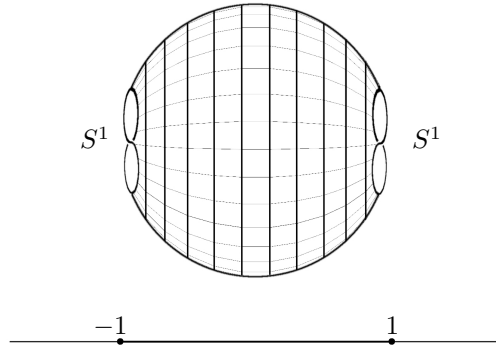


FIGURE 6. Projection of the real locus K of (S_4, σ_4) onto the interval $[-1, 1]$.

3.2. Uniqueness of models up to equivariant birational diffeomorphism. This subsection is devoted to the proof of the following result which implies Theorem 1:

Proposition 24. *Every smooth rational quasi-projective real surface with an effective $\mathbb{S}^1$ -action and whose real locus is a compact connected manifold of dimension 2 without boundary is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to one of the affine models constructed in subsection 3.1, summarized in Table 7 below.*

Real locus	$S^1 \times S^1$	S^2	$\mathbb{RP}^2$	K
Rational Mmodel	$S_1 = Q_{1, \mathbb{C}} \times \mathbb{S}_{\mathbb{C}}^1$	$S_2 = \mathbb{S}_{\mathbb{C}}^2$	$S_3 = \mathbb{S}_{\mathbb{C}}^2 / \mathbb{Z}_2$	$S_4 = \{xy^2 = 1 - z^2\}$
Real categorical quotient	$(Q_{1, \mathbb{C}}, \sigma_{Q_1})$	$(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$	$(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$	$(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$
Image of real locus	S^1	$[-1, 1]$	$[-1, 1]$	$[-1, 1]$
Real DPD-pair (D, h)	$(0, 1)$	$(0, 1 - z^2)$	$(\frac{1}{2}\{-1\}, 1 - z^2)$	$(\frac{1}{2}\{-1\} + \frac{1}{2}\{1\}, 1 - z^2)$

FIGURE 7. Rational affine models of compact surfaces with S^1 -actions. The notation $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$ refers to the underlying real algebraic variety of $\mathbb{S}^1$, that is, the complexification of the smooth affine curve in $\mathbb{A}_{\mathbb{R}}^2 = \text{Spec}(\mathbb{R}[u, v])$ with equation $u^2 + v^2 = 1$.

The scheme of the proof is the following. In Lemma 25 below, we first establish that every smooth rational quasi-projective model with $\mathbb{S}^1$ -action of the torus T , or the sphere S^2 , or the plane $\mathbb{RP}^2$ or the Klein bottle K is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to an affine one. Then in Lemma 26, we split the study of the affine case into two subcases according to the nature of the image of the real locus by the real quotient morphism. These subcases are finally studied separately in subsections 3.2.1 and 3.2.2.

Lemma 25. *Let (X, Σ) be a smooth rational real quasi-projective surface with an effective $\mathbb{S}^1$ -action and whose real locus is a compact connected manifold of dimension 2 without boundary. Then (X, Σ) is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to a smooth rational real affine surface (S, σ) with $\mathbb{S}^1$ -action.*

Proof. By Sumuhiro equivariant completion theorem [15] and equivariant desingularization results for normal surfaces with $\mathbb{G}_{m, \mathbb{C}}$ -actions [12], there exists a smooth real projective surface $(\bar{X}, \bar{\Sigma})$ with $\mathbb{S}^1$ -action and an $\mathbb{S}^1$ -equivariant open embedding $(X, \Sigma) \hookrightarrow (\bar{X}, \bar{\Sigma})$. Since (X, Σ) is rational, so is $(\bar{X}, \bar{\Sigma})$. By a result of Comessatti [5, p. 257], the real locus of $(\bar{X}, \bar{\Sigma})$ is a connected compact smooth surface without boundary, either non-orientable, or orientable and diffeomorphic to T or S^2 . Since the real locus of (X, Σ) is itself connected and compact, it follows that the real loci of (X, Σ) and $(\bar{X}, \bar{\Sigma})$ coincide, so that $(X, \Sigma) \hookrightarrow (\bar{X}, \bar{\Sigma})$ is a birational diffeomorphism. By [15, Theorem 1.6], there exists a very ample $\mathbb{S}^1$ -linearized invertible sheaf $\mathcal{L}$ on $(\bar{X}, \bar{\Sigma})$. This yields in particular a representation of $\mathbb{S}^1$ into the group of linear automorphism of $H^0(\bar{X}, \mathcal{L})$. The underlying representation of $\mathbb{G}_{m, \mathbb{C}}$ splits as a direct sum of $n_1 \geq 1$ non trivial diagonal representations of the form

$$t \cdot (x_i, y_i) = (t^{m_i} x_i, t^{-m_i} y_i), \quad m_i \in \mathbb{Z}_{>0}, \quad i = 0, \dots, n_1 - 1$$

on which the real structure is given by the composition of the involution $(x_i, y_i) \mapsto (y_i, x_i)$ with the complex conjugation, and $n_2 \geq 0$ trivial 1-dimensional representations. The $\mathbb{G}_{m, \mathbb{C}}$ -equivariant closed embedding

$$\bar{X} \hookrightarrow \mathbb{P}(H^0(\bar{X}, \mathcal{L})) \cong \text{Proj}(\mathbb{C}[x_0, y_0, \dots, x_{n_1-1}, y_{n_1-1}, z_1, \dots, z_{n_2}])$$

then becomes a real $\mathbb{S}^1$ -equivariant closed embedding for the real structure $\bar{\Sigma}$ on $\bar{X}$ and the real structure on $\mathbb{P}(H^0(\bar{X}, \mathcal{L}))$ defined as the composition of the involution

$$(x_0, y_0, \dots, x_{n_1-1}, y_{n_1-1}, z_1, \dots, z_{n_2}) \mapsto (y_0, x_0, \dots, y_{n_1-1}, x_{n_1-1}, z_1, \dots, z_{n_2})$$

with the complex conjugation. The quadric $Q \subset \mathbb{P}(H^0(\bar{X}, \mathcal{L}))$ with equation $\sum_{i=0}^{n_1-1} x_i y_i + \sum_{j=1}^{n_2} z_j^2 = 0$ is a real ample $\mathbb{S}^1$ -invariant divisor on $\mathbb{P}(H^0(\bar{X}, \mathcal{L}))$ with empty real locus. Since the real locus of $(\bar{X}, \bar{\Sigma})$ is not empty, $\bar{X}$ is not contained in Q . It follows that $(S, \sigma) = (\bar{X} \setminus Q, \bar{\Sigma}|_{\bar{X} \setminus Q})$ is a smooth rational real affine surface with $\mathbb{S}^1$ -action. By construction, the open inclusion $(S, \sigma) \hookrightarrow (\bar{X}, \bar{\Sigma})$ is an $\mathbb{S}^1$ -equivariant birational diffeomorphism. $\square$

The following lemma divides in turn the study of the affine case into two sub-cases:

Lemma 26. *Let (S, σ) be a smooth rational real affine surface with an effective $\mathbb{S}^1$ -action and whose real locus is a connected compact surface without boundary. Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be the real quotient morphism for the $\mathbb{S}^1$ -action. Then the following alternative holds:*

- a) (C, τ) is a real affine open subset of $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$ and its real locus is equal to that of $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$.
- b) (C, τ) is a real affine open subset of the real affine line $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ and its real locus is a closed interval.

Proof. The curve C is rational because S is rational. Since the real locus of (S, σ) is nonempty, connected and compact, its image by $\pi : (S, \sigma) \rightarrow (C, \tau)$ is a nonempty connected compact subset of the real locus of (C, τ) . The smooth real projective model of (C, τ) is thus isomorphic to the real projective line $(\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$. If $\mathbb{P}_{\mathbb{C}}^1 \setminus C$ contains a real point of $(\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$, then C is isomorphic to a real affine open subset of the real affine line. Being connected and compact, its real locus is then a closed interval. Otherwise, since the inclusion $(C, \tau) \hookrightarrow (\mathbb{P}_{\mathbb{C}}^1, \sigma_{\mathbb{P}_{\mathbb{R}}^1})$ is a real morphism and C is affine, $\mathbb{P}_{\mathbb{C}}^1 \setminus C$ is not empty and consists of pairs of non-real points of $\mathbb{P}_{\mathbb{C}}^1$ which are exchanged by the real structure $\sigma_{\mathbb{P}_{\mathbb{R}}^1}$. Since the complement of a pair of such points q and $\sigma_{\mathbb{P}_{\mathbb{R}}^1}(q)$ is isomorphic to the real affine quadric $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$, it follows that (C, τ) is isomorphic to a real affine open subset of $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$, and since the real locus of $\mathbb{P}_{\mathbb{C}}^1 \setminus C$ is empty, it follows that the real locus of (C, τ) is equal to that of $(Q_{1, \mathbb{C}}, \sigma_{Q_1})$. $\square$

3.2.1. *First case: (C, τ) is a real affine open subset of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ with real locus equal to S^1 .*

Proposition 27. *Let (S, σ) be a smooth rational real affine surface with an effective $\mathbb{S}^1$ -action and whose real locus is a connected compact surface without boundary. Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be the real quotient morphism and assume that (C, τ) is a real affine open subset of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ whose real locus is equal to that of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. Then (S, σ) is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to $(Q_{1,\mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ on which $\mathbb{S}^1$ acts by translations on the second factor.*

Proof. Let (D, h) be a regular real DPD-pair on (C, τ) corresponding to (S, σ) . Since h is a τ -invariant rational function on C , it is also a σ_{Q_1} -invariant rational function on $Q_{1,\mathbb{C}}$. We claim that by changing h for some rational function of the form $f\tau^*fh$, where f is a rational function on $Q_{1,\mathbb{C}}$, and changing D accordingly by $D + \text{div}(f|_C)$, we can assume that h is the restriction to (C, τ) of a real regular function on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ whose zero locus on $Q_{1,\mathbb{C}}$ consists of real points only. Indeed, since h is σ_{Q_1} -invariant, its poles on Q_1 are either real points of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ or pairs of non-real points exchanged by σ_{Q_1} . So up to changing h for $f\tau^*fh$ and D for $D + \text{div}(f|_C)$ for a suitable regular function f on $Q_{1,\mathbb{C}}$, we can assume from the very beginning that h is the restriction of a real regular function on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. Let $q = (z_1, z_2)$ and $\sigma_{Q_1}(q) = (\bar{z}_1, \bar{z}_2) = \bar{q}$ be a pair of non-real points of $Q_{1,\mathbb{C}}$ at which h vanishes. The restrictions to $Q_{1,\mathbb{C}}$ of the regular functions

$$F_q = (v - z_2) - i(u - z_1) \quad \text{and} \quad F_{\bar{q}} = (v - \bar{z}_2) + i(u - \bar{z}_1)$$

on $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec}(\mathbb{C}[u, v])$ are regular functions f_q and $f_{\bar{q}}$ on $Q_{1,\mathbb{C}}$ such that $\text{div}(f_q) = q$ and $\text{div}(f_{\bar{q}}) = \bar{q}$. Furthermore, since $\sigma_{Q_1}^* f_q = f_{\bar{q}}$ it follows that for $\delta = \text{ord}_q(h) = \text{ord}_{\bar{q}}(h)$, $f_q^{-\delta} \sigma_{Q_1}^* f_{\bar{q}}^{-\delta} h$ is a real regular function on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ which does not vanish at q and $\bar{q}$. The pair

$$(D', h') = (D - \delta \text{div}(f_q|_C), f_q^{-\delta} \sigma_{Q_1}^* f_{\bar{q}}^{-\delta} h)$$

is then a regular real DPD-pair on (C, τ) which defines a smooth real affine surface $\mathbb{S}^1$ -equivariantly isomorphic to (S, σ) by Theorem 14 1). The desired regular real DPD-pair is then obtained by applying this construction to the finitely many pairs of non-real points of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ exchanged by σ_{Q_1} at which h vanishes.

The set of non-real points q of (C, τ) such that either $D(q) \neq 0$ or $D(\tau(q)) \neq 0$ is a finite real subset Z of (C, τ) . Its complement $(U, \tau|_U)$ is a real affine open subset of (C, τ) and the restriction $(D|_U, h|_U)$ of (D, h) is a regular real DPD-pair defining a smooth real affine surface $\mathbb{S}^1$ -equivariantly isomorphic to $(\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$. Since Z consists of non-real point of (C, τ) only, the inclusion of $(\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$ in (S, σ) is an $\mathbb{S}^1$ -equivariant birational diffeomorphism. Replacing (C, τ) and (D, h) by $(U, \tau|_U)$ and $(D|_U, h|_U)$, we can therefore assume that $D(q) = \text{ord}_q(h) = 0$ for every non-real point q of C . Now let $\tilde{D}$ be the Weil $\mathbb{Q}$ -divisor on $Q_{1,\mathbb{C}}$ defined by $\tilde{D}(c) = D(c)$ if $c \in C$ and $\tilde{D}(c) = 0$ otherwise, and let $\tilde{h} = h$. Since $(C, \tau) \subset (Q_{1,\mathbb{C}}, \sigma_{Q_1})$ is a real affine open subset with the same real locus as $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$, $Q_{1,\mathbb{C}} \setminus C$ consists of finitely many pairs $\{q, \sigma_{Q_1}(q)\}$ of non-real points of $Q_{1,\mathbb{C}}$ exchanged by the real structure σ_{Q_1} . For every such pair of points, we have by construction

$$\text{ord}_q(\tilde{h}) = \text{ord}_q(h) = \tilde{D}(q) + \tilde{D}(\sigma_{Q_1}(q)) = \tilde{D}(q) + \sigma_{Q_1}^*(\tilde{D})(q)$$

and similarly for $\sigma_{Q_1}(q)$. This implies that $(\tilde{D}, \tilde{h})$ is a real DPD-pair on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ which is regular since (D, h) is regular. Let $(\tilde{S}, \tilde{\sigma})$ be the corresponding smooth real affine surface with $\mathbb{S}^1$ -action and let $\tilde{\pi} : (\tilde{S}, \tilde{\sigma}) \rightarrow (Q_{1,\mathbb{C}}, \sigma_{Q_1})$ be its real quotient morphism. We then have a cartesian square of real algebraic varieties

$$\begin{array}{ccc} (S, \sigma) & \xrightarrow{\varphi} & (\tilde{S}, \tilde{\sigma}) \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ (C, \tau) & \longrightarrow & (Q_{1,\mathbb{C}}, \sigma_{Q_1}) \end{array}$$

in which the top horizontal morphism φ is an $\mathbb{S}^1$ -equivariant open embedding of (S, σ) as the complement of the fibers of $\tilde{\pi}$ over the points of $Q_{1,\mathbb{C}} \setminus C$. Since $Q_{1,\mathbb{C}} \setminus C$ consists of pairs of non-real points of $Q_{1,\mathbb{C}}$, the real loci of (S, σ) and $(\tilde{S}, \tilde{\sigma})$ coincide, which implies that $\varphi : (S, \sigma) \rightarrow (\tilde{S}, \tilde{\sigma})$ is a birational diffeomorphism.

By construction, $(\tilde{D}, \tilde{h})$ is a regular real DPD-pair on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ such that the support of $\tilde{D}$ consists of real points of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ and such that $\tilde{h}$ is a regular function whose zero locus consists of real points only. Furthermore, the image by $\tilde{\pi} : (\tilde{S}, \tilde{\sigma}) \rightarrow (Q_{1,\mathbb{C}}, \sigma_{Q_1})$ of the real locus of $(\tilde{S}, \tilde{\sigma})$ is equal to that of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. By Lemma 28 below, $(\tilde{S}, \tilde{\sigma})$ is $\mathbb{S}^1$ -equivariantly isomorphic to $(Q_{1,\mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ on which $\mathbb{S}^1$ acts by translations on the second factor. This completes the proof. $\square$

In the proof of Proposition 27 above, we use the following auxiliary characterization of $(Q_{1,\mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ up to $\mathbb{S}^1$ -equivariant real isomorphisms:

Lemma 28. *Let (D, h) be a regular real DPD-pair on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ such that the support of D consists of real points of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ and such that h is a real regular function whose zero locus consists of real points of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ only. Let (S, σ) be the corresponding smooth real affine surface with $\mathbb{S}^1$ -action and let $\pi : (S, \sigma) \rightarrow (Q_{1,\mathbb{C}}, \sigma_{Q_1})$ be its real quotient morphism. Then the following are equivalent:*

- i) *The image by π of the real locus of (S, σ) is equal to the real locus of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$.*
- ii) *The surface (S, σ) is $\mathbb{S}^1$ -equivariantly isomorphic to $(Q_{1,\mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ on which $\mathbb{S}^1$ acts by translations on the second factor.*

Proof. The implication ii) $\Rightarrow$ i) is clear. We now proceed to the proof of i) $\Rightarrow$ ii). For every real point $c = (c_1, c_2)$ of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$, the restrictions to $Q_{1,\mathbb{C}}$ of the regular functions

$$F_c = (v - c_2) - i(u - c_1) \quad \text{and} \quad \bar{F}_c = (v - c_2) + i(u - c_1)$$

on $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec}(\mathbb{C}[u, v])$ are regular functions f_c and $\bar{f}_c$ on $Q_{1,\mathbb{C}}$ such that $\text{div}(f_c) = \text{div}(\bar{f}_c) = c$. Furthermore, we have $\sigma_{Q_1}^* f_c = \bar{f}_c$ so that $\text{div}(f_c \sigma_{Q_1}^* f_c) = 2c$. Arguing as in the proof of Lemma 18, we obtain that (S, σ) is $\mathbb{S}^1$ -equivariantly isomorphic to the surface determined by a regular real DPD-pair (D', h') on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ such that $\text{Supp}(D')$ is contained in the real locus of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$, h' is regular and vanishes at real points only, and such that for every real point c of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ exactly one of the following possibilities occurs:

- a) $D'(c) = 0$ and $\text{ord}_c(h') = 0$
- b) $D'(c) = \frac{1}{2}$ and $\text{ord}_c(h') = 1$
- c) $D'(c) = 0$ and $\text{ord}_c(h') = 1$.

Consider the restriction $h'|_{S^1} : S^1 \rightarrow \mathbb{R}$ of the real regular function h' to the real locus S^1 of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. If c_0 is a real point of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ of type b) or c) then $h'|_{S^1}$ is a continuous function on S^1 whose sign changes at c_0 . It follows that there exists a real point c of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$ such that $D'(c) = 0$ and $h'(c) < 0$. But then, it follows from the proof of Theorem 20 1) that $(\pi^{-1}(c), \sigma|_{\pi^{-1}(c)})$ is $\mathbb{S}^1$ -equivariantly isomorphic to the nontrivial $\mathbb{S}^1$ -torsor $\hat{\mathbb{S}}^1$ which has empty real locus. This is impossible since by hypothesis the real locus of (S, σ) surjects onto that of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. Thus $D'(c) = \text{ord}_c(h') = 0$ for every real point c of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. This implies in turn D' has empty support and that h' is a nowhere vanishing real regular function on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. It follows that h' is constant, with positive value $\lambda \in \mathbb{R}_+^*$ at every point since the real locus of (S, σ) surjects onto that of $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. Writing $\lambda = \alpha \tau^* \alpha = \alpha^2$ for some real number α , we deduce from Theorem 14 1) that the surface (S, σ) is $\mathbb{S}^1$ -equivariantly isomorphic to that defined by the real DPD-pair $(D', 1) = (0, 1)$ on $(Q_{1,\mathbb{C}}, \sigma_{Q_1})$. By subsection 3.1.1, the latter is $\mathbb{S}^1$ -equivariantly isomorphic to $(Q_{1,\mathbb{C}}, \sigma_{Q_1}) \times \mathbb{S}^1$ on which $\mathbb{S}^1$ acts by translations on the second factor. $\square$

3.2.2. *Second case: (C, τ) is a real affine open subset of the real affine line.*

Proposition 29. *Let (S, σ) be a smooth rational real affine surface with an effective $\mathbb{S}^1$ -action and whose real locus is a connected compact surface without boundary. Let $\pi : (S, \sigma) \rightarrow (C, \tau)$ be its real quotient morphism and assume that (C, τ) is a real affine open subset of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$. Then (S, σ) is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to one of the affine surfaces (S_2, σ_2) , (S_3, σ_3) and (S_4, σ_4) in Table 7.*

Proof. Let $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[z])$ and let (D, h) be a regular real DPD-pair on $(C, \tau) \subseteq (\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ corresponding to (S, σ) . The image by π of the real locus of (S, σ) is closed interval J of the real locus $\mathbb{R}$ of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$. By Theorem 14 1), (S, σ) is $\mathbb{S}^1$ -equivariantly isomorphic to the surface determined by a real DPD-pair of the form

$$(D + \text{div}(f|_C), (f\tau^*f)|_C h)$$

on (C, τ) , where f is any element of $\mathbb{C}(z)$. By choosing $f \in \mathbb{C}(z)$ suitably, we can assume from the very beginning that $h \in \mathbb{C}(z)$ is a real polynomial whose zero locus on $\mathbb{A}_{\mathbb{C}}^1$ is contained in J . Then arguing as in the proof of Proposition 27, we get that (S, σ) is $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to the smooth real affine surface $(\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$ determined by the real DPD pair $(D|_U, h|_U)$ on the open complement $(U, \tau|_U)$ in C of the set of non-real points q of (C, τ) such that either $D(q) \neq 0$ or $D(\tau(q)) \neq 0$. Then $(\pi^{-1}(U), \sigma|_{\pi^{-1}(U)})$ is in turn $\mathbb{S}^1$ -equivariantly birationally diffeomorphic to the smooth real affine surface $(\tilde{S}, \tilde{\sigma})$ determined by the regular real DPD-pair $(\tilde{D}, \tilde{h}) = (\tilde{D}, h)$ on $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$, where $\tilde{D}$ is the Weil $\mathbb{Q}$ -divisor defined by $\tilde{D}(c) = D(c)$ if $c \in J$ and $\tilde{D}(c) = 0$ otherwise.

By composing the real quotient morphism $\tilde{\pi} : (\tilde{S}, \tilde{\sigma}) \rightarrow (\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ by a real automorphism of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$, we can assume without loss of generality that J is equal to the interval $[-1, 1]$. For every real point c of J , $f_c = z - c$ is a real regular function f_c on $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ such that $c = \text{div}(f_c)$. Arguing as in the proof of Lemma 18, we obtain that $(\tilde{S}, \tilde{\sigma})$ is $\mathbb{S}^1$ -equivariantly isomorphic to the surface $(\hat{S}, \hat{\sigma})$ determined by a real DPD-pair $(\hat{D}, \hat{h})$ on $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ such that $\text{Supp}(\hat{D})$ is contained in J , $\hat{h}$ is a regular function whose zero locus is contained in J and such that that for every $c \in J$ exactly one of the following possibilities occurs:

- a) $\hat{D}(c) = 0$ and $\text{ord}_c(\hat{h}) = 0$
- b) $\hat{D}(c) = \frac{1}{2}$ and $\text{ord}_c(\hat{h}) = 1$
- c) $\hat{D}(c) = 0$ and $\text{ord}_c(\hat{h}) = 1$.

By composing the real quotient morphism $\hat{\pi} : (\hat{S}, \hat{\sigma}) \rightarrow (\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ by the real automorphism $z \mapsto -z$ of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$, we can further assume without loss of generality that $\hat{D}(-1) \geq \hat{D}(1)$. By Theorem 20, for a real point c of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$, the real locus of $(\hat{\pi}^{-1}(c), \hat{\sigma}|_{\hat{\pi}^{-1}(c)})$ is empty if and only if $\hat{D}(c) = \text{ord}_c(\hat{h}) = 0$ and $\hat{h}(c) < 0$. It follows that $\hat{D}(c) = \text{ord}_c(\hat{h}) = 0$ and $\hat{h}(c) < 0$ for every real point c of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ outside of J . On the other hand, the real locus of $(\hat{\pi}^{-1}(c), \hat{\sigma}|_{\hat{\pi}^{-1}(c)})$ being nonempty for every real point $c \in J$ by assumption, we have $\hat{h}(c) \geq 0$ for every $c \in J$. It follows that $\hat{h} \in \mathbb{R}[z] \subset \mathbb{C}[z]$ is a nonzero real polynomial with only simple real roots, whose restriction to the real locus $\mathbb{R}$ of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ is negative outside J and non-negative on J . This implies that $\hat{h} = \lambda(1 - z^2)$ for some $\lambda \in \mathbb{R}_+^*$ which can be further chosen equal to 1 by Theorem 14 1). It follows in turn that $\hat{D}(c) = 0$ for every real point c of $(\mathbb{A}_{\mathbb{C}}^1, \sigma_{\mathbb{A}_{\mathbb{R}}^1})$ other than -1 and 1 , and since $\text{ord}_{\pm 1} \hat{h} = 1$ and $\hat{D}(-1) \geq \hat{D}(1)$, the only remaining possibilities are the following:

- (i) $\hat{D}(-1) = \hat{D}(1) = 0$
- (ii) $\hat{D}(-1) = \frac{1}{2}$ and $\hat{D}(1) = 0$
- (iii) $\hat{D}(-1) = \hat{D}(1) = \frac{1}{2}$.

These pairs $(\hat{D}, \hat{h})$ correspond respectively to the model (S_2, σ_2) of S^2 , (S_3, σ_3) of $\mathbb{RP}^2$ and (S_4, σ_4) of K in Table 7. This completes the proof. $\square$

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