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Sub-Riemannian Geometry and Geodesics in Banach Manifolds

Sylvain Arguillère

Abstract

In this paper, we define and study sub-Riemannian structures on Banach manifolds. We obtain extensions of the Chow-Rashevski theorem for exact controllability, and give conditions for the existence of a Hamiltonian geodesic flow despite the lack of a Pontryagin Maximum Principle in the infinite dimensional setting.

Introduction

A sub-Riemannian manifold is a smooth manifold M, endowed with a so-called *horizontal* distribution of subspaces $\Delta \subset TM$ together with a smooth Riemannian metric g on Δ [13, 27]. This allows to define horizontal curves and vector fields, which are everywhere tangent to Δ. The metric then gives a natural notion of length and action for those horizontal curves, and a corresponding sub-Riemannian distance[1].

The first natural question raised by this structure is that of *controllability*: can any two points of M be joined by a horizontal curve? Equivalently, is the sub-Riemannian distance between any two points finite? This second problem is one of *optimality*: determining the shortest horizontal paths, or geodesics, between two points that can be horizontally connected. In finite dimensions, these two problems are tackled by two fundamental results in sub-Riemannian geometry:

- The *Chow-Rashevksi Theorem* [27] (generalized by Sussmann to the *orbit theorem* [32]). It states that if the iterated Lie brackets of horizontal vector fields span the whole tangent bundle, then any two points of M can be connected by a horizontal curve. This is the main tool to answer the problem of *controllability*.

- The celebrated *Pontryagin Maximum Principle* (PMP) [29] is used to find geodesics. Define the Hamiltonian of the system $H^\lambda : T^*M \oplus_M \Delta \to \mathbb{R}$ by

$$H^\lambda(q,p,v) = p(v) - \frac{\lambda}{2} g_q(v,v), \quad q \in M,\ p \in T_q^*M,\ v \in \Delta_q,$$

  with λ being 0 or 1. The PMP states that a geodesic curve $q : [0,1] \to M$ is the projection on M of a curve $t \mapsto (q(t), p(t))$ on T^*M that satisfies for almost every time t:

$$\begin{cases} \dot{q}(t) = \partial_p H^\lambda(q(t),p(t),v(t)), \\ \dot{p}(t) = -\partial_q H^\lambda(q(t),p(t),v(t)), \\ \quad 0 = \partial_v H^\lambda(q(t),p(t),v(t)), \end{cases} \tag{1}$$

  with λ in $\{0,1\}$ fixed, and $(\lambda, p(t)) \neq (0,0)$.

[1] Here, we somewhat stretch the definition of distance to allow infinite values for the distance between points that can't be horizontally connected.

1. If $\lambda = 1$, then $t \mapsto (p(t), q(t))$ is actually the solution of the Hamiltonian equation on T^*M for the *normal Hamiltonian* h, given on T^*M by $h(q,p) = \max_{v \in \Delta_q} H^1(q,p,v)$. Conversely, any projection to M of such a curve is a geodesic. In this case, $q(\cdot)$ is called a *normal geodesic*.
2. On the other hand, Equation (1) with $\lambda = 0$ means that $q(\cdot)$ is an *abnormal*. Geodesics that are abnormal are called *abnormal geodesics*. One of the more difficult aspects of sub-Riemannian geometry is the existence of those abnormals [27].

The purpose of this paper is to lay the foundation to infinite dimensional sub-Riemannian geometry for a wide range of distributions Δ (for example, Δ might be dense in infinite dimensions, as in [7, 9]), and generalize those two results. We will see both the Chow-Rashevski Theorem and the Pontryagin Maximum Principle need to be adapted, and give weaker results. For example, it is well-known that there is in general no PMP in infinite dimensions [23], though some work has been done in this direction in [18] for certain special cases.

Another problem is the possibility that g be a *weak metric*, that the norm it induces on each horizontal subspace is not complete. This makes the problem of existence of geodesics much more complicated even in the simpler case of Riemannian manifolds [1, 25].

In Section 1, we give the various definitions for a sub-Riemannian structure on a Banach manifold M. We consider horizontal distributions given by $\Delta = \xi(\mathcal{H})$, where $\mathcal{H}$ is a vector bundle over M and $\xi : \mathcal{H} \to TM$ a smooth vector bundle morphism. The metric g is then directly defined on $\mathcal{H}$ instead of Δ. A curve $q(\cdot)$ is then horizontal if it satisfies an equation of the form

$$\dot{q}(t) = \xi_{q(t)} u(t)$$

for some control $u(t) \in \mathcal{H}_{q(t)}$ for almost every t. This definition allows to consider dense distributions, and the infinite-dimensional equivalent of rank-varying distributions as defined in [2].

In the Section 2, we study the problems of approximate and exact controllability. In particular, it was already known [15] that the natural extension of the Chow-Rashevski Theorem gives approximate controllability (every point can be connected to a dense subset of M by a horizontal curve), but that in general, we cannot expect to have exact controllability. However, in the case of Hilbert structures, we do introduce a sufficient condition for exact controllability, which we call the *strong* Chow-Rashevski condition.

Theorem. *Let $(M, \mathcal{H}, g)$ be a sub-Riemannian structure, with M a Hilbert manifold, and the fibers of $\mathcal{H}$ modelled on a Hilbert space. Assume that around a point q of M, there are horizontal vector fields $X_1, \dots, X_r$, $r \in \mathbb{N}$, and a positive integer k such that*

$$T_q M = \Delta_q + \sum_{i=1}^{k} \sum_{j_1,\dots,j_i=1}^{r} [X_{j_1}, \dots, [X_{j_i}, \Delta] \dots]_q.$$

Then we have local exact controllability around q. In particular, sub-Riemannian balls centered at q are neighbourhoods of q for the intrinsic manifold topology of M.

Then, in Section 3, we investigate sub-Riemannian geodesics. We discuss the appearance of *elusive geodesics*, which cannot be characterized by a Hamiltonian equation, and therefore prevents the proof of a PMP. This is due to the fact that the differential of the *endpoint map* (which is the smooth map that associates to a control $u(\cdot)$ the final point of the corresponding horizontal curve) may have proper dense image.

We do however obtain the following partial converse to a PMP.

Proposition. *Fix $t \mapsto q(t)$ a horizontal curve with control $u(\cdot)$. Assume that there exists a lift $t \mapsto p(t) \in T^*_{q(t)}M \setminus \{0\}$ of $q(\cdot)$ to T^*M such that $(q(\cdot), p(\cdot))$ satisfies*

$$\begin{cases} \dot{q}(t) = \partial_p h(q(t), p(t)), \\ \dot{p}(t) = -\partial_q h(q(t), p(t)), \end{cases} \tag{2}$$

with $h(q,p) = \max_{u \in \mathcal{H}_q}(p(\xi_q u) - \frac{1}{2} g_q(u,u))$. Then $q(\cdot)$ is a critical point of the sub-Riemannian action with fixed endpoints.

If $(q(\cdot), p(\cdot))$ satisfies

$$\begin{cases} \dot{q}(t) = \partial_p H^0(q(t), p(t), u(t)), \\ \dot{p}(t) = -\partial_q H^0(q(t), p(t), u(t)), \\ \quad 0 = \partial_u H^0(q(t), p(t), u(t)), \end{cases}$$

with $H^0(q,p,u) = p(\xi_q u)$, then $q(\cdot)$ is an abnormal curve, that is, a critical point of the endpoint map.

Then we prove that integral curves of the Hamiltonian flow associated to h are indeed geodesics. However, if g is a weak metric, h may only be defined on a dense sub-bundle of T^*M, on which a Hamiltonian flow may not even be defined. This is already a well-known problem in Riemannian geometry, where weak metrics may not have Levi-Civita connections. However, under the assumption that h defines a Hamiltonian flow on a smooth dense sub-bundle $\tau M \subset T^*M$, we prove that curves that follow this flow do project to geodesics.

Theorem. *Let τM be a smooth dense sub-bundle of T^*M on which the normal Hamiltonian h is well-defined and admits a $\mathcal{C}^2$ symplectic gradient with respect to the restriction to τM of the canonical symplectic form on T^*M. For a strong structure, this is always true with $\tau M = T^*M$.*

Then integral curves of this symplectic gradient, that is, solutions $(q(\cdot), p(\cdot))$ of

$$\begin{cases} \dot{q}(t) = \partial_p h(q(t), p(t)), \\ \dot{p}(t) = -\partial_q h(q(t), p(t)), \end{cases}$$

project to local geodesics of the sub-Riemannian structures.

Finally, in the Section 4, we give a family of examples of weak sub-Riemannian structures on the group of diffeomorphisms of a compact manifold, and prove the existence of a Hamiltonian geodesic flow on a suitable adapted cotangent sub-bundle. Our hope is that this is a first step to the study of sub-Riemannian fluid mechanics.

1 Banach sub-Riemannian geometry

For any Banach space B, we denote the corresponding Banach norm by $\|\cdot\|_{\mathrm{B}}$

For the rest of this section, let M be a smooth connected Banach manifold of class $\mathcal{C}^\infty$, modelled on a Banach space B. For $I = [a,b]$ an interval of $\mathbb{R}$, we let $H^1(I, M)$ denote the set of continuous curves on M that are differentiable almost everywhere, and such that, for any $a' < b'$ in $[a,b]$ with $q([a',b'])$ included in some coordinate neighbourhood U, we have

$$q(b') = q(a') + \int_{a'}^{b'} \dot{q}(t) dt,$$

and

$$\int_{a'}^{b'} \|\dot{q}(t)\|_{\mathrm{B}}^2 dt < +\infty,$$

where we identified U with an open subset of B through a coordinate system. For a fixed $q_0 \in M$, we also let $H^1_{q_0}(I, M)$ the subset of curves $q(\cdot)$ in $H^1(I, M)$ with $q(a) = q_0$.

Let $\pi^{\mathcal{E}} : \mathcal{E} \to M$ be a Banach vector bundle over M with typical fiber $(\mathrm{E}, \|\cdot\|_E)$. For any subset Δ of $\mathcal{E}$, we denote $\Gamma(\Delta)$ the space of smooth sections of $\mathcal{E}$ with image in Δ.

We also define $H^1_{q_0} \times L^2(I, \mathcal{E})$ as the set

$$\{(q(\cdot), e(\cdot)) : I \to \mathcal{E} \mid q(\cdot) \in H^1_{q_0}(I, M),\ e(\cdot) \in L^2(I, \mathrm{E}) \text{ in a trivialization of } \mathcal{E} \text{ along } q(\cdot)\}. \quad (3)$$

Remark 1. Since the image of $q(\cdot)$ is compact, belonging to $L^2(I, \mathrm{E})$ is independent of the local trivialization.

Remark 2. In general, as far as we know, there is no known natural Banach manifold structure on the whole of $H^1_{q_0} \times L^2(I, \mathcal{E})$, or even simply on $H^1_{q_0}(I, M)$. This is because the usual construction of such a structure requires a *local addition* on M, see [21][Section 42.4].

1.1 Definitions

Definition 1. *A **relative tangent space** on M, is a couple $(\mathcal{H}, \xi)$, with $\mathcal{H}$ a smooth Banach vector bundle $\pi^{\mathcal{H}} : \mathcal{H} \to M$ on M, with fibers isomorphic to a fixed Banach vector space* H*, and $\xi : \mathcal{H} \to TM$ is a smooth vector bundle morphism.*

*The corresponding **horizontal distribution** is given by the image $\Delta = \xi(\mathcal{H}) \subset TM$.*

Remark 3. This notion of relative tangent bundles, also called anchored vector bundles, is often used in the study of Banach Lie Algebroids [24]. It allows to define sub-Riemannian geometries with rank-varying horizontal distribution in finite dimensions, as in [2].

Now let us fix $(\mathcal{H}, \xi)$ a relative tangent bundle on M, and $\Delta = \xi(\mathcal{H})$ the horizontal distribution. We define horizontal vector fileds and horizontal systems as follows.

Definition 2. *A **horizontal vector field** is a vector field $X \in \Gamma(TM)$ for which there exists a smooth section $u \in \Gamma(\mathcal{H})$ such that $X(q) = \xi_q u(q),\ q \in M$.*

*A **horizontal system** is a couple $(q(\cdot), u(\cdot)) \in H^1 \times L^2(I, \mathcal{H})$, with u the **control** and q the **trajectory** of the system, such that*

$$\dot{q}(t) = \xi_{q(t)} u(t), \quad a.e.\ t \in I.$$

*A curve $q(\cdot)$ in $H^1(I, M)$ is said to be a **horizontal curve** if we can find a lift $t \mapsto u(t) \in \mathcal{H}_{q(t)}$ of $q(\cdot)$ to $\mathcal{H}$ such that $(q(\cdot), u(\cdot))$ is a **horizontal system**.*

Endowing $\mathcal{H}$ with a metric, we obtain a sub-Riemannian structure which will allow the definition of sub-Riemannian length, action and distance in the next section. Much like in the Riemannian case, one distinguishes two types of metrics: weak and strong metrics.

Definition 3. *A weak sub-Riemannian structure on M is a triple $(\mathcal{H}, \xi, g)$ where $(\mathcal{H}, \xi)$ is a relative tangent space on M, and $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a smooth positive definite symmetric bilinear form on each fiber $\mathcal{H}_q$. The structure is said to be strong when g_q is a Hilbert product on each fiber, making $\mathcal{H}$ into a Hilbert bundle.*

Example 1. Let $\mathcal{H}$ be a distribution of closed subspaces on a Banach manifold M, and g be a weak Riemannian metric on M. Let i be the inclusion map $\mathcal{H} \to TM$. Then $(\mathcal{H}, i, g_{|\mathcal{H}})$ is a sub-Riemannian metric. A curve $q \in H^1(I, M)$ is horizontal when

$$\dot{q}(t) \in \mathcal{H}_{q(t)}, \quad \text{a.e. } t \in I.$$

The case where $\mathcal{H}$ admits a smooth and closed complement was partially studied in [18], in the more general setting of convenient spaces.

Example 2. Fix $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space, let $A(H)$ the space of bounded skew-symmetric operators on H, and consider the control system

$$\dot{q}(t) = u(t), \quad \dot{A}(t) = q(t) \wedge u(t), \quad \text{a.e. } t, \quad u \in L^2(0,1;H),$$

with $q \wedge u(w) = \frac{1}{2}(\langle q, w\rangle\, u - \langle u, w\rangle\, q)$.

This describes the horizontal systems for the relative tangent space $(\mathcal{H}, \xi)$ over $M = H \times A(H)$ given by the trivial bundle $\mathcal{H} = M \times H$ and

$$\xi_{q,A}(u) = (u, q \wedge u), \quad ((q, A), u) \in M \times H.$$

A metric g can be defined simply by $g(u, v) = \langle u, v\rangle$. This is a strong sub-Riemannian structure.

Example 3. Fix a d-dimensional compact manifold N, endowed with a smooth relative tangent space $(\mathcal{H}_N, \xi_N)$. Let $M = \mathcal{D}^s(N)$ the topological group of diffeomorphisms of Sobolev class H^s, $s > d/2+1$ (see for example [16] for the definition). This group is a smooth Hilbert manifold, with tangent space at φ given by $\Gamma^s(TN) \circ \varphi$ [16, 28, 31], with $\Gamma^s(TN)$ the space of H^s-vector fields on N.

Consider the relative tangent space $(\mathcal{D}^s(N) \times \Gamma^s(\mathcal{H}_N), \xi)$, with

$$\xi_\varphi(u)(x) = \xi_{N,\varphi(x)}(u(\varphi(x)).$$

A vector $X \in T_\varphi \mathcal{D}^s(N) \simeq \Gamma^s(TN) \circ \varphi$ is horizontal if and only if $X \circ \varphi^{-1}$ is a horizontal vector field of class H^s on the finite dimensional manifold N. Adding a metric of the form

$$g_\varphi(u, u) = \|u\|^2_{H^s}$$

for some H^s norm on $\Gamma^s(\mathcal{H}_N)$, we obtain a strong sub-Riemannian structure.

In this case, a horizontal curve $t \mapsto \varphi(t)$ is just the flow of a time-dependent horizontal vector field of Sobolev class H^s.

It is important to note that this relative tangent space is not smooth, as it is simply continuous with respect to φ. However, it is possible to find an equivalent relative tangent space (i.e., such that the horizontal curves are the same) that is smooth. See [7] and the last section of this paper for more details, and [6, 8, 9] for applications of such structures to shape analysis.

Let us now consider the problem of controllability.

Definition 4. *The **orbit** $\mathcal{O}_{q_0}$ of a point q_0 in manifold M endowed with a relative tangent space is the set of all points q of M that can be connected to q_0 by a horizontal curve. The structure is said to be **approximately controllable** from q_0 if $\mathcal{O}_{q_0}$ is dense in M. It is said to be **controllable** (or to have the **exact controllability** property) if $\mathcal{O}_{q_0} = M$ for some q_0.*

We trivially have $q_0 \in \mathcal{O}_{q_0}$, and $q \in \mathcal{O}_{q_0}$ implies $\mathcal{O}_{q_0} = \mathcal{O}_q$. In finite dimensions, the well-known Chow-Rashevski theorem provides easily checkable sufficient condition for controllability.

Theorem 1 (Chow-Rashevski, [27])**.** *Let M be a connected finite dimensional manifold with a smooth relative tangent space, with horizontal distribution $\Delta \subset TM$. Define the sequence of distributions of subspaces*

$$\Delta^1_q = \Delta_q, \quad , \Delta^{k+1}_q = \Delta^k + [\Gamma(\Delta), \Gamma(\Delta^k)](q), \ k \geqslant 1, q \in M,$$

with $[\cdot,\cdot]$ the usual Lie bracket on TM.

Assume that for any $q \in M$, there exists $k \geqslant 1$ such that $\Delta^k_q = T_q M$. Then any two points of M can be connected by a horizontal curve.

This theorem was improved by Sussmann [32], who proved that $\mathcal{O}_{q_0}$ is always an immersed submanifold of M, whose tangent space at q_0 contains $\cup_k \Delta^k_{q_0}$.

1.2 Length, action and distance

Let $I = [a, b] \subset \mathbb{R}$ be an interval and $(\mathcal{H}, \xi, g)$ be a weak sub-riemannian structure on the Banach manifold M.

Definition 5. *The action and length of a horizontal system $(q, u) : I \to \mathcal{H}$ are respectively defined by*

$$A(q, u) = \frac{1}{2} \int_I g_{q(t)}(u(t), u(t))dt \quad \textit{and} \quad L(q, u) \int_I \sqrt{g_{q(t)}(u(t), u(t))}dt.$$

Thanks to the smoothness (and, in particular, continuity) of the metric, any horizontal system has finite action and length.

Remark 4. Another possibility is to directly define the sub-Riemannian (semi-)norm of a horizontal vector $X = \xi_q(u) \in T_qM$: the linear map ξ_q defines on its image $\xi_q(\mathcal{H}_q)$ a seminorm $n : \xi_q(\mathcal{H}_q) \to \mathbb{R}^+$ by

$$n_q(w)^2 = \inf_{u \in \mathcal{H}_q,\ \xi_q u = w} g_q(u, u).$$

If $q : I \to M$ is a horizontal curve, its normal length and action can be respectively defined by

$$L(q) = \int_I n_{q(t)}(\dot{q}(t))dt, \quad \text{and} \quad A(q) = \frac{1}{2} \int_I n_{q(t)}(\dot{q}(t))^2 dt.$$

However, the normal action and length are much harder to study directly, because $(q, u) \mapsto n_q(u)^2$ may not be smooth. The action of horizontal systems as given in Definition 5 is better suited to the study of geodesics.

Definition 6. *The sub-Riemannian distance $d(q_0, q_1)$ between two points $q_0, q_1 \in M$ is defined by*

$$d(q_0, q_1) = \inf_{\substack{(q,u) \in L^2(0,1;\mathcal{H}), \\ (q,u) \text{ horizontal}, \\ q(0)=q_0,\ q(1)=q_1}} L(q, u).$$

We do allow infinite values for d by setting $\inf \emptyset = +\infty$. Consequently, we obtain an equivalent definition for the orbit of a point q_0 using

$$\mathcal{O}_{q_0} = \{q \in M \mid d(q_0, q) < +\infty\}.$$

Lemma 1. *The map $d : M \times M \to \mathbb{R} \cup \{+\infty\}$ is a semidistance, possibly with infinite values. When the sub-Riemannian structure is strong, d is always a true distance, that is, $d(q_0, q_1) = 0$ if and only if $q_0 = q_1$. Moreover, the topology it induces on M is not coarser than the intrinsic manifold topology.*

Proof. The mapping d being a semi-distance comes from the basic properties of horizontal systems. Indeed, reversals (i.e., $t \mapsto q(b - t)$) and concatenation of horizontal curves are also horizontal curves. The symmetry of d and the triangular inequality both follow. This proves the first part of the lemma.

For the second part, assume the metric g is strong, fix $q_0 \neq q_1 \in M$, and consider $(q(\cdot), u(\cdot)) : [a, b] \to \mathcal{H}$ a horizontal system with $q(a) = q_0$ and $q(b) = q_1$. Let us prove that $L(q, u)$ is greater than some positive constant that does not depend on the choice of $(q(\cdot), u(\cdot))$.

We work in a small neighbourhood U of q_0 with $q_1 \in M \setminus U$, such that $\mathcal{H}_{|U} \simeq U \times \mathrm{H}$. We identify U with an open subset of the Banach space B on which M is modeled.

Now, ξ is a smooth vector bundle morphism and $q \mapsto g_q$ is a smooth family of Hilbert norms on H. Hence, for $\epsilon > 0$ small enough, the B-ball $B_\varepsilon(q_0)$ centered at q_0 with radius $\varepsilon > 0$ is contained in U, and there exists $c > 0$

$$\forall (q,u) \in B_\varepsilon(q_0) \times \mathrm{H}, \quad \|\xi_q(u)\|_{\mathrm{B}} \leqslant c\sqrt{g_q(u,u)}.$$

For the horizontal system $(q,u) : I = [a,b] \to \mathcal{H}$, which starts at $q(a) = q_0$, let

$$t_e = \min(\inf\{t \in [a,b] \mid \|q(t) - q_0\|_{\mathrm{B}} \geqslant \varepsilon\}).$$

As $t \mapsto q(t)$ is continuous, and because $q(b) = q_1$ is not in $B_\varepsilon(q_0)$, t_e is well-defined and belongs to (a,b). Then

$$\varepsilon = \|q(t_e) - q_0\|_{\mathrm{B}} \leqslant \int_a^{t_e} \|\xi_{q(t)}(u(t))\|_{\mathrm{B}} dt \leqslant c \int_a^{t_e} \sqrt{g_{q(t)}(u(t),u(t))} dt \leqslant cL(q,u).$$

We get $L(q,u) \geqslant \varepsilon/c$. Since this is true for any horizontal system whose trajectory connects q_0 and q_1, we get $d(q_0,q_1) > 0$. Since this is true for any $q_0 \neq q_1$, d is a true distance.

Finally, in the previous construction, we see that for arbitrarily small $\varepsilon > 0$, the sub-Riemannian ball of center q_0 and radius ε/c is included in $B_\varepsilon(q_0)$, which proves the last part of the result. □

1.3 Geodesics

Fix a sub-Riemannian Banach manifold $(M, \mathcal{H}, \xi, g)$ modelled on a Banach space B.

Definition 7. *A **local geodesic** is a horizontal system $(q,u) : I \to M$ such that, for every $t_0 \in I$, and for every $t_1 > t_0$ with $t_1 - t_0$ small enough, there is an open neighbourhood U of $q([t_0,t_1])$ such that for any horizontal system $(q',u') : I' \to U$ with endpoints $q(t_0)$ and $q(t_1)$ and whose trajectory remains in U,*

$$L((q,u)_{|[t_0,t_1]}) \leqslant L(q',u').$$

*It is a **geodesic** if we simply have, for t_0 and t_1 close enough,*

$$L((q,u)_{|[t_0,t_1]}) = d(q(t_0), q(t_1)),$$

*and a **minimizing geodesic** if its total length is equal to the distance between its endpoints.*

We will also use the same term to describe the trajectory $q(\cdot)$ of such control system.

Remark 5. This distinction between local geodesic and plain geodesics is necessary for weak structures even in infinite dimensional Riemannian manifolds (see [21] for example). However, when the metric is strong, all local geodesics are actually geodesics. This is a trivial consequence of the proof of Lemma 1.

Remark 6. If a horizontal system (q,u) minimizes the action $A(q,u)$ among controls whose trajectories have the same endpoints, then the trajectory q is a minimizing geodesic. This is a consequence of the Cauchy-Schwartz inequality and the fact that any rectifiable curve in a length space can be parametrized with respect to arclength [14].

The existence of minimizing geodesics between two points is a difficult question in infinite dimensions. For example, even for strong Riemannian Hilbert manifolds, metric completeness does not imply geodesic completeness.

Example 4. Tgis example was first given in [19]. Consider $X = l^2(\mathbb{N})$ the Hilbert space of square-summable sequences, and let M be the ellipsoid given by

$$\left\{(x_n) \in l^2,\ \sum_{n=0}^{\infty} \frac{x_n^2}{(1+\frac{1}{n+1})^2} = 1\right\}$$

equipped M with the Riemannian metric inherited from the ambient space. M will be complete for the Riemannian distance but there will be no minimizing geodesic between $(\sqrt{2}, 0, \dots)$ and $(-\sqrt{2}, 0, \dots)$.

2 Exact and approximate controllability

The first problem when considering sub-Riemannian geometry is that of controllability, which asks in which capacity can any two points of M be connected by horizontal curves. We will consider three different notions of controllability.

Definition 8. *[33] We say that a sub-Riemannian structure $(M, \mathcal{H}, g)$ is*

- ***Approximate controllable*** *from a fixed starting point q_0 if the orbit $\mathcal{O}_{q_0}$ is dense in M.*
- ***Local controllable*** *around q_0 if for any neighbourhood $\mathcal{U}$ of the zero section of $\mathcal{H}$ around q_0, the set of endpoints of horizontal systems with control in $\mathcal{U}$ contains a neighbourhood of q_0.*
- ***Exact controllable*** *if any two points of M can be connected by a horizontal curve.*

Remark 7. Local controllability implies that any neighbourhood of q_0 contains a ball for the sub-Riemannian distance. If M is connected and the system is controllable around every point, then it is exact controllable, and the topology induced by the sub-Riemannian distance is at least as coarse as the intrinsic manifold topology.

In finite dimensions, the Chow-Rashevsky theorem [27] (and its more general version, Sussmann's orbit theorem [4, 32]) gives a nice sufficient condition for the controllability of the structure: it is controllable as soon as the iterated Lie brackets of horizontal vector fields span the entire tangent space. Moreover, the ball-box theorem also gives precise estimates on the sub-Riemannian distance in this case, showing that it is topologically equivalent to the intrinsic manifold topology of M.

We will see that it is unreasonnable to expect such conditions to hold in infinite dimensional manifolds. All we can usually expect to have dense orbits, that is, approximate controllability. We will however give some natural, stronger conditions that do ensure exact controllability.

For the rest of this section, unless stated otherwise, M is a sub-Riemannian Banach manifold endowed with a sub-Riemannian structure $(\mathcal{H}, \xi, g)$.

2.1 Finite dimensions: the Chow-Rashevski Theorem

By induction, we define the nondecreasing sequence of subsets of TM by

$$\Delta_q^0 = \{0\}, \quad \Delta_q^1 = \Delta_q = \xi_q(\mathcal{H}_q), \quad \Delta_q^{i+1} = \Delta_q^i + [\Delta, \Delta^i]_q, \quad q \in M,$$

for every q in M, where we defined

$$[\Delta, \Delta^i]_q = \{[X, Y](q),\ X, Y \in \Gamma(TM),\ X(q') \in \Delta_{q'},\ Y(q') \in \Delta_{q'}^i,\ q' \in M\}.$$

Here, $[X, Y]$ is the usual Lie bracket on M between the smooth vector fields X and Y.

Then we denote $\mathcal{L}_q = \cup_{i\in\mathbb{N}} \Delta_q^i,\ q \in M$. $\mathcal{L}$ is the *Lie algebra* of vector fields generated by Δ.

Remark 8. This definition is valid for both finite and infinite dimensional manifolds.

Definition 9. *We say that the sub-Riemannian structure satisfies the Chow-Rashevski property at $q \in M$ when $\mathcal{L}_q = T_q M$.*

Theorem 2 (**Chow-Rashevski theorem in finite dimensions** [4, 13, 27])**.** *Assume that M is finite dimensional. If the sub-Riemannian structure satisfies the Chow-Rashevski property at some q in M, then the orbit $\mathcal{O}_q$ contains a neighbourhood of q. Consequently, if M is connected and the structure satisfies the Chow-Rashevski property at every point, then the structure is controllable, that is, any two points can be connected by a horizontal curve.*

Remark 9. A more precise result is given by Sussmann's Orbit Theorem [32]. It states that each orbit $\mathcal{O}_q$ is an immersed submanifold such that $\mathcal{L}_q \subset T_q\mathcal{O}_q$, with equality for analytic structures.

The proof of Chow-Rashevski's theorem can be refined to give local estimates on the sub-Riemannian distance.

Theorem 3 (**Ball-box theorem in finite dimensions** [13, 27])**.** *Fix $q_0 \in M$, and k such that assume $T_{q_0}M = \mathcal{L}_{q_0} = \Delta^k_{q_0} \neq \Delta^{k-1}_{q_0}$. Define $r_i = \dim(\Delta^i_{q_0}) - \dim(\Delta^{i-1}_{q_0})$ for $i = 1, \dots, k$ so that $r_1 + \cdots + r_k = \dim(M)$. Then there are coordinates $q = (x_1, \dots, x_k)$, with $x_k \in \mathbb{R}^{r_i}$, around q_0 such that for some $C > 0$ such that*

$$\frac{1}{C}\sum_{i=1}^{k} |x_i|^{2/i} \leqslant d(q_0, q)^2 \leqslant C \sum_{i=1}^{k} |x_i|^{2/i}.$$

In particular, the topology induced by the sub-Riemannian distance coincides with the intrinsic manifold topology of M.

The exact statement of the Chow-Rashevski and ball-box theorems are both open problems in infinite dimensional manifolds. Moreover, even if such a result existed, it would not be as useful: it is very rare to have $\mathcal{L}_q = T_q M$, simply because $\mathcal{L}_q$ is usually dense, but almost never closed. This is expected, intuitively, because the Lie algebra generated by Δ is constructed in an algebraic way as an infinitely increasing union of brackets of horizontal vector fields. Let us give an in-depth example, which will also be useful for seeing what happens when studying geodesics, in the next section.

2.2 An Example: the ℓ^2-product of Heisenberg Groups

We take an in-depth look at the problem of controllability in a very simple example of infinite dimensional sub-Riemannian manifold, the ℓ^2-product of Heisenberg groups.

2.2.1 The 3-Dimensional Heisenberg Group

The Heisenberg group is one of the simplest examples of finite dimensional sub-Riemannian manifolds. The manifold itself is $\mathbb{H} = \mathbb{R}^3$, and the horizontal space at $q = (x, y, z)$ is spanned by

$$X(q) = (1, 0, -\frac{y}{2}) = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y(q) = (0, 1, \frac{x}{2}) = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},$$

which are orthonormal for the metric. Horizontal curves $q(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ therefore satisfy

$$\dot{x}(t) = u(t), \quad \dot{y}(t) = v(t), \quad \dot{z}(t) = \frac{1}{2}(v(t)x(t) - u(t)y(t)), \quad u, v \in L^2(I; \mathbb{R}),$$

with action

$$E^{\mathbb{H}}(q(\cdot)) = \frac{1}{2}\int_I (u(t)^2 + v(t)^2)dt.$$

Since $[X,Y](q) = \frac{\partial}{\partial z}$, so that $X, Y, [X,Y]$ span the tangent bundle pointwise, any two points can be connected by a horizontal curve, and the sub-Riemannian distance satisfies the ball-box estimates of Theorem 3

$$\frac{1}{C}(x^2 + y^2 + |z|) \leqslant d^{\mathbb{H}}(0,(x,y,z))^2 \leqslant C(x^2 + y^2 + |z|)$$

for some fixed $C > 0$.

2.2.2 The ℓ^2-product of Heisenberg groups

We now consider the Hilbert manifold $M = \ell^2(\mathbb{N}, \mathbb{R}^3)$ the space of square-summable sequences $q = (q_n)_{n\in\mathbb{N}} = (x_n, y_n, z_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^3$. We define on it the sub-Riemannian structure generated as q by the Hilbert frame

$$X_n(q) = \frac{\partial}{\partial x_n} - \frac{y_n}{2}\frac{\partial}{\partial z_n}, \quad Y_n(q) = \frac{\partial}{\partial y_n} + \frac{x_n}{2}\frac{\partial}{\partial z_n}.$$

Lie Algebra. We denote by $\mathcal{L}$ the Lie algebra of smooth vector fields generated by horizontal vector fields. Now, we have $Z_n := [X_n, Y_n] = \frac{\partial}{\partial z_n}$, so that the horizontal vector fields give a Hilbert-spanning frame of TM. In other words, any tangent vector can be written as an *infinite* linear combination with ℓ^2 coefficients of brackets of horizontal vector fields.

However, they *do not span it as a vector field*. For example take the two horizontal vector fields

$$X = \sum_{n\in\mathbb{N}} a_n X_n, \quad Y = \sum_{n\in\mathbb{N}} b_n Y_n, \quad \sum_{n\in\mathbb{N}} a_n^2 + b_n^2 < +\infty.$$

Then

$$[X_1, X_2] = \frac{1}{2}\sum_{n\in\mathbb{N}} a_n b_n \frac{\partial}{\partial z_n}.$$

But, as a product of ℓ^2-sequences, $(a_n b_n)_{n\in\mathbb{N}}$ actually belongs to the dense subspace of absolutely summable sequences $\ell^1(\mathbb{N}, \mathbb{R}) \subset \ell^2(\mathbb{N}, \mathbb{R})$.

More generally, one easily checks that any tangent vector $v \in T_qM$ at 0 belongs to the $\mathcal{L}$ if and only if it can be written

$$v = \sum_{n\in\mathbb{N}} a_n X_n(q) + b_n Y_n(q) + c_n Z_n(q), \quad \sum_{n\in\mathbb{N}} a_n^2 + b_n^2 < +\infty, \quad \sum_{n\in\mathbb{N}} |c_n| \leqslant +\infty.$$

Therefore, $\mathcal{L}$ is only *dense* in M.

Orbit of 0. Let us describe the orbit of 0 in M. A curve $t \mapsto q(t) = (q_n(t))_{n\in\mathbb{N}}$ is horizontal if and only if each curve $t \mapsto q_n(t) = (x_n(t), y_n(t), z_n(t)) \in \mathbb{R}^3 \simeq \mathbb{H}$ is horizontal for the 3-dimensional Heisenberg group. Moreover, its action is given by

$$A(q(\cdot)) = \sum_{n\in\mathbb{N}} E^{\mathbb{H}}(q_n(\cdot)).$$

Consequently, the sub-Riemannian distance between 0 and $q = (x_n, y_n, z_n)_{n\in\mathbb{N}}$ is given by

$$d(q, q')^2 = \sum_{n=0}^{+\infty} d_{\mathbb{H}}\big((x_n, y_n, z_n), (x'_n, y'_n, z'_n)\big)^2,$$

with $d_{\mathbb{H}}$ denoting the sub-Riemannian distance on the Heisenberg group $\mathbb{H}$ as described in the previous section. But we know that

$$\exists\, C > 0,\ \forall (x,y,z) \in \mathbb{H}, \quad \frac{1}{C}(x^2+y^2+|z|) \leqslant d_{\mathbb{H}}(0_{\mathbb{H}},(x,y,z))^2 \leqslant C(x^2+y^2+|z|).$$

In particular $d(0,q)$ is finite if and only if $(x_n,y_n)_n \in l^2(\mathbb{N},\mathbb{R}^2)$ while $(z_n)_n \in l^1(\mathbb{N},\mathbb{R})$. In other words,

$$\mathcal{O}_0 = l^2(\mathbb{N},\mathbb{R}^2) \times l^1(\mathbb{N},\mathbb{R}) \subset l^2(\mathbb{N},\mathbb{R}^3).$$

Moreover, the topology on $\mathcal{O}_0$ induced by the sub-Riemannian distance actually coincides with the usual Banach space topology of $l^2(\mathbb{N},\mathbb{R}^2) \times l^1(\mathbb{N},\mathbb{R})$, and we get a dense orbit in M. We also lost the Hilbert topology.

2.2.3 A Non-Locally Convex Topology

Slightly complicating our example, if we take $M = l^2(\mathbb{N},\mathbb{R}^4)$ as an infinite product of the Engel group $\mathbb{E}$ [27] (or any step-3 or higher Carnot group), we start getting even less satisfactory topologies. Indeed, on the Engel group,

$$\frac{1}{C}(x^2+y^2+|z|+|w|^{2/3}) \leqslant d_{\mathbb{H}}(0_{\mathbb{H}},(x,y,z,w))^2 \leqslant C(x^2+y^2+|z|+|w|^{2/3}).$$

Consequently, the sub-Riemannian distance on $\mathcal{O}_0$ in M is equivalent to the usual quasi-distance on $l^2(\mathbb{N},\mathbb{R}^2) \times l^1(\mathbb{N},\mathbb{R}) \times l^{2/3}(\mathbb{N},\mathbb{R})$, whose topology is *not locally convex*.

The Chow-Rashevski theorem relies on the local inversion theorem, so proving a similar result even for simple cases is probably going to be very difficult, not to mention the general case.

2.3 Approximate Controllability

As we just saw, the conditions for Chow-Rashevski's theorem are very rarely satisfied. However, it is much more common for $\mathcal{L}_q$ to be *dense* in T_qM. In this case, as was proved in [15, 20], we do have approximate controllability.

Theorem 4 ([15, 20])**.** *Assume that M is a Banach manifold, and that $\mathcal{L}_q$ is dense in T_qM for every q in M. Then each orbit $\mathcal{O}_q$ is dense in M, so that the structure is approximate controllable.*

Remark 10. This is actually true even for so-called convenient manifolds (i.e., manifolds modelled on convenient vector spaces, see [21]), as shown in [20]. This problem has also been studied with another viewpoint in [22].

2.4 Exact controllability and strong Chow-Rashevski property

The question of exact controllability is much more complex, even in simple cases as we saw in Section 2.2, and we do not often have exact controllability.

However, under a stronger bracket-generating hypothesis, we can still obtain it, although only for Hilbert structures. This is the first main result of the paper.

2.4.1 The Strong Chow-Rashevski Property

Definition 10. *The sub-Riemannian structure is said to satisfy the* ***strong Chow-Rashevski property*** *at $q \in M$ if there exists fixed horizontal vector fields $X_1,\dots,X_r \in \Delta$ and a fixed positive integer k such that*

$$T_qM = \Delta_q + \sum_{i\in\{1,\dots,r\}} [\Delta, X_i]_q + \cdots + \sum_{I\in\{1,\dots,r\}^k} [\Delta, X_I]_q, \tag{4}$$

where, for simplicity, we denoted

$$X_I = [X_{i_j}, [\dots, [X_{i_2}, X_{i_1}]\dots], \quad I = (i_1, \dots, i_j) \in \{1, \dots, r\}^j, \quad j \in \{1, \dots, k\}.$$

In this case, we can adapt the proof of the finite dimentional Chow-Rashevski theorem to the infinite dimensional context. Let us give a few examples of infinite dimensional sub-Riemannian manifolds that satisfy this property.

Example 5. If the horizontal distribution has finite codimension everywhere, then the Chow-Rashevski condition and the strong Chow-Rashevski condition are equivalent. This is the case for an infinite dimensional Heisenberg group $\mathbb{H}^\infty = \ell^2(\mathbb{N}, \mathbb{R}^2) \times \mathbb{R}$, with horizontal vector fields spanned by

$$X_n(q) = X_n(x_n, y_n, z) = \frac{\partial}{\partial x_n} - \frac{1}{2} y_n \frac{\partial}{\partial z}, \quad Y_n(q) = Y_n(x_n, y_n, z) = \frac{\partial}{\partial y_n} + \frac{1}{2} x_n \frac{\partial}{\partial z}, \quad n \in \mathbb{N}.$$

Here $T_q M = \Delta_q + [\Delta_q, X_n] = \Delta_q + [\Delta_q, Y_n]$ for any integer n.

Example 6. Consider $M = \mathbb{R} \times \ell^2(\mathbb{N}, \mathbb{R}^2)$, with Hilbert basis of horizontal vector fields given by

$$X(q) = X(x, y_n, z_n) = \frac{\partial}{\partial x}, \quad Y_n(q) = \frac{\partial}{\partial y_n} + x \frac{\partial}{\partial z_n}.$$

We have $T_q M = \Delta_q + [\Delta_q, X]$. Indeed, at $q = 0$ for example, any tangent vector can be written

$$v = a \frac{\partial}{\partial x}(0) + \sum_{n \in \mathbb{N}} b_n \frac{\partial}{\partial y_n}(0) + \sum_{n \in \mathbb{N}} c_n \frac{\partial}{\partial z_n}(0),$$

for some $a \in \mathbb{R}$, $b, c \in \ell^2(\mathbb{N}, \mathbb{R})$. Then, letting $Y = aX + \sum_{n \in \mathbb{N}} b_n Y_n$ and $Y' = \sum_{n \in \mathbb{N}} c_n Y_n$, we have

$$v = Y(0) + [Y', X](0).$$

Example 7. It was proved in [3] that the sub-Riemannian structure on the group of diffeomorphisms $\mathcal{D}^s(N)$ of a compact d-dimensional sub-Riemannian manifold N defined in Example 3 has exact controllability. It was proved in [7] that, after some work, this structure actually satisfies the strong Chow-Rashevski condition, and estimates on the corresponding sub-Riemannian distance were given.

2.4.2 Statement and Proof of the Theorem

Theorem 5. *Let $(M, \mathcal{H}, g)$ be a Hilbert manifold, endowed with a sub-Riemannian structure $(\mathcal{H}, g)$, with the fibers of $\mathcal{H}$ modelled on a Hilbert space. Assume the strong Chow-Rashevski property is satisfied at $q_0 \in M$ for some fixed horizontal vector fields $X_1, \dots, X_r$, and let k be the smallest integer such that (4) is satisfied. Then the structure is locally controllable around q_0. Consequently, if the condition is satisfied at every point, and if M is connected, then we have exact controllability, and the sub-Riemannian topology is at least as coarse as the intrinsic manifold topology.*

Remark 11. Combining this result with Lemma 1, we see that if the Chow-Rashevski condition is satisfied at every point for a strong Hilbert structure, then the two topologies coincide.

Proof. We work on a small neighbourhood V of q_0, that we identify to an open subset of a Banach space B, and on which we have a trivialization $\mathcal{H}_{|V} \simeq V_0 \times \mathrm{H}$. First of all, we can assume that ξ and each X_i is equal to zero on $M \setminus V$. This is simply done by multiplication by a smooth cut-off function, which exists thanks to the Hilbert structure of M, and does not change the hypothesis.

Moreover, if the conclusion of the theorem is true for the cut-off structure, it is trivially true for the original.

Next, let $\varphi_i^t : V \to V$ denote the flow of the X_i, $i = 1, \dots, r$, $t \in \mathbb{R}$, smooth and well-defined thanks to the cut-off function. We then define the smooth mapping $\varphi_I^t : V \to V$ by

$$\varphi_I^t(q) = \varphi_{i_j}^t \circ \cdots \circ \varphi_{i_1}^t, \quad t \in \mathbb{R}, \quad I = (i_1, \dots, i_j) \in \{1, \dots, r\}^j, \quad j \in \{1, \dots, k\}.$$

On the other hand, in our trivialization of $\mathcal{H}_{|V}$, for any $u \in H$, we define the smooth vector field $q \mapsto Y_u(q) = \xi_q(u)$. We also denote φ_u^t the corresponding flow on V.

We can now define, for $u \in \mathbb{H}$ and $t \in \mathbb{R}$,

$$\phi_I(t, u, q) := \left(\varphi^{-1}(u) \circ \varphi_I^{-t} \circ \varphi^1(u) \circ \varphi_I^t\right)(q).$$

Note that $(t, u, q) \mapsto \phi_I^t(u, q)$ is smooth, and $\phi_I^t(u, q)$ belongs to $\mathcal{O}_{q_0}$ since it is simply a obtained by a concatenation of $2j + 2$ horizontal curves. Moreover, the total length of these curves is at most equal to $C(2k + \|u\|_{\mathbb{H}})t$ for some positive constant C.

The usual formulas for commutators of flows yields

$$\phi_I(t, u, q) = q + t^i [X(u), X_I](q) + o(t^i u)$$

as $u, t \to 0$, for fixed q. Therefore, for fixed q, the mapping $U_0 \mapsto V_0$

$$\Phi_I(u, q) = \phi_I\left(\|u\|_{\mathrm{H}}^{1/i+1}, \frac{u}{\|u\|_{\mathrm{H}}^{\frac{i}{i+1}}}, q\right) \tag{5}$$

is of class $\mathcal{C}^1$, with first order limited development in u given by

$$\Phi_I(u, q) = q + \|u\|^{\frac{i}{i+1}} \left[\frac{1}{\|u\|^{\frac{i}{i+1}}} X(u), X_I\right](q) + o(u) = q + [X(u), X_I](q) + o(u).$$

This comes from the fact that $u \mapsto X_q(u)$ is a continuous linear operator, and that the Hilbert norm is smooth everywhere except at 0. From there, we easily see that the mapping

$$\begin{aligned} \Phi : \quad & U_0 \times U_0^r \times \cdots \times U_0^{kr} \to \quad V_0 \\ & \mathbf{u} = (u_{I_i})_{\substack{i=0,\dots,k, \\ I_i \in \{1,\dots,r\}^i}} \quad \mapsto \quad \left(\bigcirc_{\substack{i=1,\dots,k, \\ I_i \in \{1,\dots,r\}^i}} \Phi_I(u_{I_i}, \cdot)\right) \circ \varphi^1(u_0)(q_0) \end{aligned}$$

is of class $\mathcal{C}^1$ near 0, with

$$\Phi(\mathbf{u}) = q_0 + \sum_{\substack{i=0,\dots,k, \\ I_i \in \{1,\dots,r\}^i}} [X(u_{I_i}), X_I](q_0) + o(\mathbf{u}).$$

Then $d\Phi(0) : H \times H^r \times \cdots \times H^{r^k} \to T_{q_0}M$ is onto, thanks to the strong Chow-Rashevski condition. Hence, its range contains a neighbourhood of q_0. But since $\Phi(\mathbf{u})$ is obtained by taking the endpoint of a concatenation of horizontal curves, we see that its range is included in the orbit of q_0. Consequently, the orbit of q_0 does contain a neighbourhhod of q_0.

Moreover, let $C_1 = \max_{i=1,\dots r,\ q \in V}(n_q(X_i(q)))$ as defined in Remark 4, reducing V if necessary. Take C_2 such that $g_q(u, u) \leqslant C_2 \|u\|_{\mathrm{H}}^2$ on $V \times \mathrm{H}$ (reducing V if necessary). Finally, let $C =$

$\max(1, C_1, C_2)^2$. Then we see that any $\Phi_I(u, q)$, $I \in \{1, \dots, r\}^i$, $i = 0, \dots, k$, is obtained by taking the endpoint of a concatenation of $2i + 2$ curves of action less than $C\|u\|_{\mathrm{H}}^{2/i+1}$. Consequently, we get one side of the ball-box estimates:

$$d(q_0, \Phi(\mathbf{u}))^2 \leqslant C' \sum_{\substack{i=0,\dots,k,\\ I_i \in \{1,\dots,r\}^i}} \|u_{I_i}\|_{\mathrm{H}}^{2/i+1}.$$

In particular, any sub-Riemannian ball around q_0 includes a neighbourhood of q_0. □

Remark 12. It would be much preferable to obtain estimates on the distance of the form

$$d(q_0, \Phi(\mathbf{u}))^2 \leqslant C' \sum_{\substack{i=0,\dots,k,\\ I_i \in \{1,\dots,r\}^i}} g_{q_0}(u_{I_i}, u_{I_i})^{1/i+1}.$$

This is obviously true in the strong case. However, in the weak case, we would need to replace each instance of $\|u\|_{\mathrm{H}}$ by $\sqrt{g_{q_0}(u, u)}$ in the formula (5) for Φ_I. But then the term

$$\frac{u}{\left(\sqrt{g_{q_0}(u, u)}\right)^{i/i+1}}$$

may not go to zero as u goes to zero, which prevents the rest of the proof from working. This version of the ball-box estimates is therefore an open conjecture.

Remark 13. The Hilbert structure is necessary because of the need of a smooth norm.

Remark 14. The converse inequality (for the strong case) is still an open problem. The proof in finite dimensions uses the concept of privileged coordinates, which are much harder to construct in infinite dimensions.

Remark 15. The ℓ^2-product of Heisenberg group restricted to $\ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^1(\mathbb{N}, \mathbb{R})$ does not satisfy the strong Chow-Rashevski condition. However, it does satisfy the plain Chow-Rashevski condition. This indicates that the usual version of the condition may still hold in infinite dimensions, but the proof would most likely be very different.

3 Geodesics and the Hamiltonian Geodesic Flow

In this section we study geodesics on infinite dimensional sub-Riemannian manifolds. We first recall some properties of the canonical weak symplectic form on the cotangent bundle of a manifold, since this is the framework used to find geodesics in finite dimensions. Then we give the various possible definitions and types of geodesics in infinite dimensional sub-Riemannian geometry. We follow by defining a manifold structure on certain subsets of horizontal systems in order to investigate first order conditions for a such a system to be a geodesic.

We can then proceed with said analysis. We will in particular see that no first order *necessary* condition can be given in general. However, we will give *sufficient* conditions for a curve to be a critical point of the action with fixed endpoints, and *sufficient* conditions for a curve to be singular. We will also see that there is still a Hamiltonian flow of geodesics in the strong case, and we will give sufficient conditions for the existence of such a flow for weak sub-Riemannian structures that specialize to the well-known corresponding conditions on Riemannian manifolds (i.e., existence of a smooth Levi-Civita connection for the metric or smooth weak symplectic gradient for the Hamiltonian [1, 16, 25]). After that, we go back to the case of ℓ^2-product of Heisenberg groups, in order to highlight the problems and differences that appear in infinite dimensions.

3.1 Manifold Structure on the Set of Horizontal Systems

Let M be a Banach manifold endowed with a smooth sub-Riemannian structure $(\mathcal{H}, \xi, g)$. Let us fix $I = [0, 1]$ to simplify notations. We denote

$$\Omega^{\mathcal{H}} = \{(q, u) \in H^1 \times L^2(I, \mathcal{H}) \mid (q, u) \text{ horizontal}\}$$

the set of all horizontal systems. We also define, for q_0 in M,

$$\Omega^{\mathcal{H}}_{q_0} = \{(q, u) \in \Omega^{\mathcal{H}} \mid q(0) = q_0\}.$$

To give conditions for a curve to be a geodesic, we want to study critical points of the action among horizontal systems with fixed endpoints $q(1) = q_1$ for some $q_1 \in M$. But to define critical points, we would first need to put a manifold structure on the space of horizontal systems. However, this is harder to do in infinite dimensions (see for example the appendix of [27] for the finite dimensional case), because even the simpler set $H^1(I, M)$ may not have an obvious manifold structure if M does not possess a *local addition*, see [21][Section 42.4].

However, since the concept of geodesic is a local one, we can study a horizontal system on small enough time sub-intervals. Hence, we can restrict our study to systems whose trajectories stay within a small coordinate neighbourhood U, identified to an open subset of B. We can therefore assume that $\mathcal{H}_{|U} \simeq U \times \mathrm{H}$, and only consider horizontal systems (q, u) such that

$$(q, u) \in H^1 \times L^2(I, \mathcal{H}_{|U}) \simeq H^1(I, U) \times L^2(I, \mathrm{H}).$$

Since $H^1(I, U) \times L^2(I, \mathrm{H})$ is an open subset of the Banach space $H^1(I, B) \times L^2(I, \mathrm{H})$, $H^1 \times L^2(I, \mathcal{H}_{|U})$ does then inherit a manifold structure. Then, we have the following result, first shown in [9][Lemma 3] but whose proof we include for the sake of completeness.

Proposition 1. *We keep the notations and assumptions of the previous discussion. Fix q_0 in M. Then the space $\Omega^{\mathcal{H}}_{q_0} \cap H^1 \times L^2(I, \mathcal{H}_{|U})$ of horizontal systems whose trajectories start at q_0 and remain in U is a smooth submanifold of $H^1 \times L^2(I, \mathcal{H}_{|U})$, diffeomorphic to an open subset $\mathcal{U}$ of $L^2(I, \mathrm{H})$ through*

$$u \in \mathcal{U} \subset L^2(I, \mathrm{H}) \mapsto (q(u), u) \in H^1(I, U) \times L^2(I, \mathrm{H}) \simeq H^1 \times L^2(I, \mathcal{H}_{|U}),$$

*with the **trajectory map** $u \mapsto q(u)$ obtained by solving the Cauchy problem*

$$q(0) = q_0, \quad \dot{q}(t) = \xi_{q(t)} u(t).$$

Proof. Define $C : H^1 \times L^2(I, \mathcal{H}_{|U}) \to L^2(I, \mathrm{B})$ by

$$C(q, u)(t) = \dot{q}(t) - \xi_{q(t)} u(t).$$

Then $\Omega^h_{q_0} = C^{-1}(\{0\})$. Now fix $(q, u) \in \Omega^{\mathcal{H}}_{q_0}$. The operator $\partial_q C(q, u) : H^1(I, \mathrm{B}) \to L^2(I, \mathrm{B})$ is given by

$$(\partial_q C(q, u).\delta q)(t) = \dot{\delta q}(t) - (\partial_q \xi_{q(t)} u(t)).\delta q(t), \quad \delta q \in H^1(I, \mathrm{B}).$$

This is obviously a Banach isomorphism: for any $a \in L^2(I, \mathrm{B})$, the equation

$$\delta q(0) = 0, \quad (\partial_q C(q, u).\delta q)(t) = a(t), \quad \text{a.e. } t \in I$$

is just a linear Cauchy problem in δq, and hence admits a unique global solution $\delta q = \partial_q C(q, u)^{-1} a$. The implicit function theorem then states that $\Omega^{\mathcal{H}}_{q_0} \cap \mathcal{U} = C^{-1}(\{0\})$ is the graph of a smooth mapping $u \mapsto q(u)$ from an open subset of $L^2(I, \mathrm{H})$ onto $H^1(I, M)$. □

Consequently, we will often identify $\Omega^{\mathcal{H}}_{q_0} \cap \mathcal{U}$ with a subset of $L^2(I, \mathrm{H})$, and a control u with the corresponding horizontal system $(q(u), u)$. For the rest of this section, unless stated otherwise, we will only consider horizontal systems that belong in such a set $\mathcal{U}$.

3.2 Endpoint Mapping and Critical Points of the Action

In this section, we fix q_0 and only consider horizontal systems that remain in $\mathcal{U}$. The endpoint mapping is defined in the following trivial corollary to Proposition 1.

Corollary 1. *The so-called* ***Endpoint map*** $E : (q,u) \in \Omega_{q_0} \cap \mathcal{U} \mapsto q(u)(1)$ *is smooth. Its derivative at u in the direction $\delta u \in L^2(I,\mathrm{H})$ is equal to $\delta q(1)$, where $(q,\delta q) \in H^1(I,TM)$ and δq is obtained by solving the linear Cauchy problem*

$$\delta q(0) = 0, \quad \dot{\delta q}(t) = \partial_q(\xi_{q(t)}u(t)).\delta q(t) + \xi_{q(t)}\delta u(t).$$

Definition 11. *We say that a control u is a critical point of the action with fixed endpoint if, for any $\mathcal{C}^1$ family of controls $s \in (-\varepsilon,\varepsilon) \subset \mathbb{R} \mapsto u_s$ such that $q(u_s)(1) = q_1$ for each s and $u_0 = u$, we have*

$$\partial_s(A(u_s))_{|s=0} = 0.$$

Looking for local geodesics between q_0 and q_1 means solving the smooth constrained optimal control problem of minimizing

$$A(u) = A(q,u) = \frac{1}{2}\int_0^1 g_{q(t)}(u(t),u(t))dt$$

among all (q,u) in $\Omega_{q_0}^{\mathcal{H}}$ such that $E(q,u) = q_1$. Hence, we see that any local geodesic is such a critical point.

The three types of sub-Riemannian geodesics. Before we move on, we need to discuss the apparition in infinite dimension of a new type of geodesics, called *elusive geodesics*. They were introduced for the first time in [7].

Fix a minimizing geodesic $(q,u) = (q(u),u)$, which we identify with the corresponding optimal control u. Then, the submersion theorem implies that the smooth map $F = (A,E) : \Omega_{q_0}^{\mathcal{H}} \to \mathbb{R} \times M$ must have a derivative that is not onto. We have two possibilities:

1. The range of $dF(u)$ has positive codimension in $\mathbb{R} \times T_{q_1}M$, that is, its closure is a proper subset of $T_{q_1}M$.
2. The range of $dF(u)$ is a proper dense subset of $\mathbb{R} \times T_{q_1}M$. This can only happen when M is infinite dimensional.

Using a cotangent viewpoint, these condition can be reformulated as:

1. There exists $(\lambda,p_1) \in \{0,1\} \times T_{q_1}^*M \setminus \{(0,0)\}$ such that

   $$\lambda dA = dE(u)^*p_1,$$

   where $dE(u)^* : T_{q_1}^*M \to L^2(I,\mathrm{H})^*$ is the adjoint map of $dE(u)$. Depending on the value of λ, this splits into two subcases:

   (a) *The normal case:* $\lambda = 1$, which gives $dA = dE(u)^*p_1$. This corresponds to a *normal geodesic*, from which we will derive the Hamiltonian flow later in the section.

   (b) *The abnormal case:* $\lambda = 0$, which gives $0 = dE(u)^*p_1$ and $p_1 \neq 0$. This implies that u is an *abnormal control* (i.e., a critical point of the endpoint map), and we say that (q,u) is an *abnormal geodesic*. While there is no characterization of abnormal geodesics, even in finite dimensions, there is a nice Hamiltonian characterization yielding all abnormal controls [27, Chapter 5]. We will give the infinite dimensional version of this result.

2. $dF(u)^*$ is one-to-one, which is no different from the case of non minimizing curves. This gives no useful Hamiltonian characterization. We say that (q, u) is an *elusive geodesic*.

This elusive case is the reason why there is no Pontryagin principle in infinite dimensions [23]. It is actually a very common occurence in infinite dimensional sub-Riemannian geometry. For example, any curve in the ℓ^2 product of Heisenberg groups with no constant component is elusive.

Remark 16. As discussed in [5, 7, 9], an interpretation of this phenomenon is that the topology induced by the sub-Riemannian distance is much finer than the manifold topology. Hence, there are not enough Lagrange multipliers p_1. However, if the sub-Riemannian structure can be restricted to a smooth dense embedded submanifold $M' \subset M$, that is, a manifold modelled on a Banach space B' with dense and continuous inclusion in B, such that M' contains $\mathcal{O}_{q_0}$, then $T^*_{q_1}M \subsetneq T^*_{q_1}M'$, and we obtain additional multipliers, which turns some elusive curves into additional normal and abnormal extremums that are more easily characterized.

This is the case when $M = \ell^2(\mathbb{N}, \mathbb{R}^3)$ is the ℓ^2-product of Heisenberg groups , where one can restrict the structure to $M' = \ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^1(\mathbb{N}, \mathbb{R})$. Then $T^*_0 M' = \ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^\infty(\mathbb{N}, \mathbb{R})$, which is much bigger than $T^*_0 M = \ell^2(\mathbb{N}, \mathbb{R}^3)$.

The question of finding the "right" tangent bundle, that is, one for which there are no elusive geodesics, is open an would probably require more powerful and innovative tools to solve. For example, the structure described in Section 2.2.3 seems to indicate that the correct dense submanifold to which we should restrict the structure would be $\ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^1(\mathbb{N}, \mathbb{R}) \times \ell^{2/3}(\mathbb{N}, \mathbb{R})$. However, this is not a Banach space as it is not locally convex, and therefore has a dual space that is too small.

So there are no first order necessary conditions for a control to yield a geodesic in infinite dimensions. However, there is a partial converse to this result which does remains true. Then the following result is immediate.

Lemma 2. *Fix a control $u \in \mathcal{U}$.*

1. *If there exists $p_1 \in T^*_{q_1}M$ such that $dA(u) = dE(u)^* p_1$, then u is a critical point of the action with fixed endpoints.*

2. *If there exists $p_1 \in T^*_{q_1}M \setminus \{0\}$ such that $0 = dE(u)^* p_1$, then u is a abnormal control.*

To obtain a workable version of these conditions, we need to compute, for any control u and $p_1 \in T^*_{q_1}M$, a good expression for $\lambda dA(u) - dE(u)^* p_1$, with $\lambda = 0, 1$. This is given by a Hamiltonian formulation. Let us begin with a few reminders on weak symplectic structures.

3.3 Symplectic Gradient and Partial Symplectic Gradient

Recall that a 2-form ω on a Banach manifold N is said to be *weak symplectic* if it is closed, and if the linear mapping $v \in T_x N \mapsto \omega_x(v, \cdot) \in T^*_x N$ is one-to-one for each x in N.

We now fix a Banach manifold M, modelled on a Banach space B. Let ω be the canonical weak symplectic form on T^*M. Recall that ω is a closed 2-form on T^*M defined in canonical coordinates by

$$\omega_{q,p}(\delta q, \delta p; \delta q', \delta p') = \delta p(\delta q') - \delta p'(\delta q).$$

Remark 17. For us, "in canonical coordinates" will mean in a chart $\Psi : T^*M_{|U} \to \psi(U) \times \mathrm{B}^*$ of the form $\Psi(q, p) = (\psi(q), d\psi(q)_* p)$, with $\psi : U \mapsto \psi(U) \subset \mathrm{B}$ a coordinate chart. We then identify $T^*M_{|U} \simeq \psi(U) \times \mathrm{B}^*$ so that $(q, p) \simeq (\psi(q), d\psi(q)_* p)$ for readability.

Symplectic gradient of a function. Take a smooth function $f : T^*M \to \mathbb{R}$, and let $(q,p) \in T^*M$. We say that f admits a *symplectic gradient* at (q,p) if there exists a vector $\nabla^\omega f(q,p) \in TT^*M$ such that

$$df(q,p) = \omega(\nabla^\omega f(q,p), \cdot).$$

In infinite dimensions, not every smooth function admits a symplectic gradient (unless B is reflexive), see [21, Section 48] for example.

Now the partial derivative of f along the fiber $\partial_p f(q,p)$ is defined intrinsically:

$$\partial_p f(q,p)(\delta p) = \frac{d}{dt}\left(f(q,p+t\delta p)\right)_{|t=0}, \quad \delta p \in T_q^*M.$$

It belongs to $(T_q^*M)^* = T_q^{**}M$.

Denote by j the canonical dense inclusion $T_qM \hookrightarrow T_q^{**}M$:

$$j_q(v)(p) = p(v), \quad v \in T_qM, \quad p \in T_q^*M.$$

If $\partial_p f(q,p)$ belongs to $j(T_qM)$, it can then be identified to the vector $j^{-1}(\partial_p f(q,p)) \in T_qM$, which we also denote $\partial_p f(q,p)$.

In this case, f does admit a symplectic gradient $\nabla^\omega f(q,p)$ at (q,p), as this gradient is given in canonical coordinates by the formula

$$\nabla^\omega f(q,p) = (\partial_p f(q,p), -\partial_q f(q,p)) \in T_qM \times T_q^*M \simeq T_{(q,p)}T^*M.$$

Remark 18. While $\partial_q f(q,p)$ is not defined canonically, $\nabla^\omega f(q,p)$ is.

Restriction of ω to a dense cotangent sub-bundle.

Definition 12. *A smooth dense cotangent sub-bundle $\tau M \subset T^*M$ is a subset of T^*M that is also a Banach fiber bundle over M, with fibers modelled on a Banach space $\tau\mathrm{B}$ with dense continuous inclusion $\tau\mathrm{B} \hookrightarrow \mathrm{B}^*$, such that around any point, there is a canonical coordinate system $\Psi : T^*M_{|U} \to U \times \mathrm{B}^*$ such that $\Psi(\tau M_{|U}) = U \times \tau\mathrm{B}$*

Do note that this implies $T^{**}M \subset \tau^*M$.

Example 8. Fix N a compact manifold of dimension d, and consider for any $s > 1 + d/2$ the space $\mathcal{D}^s(N)$ of diffeomorphisms of class H^s. Then $(T^*\mathcal{D}^s(N))_{|\mathcal{D}^{s+1}(N)} \subset T^*\mathcal{D}^{s+1}(N)$ is a smooth dense cotangent sub-bundle over $\mathcal{D}^{s+1}(N)$.

We have the following trivial (but crucial) lemma.

Lemma 3. *The restriction of ω to τM, i.e., its pull-back through the inclusion map $\tau M \to T^*M$, is a weak symplectic form on τM. We still denote it ω.*

Proof. The 2-form ω on τM is closed as the pull-back of a closed 2-form. It is not degenerate because of the density of τM in T^*M. □

Canonical coordinates for τM will simply be restrictions to τM of canonical coordinates for T^*M. A smooth map $f : \tau M \to \mathbb{R}$ then admits a symplectic gradient at $(q,p) \in \tau M$ if and only if, in canonical coordinates, the following conditions are satisfied.

1. The partial derivative of f along the fiber $\partial_p f(q,p) \in \tau_q^*M$ belongs to the image of the canonical embedding $j : T_qM \hookrightarrow T_q^{**}M \subset \tau^*M$. If so, it can be identified to a vector $\partial_p f(q,p) \in T_qM$.

2. In a canonical chart, $\partial_q f(q,p)$ (which, in the chart, is an element of T_q^*M) actually belongs to the dense subspace $\tau_q M$.

In this case, in those coordinates, we can indeed write

$$\nabla^\omega f(q,p) = (\partial_p f, -\partial_q f) \in T_{(q,p)}\tau M \subset T_{(q,p)}T^*M,$$

which is a stable property under a change of canonical coordinates.

Symplectic Partial Gradient of a Function. We now consider a vector bundle $\mathcal{E}$ on M, and a smooth dense cotangent sub-bundle $\tau M \subset T^*M$. We denote by $\tau M \oplus_M \mathcal{E}$ the vector bundle with fiber $(\tau M \oplus_M \mathcal{E})_q = \tau_q M \times \mathcal{E}_q$. Let $f : \tau M \oplus_M \mathcal{E} \to \mathbb{R}$ be a smooth function. In this case, both $\partial_p f$ and $\partial_e f$ are intrinsically defined.

Now fix $(q,p,e) \in \tau M \oplus \mathcal{E}$ and consider canonical coordinates around q in M, and a local trivialization of $\mathcal{E}$. Also assume that $\partial_p f(q,p,e)$ belongs to $j(T_q M)$.

If $\partial_e f(q,p,e) = 0$, f admits an intrinsic partial differential with respect to (q,p) at (q,p,e). Indeed, for any smooth section $e : T^*M_{|U} \mapsto (T^*M \oplus \mathcal{E})_{|U}$ with $e(q,p) = e$, we have, for every $(\delta q, \delta p) \in T_{(q,p)}\tau M$

$$\begin{aligned} d(f(q,p,e(q,p)))(\delta q, \delta p) &= \partial_{(q,p)} f(q,p,e)(\delta q, \delta p) + \underbrace{\partial_e f(q,p,e)}_{=0}.de(q,p)(\delta q, \delta p) \\ &= \partial_{(q,p)} f(q,p,e). \end{aligned}$$

Then, we can look for the partial symplectic gradient $\nabla^\omega f(q,p,e)$ of f at (q,p,e). It is such that for any smooth section $e : T^*M_{|U} \mapsto (T^*M \oplus \mathcal{E})_{|U}$ with $e(q,p) = e$, we have

$$d(f(q,p,e(q,p))) = \omega(\nabla^\omega f(q,p,e), \cdot).$$

It exists if and only if in some canonical coordinates and some trivialization of $\mathcal{E}$, $\partial_p f(q,p,e) \in j(T_q M)$, and $\partial_q f(q,p,e) \in \tau M$. It is then given in canonical coordinates by

$$\nabla^\omega f(q,p,e) = (\partial_p f(q,p,e), -\partial_q f(q,p,e)).$$

3.4 Hamiltonian Formulation

We keep the same notations as in the previous sections. We define the Hamiltonian $H^\lambda : T^*M \oplus_M \mathcal{H} \to \mathbb{R}$, $\lambda = 0, 1$, of the problem by the smooth expression

$$H^\lambda(q,p,u) = p(\xi_q u) - \frac{\lambda}{2} g_q(u,u).$$

Notice that the (intrinsically defined) partial derivative of H^λ in p satisfies

$$\partial_p H^\lambda(q,p,u).\delta p = \delta p(\xi_q u), \quad \delta p \in T^*qM,$$

so that $\partial_p H^\lambda(q,p,u)$ can be identified to $\xi_q u$ through the canonical inclusion $T_q M \to T_q^{**}M$. Consequently, when $\partial_u H^\lambda(q,p,u) = 0$, H^λ admits a partial symplectic gradient $\nabla^\omega H^\lambda(q,p,u) \in T_{(q,p)}T^*M$, given in canonical coordinates by

$$\nabla^\omega H^\lambda(q,p,u) = (\partial_p H(q,p,u), -\partial_q H(q,p,u)).$$

Proposition 2. *Fix a control u in a local space of controls $\mathcal{U}$ such that $(q,u) = (q(u),u) \in \Omega^{\mathcal{H}}_{q_0}$ with $q(1) = q_1$. Then*

$$\lambda dA(u) = dE(u)^* p_1, \quad (\lambda, p_1) \in \{0,1\} \times T^*_{q_1} M \setminus \{(0,0)\}, \tag{6}$$

if and only if

$$\exists t \in I \mapsto p(t) \in T^*_{q(t)} M \text{ of class } H^1,\ p(1) = p_1, \text{ and, a.e. } t \in I,$$
$$\begin{cases} 0 & = \partial_u H^\lambda(q(t), p(t), u(t)), \\ (\dot{q}(t), \dot{p}(t)) & = \nabla^\omega H^\lambda(q(t), p(t), u(t)). \end{cases} \tag{7}$$

In this case, (q,u) is automatically a critical point of the action with fixed endpoints when $\lambda = 1$, and a critical point of the endpoint map (i.e., u is a abnormal control) when $\lambda = 0$.

Remark 19. Note that critical points, abnormal controls, and Condition(6) all require the local viewpoint we adopted. We must restrict ourselves to horizontal systems in some subset $\mathcal{U}$. However, that is not the case for (7). We can then simply use condition (7) to identify geodesics and abnormal curves, even in the global viewpoint.

The proof was given in [7] for the special case of strong structures on groups of diffeomorphisms. The general proof is almost the same.

Proof. The proof is the same as in finite dimensions. Fix $u \in L^2(0,1;\mathrm{H})$, q the corresponding trajectory, and $(\lambda, p_1) \in \{0,1\} \times T^*_{q_1} M \setminus \{(0,0)\}$. Take $\delta u \in L^2(0,1;\mathrm{H})$. We have $dE(u).\delta u = \delta q(1)$, with $\delta q \in H^1(I, \mathrm{B})$ solution of

$$\delta q(0) = 0, \quad \dot{\delta q}(t) = \partial_q(\xi_{q(t)} u(t)) + \xi_{q(t)} \delta u(t).$$

Hence

$$\lambda dA(u) - dE(u)^* p_1 = \int_0^1 \left(\lambda g_{q(t)}(u(t), \delta u(t)) + \frac{\lambda}{2} \partial_q (g_{q(t)}(u(t), u(t))).\delta q(t) \right) dt - p_1(\delta q(1)). \tag{8}$$

Now let $t \mapsto p(t) \in T^*_{q(t)} M$ solve the linear Cauchy problem

$$p(1) = p_1, \quad \dot{p}(t) = -\partial_q H^\lambda(q(t), p(t), u(t)) = -\partial_q(\xi_{q(t)} u(t))^* p(t) + \frac{\lambda}{2} \partial_q g_{q(t)}(u(t), u(t)).$$

Then $\frac{\lambda}{2} \partial_q (g_{q(t)}(u(t), u(t))) = \dot{p}(t) + \partial_q(\xi_{q(t)} u(t))^* p(t)$ so that a term $\dot{p}(t)\delta q(t)$ appears in the right-hand side of (8). An integration by part on this term, and the fact that $\delta q(0) = 0$ will yield

$$\lambda dA(u) - dE(u)^* p_1 = \int_0^1 \left(\lambda g_{q(t)}(u(t), \delta u(t)) + p(t)(\partial_q(\xi_{q(t)} u(t)).\delta q(t)) - p(t)(\dot{\delta q}(t)) \right) dt.$$

But replacing $\dot{\delta q}(t)$ with $\partial_q(\xi_{q(t)} u(t)) + \xi_{q(t)} \delta u(t)$ finally gives us

$$\lambda dA(u) - dE(u)^* p_1 = \int_0^1 \left(\lambda g_{q(t)}(u(t), \delta u(t)) - p(t)(\xi_{q(t)} \delta u(t)) \right) dt = -\int_0^1 \partial_u H^\lambda(q(t), p(t), u(t)) dt.$$

In particular,

$$\lambda dA(u) - dE(u)^* p_1 \quad \Longleftrightarrow \partial_u H^\lambda(q(t), p(t), u(t)) = 0 \text{ a.e. } t \in I.$$

□

3.5 Hamiltonian Geodesic flow

We now investigate the existence of a Hamiltonian flow for the normal geodesics. We will find that, much like in the Riemannian case, strong structures always admit such a flow, while additional assumptions are required for weak sub-Riemannian manifolds.

The strong case. We assume for now that the sub-Riemannian structure is strong. In this case, because each g_q is a Hilbert product, the equation

$$\partial_u H^1(q,p,u) = 0 = \xi_q^* p - g_q(u,\cdot)$$

has a unique solution $u(q,p) = G_q^{-1}\xi_q^* p$ for any $(q,p) \in T^*M$ (Riesz representation theorem). Here G_q^{-1} is the smooth inverse of the smooth vector bundle isomorphism $G : u \in \mathcal{H}_q \mapsto g_q(u,\cdot) \in \mathcal{H}_q^*$, also called the musical operator. This lets us define the *normal Hamiltonian* of the structure $h : T^*M \to \mathbb{R}$ by

$$h(q,p) = H^1(q,p,u(q,p)) = \frac{1}{2} g_q(u(q,p),u(q,p)). \tag{9}$$

Do note that, since H^1 is strictly concave in u, we can also write $h(q,p) = \max_{u\in\mathcal{H}_q} H^1(q,p,u)$.

Now thanks to the fact that $\partial_u H^1(q,p,u(q,p)) = 0$, h admits a smooth symplectic gradient given by

$$\nabla^\omega h(q,p) = \nabla^\omega H^1(q,p,u(q,p)), \quad (q,p) \in T^*M.$$

This gradient can be integrated into a well-defined smooth local flow that we call the *Hamiltonian geodesic flow.*

Theorem 6 (Hamiltonian geodesic flow: strong case)**.** *On a strong sub-Riemannian manifold, the normal Hamiltonian is well-defined, and for any $(q_0,p_0) \in T^*M$, there is a unique maximal solution to the normal Hamiltonian equation*

$$(\dot{q}(t),\dot{p}(t)) = \nabla^\omega h(q(t),p(t)).$$

More importantly, any such solution $t \mapsto (q(t),p(t))$ projects to a ***geodesic*** *$q(\cdot)$ on M, with control given by*

$$u(t) = u(q(t),p(t)) = G_{q(t)}^{-1}\xi_{q(t)}^* p(t).$$

We prove this theorem and the corresponding one for the weak case simultaneously in the next section.

The weak case and adapted cotangent sub-bundles. Some extra difficulties can appear when the metric is weak. More precisely, the equation $\partial_u H = 0$ may not have a solution for every $(q,p) \in T^*M$, so that the normal Hamiltonian may not be defined. Hence, we need to restrict ourselves to a subspace on which it is well-defined. We will need the following definitions.

Definition 13. *A dense Banach sub-bundle τM of T^*M is said to be* ***adapted*** *to a sub-Riemannian structure $(\mathcal{H},\xi,g)$ on M if, for every $q \in M$ and every $p \in \tau_q M$, there exists $u(q,p) \in \mathcal{H}_q$ such that*

$$\xi_q^* p = g_q(u(q,p),\cdot).$$

Note that such a $u(q,p)$ is always uniquely determined by q and p (and linear in p). Indeed, g_q is positive definite, so $u \mapsto g_q(u,\cdot)$ is injective.

Example 9. Some cases where such a τM can easily be constructed are the following.

- If g is a strong metric, then T^*M itself is of course adapted to the structure.
- In the case of a weak Riemannian structure $(\mathcal{H}, \xi) = (TM, \mathrm{Id}_{TM})$, the only adapted dense sub-bundle is $\tau M = g(TM, \cdot)$.
- More generally, if $\mathcal{H}$ is a closed subbundle of TM and g the restriction of a weak Riemannian metric to $\mathcal{H}$, one can still take $\tau M = g(TM, \cdot)$. This is the relative cotangent bundle used to find the geodesic equations in [18].

A dense sub-bundle τM is adapted to the structure means that the normal Hamiltonian can be defined as in (9) on τM. Indeed, the restriction of H^1 to τM is

$$H^1(q,p,u) = p(\xi_q u) - \frac{1}{2} g_q(u,u) = g_q\left(u(q,p) - \frac{1}{2}u, u\right),$$

so that the normal Hamiltonian

$$h(q,p) = \max_{u \in \mathcal{H}_q}(H^1(q,p,u)) = h(q,p,u(q,p)) = \frac{1}{2} g_q(u(q,p), u(q,p)) = p(\xi_q u(q,p)) \tag{10}$$

is well-defined on τM. We can now state our main theorem.

Theorem 7. *Let $(\mathcal{H}, \xi, g)$ be a weak sub-Riemannian structure on a Banach manifold M. Let $\tau M \hookrightarrow T^*M$ be an adapted dense cotangent sub-bundle, on which the normal Hamiltonian $h : \tau M \to \mathbb{R}$ is therefore well-defined. Assume that h admits a symplectic gradient $\nabla^\omega h$ on τM of class at least $\mathcal{C}^2$. For $(q_0, p_0) \in \tau M$, let $(q(\cdot), p(\cdot)) : I \to \tau M$ be the unique maximal solution to the **Hamiltonian geodesic equation***

$$(\dot{q}(t), \dot{p}(t)) = \nabla^\omega h(q(t), p(t)), \quad t \in I.$$

In canonical coordinates, this equation reads

$$\begin{cases} \dot{q}(t) = \xi_{q(t)} u(t), \\ \dot{p}(t) = \dfrac{1}{2} \partial_q g_{q(t)}(u(t), u(t)) - (\partial_q \xi_{q(t)} u(t))^* p(t), \end{cases}$$

with $u(t) = u(q(t), p(t))$ the unique element of $\mathcal{H}_{q(t)}$ such that $\xi^*_{q(t)} p(t) = g_{q(t)}(u(t), \cdot)$.

*Then the horizontal system $(q(\cdot), u(\cdot)) : I \to \mathcal{H}$ is a **local geodesic**.*

Remark 20. As discussed previously, the normal Hamiltonian admits a symplectic gradient when, in some canonical coordinates,

$$\partial_q h(q,p) = \frac{1}{2} \partial_q g_q(u(q,p), u(q,p)) - (\partial_q \xi_q u(q,p))^* p,$$

belongs to $\tau_q M$.

Remark 21. In the weak Riemannian case $(\mathcal{H}, \xi) = (TM, \mathrm{Id})$, the hypothesis is equivalent to the existence of a smooth Levi-Civita connection, see for exemple [25]. It is therefore a very natural condition to impose.

Remark 22. The fact that h admits a symplectic gradient for the relative cotangent bundle is equivalent to the assumption made in [18, Theorem 1] on the existence of a transpose operator.

Example 10. When the metric g is strong, the hypothesis is automatically satisfied for $\tau M = T^*M$, and we get Theorem 6.

Proof. First of all, if $\xi_{q_0}^* p_0 = \xi_{q(0)} p(0) = 0$, then q and p are constant curves, and therefore q is a trivial geodesic. We can therefore assume $\xi_{q_0}^* p_0 \neq 0$. Since $g(u(0), \cdot) = \xi_{q_0}^* p_0$, we also have $u_0 = u(0) \neq 0$.

Now since $\dot{q}(t) = \xi_{q(t)} u(t)$, q is obviously horizontal, so we just need to prove that it is a local geodesic. We can assume that $I = [0, 1]$ without loss of generality. We need to prove that for $\varepsilon > 0$ small enough, and any horizontal system $(\tilde{q}(\cdot), \tilde{u}(\cdot)) : [0, \varepsilon] \to \mathcal{H}$ such that $\tilde{q}(0) = q_0$ and $\tilde{q}(\varepsilon) = q(\varepsilon)$, we have

$$L((q, u)_{|[0,\varepsilon]} \leqslant L(\tilde{q}, \tilde{u}).$$

For this, we will find a *calibration* of q: 1-form $\theta \in \Gamma((T^*M)_{|U})$ on a neighbourhood U of $q_{|[0,\varepsilon]}$ in M that satisfies the following conditions.

- C1: for every $t \geqslant 0$ small enough,

$$\theta_{q(t)}(\dot{q}(t)) = c\sqrt{2h(q(t), p(t))} = c\sqrt{g_{q(t)}(u(t), u(t))}$$

  with $c > 0$ a fixed constant.

- C2: for every $(\tilde{q}, \tilde{u}) \in \mathcal{H}$ with $\tilde{q}$ close enough to q_0,

$$|\theta_{\tilde{q}}(\xi_{\tilde{q}} \tilde{u})| \leqslant c\sqrt{g_{\tilde{q}}(\tilde{u}, \tilde{u})}.$$

- C3: θ is exact near q_0.

Indeed, once θ is found, for $\varepsilon > 0$ small enough, and for any horizontal system $(\tilde{q}(\cdot), \tilde{u}(\cdot))$ in a small enough neighbourhood of $q([0, \varepsilon])$, with $\tilde{q}(0) = q_0$ and $\tilde{q}(\varepsilon) = q(\varepsilon)$,

$$\begin{aligned}
L((q, u)_{|[0,\varepsilon]}) &= \int_0^\varepsilon \sqrt{g_{q(t)}(u(t), u(t))} \\
&\underset{\text{C1}}{=} \frac{1}{c} \int_0^\varepsilon \theta_{q(t)}(\dot{q}(t)) dt \\
&\underset{\text{C3}}{=} \frac{1}{c} \int_0^\varepsilon \theta_{\tilde{q}(t)}(\dot{\tilde{q}}(t)) dt \\
&\underset{\text{C2}}{\leqslant} \int_0^\varepsilon \sqrt{g_{\tilde{q}(t)}(\tilde{u}(t), \tilde{u}(t))} dt \\
&= L(\tilde{q}, \tilde{u}).
\end{aligned}$$

We now build this calibration.

We work in canonical coordinates in a neighbourhood $U \subset M$ of q_0 in M, so that we can consider a trivialization

$$(\tau M \underset{M}{\oplus} \mathcal{H})_{|U} \simeq U \times \tau \mathrm{B} \times \mathrm{H} \subset U \times \mathrm{B}^* \times \mathrm{H},$$

as per Definition 12 on cotangent sub-bundles. We denote the local $\mathcal{C}^2$-flow of $\nabla^\omega h$ on $(\tau M)_{|U} \simeq U \times \tau \mathrm{B}$ by

$$(t, q, p) \mapsto \Phi(t, q, p) = (\Phi_M(t, q, p), \Phi_\tau(t, q, p)) \tag{11}$$

Note that by definition of Φ, $\Phi(t, q_0, p_0) = (q(t), p(t))$ for t small enough and $p_0 = p(0)$.

In our coordinate system, p_0 belongs to $\tau \mathrm{B}$ which is a subspace of B^*. We consider its kernel $\ker p_0$, which is a closed hyperplane of B. Let

$$U_0 = U \cap (q_0 + \ker p_0).$$

Note that U_0 is a neighbourhood of q_0 in $q_0 + \ker p_0$.

Reducing U_0 if needed, we can then define the map $\varphi :]-\varepsilon, \varepsilon[\times U_0 \to U$ of class $\mathcal{C}^2$ by

$$\varphi(t, \tilde{q}_0) = \Phi_M \left(t, \tilde{q}_0, \sqrt{\frac{h(q_0, p_0)}{h(\tilde{q}_0, p_0)}} p_0 \right),$$

with Φ_M as in (11). Do note that $h(q_0, p_0) = p_0(\xi_{q_0} u_0) = g_{\tilde{q}_0}(u_0, u_0) > 0$, so we can assume $h(\cdot, p_0) \neq 0$ on U_0 by continuity of h. The positive number $n(\tilde{q}_0) = \sqrt{\frac{h(q_0,p_0)}{h(\tilde{q}_0,p_0)}}$ is here to ensure

$$h(\tilde{q}_0, n(\tilde{q}_0)p_0) = h(q_0, p_0), \quad q \in U_0.$$

For $\tilde{q}_0 \in U_0$, the curve $t \mapsto \varphi(t, \tilde{q}_0)$ is the projection to M of the Hamiltonian flow starting at $\tilde{q}_0$ with initial condition $p(0) = n(\tilde{q}_0)p_0$.

Lemma 4. *Reducing U_0 if necessary, there exists $\varepsilon > 0$ such that the mapping φ is a local diffeomorphism of $(-\varepsilon, \varepsilon) \times U_0$ onto a neighbourhood of q_0.*

Proof. We just need to prove that $d\varphi(0, q_0)$ is bijective. For any $\delta\tilde{q} \in \ker p_0$, we have

$$\partial_{\tilde{q}_0} \varphi(0, q_0)\delta\tilde{q}_0 = \partial_q \Phi_M(0, q_0, p_0)\delta\tilde{q}_0 = \delta\tilde{q}_0,$$

so

$$\partial_{\tilde{q}_0} \varphi(0, q_0) = \mathrm{Id}_{\ker p_0}.$$

This is because $\Phi = (\Phi_M, \Phi_\tau)$ is the flow of a vector field, so $\partial_{(q,p)}\Phi(0, q, p) = (\mathrm{Id}_{\mathrm{B}}, \mathrm{Id}_{\tau\mathrm{B}})$. Now we just need to prove that $\partial_t \varphi(0, q_0)$ does not belong to $\ker p_0$. But

$$p(\partial_t \varphi(0, q_0)) = p(\partial_t \Phi_M(0, q_0, p_0)) = p(\dot{q}(0)) = p(\xi_{q_0} u_0) = 2h(q_0, p_0) \neq 0.$$

□

Now we reduce U to $\varphi(]-\varepsilon, \varepsilon[\times U_0)$. This mapping is of class at least $\mathcal{C}^2$, since φ itself is $\mathcal{C}^2$. This lets us define on U the one-form θ such that

$$\theta(\varphi(t, \tilde{q}_0)) = \Phi_\tau(t, \tilde{q}_0, n(\tilde{q}_0)p_0), \quad t \in (-\varepsilon, \varepsilon), \quad \tilde{q}_0 \in U_0,$$

with Φ_τ defined as in (11). In other words, θ is given by the propagation by the flow of $\nabla^\omega h$ to U of $n(\tilde{q}_0)p_0$ on U_0.

Let us prove that θ is a calibration of $q(\cdot)$. Fix $\tilde{q}$ in U. For $\tilde{q}$ in U, let $(\tilde{t}, \tilde{q}_0) = \varphi^{-1}(\tilde{q})$. Then, by construction, we have

$$\theta(\tilde{q}) = \tilde{p}(\tilde{t}),$$

where $(\tilde{q}(t), \tilde{p}(t))$ satisfies the Hamiltonian geodesic equations with $(\tilde{q}(0), \tilde{p}(0)) = (\tilde{q}_0, n(\tilde{q}_0)p_0)$. Therefore, for any $\tilde{u}$ in $\mathcal{H}_{\tilde{q}}$,

$$|\theta(\tilde{q})(\xi_{\tilde{q}}\tilde{u})| = |\tilde{p}(\tilde{t})\xi_{\tilde{q}}\tilde{u}| = |g_{\tilde{q}}(u(\tilde{q}, \tilde{p}(\tilde{t})), \tilde{u})| \leqslant \underbrace{\sqrt{g_{\tilde{q}}(u(\tilde{q}, \tilde{p}(\tilde{t})), u(\tilde{q}, \tilde{p}(\tilde{t})))}}_{=\sqrt{2h(\tilde{q}, \tilde{p}(\tilde{t}))} \text{ (see (10))}} \sqrt{g_{\tilde{q}}(\tilde{u}, \tilde{u})}.$$

As the reduced Hamiltonian is constant along the Hamiltonian flow, we have

$$h(\tilde{q}, \theta(\tilde{q})) = h(\tilde{q}, \tilde{p}(\tilde{t})) = h(\tilde{q}_0, n(\tilde{q}_0)p_0) = h(q_0, p_0),$$

so that with $c = \sqrt{2h(q_0, p_0)} > 0$, we get Condition C2, for any $(\tilde{q}, \tilde{u}) \in \mathcal{H}_{|U}$,

$$|\theta_{\tilde{q}} \xi_{\tilde{q}} \tilde{u}| \leqslant c\sqrt{g_{\tilde{q}}(\tilde{u}, \tilde{u})}.$$

Now consider the case where $\tilde{q} = q(t)$ from our initial Hamiltonian curve $t \mapsto (q(t), p(t))$ with $t \in (-\varepsilon, \varepsilon)$. Since we defined $u(t) = u(q(t), p(t))$, we get Condition C1 since

$$\begin{aligned}\theta_{q(t)}(\dot{q}(t)) = p(t)(\xi_{q(t)} u(t)) &= \underbrace{g_{q(t)}(u(t), u(t))}_{=2h(q(t),p(t))} \\ &= \underbrace{\sqrt{2h(q(t), p(t))}}_{=\sqrt{2h(q_0,p_0)}=c}\sqrt{g_{q(t)}(u(t), u(t))} = c\sqrt{g_{q(t)}(u(t), u(t))}.\end{aligned}$$

Therefore, if we can prove that θ is exact near q_0, we do have that θ calibrates $q(\cdot)$. Since φ is a diffeomorphism, we just need to prove the following lemma.

Lemma 5. *We have $\varphi^*\theta = c^2 dt$ on $]-\varepsilon, \varepsilon[\times U_0$.*

Proof. Fix some $(t_0, \tilde{q}_0)$ in $]-\varepsilon, \varepsilon[\times U_0$, and let $t \mapsto (\tilde{q}(t), \tilde{p}(t))$ follow the Hamiltonian flow with initial condition $\tilde{q}(0) = \tilde{q}_0$ and $\tilde{p}(0) = n(q_0)p_0$. In particular, $\theta(\varphi(t, \tilde{q}_0)) = \tilde{p}(t)$. We will also denote $\tilde{u}(t) = u(\tilde{q}(t), \tilde{p}(t))$ the corresponding control.

Then for every $(\delta t, \delta q_0) \in \mathbb{R} \times \ker p_0$,

$$\begin{aligned}(\varphi^*\theta)_{(t_0,\tilde{q}_0)}(\delta t, \delta q_0) &= \theta(\varphi(t_0, \tilde{q}_0))(\partial_t \varphi(t_0, \tilde{q}_0)\delta t + \partial_{\tilde{q}_0}\varphi(t_0, \tilde{q}_0)\delta q_0) \\ &= \Phi_\tau(t_0, \tilde{q}_0, n(\tilde{q}_0)p_0)(\partial_t \varphi(t_0, \tilde{q}_0)\delta t + \partial_{\tilde{q}_0}\varphi(t_0, \tilde{q}_0)\delta q_0) \\ &= \tilde{p}(t_0)(\partial_t \varphi(t_0, \tilde{q}_0)\delta t) + \tilde{p}(t_0)(\partial_{\tilde{q}_0}\varphi(t, \tilde{q}_0)\delta q_0).\end{aligned} \tag{12}$$

Now recall that $\partial_t \varphi(t, \tilde{q}_0) = \dot{\tilde{q}}(t) = \xi_{\tilde{q}(t)}\tilde{u}(t)$, so that for every time t,

$$\begin{aligned}\tilde{p}(t)(\partial_t \varphi(t, \tilde{q}_0)) &= g_{\tilde{q}(t)}(\tilde{u}(t), \tilde{u}(t)) = 2h(\tilde{q}(t), \tilde{p}(t)) \\ &= 2h(\tilde{q}(0), \tilde{p}(0)) = 2h(\tilde{q}_0, n(\tilde{q}_0)p_0) = 2h(q_0, p_0) = c^2.\end{aligned} \tag{13}$$

Hence, from (12) we get

$$(\varphi^*\theta)_{(t_0,\tilde{q}_0)} = c^2 dt + \partial_{\tilde{q}_0}\varphi(t_0, \tilde{q}_0)^*\tilde{p}(t_0).$$

Now let us check that $\tilde{p}(t)(\partial_{\tilde{q}_0}\varphi(t, \tilde{q}_0)\delta q_0) = 0$ for every δq_0 in $\ker p_0$ and t in $]-\varepsilon, \varepsilon[$. For small $s > 0$ and $t \in]-\varepsilon, \varepsilon[$, denote $\tilde{q}(s, t) = \varphi(t, \tilde{q}_0 + s\delta q_0)$. For each s, $t \mapsto \tilde{q}(s, t)$ is horizontal, with associated control $t \mapsto \tilde{u}(s, t)$ that can be taken $\mathcal{C}^2$ in s and such that $g_{\tilde{q}(s,t)}(\tilde{u}(s, t), \tilde{u}(s, t)) = c^2$ for every (s, t) [2]. Let

$$\delta q(t) = \partial_s \tilde{q}(0, t) = \partial_{\tilde{q}_0}\varphi(t, \tilde{q}_0)\delta q_0$$

and $\delta\tilde{u} = \partial_s \tilde{u}(s, t)_{s=0}$. Since $(t, s) \mapsto \tilde{q}(s, t)$ is of class at least $\mathcal{C}^2$, we have

$$\dot{\delta q}(t) = \partial_{\tilde{q}}(\xi_{\tilde{q}(t)}\tilde{u}(t))\delta q(t) + \xi_{\tilde{q}(t)}\delta\tilde{u}(t).$$

Hence, for any t in $[0, t_0]$

$$\frac{d}{dt}\left(\tilde{p}(t)(\delta q(t))\right) = -\partial_q h(\tilde{q}(t), \tilde{p}(t))\delta q(t) + \tilde{p}(t)\partial_q(\xi_{\tilde{q}(t)}u(t))\delta q(t) + \tilde{p}(t)\xi_{\tilde{q}(t)}\delta\tilde{u}(t).$$

[2] Each $\tilde{q}(s, \cdot)$ is the projection to M of a curve $(\tilde{q}(s, \cdot), \tilde{p}(s, \cdot))$ that follows the Hamiltonian flow with initial condition $(\tilde{q}_0 + s\delta q_0, n(\tilde{q}_0 + s\delta q_0)p_0)$. Then $\tilde{u}(s, t) = u(\tilde{q}(s, t), \tilde{p}(s, t))$, which is $\mathcal{C}^2$ in (t, s). Moreover, $g_{\tilde{q}(s,t)}(\tilde{u}(s, t), \tilde{u}(s, t)) = 2h(\tilde{q}(s, t), \tilde{p}(s, t)) = 2h(q_0, p_0) = c^2$.

But

$$\partial_q h(\tilde{q}(t),\tilde{p}(t)) = \partial_q H^1(\tilde{q}(t),\tilde{p}(t),\tilde{u}(t)) = \tilde{p}(t)\partial_q(\xi_{\tilde{q}(t)}\tilde{u}(t)) - \frac{1}{2}(\partial_q g_{\tilde{q}(t)})(\tilde{u}(t),\tilde{u}(t)),$$

and

$$\tilde{p}(t)\xi_{\tilde{q}(t)}\delta u(t) = g_{\tilde{q}(t)}(\tilde{u}(t),\delta u(t)),$$

so that for any t in $[0,t_0]$,

$$\begin{aligned}\frac{d}{dt}\left(\tilde{p}(t)(\delta\tilde{q}(t))\right) &= \frac{1}{2}\partial_q g_{\tilde{q}(t)}(\tilde{u}(t),\tilde{u}(t))\delta\tilde{q}(t) + g_{\tilde{q}(t)}(\tilde{u}(t),\delta\tilde{u}(t)) \\ &= \partial_s\left(\frac{1}{2}g_{\tilde{q}(s,t)}(\tilde{u}(s,t),\tilde{u}(s,t))\right)_{|s=0} \\ &= \partial_s(c^2)_{|s=0} = 0.\end{aligned}$$

We get that $\tilde{p}(t_0)(\delta\tilde{q}(t_0)) = \tilde{p}(0)(\delta\tilde{q}_0)$. Since $\tilde{p}(0) = n(\tilde{q}_0)p_0$ and δq_0 is in $\ker p_0$, we finally get $\tilde{p}(t_0)(\delta\tilde{q}(t_0)) = 0$, that is,

$$\tilde{p}(t)(\partial_{\tilde{q}_0}\varphi(t,\tilde{q}_0)\delta\tilde{q}_0) = 0.$$

We finally conclude

$$\varphi^*\theta(t_0,\tilde{q}_0) = c^2 dt.$$

Since c is constant and this is true for any $(t_0,\tilde{q}_0)$ in $(-\varepsilon,\varepsilon)\times U_0$, we completed the proof. □

Since φ is a diffeomorphism, θ is indeed exact on $\varphi((-\varepsilon,\varepsilon)\times U_0) = U$, which concludes the proof. □

4 Weak Hamiltonian Geodesic Flows on Groups of Diffeomorphisms

In this section, we give two classes of examples of geodesic flows for weak sub-Riemannian metrics on manifolds that are of particular interest: groups of diffeomorphisms. In order to simplify notations, we will work on diffeomorphisms on some $\mathbb{R}^d$, but the results still hold for diffeomorphisms on compact manifolds (see [7, 17, 31] for more information on the corresponding manifold structures of spaces of mapping between such spaces).

The purpose of these examples, in addition to providing insight on what an adapted cotangent subbundle might be, is to be a starting point to the development of fluid mechanics in sub-Riemannian manifolds, that is, fluids whose particles are subjected to non-holonomic constraints.

Remark 23. The case of strong, right-invariant sub-Riemannian structures on such groups has been treated in [7].

4.1 The space of sobolev diffeomorphisms of class H^s

We work on $\mathbb{R}^d$, $d\in\mathbb{N}^*$. Denote by $\mathcal{D}(\mathbb{R}^d)$ the set of diffeomorphisms of $\mathbb{R}^d$ of class $\mathcal{C}^1$.

We define, for an integer s greater than $d/2+1$, the group of diffeomorphisms $\mathcal{D}^s(\mathbb{R}^d)$ of Sobolev class H^s as the set of all H^s-diffeomorphisms φ of $\mathbb{R}^d$ such that $\varphi - e \in H^s(\mathbb{R}^d,\mathbb{R}^d)$, that is,

$$\mathcal{D}^s(\mathbb{R}^d) = \left(\mathrm{Id}_{\mathbb{R}^d} + H^s(\mathbb{R}^d,\mathbb{R}^d)\right)\cap\mathcal{D}(\mathbb{R}^d).$$

It is an open subset of the Hilbert affine space $\mathrm{Id}_{\mathbb{R}^d} + H^s(\mathbb{R}^d,\mathbb{R}^d)$, and therefore a Hilbert manifold.

$\mathcal{D}^s(\mathbb{R}^d)$ is also a topological group for the composition of maps $(\varphi,\psi)\mapsto\varphi\circ\psi$. The composition on the right $R_\varphi : \psi\mapsto\psi\circ\varphi$ is smooth (and is, in fact, the restriction of an affine continuous map).

However, the composition on the left $L_\varphi : \psi \mapsto \varphi \circ \psi$ and the inversion $\varphi \mapsto \varphi^{-1}$ are only continuous, so this is not a Lie group [16, 17, 28, 31].

Remark 24. If φ actually belongs to $\mathcal{D}^{s+k}(\mathbb{R}^d)$, then L_φ is actually of class $\mathcal{C}^k$. This property allows to give $\mathcal{D}^\infty(\mathbb{R}^d)$ an inverse limit Hilbert Lie group structure [17, 28, 31].

Since the composition on the right is smooth, we can nonetheless define right-invariant vector fields: a vector field

$$\begin{aligned} X : \quad \mathcal{D}^s(\mathbb{R}^d) \to \quad & T\mathcal{D}^s(\mathbb{R}^d) = \mathcal{D}^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}^d) \\ \varphi \mapsto \quad & (\varphi, X(\varphi)) \end{aligned}$$

is called *right-invariant* if, for every φ in $\mathcal{D}^s(\mathbb{R}^d)$,

$$X(\varphi) = X(\mathrm{Id}_{\mathbb{R}^d}) \circ \varphi = R_\varphi X(\mathrm{Id}_{\mathbb{R}^d}).$$

Remark 25. Again, in general, $\varphi \mapsto X(\varphi)$ is only continuous, but if $X(\mathrm{Id}_{\mathbb{R}^d}) \in H^{s+k}(\mathbb{R}^d, \mathbb{R}^d)$, then $\varphi \mapsto X(\varphi)$ is of class $\mathcal{C}^k$.

4.2 The naive H^r-norms

Sub-Riemannian structures on $\mathbb{R}^d$. Consider *smooth* vector fields $f_1, \dots, f_k : \mathbb{R}^d \mapsto \mathbb{R}^d$, with derivatives bounded at every order. Such vector fields induce a sub-Riemannian structure $\mathcal{S} = (f, \mathbb{R}^d \times \mathbb{R}^k, g)$ on $\mathbb{R}^d$ by

$$f_x(u) = u_1 f_1(x) + \cdots + u_k f_k(x), \quad g_x(u,u) = u_1^2 + \cdots + u_k^2, \quad (x,u) \in \mathbb{R}^d \times \mathbb{R}^k,$$

as described in [27].

We want to define a certain sub-Riemannian structure $(\mathcal{H}, \xi, g)$ on $\mathcal{D}^s(\mathbb{R}^d)$ that is induced by $\mathcal{S}$. A natural thing to ask is that for any initial point x in $\mathbb{R}^d$, and any horizontal curve $t \mapsto \varphi(t)$ in $\mathcal{D}^s(\mathbb{R}^d)$, the curve $t \mapsto \varphi(t,x)$ also be horizontal with respect to $\mathcal{S}$. In other words, for every t, x, there is some $u(t,x) = (u_1(t,x), \dots, u_k(t,x)) \in \mathbb{R}^k$ such that

$$\partial_t \varphi(t,x) = f_{x(t)}(u(t,x)) = u_1(t,x) f_1(\varphi(t,x)) + \cdots + u_k(t,x) f_k(\varphi(t,x)).$$

Induced relative tangent space on $\mathcal{D}^s(\mathbb{R}^d)$. This leads us to define the relative tangent space $(\mathcal{H}, \xi)$, with $\mathcal{H} = \mathcal{D}^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}^k)$ and

$$\xi_\varphi(u)(x) = f_{\varphi(x)}(u(x)) = u_1(x) f_1(\varphi(x)) + \cdots + u_k(x) f_k(\varphi(x)), \tag{14}$$

where

$$u = (u_1, \dots, u_k) \in H^s(\mathbb{R}^d, \mathbb{R}^k), \quad x \in \mathbb{R}^d.$$

As the f_is are $\mathcal{C}^\infty$ and have bounded derivatives at every order, $\xi : \mathcal{H} \to T\mathcal{D}^s(\mathbb{R}^d)$ is indeed smooth as mentionned in Remark 25.

Naive H^r-metrics. We now define a family of metrics on this relative tangent space, in order to obtain various sub-Riemannian structures. For $r \in \{0, \dots, s\}$, we define the metric

$$g_r(u,u) = \sum_{i=1}^k \int_{\mathbb{R}^d} u_i(x) L_r u_i(x) dx = \sum_{i=1}^k (L_r u | u)_{H^{s-2r}, H^s},$$

where L_r is the differential operator $1+(-\Delta)^r$, with Δ the Laplace operator on $\mathbb{R}^d$. In particular, $g_r(u,u)$ is (equivalent to) the squared H^r-norm of u, and we can write

$$\begin{aligned} g_r(u,v) &= \sum_{i=1}^{k} \int_{\mathbb{R}^d} \left(u_i(x)v_i(x) + d^r u_i(x) \cdot d^r v_i(x)\right) dx \\ &= \sum_{i=1}^{k} \int_{\mathbb{R}^d} \left(u_i(x)v_i(x) + \sum_{m_1+\cdots+m_d=r} \frac{\partial^r u_i(x)}{\partial x_1^{m_1} \dots \partial x_d{}^{m_d}} \frac{\partial^r v_i(x)}{\partial x_1^{m_1} \dots \partial x_d{}^{m_d}} \right) dx. \end{aligned}$$

Note that g_r is a strong metric if and only if s is an integer and $s=r$.

Now g_r is obviously smooth, so that $(\mathcal{H}, \xi, g)$ is indeed a smooth sub-Riemannian structure on $\mathcal{D}^s(\mathbb{R}^d)$. We now need to find an adapted cotangent bundle when $r<s$ and the metric is weak.

The cotangent bundle. As $\mathcal{D}^s(\mathbb{R}^d)$ is an open subset of $H^s(\mathbb{R}^d,\mathbb{R}^d)$, the identity gives global canonical coordinates, so that

$$T^*\mathcal{D}^s(\mathbb{R}^d) = \mathcal{D}^s(\mathbb{R}^d) \times H^{-s}(\mathbb{R}^d,\mathbb{R}^d).$$

An element $p=(p_1,\dots,p_d)$ of $H^{-s}(\mathbb{R}^d,\mathbb{R}^d) = H^{-s}(\mathbb{R}^d)^d$ can be seen as 1-form on $\mathbb{R}^d$ whose coordinates p_i are distributions that belong to $H^{-s}(\mathbb{R}^d)$, the topological dual space of $H^s(\mathbb{R}^d)$. For $v=(v^1,\dots,v^d) \in H^s(\mathbb{R}^d,\mathbb{R}^d)$, we can write

$$(p|v)_{H^{-s},H^s} = \sum_{i=1}^{d} \left(p_i|v^i\right)_{H^{-s},H^s}.$$

Note that if v is of Sobolev class either H^s or $W^{\infty,s}$, with $s>d/2$, then the pointwise product of any v^i with a function $u \in H^s(\mathbb{R}^d)$ is still in $H^s(\mathbb{R}^d)$. Moreover both multiplications mappings $W^{s,\infty} \times H^s \to H^s$ and $H^s \times H^s \to H^s$ are continuous bilinear mappings. In particular, they are smooth.

Hence, we can define $p \cdot v := \sum p_i v^i \in H^{-s}(\mathbb{R}^d)$ by

$$(p \cdot v|u)_{H^{-s},H^s} = \sum_{i=1}^{d} (p_i|v^i u)_{H^{-s},H^s}.$$

Finding an adapted cotangent bundle. If $r=s$, the metric is strong, and yields a smooth Hamiltonian on $T^*\mathcal{D}^s(\mathbb{R}^d)^s(\mathbb{R}^d)$, whose symplectic gradient is well-defined and smooth. This of course yields smooth normal geodesics.

However, when $r<s$, this regular cotangent bundle is not adapted to the sub-Riemannian structure (no p in $H^{-s}(\mathbb{R}^d,\mathbb{R}^d) \setminus H^{-r}(\mathbb{R}^d,\mathbb{R}^d)$ can be of the form $L_r v$ with v in H^s). Now, let $\varphi \in \mathcal{D}^s(\mathbb{R}^d)$ and

$$p=(p_1,\dots,p_k) \in T^*_\varphi \mathcal{D}^s(\mathbb{R}^d) = H^{-s}(\mathbb{R}^d,\mathbb{R}^d)$$

Let us give a necessary condition for p to belong to an adapted relative cotangent bundle. In other words, a condition for $\xi^*_\varphi p$ to be of the form $g_r(u,\cdot)$.

Let us start by computing $\xi^*_\varphi p$. For $u=(u_1,\dots,u_k) \in H^s(\mathbb{R}^d,\mathbb{R}^k)$,

$$\begin{aligned} (\xi^*_\varphi p|u)_{H^{-s},H^s} &= \sum_{i=1}^{k} (p|u_i f_i \circ \varphi)_{H^{-s},H^s} = \sum_{i=1}^{k}\sum_{j=1}^{d} (p_j|u_i f_i^j \circ \varphi)_{H^{-s},H^s} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{d} (p_j(f_i^j \circ \varphi)|u_i)_{H^{-s},H^s} = \sum_{i=1}^{k} (p \cdot (f_i \circ \varphi)|u_i)_{H^{-s},H^s}, \end{aligned}$$

where $f_i = (f_i^1, \dots, f_i^d)$. Since each f_i^j is smooth with bounded derivatives at every order and $\varphi - \mathrm{Id}_{\mathbb{R}^d}$ belongs to H^s, we get $f_i \circ \varphi \in W^{s,\infty}$. The (distributional) product $p_j(f_i^j \circ \varphi)$ is well-defined, and belongs to $H^{-s}(\mathbb{R}^d)$, and so does $p \cdot (f \circ \varphi)$. Now for p to belong to an adapted contangent bundle, we need a condition that ensures $\xi_\varphi^* p = g_r(u, \cdot)$ for some $u = (u_1, \dots, u_k) \in H^s(\mathbb{R}^d, \mathbb{R}^k)$.

This is not difficult. We know that

$$g_r(u, v) = (L_r u, v)_{H^{s-2r}, H^s} = (L_r u, v)_{H^{-s}, H^s}, \quad u, v \in H^s(\mathbb{R}^d, \mathbb{R}^k),$$

and $L_r : H^s \to H^{s-2r}$ is an elliptic, bijective linear operator of order $2r$. Hence, the image of $u \mapsto g_r(u, \cdot)$ is H^{-s}. We get the following result.

Lemma 6. *We have $\xi_\varphi^* p = g_r(u, \cdot)$ for some $u = (u_1, \dots, u_k)$ if and only if $u_i = L_r^{-1}(p \cdot f_i \circ \varphi)$, or*

$$u = L_r^{-1}(p \cdot f \circ \varphi)$$

for short. This is possible if and only if each $p \cdot f_i \circ \varphi$ actually belongs to $H^{s-2r}(\mathbb{R}^d)$.

This leads us to consider the dense cotangent sub-bundle

$$\tau_r \mathcal{D}^s(\mathbb{R}^d) = \mathcal{D}^s(\mathbb{R}^d) \times H^{s-2r}(\mathbb{R}^d, \mathbb{R}^d).$$

The natural inclusion $H^{s-2r}(\mathbb{R}^d, \mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d, \mathbb{R}^d)$ has dense image. Moreover, it is adapted to the structure, because if $p \in H^{s-2r}(\mathbb{R}^d, \mathbb{R}^d)$, then so does $p \cdot (f_i \circ \varphi)$, since $f_i \circ \varphi$ is of class $W^{s,\infty}$.

Reduced Hamiltonian. For every $(\varphi, p) \in \tau_r \mathcal{D}^s(\mathbb{R}^d)$, the only element $u(\varphi, p) \in H^s(\mathbb{R}^d, \mathbb{R}^k)$ such that

$$g_{r,\varphi}(u(\varphi, p), \cdot) = \xi_\varphi^* p$$

is given by

$$u(\varphi, p) = L_r^{-1}(p \cdot (f \circ \varphi)),$$

which is smooth in (φ, p) as a composition of smooth maps.

Indeed, all derivatives of each f_i are bounded, so the mapping

$$\begin{aligned} \alpha_{f_i} : \mathcal{D}^s(\mathbb{R}^d) &\to W^{s,\infty}(\mathbb{R}^d, \mathbb{R}^d) \\ \varphi &\mapsto f_i \circ \varphi \end{aligned}$$

is smooth thanks to the α-lemma [16, 28, 31]. The other mappings involved are either linear continuous or bilinear continuous, and independant of φ, and therefore smooth as well.

The reduced Hamiltonian is therefore smooth, and given by

$$\begin{aligned} h(\varphi, p) &= \frac{1}{2} g_r\left(u(\varphi, P), u(\varphi, P)\right) \\ &= \frac{1}{2} \sum_{i=1}^k \left(L_r L_r^{-1}(p \cdot (f_i \circ \varphi)) | L_r^{-1}(p \cdot (f_i \circ \varphi))\right)_{H^{s-2r}, H^s} \\ &= \frac{1}{2} \sum_{i=1}^k \left(p \cdot (f_i \circ \varphi) | L_r^{-1}(p \cdot (f_i \circ \varphi))\right)_{H^{s-2r}, H^s}. \end{aligned}$$

Existence of a smooth symplectic gradient. All that is left now is to check that h admits a symplectic gradient of class $\mathcal{C}^2$. For this, we must first make sure that $\partial_\varphi H(\varphi,p)$ actually belongs to $\tau_\varphi\mathcal{D}^s(\mathbb{R}^d) = H^{s-2r}(\mathbb{R}^d,\mathbb{R}^d)$. As shown in the previous section, this will imply that h has a symplectic gradient in the relative cotangent bundle $T\tau\mathcal{D}^s(\mathbb{R}^d)$.

Now recall that $\partial_\varphi h(\varphi,p) = \partial_\varphi H^1(\varphi,p,u)$ with $u = u(\varphi,p)$. But with

$$H^1(\varphi,P,u) = \sum_{i=1}^k (p|u_i f_i\circ\varphi)_{H^{s-2r},H^s} - g_r(u,u),$$

and the fact that (see [16, 28, 31] for example),

$$(d\alpha_{f_i}(\varphi)\delta\varphi)(x) = df_i(\varphi(x))\delta\varphi(x), \quad i=1,\dots,k,\ \varphi\in\mathcal{D}^s(\mathbb{R}^d),\ \delta\varphi\in H^s(\mathbb{R}^d,\mathbb{R}^d),\ x\in\mathbb{R}^d,$$

we get

$$\partial_\varphi H^1(\varphi,P,u)\delta\varphi = \sum_{i=1}^k (p|(df_i\circ\varphi)u_i\delta\varphi)_{H^{s-2r},H^s},$$

for any $\delta\varphi\in H^s(\mathbb{R}^d,\mathbb{R}^d)$, with

$$((df_i\circ\varphi)u_i\delta\varphi)(x) = df_i(\varphi(x))u_i(x)\delta\varphi(x),\ x\in\mathbb{R}^d.$$

Since each f_i has all derivatives globally bounded and $\varphi - \mathrm{Id}_{\mathbb{R}^d}\in H^s(\mathbb{R}^d,\mathbb{R}^d)$, we can write

$$(p|df_i\circ\varphi u_i\delta\varphi)_{H^{s-2r},H^s} = (p\cdot(df_i\circ\varphi)\,|u_i\delta\varphi)_{H^{s-2r},H^s} = (p\cdot(df_i\circ\varphi)\,u_i|\delta\varphi)_{H^{s-2r},H^s},$$

with

$$p\cdot(df_i\circ\varphi) \underset{def}{:=} \sum_{j=1}^d \underbrace{p_j}_{\in H^{s-2r}(\mathbb{R}^d,\mathbb{R}^d)} \underbrace{\left(\frac{\partial f_i}{\partial x^j}\circ\varphi\right)}_{\in W^{s,\infty}(\mathbb{R}^d,\mathbb{R}^d)} \in H^{s-2r}(\mathbb{R}^d,\mathbb{R}^d).$$

Recalling that $u(\varphi,p) = L_r^{-1}p\cdot(f_i\circ\varphi)$, we finally get

$$\partial_\varphi h(\varphi,p) = \sum_{i=1}^k \underbrace{(p\cdot(df_i\circ\varphi))}_{\in H^{s-2r}}\underbrace{L_r^{-1}(p\cdot(f_i\circ\varphi))}_{\in H^s} \quad \in H^{s-2r}(\mathbb{R}^d,\mathbb{R}^d) = \tau_\varphi\mathcal{D}^s(\mathbb{R}^d).$$

Moreover, we already know that

$$\partial_p h(\varphi,p) = \xi_\varphi u(\varphi,p) = \sum_{i=1}^k L_r^{-1}(p\cdot f_i\circ\varphi)f_i\circ\varphi.$$

And, again, because all the f_is have globally bounded derivatives at every order, both of these derivatives are of class $\mathcal{C}^\infty$ in (φ,p) as composition of smooth mappings, linear continuous operators, and bilinear continuous operators.

We can then apply Theorem 7 and get the following result.

Proposition 3. *For any $(\varphi_0,p_0)\in\mathcal{D}^s(\mathbb{R}^d)\times H^{s-2r}(\mathbb{R}^d,\mathbb{R}^d)$, there is a unique maximal solution $(\varphi,p): I\ni 0\to(\varphi(t),p(t))$ to the Hamiltonian equation*

$$\begin{cases} \dot\varphi(t,x) = \displaystyle\sum_{i=1}^k u_i(t)f_i(\varphi(t)), \\ \dot p(t) = -\displaystyle\sum_{i=1}^k p(t)\cdot df_i(\varphi(t))u_i(t), \\ u_i(t) = L_r^{-1}\left(p(t)\cdot(f_i\circ\varphi(t))\right), \end{cases} \qquad t\in I,$$

such that $\varphi(0) = \varphi_0$ and $p(0) = p_0$. Then (φ, u) is a local geodesic for the sub-Riemannian structure $(\mathcal{H}, \xi, g_r)$.

Moreover, if $r = s$, the structure is strong and therefore (φ, u) is just a geodesic.

The case $r = 0$. This case is of particular interest. We will take $L_0 = 1$ instead of $1+(-\Delta)^0 = 2$. In this case, the action of a horizontal curve $\varphi : [0,1] \to \mathcal{D}^s(\mathbb{R}^d)$, with $\dot{\varphi} = \xi_\varphi u$, is given by

$$A(\varphi) = \frac{1}{2}\sum_{i=1}^{k}\int_0^1\int_{\mathbb{R}^d} u_i(t,x)^2 dxdt = \int_{\mathbb{R}^d}\left(\frac{1}{2}\int_0^1\sum_{i=1}^{k} u_i(t,x)^2 dt\right)dx = \int_{\mathbb{R}^d} A^{\mathcal{S}}(\varphi(\cdot,x))dx,$$

where $A^{\mathcal{S}}$ denotes the action of horizontal curves on $\mathbb{R}^d$ for the sub-Riemannian structure $\mathcal{S}$ induced by the f_is. So $A(\varphi)$ is just the integral of all the actions over every horizontal curve on $\mathbb{R}^d$ obtained by following φ. In particular, if $d^{\mathcal{S}}$ is the corresponding sub-Riemannian distance on $\mathbb{R}^d$, then

$$A(\varphi) \geqslant \frac{1}{2}\int_{\mathbb{R}^d} d^{\mathcal{S}}(x, \varphi(1,x))^2 dx,$$

with equality if and only if, for every $x \in \mathbb{R}^d$, $t \mapsto \varphi(t,x)$ is a minimizing geodesic for $\mathcal{S}$.

On the other hand, the adapted cotangent bundle is given by

$$\tau\mathcal{D}^s(\mathbb{R}^d) = \mathcal{D}^s(\mathbb{R}^d) \times \underbrace{H^{s-2\times 0}(\mathbb{R}^d, \mathbb{R}^d)}_{=H^s(\mathbb{R}^d,\mathbb{R}^d)},$$

so that a Hamiltonian geodesic $(\varphi, p) : I \to (\varphi(t), p(t))$ has momentum $p(t)$ belonging to $H^s(\mathbb{R}^d, \mathbb{R}^d)$. Also, $L_0 = L_0^{-1} = 1$ and we obtain

$$\begin{cases} \dot{\varphi}(t,x) = \displaystyle\sum_{i=1}^{k}(p(t,x)\cdot f_i(\varphi(t,x)))f_i(\varphi(t,x)), \\ \dot{p}(t,x) = -\displaystyle\sum_{i=1}^{k} p(t,x)\cdot f_i(\varphi(t,x))p(t,x)\cdot df_i(\varphi(t,x)). \end{cases}$$

We recognize ([27]), for each x in $\mathbb{R}^d$, that the curve $t \mapsto (\varphi(t,x), p(t,x))$ actually follows the Hamiltonian geodesic equation for the structure $\mathcal{S}$:

$$\begin{cases} \dot{\varphi}(t,x) = \partial_p h^{\mathcal{S}}(\varphi(t,x), p(t,x)), \\ \dot{p}(t,x) = -\partial_x h^{\mathcal{S}}(\varphi(t,x), p(t,x)). \end{cases}$$

Remark 26. This result is similar to, although much weaker than, that of [30], which considers optimal transport problems with sub-Riemannian distances for costs.

Remark 27. Another type of metrics one can put on this structure are the so-called right-invariant H^r metrics, see [7] for examples in the strong case $r = s$. This type of metrics is well-known in the Riemannian case and their geodesic flows in various cases have been studied in [10, 11, 12] for example, and the Fredholm property of the Riemannian exponential has even been studied in [26].

Conclusion

We gave a very general definition of infinite dimensional sub-Riemannian manifolds, and provided some tools to study them, with generalizations of the Chow-Rashevski Theorem and the Hamiltonian geodesic flow. However, many problems remain open.

The first step is to find more powerful tools to caracterize orbits and study their local behavior for more general structures. In particular, try to generalize in some way Sussmann's orbit theorem, and find out what the tangent space to an orbit is. The simple example of the ℓ^2-product of Engel groups from Section 2.2.3 shows that we need to consider spaces that are not even locally convex. However, perhaps one can at least give a general answer for 2-steps structures, i.e., those such that $\mathcal{L}(\Delta) = \Delta^2$.

The next problem is to find new tools to investigate elusive geodesics. We saw that for a strong structure, if $M' \subset M$ is a dense submanifold, modelled on a dense subset $\mathrm{B}' \subset \mathrm{B}$, and on which the sub-Riemannian structure of M can be restricted, then we get more Hamiltonian geodesics working on M' than on M. Indeed, the cotangent fibers of M' are bigger than those of M, giving more choice for inital momentum p. For example, for the infinite product of Heisenberg groups from Section 2.2.2, the structure can clearly be restricted to the orbit of 0, $\ell^2(\mathbb{R}^2) \times \ell^1(\mathbb{R}) \subset \ell^2(\mathbb{R}^3)$. The dual of $\ell^2(\mathbb{R}^2) \times \ell^1(\mathbb{R})$ is much bigger than that of $\ell^2(\mathbb{R}^3)$, giving new initial momenta p_0 and new Hamiltonian geodesics. These curves still existed in the original space $\ell^2(\mathbb{R}^3)$, and were still geodesics, but they could not be obtained by the Hamiltonian flow: they were neither normal or abnormal, and were therefore elusive geodesics. However, by enlarging the cotangent fibers, some elusive geodesics became normal geodesics.

One then naturally wonders: is it possible to find some ideal momentum space, for which there are no elusive geodesics, i.e., can we prove a new version of a Pontryagin Maximum Principle. This seems, again, extremely difficult to do in general. However, studying some simple but non-trivial cases might lead to clues on how to proceed.

Finally, the link between L^2-metrics in $\mathcal{D}^s(N)$, N a compact manifold, and the Euler equations of fluid dynamics is well-known [16]. The example given in Section 4.2 is a first step toward a possible study of sub-Riemannian fluids, or, more generally, fluid mechanics in which the fluid particules are subject to very general constraints.

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