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Mean field limits for interacting Hawkes processes in a diffusive regime

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Abstract: We consider a sequence of systems of Hawkes processes having mean field interactions in a diffusive regime. The stochastic intensity of each process is a solution of a stochastic differential equation driven by N independent Poisson random measures. We show that, as the number of interacting components N tends to infinity, this intensity converges in distribution in Skorohod space to a CIR-type diffusion. Moreover, we prove the convergence in distribution of the Hawkes processes to the limit point process having the limit diffusion as intensity. To prove the convergence results, we use analytical technics based on the convergence of the associated infinitesimal generators and Markovian semigroups.

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Introduction

Hawkes processes were originally introduced by [18] to model the appearance of earthquakes in Japan. Since then these processes have been successfully used in many fields to model various physical, biological or economical phenomena exhibiting self-excitation or -inhibition and interactions, such as seismology ([20], [37], [31], [2]), financial contagion ([1]), high frequency financial order books arrivals ([28], [3], [21]), genome analysis ([36]) and interactions in social networks ([38]). In particular, multivariate Hawkes processes are extensively used in neurosciences to model temporal arrival of spikes in neural networks ([17], [32], [33], [35]) since they provide good models to describe the typical temporal decorrelations present in spike trains of the neurons as well as the functional connectivity in neural nets.

In this paper, we consider a sequence of multivariate Hawkes processes $(Z^N)_{N \in \mathbb{N}^*}$ of the form $Z^N = (Z_t^{N,1}, \dots, Z_t^{N,N})_{t \geq 0}$. Each Z^N is designed to describe the behaviour of some interacting system with N components, as for example a neural network of N neurons. More precisely, Z^N is a multivariate counting process where each $Z^{N,i}$ records the number of events related to the i -th component, as for example the number of spikes of the i -th neuron. These counting processes are interacting, that is, any event of type i is able to trigger or to inhibit future events of all other types j . The process $(Z^{N,1}, \dots, Z^{N,N})$ is informally defined via its stochastic intensity process $\lambda^N = (\lambda^{N,1}(t), \dots, \lambda^{N,N}(t))_{t \geq 0}$ through the relation

$$\mathbb{P}(Z^{N,i} \text{ has a jump in }]t, t + dt] | \mathcal{F}_t) = \lambda^{N,i}(t)dt, \quad 1 \leq i \leq N,$$

where $\mathcal{F}_t = \sigma(Z_s^N : 0 \leq s \leq t)$. The stochastic intensity of a Hawkes process is given by

$$\lambda^{N,i}(t) = f_i^N \left(\sum_{j=1}^N \int_{-\infty}^t h_{ij}^N(t-s) dZ^{N,j}(s) \right). \quad (1)$$

Here, h_{ij}^N models the action or the influence of events of type j on those of type i , and how this influence decreases as time goes by. The function f_i^N is called the jump rate function of $Z^{N,i}$.

Since the founding works of [18] and [19], many probabilistic properties of Hawkes processes have been well-understood, such as ergodicity, stationarity and long time behaviour (see [5], [9], [8], [34] and [16]). A number of authors studied the statistical inference for Hawkes processes ([30] and [36]). Another field of study, very active nowadays, concerns the behaviour of the Hawkes process when the number of components N goes to infinity. During the last decade, large population limits of systems of interacting Hawkes processes have been studied in [15], [12] and [13].

[12] consider a general class of Hawkes processes whose interactions are given by a graph. In the case where the interactions are of mean field type and scaled in N^{-1} , namely $h_{ij}^N = N^{-1}h$ and $f_i^N = f$ in (1), they show that the Hawkes processes can be approximated by an i.i.d. family of inhomogeneous Poisson processes. They observe that for each fixed integer k , the joint law of k components converges to a product law as N tends to infinity, which is commonly referred to as the propagation of chaos. [13] generalize this result to a multi-population frame and show how oscillations emerge in the large population limit. Note again that the interactions in both papers are scaled in N^{-1} , which leads to limit point processes with deterministic intensity.

The purpose of this paper is to study the large population limit (when N goes to infinity) of the multivariate Hawkes processes $(Z^{N,1}, \dots, Z^{N,N})$ with mean field interactions scaled in $N^{-1/2}$. Contrarily to the situation considered in [12] and [13], this scaling leads to a non-chaotic limiting process with stochastic intensity. As we consider interactions scaled in $N^{-1/2}$, we have to center the terms of the sum in (1) to make the intensity process converge according to some kind of central limit theorem. To this end, we consider intensities with stochastic jump heights. Namely, in this model, the multivariate Hawkes processes $(Z^{N,i})_{1 \leq i \leq N}$ ($N \in \mathbb{N}^*$) are of the form

$$Z_t^{N,i} = \int_{]0,t] \times \mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\{z \leq \lambda_s^N\}} d\pi_i(s, z, u), \quad 1 \leq i \leq N, \quad (2)$$

where $(\pi_i)_{i \in \mathbb{N}^*}$ are i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ of intensity $ds dz d\mu(u)$ and μ is a centered probability measure on \mathbb{R} having a finite second moment σ^2 . The stochastic intensity of $Z^{N,i}$ is given by

$$\lambda_t^{N,i} = \lambda_t^N = f(X_{t-}^N),$$

where

$$X_t^N = \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{]0,t] \times \mathbb{R}_+ \times \mathbb{R}} h(t-s) u \mathbb{1}_{\{z \leq f(X_{s-}^N)\}} d\pi_j(s, z, u).$$

Moreover we consider a function h of the form $h(t) = e^{-\alpha t}$ so that the process $(X_t^N)_t$ is a piecewise deterministic Markov process. In the framework of neurosciences, X_t^N represents the membrane potential of the neurons at time t . The random jump heights u , chosen according to the measure μ , model random synaptic weights and the jumps of $Z^{N,j}$ represent the spike times of

neuron j . If neuron j spikes at time t , an additional random potential height u/\sqrt{N} is given to all other neurons in the system. As a consequence, the process X^N has the following dynamic

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{t-}^N)\}} d\pi_j(t, z, u).$$

Its infinitesimal generator is given by

$$A^N g(x) = -\alpha x g'(x) + N f(x) \int_{\mathbb{R}} \left[g\left(x + \frac{u}{\sqrt{N}}\right) - g(x) \right] \mu(du),$$

for sufficiently smooth functions g . As N goes to infinity, the above expression converges to

$$\bar{A}g(x) = -\alpha x g'(x) + \frac{\sigma^2}{2} f(x) g''(x),$$

which is the generator of a CIR-type diffusion solution of the SDE

$$d\bar{X}_t = -\alpha \bar{X}_t dt + \sigma \sqrt{f(\bar{X}_t)} dB_t. \quad (3)$$

It is classical to show in this framework that the convergence of generators implies the convergence of X^N to \bar{X} in distribution in Skorohod space. In this article we establish explicit bounds for the weak error for this convergence by means of a Trotter-Kato like formula. Moreover we establish for each i , the convergence in distribution in Skorohod space of the associated counting process $Z^{N,i}$ to the limit counting process \bar{Z}^i which has intensity $(f(\bar{X}_t))_t$. Conditionally on \bar{X} , the $\bar{Z}^i, i \geq 1$, are independent. This property can be viewed as a conditional propagation of chaos-property, which has to be compared to [12] and [13] where the intensity of the limit process is deterministic and its components are truly independent, and to [6], [11] and [27] where all interacting components are subject to common noise. In our case, the common noise, that is, the Brownian motion B of (3), emerges in the limit as a consequence of the central limit theorem.

To obtain a precise control of the speed of convergence of X^N to \bar{X} we use analytical methods showing first the convergence of the generators from which we deduce the convergence of the semigroups via the formula

$$\bar{P}_t g(x) - P_t^N g(x) = \int_0^t P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x) ds. \quad (4)$$

Here $\bar{P}_t g(x) = \mathbb{E}_x [g(\bar{X}_t)]$ and $P_t^N g(x) = \mathbb{E}_x^N [g(X_t^N)]$ denote the Markovian semigroups of \bar{X} and X^N . This formula is well-known in the classical semigroup theory setting where the generators are strong derivatives of semigroups in the Banach space of continuous bounded functions (see Lemma 1.6.2 of [14]). In our case, we have to consider extended generators (see [10] or [29]), i.e. $A^N g(x)$ is the point-wise derivative of $t \mapsto P_t^N g(x)$. The version of formula (4) for extended generators is stated and proved in Appendix (Proposition 5.4).

It is well-known that under suitable assumptions on f , the solution of (3) admits a unique invariant measure π whose density is explicitly known. Thus, a natural question is to consider the limit of the law $\mathcal{L}(X_t^N)$ of X_t^N when t and N go simultaneously to infinity. We prove that the limit of $\mathcal{L}(X_t^N)$ is π , for $(N, t) \rightarrow (\infty, \infty)$, under suitable conditions on the joint convergence of (N, t) . We also prove that there exists a parameter α^* such that for all $\alpha > \alpha^*$, this converges holds

whenever $(N, t) \rightarrow (\infty, \infty)$ jointly, without any further condition, and we provide a control of the error (Theorem 1.6).

The paper is organized as follows: in Section 1, we state the assumptions and formulate the main results. Section 2 is devoted to the proof of the convergence of the semigroup of X^N to that of \bar{X} (Theorem 1.4.(i)), and Section 3 to the study of the limit of the law of X_t^N as $N, t \rightarrow \infty$ (Theorem 1.6). In Section 4, we prove the convergence of the systems of point processes $(Z^{N,i})_{1 \leq i \leq N}$ to $(\bar{Z}^i)_{i \geq 1}$ (Theorem 1.7). Finally in Appendix, we prove some results on the extended generators, and some other technical results that we use throughout the paper.

1. Notation, assumptions and main results

1.1. Notation

The following notation are used throughout the paper:

- If X is a random variable, we note $\mathcal{L}(X)$ its distribution.
- If g is a real-valued function which is n times differentiable, we note $\|g\|_{n,\infty} = \sum_{k=0}^n \|g^{(k)}\|_{\infty}$.
- If $g : \mathbb{R} \rightarrow \mathbb{R}$ is a real-valued measurable function and π a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that g is integrable with respect to π , we write πg for $\int_{\mathbb{R}} g d\pi$.
- We write $C_b^n(\mathbb{R})$ for the set of the functions g which are n times continuously differentiable such that $\|g\|_{n,\infty} < +\infty$, and we write for short $C_b(\mathbb{R})$ instead of $C_b^0(\mathbb{R})$. Finally, $C^n(\mathbb{R})$ denotes the set of n times continuously differentiable functions that are not necessarily bounded nor have bounded derivatives.
- If g is a real-valued function and I is an interval, we note $\|g\|_{\infty,I} = \sup_{x \in I} |g(x)|$.
- We write $C_c^n(\mathbb{R})$ for the set of functions that are n times continuously differentiable and that have a compact support.
- We write $D(\mathbb{R}_+, \mathbb{R})$ for the Skorohod space of càdlàg functions from \mathbb{R}_+ to \mathbb{R} , endowed with Skorohod metric (see Chapter 3 Section 16 of [4]), and $D(\mathbb{R}_+, \mathbb{R}_+)$ for this space restricted to non-negative functions.
- $\mathcal{M}^\#$ denotes the space of locally finite measures on $\mathbb{R}_+ \times \mathbb{R}_+$ endowed with the topology of the weak convergence, and $\mathcal{N}^\#$ the subspace that contains only the simple point measures.
- α is a positive constant, L, σ and m_k ($1 \leq k \leq 4$) are fixed parameters defined in Assumptions 1, 2 and 3 below. Finally, we note C any arbitrary constant, so the value of C can change from line to line in an equation. Moreover, if C depends on some non-fixed parameter θ , we write C_θ .

1.2. Assumptions

Let X^N satisfy

$$\begin{cases} dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{t-}^N)\}} d\pi_j(t, z, u), \\ X_0^N \sim \nu_0^N, \end{cases} \quad (5)$$

where ν_0^N is a probability measure on \mathbb{R} . Under natural assumptions on f , the SDE (5) admits a unique non-exploding strong solution (see Proposition 5.6).

The aim of this paper is to provide explicit bounds for the convergence of X^N in Skorokhod space to the limit process $(\bar{X}_t)_{t \in \mathbb{R}_+}$ which is solution to the SDE

$$\begin{cases} d\bar{X}_t = -\alpha\bar{X}_t dt + \sigma\sqrt{f(\bar{X}_t)}dB_t, \\ \bar{X}_0 \sim \bar{\nu}_0, \end{cases} \quad (6)$$

where σ^2 is the variance of μ , $(B_t)_{t \in \mathbb{R}_+}$ is a one-dimensional standard Brownian motion, and $\bar{\nu}_0$ a suitable probability measure on \mathbb{R} .

To prove our results, we need to introduce the following assumptions.

Assumption 1. \sqrt{f} is a positive and Lipschitz continuous function, having Lipschitz constant L .

Under Assumption 1, it is classical that the SDE (6) admits a unique non-exploding strong solution (see remark IV.2.1, Theorems IV.2.3, IV.2.4 and IV.3.1 of [22]).

Assumption 2.

- $\int_{\mathbb{R}} x^4 d\bar{\nu}_0(x) < \infty$ and for every $N \in \mathbb{N}^*$, $\int_{\mathbb{R}} x^4 d\nu_0^N(x) < \infty$.
- μ is a centered probability measure having a fourth moment, we note σ^2 its variance.

Assumption 2 allows us to control the moments up to order four of the processes $(X_t^N)_t$ and $(\bar{X}_t)_t$ (see Lemma 2.1) and to prove the convergence of the generators of the processes $(X_t^N)_t$ (see Proposition 2.3).

Assumption 3. We assume that f is C^4 and for each $1 \leq k \leq 4$, $(\sqrt{f})^{(k)}$ is bounded by some constant m_k .

Remark 1.1. By definition $m_1 = L$, since $m_1 := \|(\sqrt{f})'\|_{\infty}$ and L is the Lipschitz constant of \sqrt{f} .

Assumption 3 guarantees that the stochastic flow associated to (6) has regularity properties with respect to the initial condition $\bar{X}_0 = x$. This will be the main tool to obtain uniform in time estimates of the limit semigroup, see Proposition 2.4.

Example 1.2. The functions $f(x) = 1 + x^2$, $f(x) = \sqrt{1 + x^2}$ and $f(x) = (\pi/2 + \arctan x)^2$ satisfy Assumptions 1 and 3.

Assumption 4. X_0^N converges in distribution to \bar{X}_0 .

Obviously, Assumption 4 is a necessary condition for the convergence in distribution of X^N to \bar{X} .

1.3. Main results

Our first main result is the convergence of the process X^N to \bar{X} in distribution in Skorokhod space, with an explicit rate of convergence for their semigroups. This rate of convergence will be expressed in terms of the following parameters

$$\beta := \max\left(\frac{1}{2}\sigma^2 L^2 - \alpha, 2\sigma^2 L^2 - 2\alpha, \frac{7}{2}\sigma^2 L^2 - 3\alpha\right) \quad (7)$$

and, for any $T > 0$ and any fixed $\varepsilon > 0$,

$$K_T := (1 + 1/\varepsilon) \int_0^T (1 + s^2) e^{\beta s} \left(1 + e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)(T-s)}\right) ds. \quad (8)$$

Remark 1.3. If $\alpha > 7/6 \sigma^2 L^2$, then $\beta < 0$, and one can choose $\varepsilon > 0$ such that $\sigma^2 L^2 - 2\alpha + \varepsilon < 0$, implying that $\sup_{T>0} K_T < \infty$.

Theorem 1.4. If Assumptions 1 and 2 hold, then the following assertions are true.

(i) Under Assumption 3, for all $T \geq 0$, for each $g \in C_b^3(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\sup_{0 \leq t \leq T} |P_t^N g(x) - \bar{P}_t g(x)| \leq C(1+x^2)K_T \|g\|_{3,\infty} \frac{1}{\sqrt{N}}.$$

In particular, if $\alpha > \frac{7}{6} \sigma^2 L^2$, then

$$\sup_{t \geq 0} |P_t^N g(x) - \bar{P}_t g(x)| \leq C(1+x^2) \|g\|_{3,\infty} \frac{1}{\sqrt{N}}.$$

(ii) If in addition Assumption 4 holds, then $(X^N)_N$ converges in distribution to \bar{X} in $D(\mathbb{R}_+, \mathbb{R})$.

We refer to Proposition 2.4 for the form of β given in (7). Theorem 1.4 is proved in the end of Subsection 2.2. (ii) is a consequence of Theorem IX.4.21 of [23], using that X^N is a semimartingale. Alternatively, it can be proved as a consequence of (i), using that X^N is a Markov process.

Below we give some simulations of the trajectories of the process $(X_t^N)_{t \geq 0}$ in Figure 1.

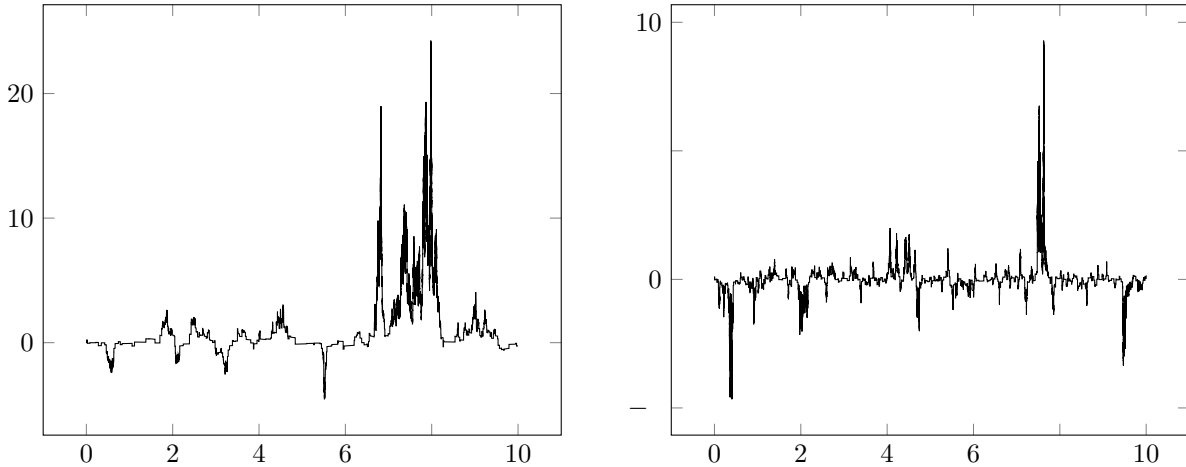


FIGURE 1. Simulation of trajectories of $(X_t^N)_{0 \leq t \leq 10}$ with $X_0^N = 0$, $\alpha = 1$, $\mu = \mathcal{N}(0, 1)$, $f(x) = 1 + x^2$, $N = 10$ (left picture) and $N = 50$ (right picture).

Remark 1.5. Theorem 1.4.(ii) states the convergence of X^N to \bar{X} in Skorohod topology. Since \bar{X} is almost surely continuous, this implies the, a priori stronger, convergence in distribution in the topology of the uniform convergence on compact sets. Indeed, according to Skorohod's representation theorem (see Theorem 6.7 of [4]), we can assume that X^N converges almost surely to \bar{X} in Skorohod space, and this classically entails the uniform convergence on every compact set (see the discussion at the bottom of page 124 in Section 12 of [4]).

Under our assumptions, \bar{P} admits an invariant probability measure π , and we can even control the speed of convergence of $P_t^N g(x)$ to $\pi(g)$, as (N, t) goes to infinity, for suitable conditions on the joint convergence of N and t .

Theorem 1.6. *Under Assumptions 1 and 2, \bar{X} is recurrent in the sense of Harris, having invariant probability measure $\pi(dx) = p(x)dx$ with density*

$$p(x) = C \frac{1}{f(x)} \exp\left(-\frac{2\alpha}{\sigma^2} \int_0^x \frac{y}{f(y)} dy\right).$$

Besides, if Assumption 3 holds, then for all $g \in C_b^3(\mathbb{R})$ and $x \in \mathbb{R}$,

$$|P_t^N g(x) - \pi g| \leq C \|g\|_{3,\infty} (1+x^2) \left(\frac{K_t}{\sqrt{N}} + e^{-\gamma t} \right),$$

where C and γ are positive constants independent of N and t , and where K_t is defined in (8). In particular, $P_t^N(x, \cdot)$ converges weakly to π as $(N, t) \rightarrow (\infty, \infty)$, provided $K_t = o(\sqrt{N})$.

If we assume, in addition, that $\alpha > \frac{7}{6}\sigma^2 L^2$, then $P_t^N(x, \cdot)$ converges weakly to π as $(N, t) \rightarrow (\infty, \infty)$ without any condition on the joint convergence of (t, N) , and we have, for any $g \in C_b^3(\mathbb{R})$ and $x \in \mathbb{R}$,

$$|P_t^N g(x) - \pi g| \leq C \|g\|_{3,\infty} (1+x^2) \left(\frac{1}{\sqrt{N}} + e^{-\gamma t} \right).$$

Theorem 1.6 is proved in the end of Section 3.

Finally, using Theorem 1.4.(ii), we show the convergence of the point processes $Z^{N,i}$ defined in (2) to limit point processes \bar{Z}^i having stochastic intensity $f(\bar{X}_t)$ at time t . To define the processes \bar{Z}^i ($i \in \mathbb{N}^*$), we fix a Brownian motion $(B_t)_{t \geq 0}$ on some probability space different from the one where the processes X^N ($N \in \mathbb{N}^*$) and the Poisson random measures π_i ($i \in \mathbb{N}^*$) are defined. Then we fix a family of i.i.d. Poisson random measures $\bar{\pi}_i$ ($i \in \mathbb{N}^*$) on the same space as $(B_t)_{t \geq 0}$, independent of $(B_t)_{t \geq 0}$. The limit point processes \bar{Z}^i are then defined by

$$\bar{Z}_t^i = \int_{]0,t] \times \mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\{z \leq f(\bar{X}_s)\}} d\bar{\pi}_i(s, z, u). \quad (9)$$

Theorem 1.7. *Under Assumptions 1, 2 and 4, the sequence $(Z^{N,1}, Z^{N,2}, \dots, Z^{N,k}, \dots)_N$ converges to $(\bar{Z}^1, \bar{Z}^2, \dots, \bar{Z}^k, \dots)$ in distribution in $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$ for the product topology.*

Let us give a brief interpretation of the above result. Conditionally on \bar{X} , for any $k > 1$, $\bar{Z}^1, \dots, \bar{Z}^k$ are independent. Therefore, the above result can be interpreted as a conditional propagation of chaos property (compare to [6] dealing with the situation where all interacting components are subject to common noise). In our case, the common noise, that is, the Brownian motion B driving the dynamic of \bar{X} , emerges in the limit as a consequence of the central limit theorem. Theorem 1.7 is proved in the end of Section 4.

Remark 1.8. *In Theorem 1.7, we implicitly define $Z^{N,i} := 0$ for each $i \geq N + 1$.*

2. Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. To prove the convergence of the semigroups of $(X^N)_N$, we show in a first time the convergence of their generators. We start with useful a priori bounds on the moments of X^N and \bar{X} .

Lemma 2.1. *Under Assumptions 1 and 2, the following holds.*

- (i) *For all $\varepsilon > 0$, $t > 0$ and $x \in \mathbb{R}$, $\mathbb{E}_x [(X_t^N)^2] \leq C(1 + 1/\varepsilon)(1 + x^2)(1 + e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)t})$, for some $C > 0$ independent of N, t, x and ε .*
- (ii) *For all $\varepsilon > 0$, $t > 0$ and $x \in \mathbb{R}$, $\mathbb{E}_x [(\bar{X}_t)^2] \leq C(1 + 1/\varepsilon)(1 + x^2)(1 + e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)t})$, for some $C > 0$ independent of t, x and ε .*
- (iii) *For all $N \in \mathbb{N}^*$, $T > 0$, $\mathbb{E} [(\sup_{0 \leq t \leq T} |X_t^N|)^2] < +\infty$ and $\mathbb{E} [(\sup_{0 \leq t \leq T} |\bar{X}_t|)^2] < +\infty$.*
- (iv) *For all $T > 0$, $N \in \mathbb{N}^*$, $\sup_{0 \leq t \leq T} \mathbb{E}_x [(X_t^N)^4] \leq C_T(1 + x^4)$ and $\sup_{0 \leq t \leq T} \mathbb{E}_x [(\bar{X}_t)^4] \leq C_T(1 + x^4)$.*

We postpone the proof of Lemma 2.1 to Appendix.

2.1. Convergence of the generators

Throughout this paper, we consider extended generators similar to those used in [29] and in [10], because the classical notion of generator does not suit to our framework (see the beginning of Section 5.1). As this definition slightly differs from one reference to another, we define explicitly the extended generator in Definition 5.1 below and we prove the results on extended generators that we need in this paper. We note A^N the extended generator of X^N and \bar{A} the one of \bar{X} , and $\mathcal{D}'(A^N)$ and $\mathcal{D}'(\bar{A})$ their extended domains. The goal of this section is to prove the convergence of $A^N g(x)$ to $\bar{A}g(x)$ and to establish the rate of convergence for test functions $g \in C_b^3(\mathbb{R})$. Before proving this convergence, we state a lemma which characterizes the generators for some test functions. This lemma is a straightforward consequence of Ito's formula and Lemma 2.1.(i).

Lemma 2.2. $C_b^2(\mathbb{R}) \subseteq \mathcal{D}'(\bar{A})$, and for all $g \in C_b^2(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$\bar{A}g(x) = -\alpha x g'(x) + \frac{1}{2} \sigma^2 f(x) g''(x).$$

Moreover, $C_b^1(\mathbb{R}) \subseteq \mathcal{D}'(A^N)$, and for all $g \in C_b^1(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$A^N g(x) = -\alpha x g'(x) + N f(x) \int_{\mathbb{R}} \left[g \left(x + \frac{u}{\sqrt{N}} \right) - g(x) \right] d\mu(u).$$

Now we can prove the main result of this subsection.

Proposition 2.3. *If Assumptions 1 and 2 hold, then for all $g \in C_b^3(\mathbb{R})$,*

$$|\bar{A}g(x) - A^N g(x)| \leq f(x) \|g'''\|_{\infty} \frac{1}{6\sqrt{N}} \int_{\mathbb{R}} |u|^3 d\mu(u).$$

Proof. For $g \in C_b^3(\mathbb{R})$, if we note U a random variable having distribution μ , we have, since $\mathbb{E}[U] = 0$,

$$\begin{aligned} |A^N g(x) - \bar{A}g(x)| &\leq f(x) \left| N \mathbb{E} \left[g \left(x + \frac{U}{\sqrt{N}} \right) - g(x) \right] - \frac{1}{2} \sigma^2 g''(x) \right| \\ &= f(x) N \left| \mathbb{E} \left[g \left(x + \frac{U}{\sqrt{N}} \right) - g(x) - \frac{U}{\sqrt{N}} g'(x) - \frac{U^2}{2N} g''(x) \right] \right| \\ &\leq f(x) N \mathbb{E} \left[\left| g \left(x + \frac{U}{\sqrt{N}} \right) - g(x) - \frac{U}{\sqrt{N}} g'(x) - \frac{U^2}{2N} g''(x) \right| \right]. \end{aligned}$$

Using Taylor-Lagrange's inequality, we obtain the result. \square

2.2. Convergence of the semigroups

Once the convergence $A^N g(x) \rightarrow \bar{A}g(x)$ is established, together with a control of the speed of convergence, our strategy is to rely on the following representation

$$(\bar{P}_t - P_t^N)g(x) = \int_0^t P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x) ds, \quad (10)$$

which is valid under suitable assumptions on X^N and \bar{X} and for suitable test functions g . The precise assumptions needed for this formula to hold and the proof of it are postponed to Proposition 5.4 below.

Obviously, to be able to apply the above formula, we need to ensure the regularity of $x \mapsto \bar{P}_s g(x)$, together with a control of the associated norm $\|(\bar{P}_s g)'''\|_\infty$. This is done in the next proposition.

Proposition 2.4. *If Assumptions 1, 2 and 3 hold, then for all $t \geq 0$ and for all $g \in C_b^3(\mathbb{R})$, the function $x \mapsto \bar{P}_t g(x)$ belongs to $C_b^3(\mathbb{R})$ and satisfies*

$$\left\| (\bar{P}_t g)''' \right\|_\infty \leq C \|g\|_{3,\infty} (1 + t^2) e^{\beta t},$$

with $\beta = \max(\frac{1}{2}\sigma^2 L^2 - \alpha, 2\sigma^2 L^2 - 2\alpha, \frac{7}{2}\sigma^2 L^2 - 3\alpha)$.

The proof of Proposition 2.4 requires some detailed calculus to obtain the explicit expression for β , so we postpone it to Appendix.

Proof of Theorem 1.4. Step 1. The main part of the proof of the point (i) will be to check that the conditions stated in Proposition 5.4 below allowing to obtain formula (10) are satisfied. This will be done in Step 2 below. Indeed, once this is shown, the rest of the proof will be a straightforward consequence of Proposition 2.3, since, applying formula (10),

$$\begin{aligned} |\bar{P}_t g(x) - P_t^N g(x)| &= \left| \int_0^t P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x) ds \right| \\ &\leq \int_0^t \mathbb{E}_x^N [|\bar{A}(\bar{P}_s g)(X_{t-s}^N) - A^N(\bar{P}_s g)(X_{t-s}^N)|] ds \\ &\leq C \frac{1}{\sqrt{N}} \int_0^t \left\| (\bar{P}_s g)''' \right\|_\infty \mathbb{E}_x [f(X_{t-s}^N)] ds \\ &\leq C \frac{1}{\sqrt{N}} \|g\|_{3,\infty} \int_0^t \left((1 + s^2) e^{\beta s} \left(1 + \mathbb{E}_x [(X_{t-s}^N)^2] \right) \right) ds \\ &\leq C \left(1 + \frac{1}{\varepsilon} \right) \frac{1}{\sqrt{N}} \|g\|_{3,\infty} (1 + x^2) \int_0^t (1 + s^2) e^{\beta s} \left(1 + e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)(t-s)} \right) ds, \end{aligned}$$

where we have used respectively Proposition 2.4 and Lemma 2.1.(i) to obtain the two last inequalities above, and ε is any positive constant.

Step 2. Now we show that X^N and \bar{X} satisfy the hypotheses of Proposition 5.4. To begin with we know that \bar{X} and X^N satisfy the hypotheses (i), (ii) and (iii), using Lemma 2.1. Then the hypothesis (iv) can be proved for the processes X^N and \bar{X} solving the SDEs (5) and (6) with straightforward calculations using Lemma 2.1. We know that \bar{P} satisfies hypothesis (v) thanks to Proposition 2.4.

Besides one can note that \bar{P} satisfies hypothesis (vi) using the calculations of the proof of Proposition 2.4. Then using Lemma 2.2, we see directly that \bar{A} and A^N satisfy the hypotheses (vii) and (ix). In addition (viii) is straightforward for \bar{A} , and it is a consequence of Lemma 2.5 below for A^N . The only remaining hypothesis (x) is a straightforward consequence of Lemma 2.6 below.

Step 3. We finally give the proof of the point (ii) of the theorem. With the notation of Theorem IX.4.21 of [23], we have $K^N(x, dy) := Nf(x)\mu(\sqrt{N}dy)$, $b'^N(x) = -\alpha x + \int K^N(x, dy)y = -\alpha x$, and $c'^N(x) = \int K^N(x, dy)y^2 = \sigma^2 f(x)$. Then, an immediate adaptation of Theorem IX.4.21 of [23] to our frame implies the result. \square

Lemma 2.5. *For all $g \in C_c^2(\mathbb{R})$ such that $\text{Supp } g \subseteq [-M, M]$, we have*

$$\left\| (A^N g)' \right\|_{\infty} \leq C \|g\|_{1, \infty} (1 + M^2),$$

for some constant $C > 0$.

Proof. We have

$$\begin{aligned} (A^N g)'(x) &= -\alpha g(x) - \alpha x g'(x) - Nf'(x)g(x) - Nf(x)g'(x) \\ &\quad + Nf'(x)\mathbb{E}\left[g\left(x + \frac{U}{\sqrt{N}}\right)\right] + Nf(x)\mathbb{E}\left[g'\left(x + \frac{U}{\sqrt{N}}\right)\right]. \end{aligned}$$

Then it is clear that for all $x \in \mathbb{R}$, we have

$$\left| (A^N g)'(x) \right| \leq C \|g\|_{1, \infty} (1 + M^2) + \left| Nf'(x)\mathbb{E}\left[g\left(x + \frac{U}{\sqrt{N}}\right)\right] \right| + \left| Nf(x)\mathbb{E}\left[g'\left(x + \frac{U}{\sqrt{N}}\right)\right] \right|. \quad (11)$$

We bound the jump terms using the subquadraticity of f and f' (indeed with Assumptions 1 and 3, we know that f' is sublinear, and consequently subquadratic). We can write:

$$\begin{aligned} \mathbb{E}\left[\left|g'\left(x + \frac{U}{\sqrt{N}}\right)\right|\right] &\leq \|g'\|_{\infty} \mathbb{E}\left[\mathbf{1}_{\{|x+U/\sqrt{N}| \leq M\}}\right] \\ &= \|g'\|_{\infty} \mathbb{P}\left(\left\{x + \frac{U}{\sqrt{N}} \geq -M\right\} \cap \left\{x + \frac{U}{\sqrt{N}} \leq M\right\}\right) \\ &\leq \|g'\|_{\infty} \mathbb{P}\left(\left\{x + \frac{|U|}{\sqrt{N}} \geq -M\right\} \cap \left\{x - \frac{|U|}{\sqrt{N}} \leq M\right\}\right) \\ &= \|g'\|_{\infty} \mathbb{P}\left(\left\{|U| \geq -\sqrt{N}(M+x)\right\} \cap \left\{|U| \geq \sqrt{N}(x-M)\right\}\right). \end{aligned}$$

Then for $x > M + 1$, using that $f(x) \leq C(1 + x^2)$, and for a constant C that may change from line to line,

$$\begin{aligned} \left|f(x)\mathbb{E}\left[g'\left(x + \frac{U}{\sqrt{N}}\right)\right]\right| &\leq C \|g'\|_{\infty} (1 + x^2) \mathbb{P}\left(|U| \geq \sqrt{N}(x - M)\right) \\ &\leq C \frac{1}{N} \mathbb{E}[U^2] \|g'\|_{\infty} \frac{1 + x^2}{(x - M)^2} \leq C \|g'\|_{\infty} (1 + M^2). \end{aligned}$$

The last inequality comes from the fact that the function $x \in [M+1, +\infty[\mapsto \frac{1+x^2}{(x-M)^2}$ is bounded by $1+(M+1)^2$. With the same reasoning, we know that for all $x < -M-1$, we have

$$\left| f(x) \mathbb{E} \left[g' \left(x + \frac{U}{\sqrt{N}} \right) \right] \right| \leq C \|g'\|_\infty (1+M^2).$$

This concludes the proof. \square

Lemma 2.6. *Let $(g_k)_k$ be a sequence of $C_b^1(\mathbb{R})$ satisfying $\sup_k \|g'_k\|_\infty < \infty$, and for all $x \in \mathbb{R}$, $g_k(x) \rightarrow 0$ as $k \rightarrow \infty$.*

Then for all bounded sequences of real numbers $(x_k)_k$, $g_k(x_k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Let $(x_k)_k$ be a bounded sequence. In a first time, we suppose that $(x_k)_k$ converges to some $x \in \mathbb{R}$. Then we have $|g_k(x_k)| \leq \|g'_k\|_\infty |x - x_k| + |g_k(x)|$ which converges to zero as k goes to infinity. In the general case, we show that for all subsequence of $(g_k(x_k))_k$, there exists a subsequence of the first one that converges to 0 (the second subsequence has to be chosen such that x_k converges). \square

3. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. We begin by proving some properties of the invariant measure of \bar{P}_t . In what follows we use the total variation distance between two probability measures ν_1 and ν_2 defined by

$$\|\nu_1 - \nu_2\|_{TV} = \frac{1}{2} \sup_{g: \|g\|_\infty \leq 1} |\nu_1(g) - \nu_2(g)|.$$

Proposition 3.1. *If Assumptions 1 and 2 hold, then the invariant measure π of $(\bar{P}_t)_t$ exists and is unique. Its density is given, up to multiplication with a constant, by*

$$p(x) = C \frac{1}{f(x)} \exp \left(-\frac{2\alpha}{\sigma^2} \int_0^x \frac{y}{f(y)} dy \right).$$

In addition, if Assumption 3 holds, then for every $0 < q < 1/2$, there exists some $\gamma > 0$ such that, for all $t \geq 0$,

$$\|\bar{P}_t(x, \cdot) - \pi\|_{TV} \leq C (1+x^2)^q e^{-\gamma t}.$$

Proof. In a first time, let us prove the positive Harris recurrence of \bar{X} implying the existence and uniqueness of π . According to Example 3.10 of [24] it is sufficient to show that $S(x) := \int_0^x s(y) dy$ goes to $+\infty$ (resp. $-\infty$) as x goes to $+\infty$ (resp. $-\infty$), where

$$s(x) := \exp \left(\frac{2\alpha}{\sigma^2} \int_0^x \frac{v}{f(v)} dv \right).$$

For $x > 0$, and using that f is subquadratic,

$$s(x) \geq \exp \left(C \int_0^x \frac{2v}{1+v^2} dv \right) = \exp(C \ln(1+y^2)) = (1+y^2)^C \geq 1,$$

implying that $S(x)$ goes to $+\infty$ as x goes to $+\infty$. With the same reasoning, we obtain that $S(x)$ goes to $-\infty$ as x goes to $-\infty$. Finally, the associated invariant density is given, up to a constant, by

$$p(x) = \frac{C}{f(x)s(x)}.$$

For the second part of the proof, take $V(x) = (1 + x^2)^q$, for some $q < 1/2$, then

$$V'(x) = 2qx(1 + x^2)^{q-1}, V''(x) = 2q(1 + x^2)^{q-2}[2x^2(q-1) + (1 + x^2)].$$

As $q < \frac{1}{2}$, $V''(x) < 0$ for x^2 sufficiently large, say, for $|x| \geq K$. In this case, for $|x| \geq K$,

$$\bar{A}V(x) \leq -2\alpha qx^2(1 + x^2)^{q-1} \leq -2\alpha q \frac{x^2}{1 + x^2} V(x) \leq -2\alpha q \frac{K^2}{1 + K^2} V(x) = -cV(x).$$

So we obtain all in all for suitable constants c and d that, for any $x \in \mathbb{R}$,

$$\bar{A}V(x) \leq -cV(x) + d. \quad (12)$$

Obviously, for any fixed $T > 0$, the sampled chain $(\bar{X}_{kT})_{k \geq 0}$ is Feller and π -irreducible. The support of π being \mathbb{R} , Theorem 3.4 of [29] implies that every compact set is petite for the sampled chain. Then, as (12) implies the condition (CD3) of Theorem 6.1 of [29], we have the following bound: introducing for any probability measure μ the weighted norm

$$\|\mu\|_V := \sup_{g: |g| \leq 1+V} |\mu(g)|,$$

there exist $C, \gamma > 0$ such that

$$\|\bar{P}_t(x, \cdot) - \pi\|_V \leq C(1 + V(x))e^{-\gamma t}.$$

This implies the result, since $\|\cdot\|_{TV} \leq \|\cdot\|_V$. \square

Now the proof of Theorem 1.6 is straightforward.

Proof of Theorem 1.6. The first part of the theorem has been proved in Proposition 3.1. For the second part, for any $g \in C_b^3(\mathbb{R})$,

$$\begin{aligned} |P_t^N g(x) - \pi g| &\leq |P_t^N g(x) - \bar{P}_t g(x)| + |\bar{P}_t g(x) - \pi g| \\ &\leq \frac{K_t}{\sqrt{N}}(1 + x^2) \|g\|_{3,\infty} + \|g\|_{\infty} \|\bar{P}_t(x, \cdot) - \pi\|_{TV} \\ &\leq \|g\|_{3,\infty} C \left(\frac{K_t}{\sqrt{N}}(1 + x^2) + e^{-\gamma t}(1 + x^2)^q \right), \end{aligned}$$

where we have used Theorem 1.4 and Proposition 3.1. Since $(1 + x^2)^q \leq 1 + x^2$, q being smaller than $1/2$, this implies the result. \square

4. Proof of Theorem 1.7

To prove the convergence in distribution of $(Z^{N,i})_{i \geq 1}$ to $(\bar{Z}^i)_{i \geq 1}$ as N goes to infinity, it is sufficient to prove that, for any $k \in \mathbb{N}^*$, $(Z^{N,i})_{1 \leq i \leq k}$ converges to $(\bar{Z}^i)_{1 \leq i \leq k}$ in $D(\mathbb{R}_+, \mathbb{R})^k$. This last condition is weaker than the convergence in $D(\mathbb{R}_+, \mathbb{R}^k)$, that we prove using Theorem IX.4.15 of [23].

Let $k \in \mathbb{N}^*$, let us note $Y^N := (X^N, Z^{N,1}, \dots, Z^{N,k})$ and $\bar{Y} := (\bar{X}, \bar{Z}^1, \dots, \bar{Z}^k)$. Using the notation of Theorem IX.4.15 of [23] with the semimartingales Y^N ($N \in \mathbb{N}^*$) and \bar{Y} and denoting e^j ($0 \leq j \leq k$) the j -th unit vector, we have:

- $b^{N,0}(x) = b^0(x) = -\alpha x$ and $b^{N,i}(x) = b^i(x) = 0$ for $1 \leq i \leq k$,
- $\tilde{c}^{N,0,0}(x) = c^{0,0}(x) = \sigma^2 f(x^0)$ and $c^{N,i,j}(x) = c^{i,j}(x) = 0$ for $(i, j) \neq (0, 0)$,
- $g * K^N(x) = f(x^0) \sum_{j=1}^k \int_{\mathbb{R}} g\left(\frac{u}{\sqrt{N}}e^0 + e^j\right) d\mu(u) + (N-k) \int_{\mathbb{R}} g\left(\frac{u}{\sqrt{N}}e^0\right) d\mu(u)$,
- $g * K(x) = f(x^0) \sum_{j=1}^k g(e^j)$.

The only condition of Theorem IX.4.15 that is not straightforward is the convergence of $g * K^N$ to $g * K$ for $g \in C_1(\mathbb{R}^{k+1})$. The complete definition of $C_1(\mathbb{R}^{k+1})$ is given in VII.2.7 of [23], but here, we just use the fact that $C_1(\mathbb{R}^{k+1})$ is a subspace of $C_b(\mathbb{R}^{k+1})$ containing functions which are zero around zero. This convergence follows from the fact that any $g \in C_1(\mathbb{R}^{k+1})$ can be written as $g(x) = h(x)\mathbb{1}_{\{|x|>\varepsilon\}}$ where $h \in C_b(\mathbb{R}^{k+1})$ and $\varepsilon > 0$. This allows to show that, for this kind of function g ,

$$\begin{aligned} \left| (N-k)f(x^0) \int_{\mathbb{R}} g\left(\frac{u}{\sqrt{N}}e^0\right) d\mu(u) \right| &\leq (N-k)f(x^0) \|h\|_{\infty} \int_{\mathbb{R}} \mathbb{1}_{\{|u|>\varepsilon\sqrt{N}\}} d\mu(u) \\ &\leq f(x^0) C \frac{N-K}{N^2} \leq C f(x^0) N^{-1}, \end{aligned}$$

where the second inequality follows from the fact that we assume that μ is a probability measure having a fourth moment.

5. Appendix

5.1. Extended generators

In this subsection, we define precisely the notion of generators we use and we prove the results needed to prove formula (4). In the general theory of semigroups, one defines the generators on some Banach space. In the frame of semigroups related to Markov processes, one generally considers $(C_b(\mathbb{R}), \|\cdot\|_{\infty})$. In this context, the generator A of a semigroup $(P_t)_t$ is defined on the set of functions $\mathcal{D}(A) = \{g \in C_b(\mathbb{R}) : \exists h \in C_b(\mathbb{R}), \|\frac{1}{t}(P_t g - g) - h\|_{\infty} \rightarrow 0 \text{ as } t \rightarrow 0\}$. Then one denotes the previous function h as Ag . In general, we can only guarantee that $\mathcal{D}(A)$ contains the functions that have a compact support, but to prove Proposition 5.4, we need to apply the generators of the processes $(X_t^N)_t$ and $(\bar{X}_t)_t$ to functions of the type $\bar{P}_s g$, and we cannot guarantee that $\bar{P}_s g$ has compact support even if we assume g to be in $C_c^{\infty}(\mathbb{R})$.

This is why we consider extended generators (see for instance [29] or [10]). These extended generators are defined by the point-wise convergence on \mathbb{R} instead of the uniform convergence. Moreover, they verify the fundamental martingale property, which allows us to define the generator on $C_b^n(\mathbb{R})$ for suitable $n \in \mathbb{N}^*$ and to prove that some properties of the classical theory of semigroups still hold for this larger class of functions.

Definition 5.1. *Let $(X_t)_t$ be a Markov process on \mathbb{R} . We define $P_t g(x) = \mathbb{E}_x[g(X_t)]$ for all measurable functions g such that $\mathbb{E}_x[|g(X_t)|]$ is finite for every $x \in \mathbb{R}$. Then we define $\mathcal{D}'(A)$ to be the set of measurable functions g such that for each $x \in \mathbb{R}$, $\frac{1}{t}(P_t g(x) - g(x))$ converges as t goes to 0 to some limit that we note $Ag(x)$ and such that:*

- for all $t \geq 0$, for all x , $\int_0^t |Ag(X_s)| ds$ is \mathbb{P}_x -almost surely defined and \mathbb{P}_x -almost surely finite,
- $g(X_t) - g(X_0) - \int_0^t Ag(X_s) ds$ is a \mathbb{P}_x -martingale for all x .

Remark 5.2. One can note that if we assume the function $t \mapsto P_t Ag(x)$ to be continuous and that $\int_0^t \mathbb{E}_x [|Ag(X_s)|] ds$ is finite for any $x \in \mathbb{R}$, then the martingale property of Definition 5.1 implies that $Ag(x)$ is the derivative of $t \mapsto P_t g(x)$ at $t = 0$.

In our Definition 5.1, contrarily e.g. to [29], we impose this additional condition of differentiability. We do this because this is the main property we need in our applications. In practice, this differentiability condition is satisfied for C^2 functions g and for "common" Markov processes X with finite $g^{(k)}$ -moments ($0 \leq k \leq 2$).

We note $\mathcal{D}'(A)$ the domain of the extended generator to avoid confusions with $\mathcal{D}(A)$ which is reserved for the domain of A for the uniform convergence.

Now we generalize a classical result for generators defined with respect to the uniform convergence to extended generators. The difference is that here we have to replace the uniform convergence by point-wise convergence, hence we need boundedness assumptions on the Markov process.

Lemma 5.3. Let $(X_t)_t$ be a Markov process with semigroup $(P_t)_t$ and extended generator A .

- (1) Let $g \in \mathcal{D}'(A)$ and $x \in \mathbb{R}$ such that for all $t \geq 0$, $\mathbb{E}_x [\sup_{0 \leq s \leq t} |P_s Ag(X_t)|]$ is finite. Then the function $t \mapsto P_t g(x)$ is right differentiable at every $t \geq 0$, and we have

$$\frac{d^+}{dt} (P_t g(x)) = P_t Ag(x).$$

In addition, if $P_t g \in \mathcal{D}'(A)$, then $AP_t g(x) = P_t Ag(x)$.

- (2) Let $g \in \mathcal{D}'(A)$ and $x \in \mathbb{R}$ such that there exists some non-negative function $M : \mathbb{R} \rightarrow \mathbb{R}_+$ such that for all $t \geq 0$, $\sup_{0 \leq s \leq t} \mathbb{E}_x [M(X_s)]$ is finite and such that for all $0 \leq t \leq 1$ and $y \in \mathbb{R}$, we have $|P_t Ag(y) - Ag(y)| \leq CM(y)\varepsilon(t)$ for some constant C that is allowed to depend on g , where $\varepsilon(t)$ vanishes when t goes to 0. Then the function $t \mapsto P_t g(x)$ is left differentiable at every $t > 0$, and we have

$$\frac{d^-}{dt} (P_t g(x)) = P_t Ag(x).$$

Proof. For the point (1), we know that for all $h > 0$, we have:

$$\left| \frac{1}{h} (P_{t+h} g(x) - P_t g(x)) - P_t Ag(x) \right| \leq \mathbb{E}_x \left[\left| \frac{1}{h} (P_h g(X_t) - g(X_t)) - Ag(X_t) \right| \right].$$

As the expression appearing within the expectation above vanishes almost surely when h goes to 0 (since $g \in \mathcal{D}'(A)$), and as we can bound it by $\sup_{0 \leq s \leq t} |P_s Ag(X_t)| + |Ag(X_t)|$ (using the fact that $P_h g(y) - g(y) = \int_0^h P_s Ag(y) ds$ which is a consequence of the last point of Definition 5.1), we know that this expectation vanishes as h goes to 0 by dominated convergence. This means exactly that $\frac{d^+}{dt} (P_t g(x))$ exists and is $P_t Ag(x)$.

If we suppose in addition that $P_t g \in \mathcal{D}'(A)$, then $AP_t g(x)$ is the limit of $h^{-1}(P_{t+h} g(x) - P_t g(x))$, which is $\frac{d^+}{dt} P_t g(x) = P_t Ag(x)$.

Now we prove the point (2) of the lemma. Let h be some positive number. We know that

$$\left| \frac{1}{-h} (P_{t-h} g(x) - P_t g(x)) - P_t Ag(x) \right|$$

is upper bounded by

$$\mathbb{E}_x \left[\left| \frac{1}{h} (P_h g(X_{t-h}) - g(X_{t-h})) - Ag(X_{t-h}) \right| \right] + \mathbb{E}_x [|Ag(X_{t-h}) - P_h Ag(X_{t-h})|]$$

$$\leq \mathbb{E}_x \left[\sup_{0 \leq s \leq h} |Ag(X_{t-h}) - P_s Ag(X_{t-h})| \right] + \mathbb{E}_x [|Ag(X_{t-h}) - P_h Ag(X_{t-h})|].$$

Then we just have to show that $\mathbb{E}_x \left[\sup_{0 \leq s \leq h} |Ag(X_{t-h}) - P_s Ag(X_{t-h})| \right]$ vanishes when h goes to 0. But this follows from the fact that it is upper bounded by $C(\sup_{0 \leq s \leq h} \varepsilon(s))(\sup_{0 \leq r \leq t} \mathbb{E}_x [M(X_r)])$. \square

The goal of the next proposition is to obtain a control of the difference between the semigroups of two Markov processes, provided we dispose already of a control of the distance between the two generators. This proposition is an adaptation of Lemma 1.6.2 from [14] to the notion of extended generators defined by the point-wise convergence.

Proposition 5.4. *Let $(Y_t^N)_{t \in \mathbb{R}_+}$ and $(\bar{Y}_t)_{t \in \mathbb{R}_+}$ be Markov processes whose semigroups and (extended) generators are respectively P^N, A^N and \bar{P}, \bar{A} . We suppose that:*

- (i) *for all $x \in \mathbb{R}$ and $T > 0$, $\sup_{0 \leq t \leq T} \mathbb{E}_x [(\bar{Y}_t)^4] \leq C_T(1 + x^4)$ and $\sup_{0 \leq t \leq T} \mathbb{E}_x^N [(Y_t^N)^4] \leq C_T(1 + x^4)$ for some constant $C_T > 0$ not depending on N .*
- (ii) *for all $T > 0$, $\mathbb{E} [(\sup_{0 \leq t \leq T} |Y_t^N|)^2] < +\infty$.*
- (iii) *for all $0 \leq s, t \leq T$ and $x \in \mathbb{R}$,*

$$\mathbb{E}_x [(\bar{Y}_t - \bar{Y}_s)^2] \leq C_{T,x} \varepsilon(|t - s|) \text{ and } \mathbb{E}_x^N [(Y_t^N - Y_s^N)^2] \leq C_{T,x} \varepsilon(|t - s|),$$

where $\varepsilon(h)$ vanishes when h goes to 0, and where $C_{T,x}$ is some constant that depends only on T and x .

- (iv) *for all $g \in C_b^3(\mathbb{R})$, $\bar{P}_t g \in C_b^3(\mathbb{R})$, and for all $T > 0$, $\sup_{0 \leq t \leq T} \|\bar{P}_t g\|_{3,\infty} \leq Q_T \|g\|_{3,\infty}$ for some $Q_T > 0$.*
- (v) *for all $g \in C_b^3(\mathbb{R})$, $i \in \{0, 1, 2\}$ and $x \in \mathbb{R}$, $s \mapsto (\bar{P}_s g)^{(i)}(x) = \frac{\partial^i}{\partial x^i} (\bar{P}_s g(x))$ is continuous.*
- (vi) *$C_b^3(\mathbb{R}) \subseteq \mathcal{D}'(A^N) \cap \mathcal{D}'(\bar{A})$. For all $g \in C_b^3(\mathbb{R})$ and $x \in \mathbb{R}$, $|Ag(x)| \leq C \|g\|_{2,\infty} (1 + x^2)$ and $|A^N g(x)| \leq C \|g\|_{2,\infty} (1 + x^2)$.*
- (vii) *for all $g \in C_c^3(\mathbb{R})$ such that $\text{Supp } g \subseteq [-M, M]$, $\|(\bar{A}g)'\|_\infty \leq C \|g\|_{3,\infty} (1 + M^2)$ and $\|(A^N g)'\|_\infty \leq C \|g\|_{3,\infty} (1 + M^2)$.*
- (viii) *there exists some $C > 0$ such that for all $x, y \in \mathbb{R}$, for all $g \in C_b^3(\mathbb{R})$, $|\bar{A}g(x) - \bar{A}g(y)| \leq C(1 + x^2 + y^2)|x - y|$ and $|A^N g(x) - A^N g(y)| \leq C(1 + x^2 + y^2)|x - y|$.*
- (ix) *we assume that $\lim_{k \rightarrow \infty} \bar{A}g_k(x_k) = \lim_{k \rightarrow \infty} A^N g_k(x_k) = 0$, for any bounded sequence of real numbers $(x_k)_k$, and for any sequence $(g_k)_k$ of $C_b^3(\mathbb{R})$ satisfying*
 - (1) $\forall i \in \{0, 1, 2\}, \forall x \in \mathbb{R}, g_k^{(i)}(x) \xrightarrow[k \rightarrow \infty]{} 0$,
 - (2) $\forall i \in \{0, 1, 2, 3\}, \sup_k \|g_k^{(i)}\|_\infty < \infty$.

Then we have for each $g \in C_b^3(\mathbb{R})$, $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$:

$$(\bar{P}_t - P_t^N)g(x) = \int_0^t P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x) ds. \quad (13)$$

Remark 5.5. *Notice that the conditions of Proposition 5.4 are not all symmetric with respect to the processes \bar{Y} and Y^N . Indeed, the regularity hypothesis of the semigroup with respect to the initial condition only concerns \bar{P} (see hypothesis (iv) and (v)). Moreover, hypothesis (ii) provides a stronger control on Y^N than what is needed for \bar{Y} .*

Proof. To begin with, let us emphasize the fact that hypothesis (i) implies

$$\sup_{0 \leq t \leq T} \mathbb{E}_x [(\bar{Y}_t)^2] \leq C_T(1+x^2) \text{ and } \sup_{0 \leq t \leq T} \mathbb{E}_x^N [(Y_t^N)^2] \leq C_T(1+x^2), \quad (14)$$

since

$$\mathbb{E}_x [(\bar{Y}_t)^2] \leq \mathbb{E}_x [(\bar{Y}_t)^4]^{1/2} \leq C_T \sqrt{1+x^4} \leq C_T (1+x^2).$$

We fix $t \geq 0, N \in \mathbb{N}^*, g \in C_b^3(\mathbb{R}), x \in \mathbb{R}$ in the rest of the proof. We note $u(s) = P_{t-s}^N \bar{P}_s g(x)$. Firstly we show that $s \mapsto \bar{P}_s g(x)$ and $s \mapsto P_s^N h(x)$ are differentiable for all $h \in C_b^3(\mathbb{R})$, by showing that \bar{P} and P^N satisfy the hypothesis of Lemma 5.3. The condition of the point (1) of the lemma is a straightforward consequence of (14) and hypothesis (vi), and the conditions of the point (2) are satisfied for $M(x) = \sqrt{1+x^4}$ using (14) and hypothesis (i), (iii) and (viii). As a consequence, and thanks to hypothesis (iv), u is differentiable and

$$\begin{aligned} u'(s) &= -\frac{d}{du} (P_u^N \bar{P}_s g(x)) \Big|_{u=t-s} + \frac{d}{du} (P_{t-s}^N \bar{P}_u g(x)) \Big|_{u=s} \\ &= -P_{t-s}^N A^N \bar{P}_s g(x) + P_{t-s}^N \bar{P}_s \bar{A} g(x) \\ &= P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x). \end{aligned}$$

The second equality comes from the fact that \bar{P} satisfy the additional assumption of the point (1) of Lemma 5.3 (see hypothesis (iv) and (vi)).

Now we show that u' is continuous. Indeed if it is the case, then we will have

$$u(t) - u(0) = \int_0^t u'(s) ds,$$

which is exactly the assertion. In order to prove the continuity of u' , we consider a sequence $(s_k)_k$ that converges to some $s \in [0, t]$, and we write

$$\begin{aligned} |P_{t-s}^N (\bar{A} - A^N) \bar{P}_s g(x) - P_{t-s_k}^N (\bar{A} - A^N) \bar{P}_{s_k} g(x)| &\leq |(P_{t-s}^N - P_{t-s_k}^N) (\bar{A} - A^N) g_s(x)| \quad (15) \\ &\quad + |P_{t-s_k}^N (\bar{A} - A^N) (\bar{P}_s - \bar{P}_{s_k}) g(x)|, \quad (16) \end{aligned}$$

where $g_s = \bar{P}_s g \in C_b^3(\mathbb{R})$.

To show that the term (15) vanishes when k goes to infinity, we introduce, for all $M > 0$ the function $\varphi_M(g_s)(y) = g_s(y) \cdot \xi_M(y)$ where $\xi_M : \mathbb{R} \rightarrow [0, 1]$ is C^∞ , and $\forall |y| \leq M, \xi_M(y) = 1$ and $\forall |y| \geq M+1, \xi_M(y) = 0$. We note that the term (15) is bounded by

$$|(P_{t-s}^N - P_{t-s_k}^N) (\bar{A} - A^N) \varphi_M(g_s)(x)| + |(P_{t-s}^N - P_{t-s_k}^N) (\bar{A} - A^N) (g_s - \varphi_M(g_s))(x)| =: A_1 + A_2.$$

If we consider the function $h_{M,s} = (\bar{A} - A^N) \varphi_M(g_s)$, using hypothesis (iii), (iv) and (vii), we have

$$\begin{aligned} A_1 &\leq \mathbb{E}_x^N [|h_{M,s}(Y_{t-s}^N) - h_{M,s}(Y_{t-s_k}^N)|] \\ &\leq \|h'_{M,s}\|_\infty \mathbb{E}_x^N [|Y_{t-s}^N - Y_{t-s_k}^N|] \leq C(1+M^2) \|g\|_{3,\infty} \varepsilon(|s-s_k|)^{1/2}. \end{aligned}$$

Choosing $M = M_k = \varepsilon(|s-s_k|)^{-1/5}$, it follows that $\lim_{k \rightarrow \infty} A_1 = 0$. To see that the term A_2 vanishes, it is sufficient to notice that A_2 is bounded by

$$\mathbb{E}_x^N [|(\bar{A} - A^N) (g_s - \varphi_{M_k}(g_s)) (Y_{t-s}^N)|] + \mathbb{E}_x^N [|(\bar{A} - A^N) (g_s - \varphi_{M_k}(g_s)) (Y_{t-s_k}^N)|].$$

We know that the expressions in the expectations vanish almost surely (using hypothesis (ix)), and then we can apply dominated convergence (using hypothesis (ii) and (vi)).

We just proved that the term (15) vanishes. To finish the proof, we need to show that the term (16) vanishes. We note that the term (16) is bounded by:

$$\mathbb{E}_x^N [|\bar{A}g_k(Y_{t-s_k}^N)|] + \mathbb{E}_x^N [A^N g_k(Y_{t-s_k}^N)],$$

where $g_k = (\bar{P}_s - \bar{P}_{s_k})g \in C_b^3(\mathbb{R})$.

We have to show that the terms in the sum above vanish as k goes to infinity. Firstly we know that $\bar{A}g_k(Y_{t-s_k}^N)$ and $A^N g_k(Y_{t-s_k}^N)$ vanish almost surely when k goes to infinity (see hypothesis (ii), (iv), (v) and (ix)). Dominated convergence, using (14) and hypothesis (ii), (iv) and (vi), then implies the result. \square

5.2. Existence and uniqueness of the process $(X_t^N)_t$

Proposition 5.6. *If Assumptions 1 and 2 hold, the equation (5) admits a unique non-exploding strong solution.*

Proof. It is well known that if f is bounded, there is a unique strong solution of (5) (see Theorem IV.9.1 of [22]). In the general case we reason in a similar way as in the proof of Proposition 2 in [15]. Consider the solution $(X_t^{N,K})_{t \in \mathbb{R}_+}$ of the equation (5) where f is replaced by $f_K : x \in \mathbb{R} \mapsto f(x) \wedge \sup_{|y| \leq K} f(y)$ for some $K \in \mathbb{N}^*$. Introduce moreover the stopping time

$$\tau_K^N = \inf \left\{ t \geq 0 : |X_t^{N,K}| \geq K \right\}.$$

Since for all $t \in [0, \tau_K^N \wedge \tau_{K+1}^N]$, $X_t^{N,K} = X_t^{N,K+1}$, we know that $\tau_K^N(\omega) \leq \tau_{K+1}^N(\omega)$ for all ω . Then we can define τ^N as the non-decreasing limit of τ_K^N . With a classical reasoning relying on Ito's formula and Grönwall's lemma, we can prove that

$$\mathbb{E} \left[\left(X_{t \wedge \tau_K^N}^{N,K} \right)^2 \right] \leq C_t (1 + x^2), \quad (17)$$

where $C_t > 0$ does not depend on K . As a consequence, we know that almost surely, $\tau^N = +\infty$. So we can simply define X_t^N as the limit of $X_t^{N,K}$, as K goes to infinity. Now we show that X^N satisfies equation (5). Consider some $\omega \in \Omega$ and $t > 0$, and choose K such that $\tau_K^N(\omega) > t$. Then we know that for all $s \in [0, t]$, $X_s^N(\omega) = X_s^{N,K}(\omega)$ and $f(X_s^N(\omega)) = f_K(X_s^{N,K}(\omega))$. Moreover, as $X^{N,K}(\omega)$ satisfies the equation (5) with f replaced by f_K , we know that $X^N(\omega)$ verifies the equation (5) on $[0, t]$. This holds for all $t > 0$. As a consequence, we know that X^N satisfies the equation (5). This proves the existence of strong solution. The uniqueness is a consequence of the uniqueness of strong solutions of (5), if we replace f by f_K in (5), and of the fact that any strong solution $(Y_t^N)_t$ equals necessarily $(X_t^{N,K})_t$ on $[0, \tau_K^N]$. \square

5.3. Proof of Lemma 2.1

We begin with the proof of (i). Let $\Phi(x) = x^2$ and A^N be the extended generator of $(X_t^N)_{t \geq 0}$. One can note that, applying Fatou's lemma to the inequality (17), one obtains for all $t \geq 0$, $\mathbb{E} [(X_t^N)^2]$

is finite. As a consequence $\Phi \in \mathcal{D}'(A^N)$ (in the sense of Definition 5.1), and for all $x \in \mathbb{R}$,

$$\begin{aligned} A^N \Phi(x) &= -2\alpha\Phi(x) + \sigma^2 f(x) \leq -2\alpha\Phi(x) + \sigma^2 \left(L|x| + \sqrt{f(0)} \right)^2 \\ &\leq (\sigma^2 L^2 - 2\alpha)\Phi(x) + 2\sigma^2 L|x| \sqrt{f(0)} + \sigma^2 f(0). \end{aligned}$$

Let $\varepsilon > 0$ be fixed, and $\eta_\varepsilon = 2\sigma^2 L \sqrt{f(0)}/\varepsilon$. Using that, for every $x \in \mathbb{R}$, $|x| \leq x^2/\eta_\varepsilon + \eta_\varepsilon$, we have

$$A^N \Phi(x) \leq c_\varepsilon \Phi(x) + d_\varepsilon, \quad (18)$$

with $c_\varepsilon = \sigma^2 L^2 - 2\alpha + \varepsilon$ and $d_\varepsilon = O(1/\varepsilon)$. Let us assume that $c_\varepsilon \neq 0$, possibly by reducing $\varepsilon > 0$. Considering $Y_t^N := e^{-c_\varepsilon t} \Phi(X_t^N)$, by Ito's formula,

$$\begin{aligned} dY_t^N &= -c_\varepsilon e^{-c_\varepsilon t} \Phi(X_t^N) dt + e^{-c_\varepsilon t} d\Phi(X_t^N) \\ &= -c_\varepsilon e^{-c_\varepsilon t} \Phi(X_t^N) dt + e^{-c_\varepsilon t} A^N \Phi(X_t^N) dt + e^{-c_\varepsilon t} dM_t, \end{aligned}$$

where $(M_t)_{t \geq 0}$ is a \mathbb{P}_x -martingale. Using (18), we obtain

$$dY_t^N \leq d_\varepsilon e^{-c_\varepsilon t} dt + e^{-c_\varepsilon t} dM_t,$$

implying

$$\mathbb{E}_x [Y_t^N] \leq \mathbb{E}_x [Y_0^N] + \frac{d_\varepsilon}{c_\varepsilon} e^{(-c_\varepsilon t + 1)}.$$

One deduces

$$\mathbb{E}_x \left[(X_t^N)^2 \right] \leq x^2 e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)t} + \frac{C}{\varepsilon} \left(e^{(\sigma^2 L^2 - 2\alpha + \varepsilon)t} + 1 \right), \quad (19)$$

for some constant $C > 0$ independent of t, ε, N .

The proof of (ii) is analogous and therefore omitted.

Now we prove (iii). From

$$X_t^N = X_0^N - \alpha \int_0^t X_s^N ds + \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{]0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{s-}^N)\}} d\pi_j(s, z, u),$$

we deduce

$$\begin{aligned} \left(\sup_{0 \leq s \leq t} |X_s^N| \right)^2 &\leq 3(X_0^N)^2 + 3\alpha^2 t \int_0^t (X_s^N)^2 ds \\ &\quad + 3 \sum_{j=1}^N \left(\sup_{0 \leq s \leq t} \left| \int_{]0,s] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbf{1}_{\{z \leq f(X_{r-}^N)\}} d\pi_j(r, z, u) \right| \right)^2. \end{aligned} \quad (20)$$

Applying Burkholder-Davis-Gundy inequality to the last term above in (20), we can bound its expectation by

$$\begin{aligned} 3N \mathbb{E} \left[\int_{]0,t] \times \mathbb{R}_+ \times \mathbb{R}} u^2 \mathbf{1}_{\{z \leq f(X_{s-}^N)\}} d\pi_j(s, z, u) \right] &\leq 3N\sigma^2 \int_0^t \mathbb{E} [f(X_{s-}^N)] ds \\ &\leq 3N\sigma^2 C \int_0^t (1 + \mathbb{E} [(X_s^N)^2]) ds. \end{aligned} \quad (21)$$

Now, bounding (20) by (21), and using point (i) of the lemma we conclude the proof of (iii). Finally, (iv) can be proved in classical way, applying Ito's formula and Grönwall's lemma.

5.4. Proof of Proposition 2.4

To begin with, we use Theorem 1.4.1 of [25] to prove that the flow associated to the SDE (6) admits a modification which is C^3 with respect to the initial condition x (see also Theorem 4.6.5 of [26]). Indeed the local characteristics of the flow are given by

$$b(x, t) = -\alpha x \quad \text{and} \quad a(x, y, t) = \sigma^2 \sqrt{f(x)f(y)},$$

and, under Assumptions 1 and 3, they satisfy the conditions of Theorem 1.4.1 of [25]:

- $\exists C, \forall x, y, t, |b(x, t)| \leq C(1 + |x|)$ and $|a(x, y, t)| \leq C(1 + |x|)(1 + |y|)$.
- $\exists C, \forall x, y, t, |b(x, t) - b(y, t)| \leq C|x - y|$ and $|a(x, x, t) + a(y, y, t) - 2a(x, y, t)| \leq C|x - y|^2$.
- $\forall 1 \leq k \leq 4, 1 \leq l \leq 4 - k, \frac{\partial^k}{\partial x^k} b(x, t)$ and $\frac{\partial^{k+l}}{\partial x^k \partial y^l} a(x, y, t)$ are bounded.

In the following, we consider the process $(\bar{X}_t^{(x)})_t$, solution of the SDE (6) and satisfying $\bar{X}_0^{(x)} = x$. Then we can consider a modification of the flow $\bar{X}_t^{(x)}$ which is C^3 with respect to the initial condition $x = \bar{X}_0^{(x)}$. It is then sufficient to control the moment of the derivatives of $\bar{X}_t^{(x)}$ with respect to x , since with those controls we will have

$$\begin{aligned} \bar{P}_t g(x) &= \mathbb{E} \left[g \left(\bar{X}_t^{(x)} \right) \right], \quad (\bar{P}_t g)'(x) = \mathbb{E} \left[\frac{\partial \bar{X}_t^{(x)}}{\partial x} g' \left(\bar{X}_t^{(x)} \right) \right], \\ (\bar{P}_t g)''(x) &= \mathbb{E} \left[\frac{\partial^2 \bar{X}_t^{(x)}}{\partial x^2} g' \left(\bar{X}_t^{(x)} \right) + \left(\frac{\partial \bar{X}_t^{(x)}}{\partial x} \right)^2 g'' \left(\bar{X}_t^{(x)} \right) \right], \\ (\bar{P}_t g)'''(x) &= \mathbb{E} \left[\frac{\partial^3 \bar{X}_t^{(x)}}{\partial x^3} g' \left(\bar{X}_t^{(x)} \right) + 3 \frac{\partial^2 \bar{X}_t^{(x)}}{\partial x^2} \cdot \frac{\partial \bar{X}_t^{(x)}}{\partial x} g'' \left(\bar{X}_t^{(x)} \right) + \left(\frac{\partial \bar{X}_t^{(x)}}{\partial x} \right)^3 g''' \left(\bar{X}_t^{(x)} \right) \right]. \end{aligned} \quad (22)$$

We start with the representation

$$\bar{X}_t^{(x)} = x e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} \sqrt{f \left(\bar{X}_s^{(x)} \right)} dB_s.$$

This implies

$$\frac{\partial \bar{X}_t^{(x)}}{\partial x} = e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} \frac{\partial \bar{X}_s^{(x)}}{\partial x} \left(\sqrt{f} \right)' \left(\bar{X}_s^{(x)} \right) dB_s. \quad (23)$$

Writing $U_t = e^{\alpha t} \frac{\partial \bar{X}_t^{(x)}}{\partial x}$ and

$$M_t = \int_0^t \sigma \left(\sqrt{f} \right)' \left(\bar{X}_s^{(x)} \right) dB_s, \quad (24)$$

we obtain $U_t = 1 + \int_0^t U_s dM_s$, whence

$$U_t = \exp \left(M_t - \frac{1}{2} \langle M \rangle_t \right).$$

Notice that this implies $U_t > 0$ almost surely, whence $\frac{\partial \bar{X}_t^{(x)}}{\partial x} > 0$ almost surely. Hence

$$U_t^p = e^{pM_t - \frac{p}{2}\langle M \rangle_t} = \exp\left(pM_t - \frac{1}{2}p^2 \langle M \rangle_t\right) e^{\frac{1}{2}p(p-1)\langle M \rangle_t} = \mathcal{E}(M)_t e^{\frac{1}{2}p(p-1)\langle M \rangle_t}.$$

Since $(\sqrt{f})'$ is bounded, M_t is a martingale, thus $\mathcal{E}(M)$ is an exponential martingale with expectation 1, implying that

$$\mathbb{E}U_t^p \leq e^{\frac{1}{2}p(p-1)\sigma^2 m_1^2 t}, \quad (25)$$

where m_1 is the bound of $(\sqrt{f})'$ introduced in Assumption 3. In particular we have

$$\mathbb{E}\left[\left(\frac{\partial \bar{X}_t^{(x)}}{\partial x}\right)^2\right] \leq e^{(\sigma^2 m_1^2 - 2\alpha)t} \text{ and } \mathbb{E}\left[\left|\frac{\partial \bar{X}_t^{(x)}}{\partial x}\right|^3\right] \leq e^{(3\sigma^2 m_1^2 - 3\alpha)t}. \quad (26)$$

Differentiating (23) with respect to x , we obtain

$$\frac{\partial^2 \bar{X}_t^{(x)}}{\partial x^2} = \sigma \int_0^t e^{-\alpha(t-s)} \left[\frac{\partial^2 \bar{X}_s^{(x)}}{\partial x^2} (\sqrt{f})'(\bar{X}_s^{(x)}) + \left(\frac{\partial \bar{X}_s^{(x)}}{\partial x}\right)^2 (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) \right] dB_s. \quad (27)$$

We introduce $V_t = \frac{\partial^2 \bar{X}_t^{(x)}}{\partial x^2} e^{\alpha t}$ and deduce from this that

$$V_t = \sigma \int_0^t \left[V_s (\sqrt{f})'(\bar{X}_s^{(x)}) + e^{-\alpha s} U_s^2 (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) \right] dB_s,$$

which can be rewritten as

$$dV_t = V_t dM_t + Y_t dB_t, \quad V_0 = 0, \quad Y_t = \sigma e^{-\alpha t} U_t^2 (\sqrt{f})^{(2)}(\bar{X}_t^{(x)}),$$

with M_t as in (24). Applying Ito's formula to $Z_t := V_t/U_t$ (recall that $U_t > 0$), we obtain

$$dZ_t = \frac{Y_t}{U_t} dB_t - \frac{Y_t}{U_t} d\langle M, B \rangle_t,$$

such that, by the precise form of Y_t and since $Z_0 = 0$,

$$Z_t = \sigma \int_0^t e^{-\alpha s} U_s (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) dB_s - \sigma^2 \int_0^t e^{-\alpha s} U_s (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) (\sqrt{f})'(\bar{X}_s^{(x)}) ds.$$

Using Jensen's inequality, (25) and Burkholder-Davis-Gundy inequality, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}[Z_t^4] &\leq C \left(\mathbb{E}\left[\left(\int_0^t e^{-\alpha s} U_s (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) dB_s\right)^4\right] \right. \\ &\quad \left. + \mathbb{E}\left[\left(\int_0^t e^{-\alpha s} U_s (\sqrt{f})'(\bar{X}_s^{(x)}) (\sqrt{f})^{(2)}(\bar{X}_s^{(x)}) ds\right)^4\right] \right) \end{aligned}$$

$$\begin{aligned} &\leq C(t+t^3) \int_0^t e^{(6\sigma^2 m_1^2 - 4\alpha)s} ds \\ &\leq C(t+t^3) \left(1+t+e^{(6\sigma^2 m_1^2 - 4\alpha)t}\right). \end{aligned} \quad (28)$$

We deduce that

$$\mathbb{E}[V_t^2] \leq \mathbb{E}[Z_t^4]^{1/2} \mathbb{E}[U_t^4]^{1/2} \leq C(t^{1/2} + t^2)e^{3\sigma^2 m_1^2 t},$$

whence

$$\mathbb{E} \left[\left(\frac{\partial^2 \bar{X}_t^{(x)}}{\partial x^2} \right)^2 \right] \leq C(t^{1/2} + t^2)e^{(3\sigma^2 m_1^2 - 2\alpha)t}. \quad (29)$$

Finally, differentiating (27), we get

$$\begin{aligned} \frac{\partial^3 \bar{X}_t^{(x)}}{\partial x^3} = \sigma \int_0^t e^{-\alpha(t-s)} &\left[\frac{\partial^3 \bar{X}_s^{(x)}}{\partial x^3} (\sqrt{f})' (\bar{X}_s^{(x)}) + 3 \frac{\partial^2 \bar{X}_s^{(x)}}{\partial x^2} \frac{\partial \bar{X}_s^{(x)}}{\partial x} (\sqrt{f})^{(2)} (\bar{X}_s^{(x)}) \right. \\ &\left. + \left(\frac{\partial \bar{X}_s^{(x)}}{\partial x} \right)^3 (\sqrt{f})^{(3)} (\bar{X}_s^{(x)}) \right] dB_s. \end{aligned}$$

Introducing $W_t = e^{\alpha t} \frac{\partial^3 \bar{X}_t^{(x)}}{\partial x^3}$, we obtain

$$W_t = \sigma \int_0^t \left[W_s (\sqrt{f})' (\bar{X}_s^{(x)}) + 3e^{-\alpha s} U_s V_s (\sqrt{f})^{(2)} (\bar{X}_s^{(x)}) + e^{-2\alpha s} U_s^3 (\sqrt{f})^{(3)} (\bar{X}_s^{(x)}) \right] dB_s.$$

Once again we can rewrite this as

$$dW_t = W_t dM_t + Y_t' dB_t, W_0 = 0,$$

where

$$Y_t' = \sigma \left(3e^{-\alpha t} U_t V_t (\sqrt{f})^{(2)} (\bar{X}_t^{(x)}) + e^{-2\alpha t} U_t^3 (\sqrt{f})^{(3)} (\bar{X}_t^{(x)}) \right),$$

whence, introducing $Z_t' = \frac{W_t}{U_t}$,

$$Z_t' = \int_0^t \frac{Y_s'}{U_s} dB_s - \int_0^t \frac{Y_s'}{U_s} d \langle M, B \rangle_s.$$

As previously, we obtain,

$$\begin{aligned} \mathbb{E} \left[(Z_t')^2 \right] &\leq C(1+t) \int_0^t \mathbb{E} \left[\left(\frac{Y_s'}{U_s} \right)^2 \right] ds \\ &\leq C(1+t) \int_0^t (e^{-2\alpha s} \mathbb{E}[V_s^2] + e^{-4\alpha s} \mathbb{E}[U_s^4]) ds \\ &\leq C(1+t) \int_0^t \left((s^{1/2} + s^2)e^{(3\sigma^2 m_1^2 - 2\alpha)s} + e^{(6\sigma^2 m_1^2 - 4\alpha)s} \right) ds \end{aligned}$$

$$\begin{aligned}
&\leq C(1+t^3) \int_0^t \left(e^{(3\sigma^2 m_1^2 - 2\alpha)s} + e^{(6\sigma^2 m_1^2 - 4\alpha)s} \right) ds \\
&\leq C(1+t^3) \int_0^t \left(1 + e^{(6\sigma^2 m_1^2 - 4\alpha)s} \right) ds \leq C(1+t^4) \left(1 + e^{(6\sigma^2 m_1^2 - 4\alpha)t} \right). \quad (30)
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\mathbb{E}[|W_t|] &\leq \mathbb{E}[(Z'_t)^2]^{1/2} \mathbb{E}[U_t^2]^{1/2} \leq C(1+t^2) \left(1 + e^{(3\sigma^2 m_1^2 - 2\alpha)t} \right) e^{\frac{1}{2}\sigma^2 m_1^2 t} \\
&\leq C(1+t^2) \left(e^{\frac{1}{2}\sigma^2 m_1^2 t} + e^{(\frac{7}{2}\sigma^2 m_1^2 - 2\alpha)t} \right),
\end{aligned}$$

implying

$$\mathbb{E} \left[\left| \frac{\partial^3 \bar{X}_t^{(x)}}{\partial^3 x} \right| \right] \leq C(1+t^2) \left(e^{(\frac{1}{2}\sigma^2 m_1^2 - \alpha)t} + e^{(\frac{7}{2}\sigma^2 m_1^2 - 3\alpha)t} \right). \quad (31)$$

Finally, using Cauchy-Schwarz inequality, and inserting (26), (29) and (31) in (22),

$$\left\| (\bar{P}_t g)''' \right\|_{\infty} \leq C \|g\|_{3,\infty} (1+t^2) \left(e^{(\frac{1}{2}\sigma^2 m_1^2 - \alpha)t} + e^{2(\sigma^2 m_1^2 - \alpha)t} + e^{(\frac{7}{2}\sigma^2 m_1^2 - 3\alpha)t} \right),$$

which proves the proposition.

Bibliography

- [1] Aït-Sahalia, Y., Cacho-Diaz, J., and Laeven, R. J. A. (2015). Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117:585–606.
- [2] Bacry, E. and Muzy, J. F. (2016). Second order statistics characterization of Hawkes processes and non-parametric estimation. *Trans. in Inf. Theory*, 2.
- [3] Bauwens, L. and Hautsch, N. (2009). *Modelling financial high frequency data using point processes*. Springer Berlin Heidelberg.
- [4] Billingsley, P. (1999). *Convergence of Probability Measures*. Wiley Series In Probability And Statistics, second edition.
- [5] Brémaud, P. and Massoulié, L. (1996). Stability of Nonlinear Hawkes Processes. *The Annals of Probability*, 24(3):1563–1588.
- [6] Carmona, R., Delarue, F., and Lacker, D. (2016). Mean field games with common noise. *Ann. Probab.*, 44(6):3740–3803.
- [7] Clinet, S. and Yoshida, N. (2017). Statistical inference for ergodic point processes and application to Limit Order Book. *Stochastic Processes and their Applications*, 127(6):1800–1839.
- [8] Costa, M., Graham, C., Marsalle, L., and Tran, V. C. (2018). Renewal in Hawkes processes with self-excitation and inhibition. *arXiv e-prints*, page arXiv:1801.04645.
- [9] Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods*. Springer, second edition.
- [10] Davis, M. H. A. (1993). *Markov Models and Optimization*. Springer Science+Business Media Dordrecht, first edition.
- [11] Dawson, D. and Vaillancourt, J. (1995). Stochastic McKean-Vlasov equations. *Nonlinear Differential Equations and Applications NoDEA*, 2(2):199–229.
- [12] Delattre, S., Fournier, N., and Hoffmann, M. (2016). Hawkes processes on large networks. *Ann. Appl. Probab.*, 26(1):216–261.

- [13] Ditlevsen, S. and Löcherbach, E. (2017). Multi-class Oscillating Systems of Interacting Neurons. *Stochastic Processes and their Applications*, 127(6):1840–1869.
- [14] Ethier, S. and Kurtz, T. (2005). *Markov Processes. Characterization and Convergence*. Wiley Series In Probability And Statistics.
- [15] Fournier, N. and Löcherbach, E. (2016). On a toy model of interacting neurons. *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 52:1844–1876.
- [16] Graham, C. (2019). Regenerative properties of the linear Hawkes process with unbounded memory. *arXiv:1905.11053 [math, stat]*.
- [17] Grün, S., Diedsmann, M., and Aertsen, A. M. (2010). *Analysis of Parallel Spike Trains*. Rotter, Springer series in computational neurosciences.
- [18] Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58:83–90.
- [19] Hawkes, A. G. and Oakes, D. (1974). A Cluster Process Representation of a Self-Exciting Process. *Journal of Applied Probability*, 11(3):493–503.
- [20] Helmstetter, A. and Sornette, D. (2002). Subcritical and supercritical regimes in epidemic models of earthquake aftershocks. *Journal of Geophysical Research*, 107.
- [21] Hewlett, P. (2006). Clustering of order arrivals, price impact and trade path optimisation. In Workshop on Financial Modeling with Jump processes. Ecole Polytechnique.
- [22] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*. North-Holland Publishing Company, second edition.
- [23] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer-Verlag BerlinHeidelberg NewYork, second edition.
- [24] Khasminskii, R. (2012). *Stochastic stability of differential equations*. Springer, second edition.
- [25] Kunita, H. (1986). Lectures on Stochastic Flows and Applications for the Indian Institute Of Science Bangalore.
- [26] Kunita, H. (1990). *Stochastic flows and stochastic differential equations*. Cambridge University Press.
- [27] Kurtz, T. G. and Xiong, J. (1999). Particle representations for a class of nonlinear SPDEs. *Stochastic Processes and their Applications*, 83(1):103–126.
- [28] Lu, X. and Abergel, F. (2018). High dimensional Hawkes processes for limit order books modelling, empirical analysis and numerical calibration. *Quantitative Finance*, pages 1–16.
- [29] Meyn, S. P. and Tweedie, R. L. (1993). Stability of Markovian Processes III: Foster-Lyapunov Criteria for Continuous-Time Processes. *Applied Probability Trust*, 25:518–548.
- [30] Ogata, Y. (1978). The asymptotic behavior of maximum likelihood estimators for stationary point processes. *Annals of the Institute of Statistical Mathematics*, 30:243–261.
- [31] Ogata, Y. (1999). Seismicity analysis through point-process modeling: A review. *Pure and applied geophysics*, 155:471–507.
- [32] Okatan, M., A Wilson, M., and N Brown, E. (2005). Analyzing functional connectivity using a network likelihood model of ensemble neural spiking activity. *Neural computation*, 17:1927–61.
- [33] Pillow, J. W., Wilson, M. A., and Brown, E. N. (2008). Spatio-temporal correlations and visual signalling in a complete neuronal population. *Nature*, 454:995–999.
- [34] Raad, M. B. (2019). Renewal Time Points for Hawkes Processes. *arXiv:1906.02036 [math]*.
- [35] Reynaud-Bouret, P., Rivoirard, V., Grammont, F., and Tuleau-Malot, C. (2014). Goodness-of-Fit Tests and Nonparametric Adaptive Estimation for Spike Train Analysis. *Journal of Mathematical Neuroscience*, 4:3 – 330325.
- [36] Reynaud-Bouret, P. and Schbath, S. (2010). Adaptive estimation for Hawkes processes; appli-

- cation to genome analysis. *Ann. Statist.*, 38(5):2781–2822.
- [37] Y. Kagan, Y. (2009). Statistical Distributions of Earthquake Numbers: Consequence of Branching Process. *Geophys. J. Int.*, 180.
- [38] Zhou, K., Zha, H., and Song, L. (2013). Learning triggering kernels for multi-dimensional Hawkes processes. Proceedings of the 30th International Conf. on Machine Learning (ICML).