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Confluence in (Un)Typed Higher-Order Theories I

Gaspard Férey and Jean-Pierre Jouannaud

INRIA, Dedukti and ENS-Cachan, LSV, France

Abstract

In a series of papers, we develop techniques based on van Oostrom’s decreasing diagrams that reduce confluence proofs to the checking of various forms of critical pairs for higher-order rewrite rules extending β -reduction on pure λ -terms. The present paper concentrates on the case of left-linear rewrite rules, assuming that critical pairs can be joined without beta-reduction steps.

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1 Introduction

User-defined higher-order rewrite rules are becoming a standard in proof assistants based on intuitionistic type theory. This is true of DEDUKTI and Agda, and will be true of Coq next. This raises the question of proving that these rules preserve consistency of the type theory.

Consistency of a type theory relies on three other properties: confluence, type preservation and strong normalization. In dependent type theories, confluence is needed to prove type preservation and strong normalization, making all three properties interdependent if termination is used in the confluence proof. This circularity can be broken by proving these properties within a single induction [9], or by proving confluence on untyped terms first, and then, successively, type preservation, confluence on typed terms, and strong normalization.

We focus here on confluence on untyped terms. The format of the higher-order rules is classical: left-hand sides must be patterns [16, 15], which are extremely useful for describing encodings of a type theory in another, a keen application to us. Considering untyped terms requires simple adaptations of the usual higher-order rewriting definitions.

There are three main tools for analyzing confluence of a rewrite relation: Newman’s Lemma [17], Hindley-Rosen’s Lemma [11], and van Oostrom’s decreasing diagrams Theorem [20, 22]. Since beta rewrites are non-terminating in pure lambda calculus, Newman’s Lemma does not apply. And if the rules have non-trivial critical pairs, then Hindley-Rosen’s Lemma does not apply either. Even its subtle use allowing for development-closed critical pairs [21] is too restrictive for practical usage. One way out is the use of decreasing diagrams.

Van Oostrom’s theorem is abstract, its application to non-terminating term rewriting relations conceals many difficulties [2]. Further, neither confluence nor termination are preserved by adding a confluent and terminating set of rewrite rules to a λ -calculus. A counter-example to termination in the simply typed λ -calculus is given in [18]. Numerous counter-examples to confluence in the pure λ -calculus, due to Klop, are given in [14].

Our contribution is the description of the conditions under which sets of left-linear higher-order rules preserve confluence of the untyped λ -calculus. Left-linear rules that are terminating (along with β^0) preserve confluence on all terms if their critical pairs are joinable. When they are not, then their parallel critical pairs must be joinable. In both these cases, $\beta^{\neq 0}$ -steps can’t be used. When $\beta^{\neq 0}$ -steps are needed, then *nested critical pairs* must be checked, a result proved in separate papers [1, 8]. The case of left-linear rules is therefore now fully understood. Examples illustrate our results.

We recall the notion of labeled reduction and decreasing diagram in Section 2, and describe our higher-order setting in Section 3. Our two results come separately in Sections 4 and 5, along with examples. Confluence of non-left-linear rules are discussed in conclusion.

2 Labeled reductions

2.1 Reductions

Given a binary relation \longrightarrow on terms, called *rewriting*, we use: \longleftarrow for its inverse, \Longrightarrow for its parallelization, allowing one to rewrite at once several subterms of a given term, when none is a subterm of another, and \longleftrightarrow , $\longrightarrow\!\!\!\!\!\rightarrow$, and $\longleftarrow\!\!\!\!\!\leftarrow$, for its closures by, respectively, symmetry; reflexivity and transitivity; and reflexivity, symmetry and transitivity (called *convertibility*).

Rewriting terms extends to substitutions as expected.

A term s is in *normal form* if there is no t such that $s \longrightarrow t$. We define a *normal form* for an arbitrary term s as a term t in normal form, denoted by $s \downarrow$, such that $s \longrightarrow\!\!\!\!\!\rightarrow t$. Termination is the impossibility of an infinite rewriting sequence $t_0 \longrightarrow t_1 \longrightarrow \dots \longrightarrow t_n \longrightarrow \dots$. Termination guarantees the existence of normal forms for every term. A *local peak* is a triple of terms (s, u, t) such that $s \longleftarrow u \longrightarrow t$; u is the *source* and s, t are its *reducts*. Two terms s, t are *joinable* if $s \longrightarrow\!\!\!\!\!\rightarrow v \longleftarrow\!\!\!\!\!\leftarrow t$ for some v , making the peak $s \longleftarrow u \longrightarrow t$ itself *joinable*. The property that every two convertible terms are joinable is called *confluence* (or *Church-Rosser*). Confluence guarantees the unicity of normal forms for every term.

When rewriting terminates, it is well-known that the joinability of all local peaks implies the confluence property, this is the so-called Newman's lemma. When it does not, it is then necessary to strengthen joinability, this is the rôle of *decreasing diagrams*.

2.2 Decreasing diagrams

In the following, we consider rewrite relations whose all elementary steps are equipped with a label belonging to some well-founded set whose partial order is denoted by \triangleright .

► **Definition 1** (Decreasing diagram [20]). Given a labeled relation \longrightarrow on an abstract set, we denote by $DS(m, n)$ the set of *decreasing rewrite sequences* of the form $u \xrightarrow{\alpha} s \xrightarrow{n} t \xrightarrow{\delta} v$ such that the labels in α and β are strictly smaller than, respectively, m , and, m or n .

Given a local peak $v \xleftarrow{m} u \xrightarrow{n} w$, a *decreasing (rewrite) diagram* is a pair of decreasing derivations $v \xrightarrow{\alpha} s \xrightarrow{n} t \xrightarrow{\delta} w$ w.r.t. (m, n) and $w \xleftarrow{\alpha'} s \xrightarrow{m} t \xrightarrow{\delta'} v$ w.r.t. (n, m) . The steps labeled by α, α' , by δ, δ' , and by m, n are called the *side* steps, *facing* steps and *middle* steps of the decreasing diagram, respectively.

Decreasing rewrite diagrams are abbreviated as DDs. Note that a facing step of a decreasing diagram may be missing, its side steps are then absorbed by the middle ones. Local peaks whose DDs have no side nor middle steps are called *strongly joinable*.

► **Theorem 2** ([22]). *A labeled relation is Church-Rosser if all its local peaks have DDs.*

A relation whose all local peaks are strongly joinable is called *strongly confluent*.

van Oostrom's theorem generalizes to rewriting modulo an equational theory, for example $=_{\alpha}$ in the λ -calculus, in which case \triangleright must be compatible with the equational theory [13]. Further, equational steps must have a minimal label.

3 Higher-order rewriting

Given now a set R of user-defined rewrite rules, we are interested in rewriting terms of an *untyped* lambda calculus generated by three pairwise disjoint sets, a signature \mathcal{F} of *function symbols*, a set \mathcal{X} of *variables*, and a set \mathcal{Z} of *meta-variables*. We use small letters

$x, y, z, \dots, f, g, h, \dots$ for variables and function symbols, and reserve capital letters X, Y, Z, \dots for meta-variables. The *arity* $|a|$ of $a \in \mathcal{F} \cup \mathcal{Z}$ is fixed. We use the same notation $|_|_$ to denote various other quantities, such as the length of a list, the size of an expression or the cardinality of a set.

Were we to analyze the confluence of R alone, then, the situation would be similar to the first-order case, at least when left-hand sides of rules are patterns in the sense of Miller [16]. Unfortunately, as is stressed in [2], modularity results do not scale to higher-order reductions, preventing the possibility to deduce the confluence of $R \cup \beta$ from that of R . Our purpose in the rest of this section is to analyze the calculus $\lambda\mathcal{F}$, made of R -reductions and β -reductions.

3.1 Terms in $\lambda\mathcal{F}$

$\lambda\mathcal{F}$ is a mix of the pure lambda-calculus and Klop's combinatory reduction systems [14]. Terms are those of the pure lambda calculus enriched with \mathcal{F} -headed terms of the form $f(\bar{u})$ with $f \in \mathcal{F}$, and *meta-terms* of the form $Z(\bar{u})$ with $Z \in \mathcal{Z}$, where \bar{u} is a list of terms whose length equals the arity of f or Z respectively. Only variables can be abstracted over. $\mathcal{V}ar(M)$ and $M\mathcal{V}ar(M)$ are the sets of free variables and of meta-variables respectively, that occur in M . A term M is *closed* if $M\mathcal{V}ar(M) = \emptyset$, *ground* if $\mathcal{V}ar(M) = \emptyset$. The grammar of terms is:

$$u, v := x \in \mathcal{X} \mid (u v) \mid \lambda x.u \mid f(\bar{u}) \mid Z(\bar{v}) \quad \text{where } |\bar{u}| = |f|, Z \in \mathcal{Z} \text{ and } |\bar{v}| = |Z|$$

Positions in terms are words over the natural numbers (assuming $|\lambda x._| = 1$), using \cdot for concatenation, Λ for the empty word, $\geq_{\mathcal{P}}$ for the prefix order on positions (*below*), $\leq_{\mathcal{P}}$ for its inverse (*above*), $>_{\mathcal{P}}$ for its strict part, and $p \# q$ for $\neg(>_{\mathcal{P}} \vee \leq_{\mathcal{P}})$ (*parallel*).

Given a term M , we use $M(p)$ for the symbol at positions p in M , and $\mathcal{P}os(M)$, $\mathcal{V}\mathcal{P}os(M)$, $M\mathcal{P}os(M)$ for the following respective sets of positions of M : all positions, the positions of free variables, and of meta-variables. A term M is linear if $|M\mathcal{P}os(M)| = |M\mathcal{V}ar(M)|$.

Substitutions are *capture-avoiding* homomorphisms written as $\sigma = \{x_1 \mapsto M_1, \dots, x_n \mapsto M_n\}$, or $\sigma = \{\bar{x} \mapsto \bar{M}\}$, where M_i has the form $\lambda \bar{y}_i.N_i$ if x_i has arity $|\bar{y}_i|$. The substitution σ is \mathcal{F} -headed if its *values* N_i are \mathcal{F} -headed. $\mathcal{D}om(\sigma) = \{x_1, \dots, x_n\} \subseteq \mathcal{X} \cup \mathcal{Z}$ is the *domain* of σ while $\mathcal{R}an(\sigma) = \bigcup_{i=1}^n \mathcal{V}ar(M_i)$ is its *image*. Substitutions are extended to sequences of terms or to substitutions in the natural way. We will use postfix notation for the application of σ to a term t , writing $t\sigma$, or to a vector of terms \bar{t} , writing $\bar{t}\sigma$, or to a substitution τ , writing $\tau\sigma$, and call $t\sigma$ (resp., $\bar{t}\sigma$, $\tau\sigma$) the *instance* of t (resp., \bar{t} , τ) by σ . The notation $\mathcal{P}os(\sigma)$ will have the obvious meaning of a sequence of $\mathcal{D}om(\sigma)$ -indexed sets of positions.

As in λ -calculus, substituting in terms requires renaming bound variables to avoid capturing free variables: $x_i\sigma = t_i$ and $y\sigma = y$ if $y \notin \mathcal{D}om(\sigma)$; $f(\bar{t})\sigma = f(\bar{t}\sigma)$; $(u v)\sigma = (u\sigma v\sigma)$; and $(\lambda x.u)\sigma = \lambda x.u\sigma$ if $x \notin \mathcal{D}om(\sigma) \cup \mathcal{R}an(\sigma)$ (otherwise, x must be renamed.) The additional rule for meta-variables is as follows: if $Z \mapsto \lambda \bar{x}.s \in \sigma$, then $(Z(\bar{u}))\sigma = s\{\bar{x} \mapsto \sigma(\bar{u})\}$. Since only variables can be abstracted over, substituting a meta-variable ends up in substituting (possibly several) variables. These substitutions compose as usual.

Given a term u and a list $P = \{p_i\}_{i=1}^n$ of parallel positions in u , we define the term obtained by *splitting* u along P as $\underline{u}_P = u[Z_1(\bar{x}_1)]_{p_1} \dots [Z_n(\bar{x}_n)]_{p_n}$ (u is cut below P) and its associated substitution by $\bar{u}^P = \{Z_i \mapsto \lambda \bar{x}_i.u|_{p_i}\}_{i=1}^n$ (u is cut above P), where, for all $i \in [1, n]$, \bar{x}_i is the list of all variables of $u|_{p_i}$ bound in u above p_i and Z_i is a fresh meta-variable of arity $|\bar{x}_i|$. Klop's definition of substitution for meta-variables ensures that $\underline{u}_P \bar{u}^P = u$, as with first-order terms, which justifies writing $u = u|_P|_P$ as a familiar shorthand.

In this paper, meta-variables are used for expressing splitting, which will play a major rôle, as well as free variables in rewrite rules, and nested critical pairs. Apart from these three different uses, terms will always be closed.

3.2 Functional reductions

Two different kinds of reductions coexist in $\lambda\mathcal{F}$, functional and higher-order reductions.

Arrow signs are often decorated by the position p at which rewriting takes place, as in $s \xrightarrow{p} t$ or by a property that this position satisfies, as in $u \xrightarrow{\geq \mathcal{P}^p} v$ and in $u = v \downarrow^{\geq \mathcal{P}^P}$ (u is obtained from v by normalizing its subterms $v|_{p \in P}$.)

Functional reduction is the relation on terms generated by the rule $(\lambda x.u v) \xrightarrow{\beta_\alpha} u\{x \mapsto v\}$.

The usually omitted α -index stresses that renaming bound variables is built-in. We also use the particular cases β^0 and $\beta^{\neq 0}$, for which v is a variable and is not a variable, respectively. Note the difficulty that instantiating a β^0 -steps may yield a $\beta^{\neq 0}$ -step. Note also that β^0 defines a confluent and terminating rewrite relation. Let $M\downarrow$ denote the normal form of M with respect to β^0 . A term M is *normal* if $M = M\downarrow$. A substitution is normal if its values are normal. A substitution σ *preserves* a normal term s if $s\sigma$ is normal, and is *preserving* if it preserves all normal terms. Normal substitutions whose values are not abstractions are preserving.

3.3 Higher-order reductions

Higher-order reductions result from two kinds of rules, whose left-hand sides are either higher-order patterns in Miller's sense [15] or algebraic expressions as in recursor rules [4]. Miller's patterns require higher-order pattern matching to fire rules, while first-order pattern matching suffices for algebraic left-hand sides. Our definition of pattern captures both.

► **Definition 3** (Pattern). A *pattern* is a β -normal, \mathcal{F} -headed, ground term whose all meta-variables occur in maximal subterms, called *pre-redexes*, of the form $(Z(\bar{x}) \bar{y})$, where $\bar{x}\bar{y}$ is a list of pairwise distinct variables.

It is important to assume, as we shall see, that patterns are β -normal, not only normal. The notion of *fringe* plays a key rôle:

► **Definition 4** (Fringe). The *fringe* F_L of a pattern L is the set of parallel positions of its pre-redexes. We denote by $\mathcal{FPos}(L) = \{p \in \mathcal{Pos}(L) : p <_{\mathcal{P}} F_L\}$ the set of *functional positions* of the pattern L , and by $MVar(L, o)$, for $o \in F_L$, the meta-variable Z such that $L|_o = (Z(\bar{x}) \bar{y})$. We also define $F_\beta = \{1, 2\}$ for convenience.

The terms $(X(x, y) z)$ and $Y(y, z)$ are the pre-redexes of the pattern $L = f(\lambda xyz.g((X(x, y) z), (Y(y, z) a)))$, whose fringe is the set $F_L = \{1^5, 1^4 2^1\}$. Note that the set of functional positions coincides with the usual notion for first-order terms.

Since patterns are ground terms, we have:

► **Lemma 5.** *Given a pattern L , let $p \in F_L$ and $L|_p = (Z(\bar{x}) \bar{y})$ be a pre-redex. Then, all variables in \bar{x}, \bar{y} are bound above in L .*

Instantiating the meta-variables in a pattern, as is well-known, produces β^0 -redexes which may propagate upwards until they hit the *fringe*:

► **Lemma 6.** *Let L be a pattern and γ a normal substitution. Then, $L\gamma \xrightarrow[\beta^0]{\geq \mathcal{P}^{F_L}} (L\gamma)\downarrow$.*

Proof. The proof is by induction on the size of $v = L\gamma$. If v is normal, we are done. Otherwise, by confluence of β^0 , $v = L\gamma \xrightarrow[\beta^0]{p} v' \xrightarrow[\beta^0]{\rightarrow} (L\gamma)\downarrow$. Since $p \not\leq_{\mathcal{P}} F_L$ would contradict the fact that v is normal at all positions which are not below F_L , $p \geq_{\mathcal{P}} F_L$. We then conclude by induction on v' . ◀

We can now define higher-order rules and rewriting:

► **Definition 7 (Rule).** A (higher-order) *rule* is a triple $i : L \rightarrow R$, whose *index* i is a natural number, left-hand side L is a pattern, and right-hand side R is a β -normal term such that $MVar(R) \subseteq MVar(L)$.

► **Definition 8 (Higher-order untyped rewriting).** Given a term u , a position $p \in Pos(u)$ such that $u|_p$ is normal, and a rule $i : L \rightarrow R$, then u rewrites with i at p , written $u \xrightarrow[p]{i} v$, iff $u|_p =_{\beta^0} L\gamma$ for some normal substitution γ , and $v = u[X(\bar{x})]_p \{X \mapsto \lambda \bar{x}. R\gamma\} = u[R\gamma]_p$, where \bar{x} is the list of variables of $u|_p$ which are bound above the position p in u .

Note that we do *not* assume that u is normal, nor do we assume that v is normal, or even normal up to position p . The reason is that we need monotonicity and stability properties:

► **Lemma 9 (Monotonicity).** Let $s \xrightarrow[L \rightarrow R]{p} t$ and u a term such that $q \in Pos(u)$. Then, $u[s]_q \xrightarrow[L \rightarrow R]{q \cdot p} u[t]_q$.

While monotonicity follows directly from the definition, stability is more delicate:

► **Lemma 10 (Stability).** Let s, t be terms such that $s \xrightarrow[L \rightarrow R]{p} t$ and σ a closed substitution. Then:

If σ is preserving s , then $s\sigma \xrightarrow[L \rightarrow R]{p} t\sigma$; Otherwise, $s\sigma \downarrow^{\geq p p} \xrightarrow[L \rightarrow R]{p} \xrightarrow[\beta^0]{\geq p p} t\sigma \downarrow^{\geq p p}$.

Proof. By definition of higher-order rewriting, $s|_p =_{\beta^0} L\gamma$ for some normal substitution γ , and $t = s[R\gamma]_p$. By Lemma 6, $L\gamma \xrightarrow[\beta^0]{\geq p FL} s|_p$. Since γ is normal, all β^0 -redexes in this derivation originate from the instantiation by γ of a pre-redex of L , hence are of the form $(\lambda z.w \ x)$ where x is bound above in L by Lemma 5. It follows that $L\gamma\sigma \xrightarrow[\beta^0]{\geq p FL} (s|_p)\sigma = s\sigma|_p \xrightarrow[\beta^0]{\geq p FL} (s\sigma|_p)\downarrow$.

Hence $(s\sigma|_p)\downarrow =_{\beta^0} L((\gamma\sigma)\downarrow) \xrightarrow[L \rightarrow R]{\Lambda} R((\gamma\sigma)\downarrow)$, and $s\sigma \downarrow^{\geq p p} \xrightarrow[L \rightarrow R]{p} s\sigma[R((\gamma\sigma)\downarrow)]_p$ by monotonicity.

We now conclude. If σ preserves s , then σ must preserve γ , hence $\gamma\sigma$ is normal. Hence, $R((\gamma\sigma)\downarrow) = (R\gamma)\sigma$, and $s\sigma \xrightarrow[L \rightarrow R]{p} t\sigma$. Otherwise, $R((\gamma\sigma)\downarrow) \xrightarrow[\beta^0]{\geq p FL} (R\gamma\sigma)\downarrow$, yielding the result. ◀

Monotonicity and stability extend to rewriting at a set of parallel positions P .

3.4 Rewrite theories

► **Definition 11.** We call *higher-order (untyped) rewriting system* a set \mathcal{R} of higher-order rules and denote by $\rightarrow_{\mathcal{R}}$ the rewrite relation generated by \mathcal{R} .

A $\lambda\mathcal{F}$ -rewrite theory is a pair $(\mathcal{F}, \mathcal{R})$ of a user's signature \mathcal{F} and a higher-order rewrite system \mathcal{R} , defining the rewrite relation $\xrightarrow[\mathcal{R} \cup \beta_\alpha]{\mathcal{F}}$ also denoted by $\xrightarrow[\lambda\mathcal{F}]{}.$

The main question addressed in this paper is whether a $\lambda\mathcal{F}$ -rewrite theory is Church-Rosser (or confluent), and how to show confluence by calculating and inspecting critical pairs of some form. We shall focus on rewrite theories for which the set of rules \mathcal{R} satisfies linearity assumptions. We say that $\lambda\mathcal{F}$ is : a *left-linear* theory if \mathcal{R} is a set of left-linear rules; a *right-linear* theory if \mathcal{R} is a set of right-linear rules; a *semi-linear* theory if \mathcal{R} is made of rules which are of either kind. We restrict our attention here to left-linear theories $(\mathcal{F}, \mathcal{R}_{ll})$.

3.5 Local ancestor peaks in rewrite theories

Rewrite theories have two kinds of local ancestor peaks, *homogeneous* ones, between functional or higher-order reductions, and *heterogeneous* ones, which mix both kinds of reductions. Our goal here is to analyze which local ancestor peaks enjoy *decreasing diagrams for free*.

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A major property of plain first-order rewriting is that there are three possible kinds of local peaks depending on the respective positions of the rewrites that define the peak. This property generalizes to higher-order rewrites with our definition of $\mathcal{FPos}(L)$ when L is a pattern:

► **Lemma 12.** *Given terms s, t such that $s \xleftarrow[p]{i:L \rightarrow R} u \xrightarrow[q]{j:G \rightarrow D} t$, then, either $p \# q$ (disjoint peak case), or $q \geq_P p \cdot F_L$ or $p \geq_P q \cdot F_L$ (ancestor peak case), or $p = q \cdot o$ with $o \in \mathcal{FPos}(L)$ or $q = p \cdot o$ with $o \in \mathcal{FPos}(G)$ (overlapping peak case).*

In the case of plain rewriting, two non-overlapping rewrite steps issuing from a same term commute, a major component of any confluence proof. When the steps occur at disjoint positions, this property, which holds for any monotonic relation, remains true for rewriting modulo a theory, hence all disjoint peaks have decreasing diagrams for free. This is not the case, however, when the steps occur at positions whose one is an ancestor of the other, because the modulo part of the above rewrite may interact with the rewrite below. Our definition of higher-order rewriting, however, enjoys a similar property, because the fringe of a rewrite step protects positions below it. These local peak properties of higher-order rewriting are shown at Figures 1 and 3. The β^0 -steps in the lower part of the pictures serve normalizing terms, in particular before a higher-order rewriting step can be applied.

Figure 3 shows the form of an ancestor peak with a β -step above a higher-order step, a property called (LAP β a), where “LAP” stands for *linear ancestor peak* and “a” for *above*. There are two cases: if the higher-order rewrite applies to the argument, the ancestor peak is joinable as in the first-order case; if it applies to the body, some β^0 -steps are needed since the instance of a higher-order redex requires being normalized before being fired. The lemma below adds extra β^0 -steps so as to capture both cases into a single one:

► **Lemma 13 (LAP β a).** *Let u be a term, $p, q \in \mathcal{Pos}(u)$ such that $q \geq_P p \cdot F_\beta$ and $s \xleftarrow[p]{\beta} u \xrightarrow[q]{j \in \mathcal{R}} t$. Then $\xrightarrow[\beta^0]{j} \xrightarrow[\beta^0]{Q} \xrightarrow[\beta^0]{j} \xleftarrow[\beta^0]{\beta} \xleftarrow[\beta^0]{\beta} \xleftarrow[\beta^0]{\beta} t$ for some set Q of parallel positions of s such that $Q \geq_P p$.*

Proof. By assumption, $u|_p = (\lambda x.M N)$. The case where $q = p \cdot 2 \cdot q'$ does not require β^0 -steps, but several j -steps at the parallel positions of x in M . Otherwise, $q = p \cdot 1^2 \cdot q'$, that is, $M|_{q'} = u|_q \xrightarrow{j} t|_q$. Then, $v = u[M\sigma]_p = u[M\sigma[u|_q\sigma]_{q'}]_p$, where $\sigma = \{x \mapsto N\}$ is in general not a preserving substitution. By Lemma 10, $(u|_q\sigma) \downarrow \xrightarrow{j} \xrightarrow[\beta^0]{j} (t|_q\sigma) \downarrow$, hence $s \xrightarrow[\beta^0]{j} u[M\sigma[(u|_q\sigma) \downarrow]_{q'}]_p \xrightarrow{j} \xrightarrow[\beta^0]{j} u[P\sigma[(t|_q\sigma) \downarrow]_{q'}]_p$.

On the other hand, $t = u[(\lambda x.P N)]_p$, where $P = s[t|_q]_{q'}$, hence $t \xrightarrow[p]{\beta} u[s[t|_q]_{q'}\sigma]_p = u[P\sigma[t|_q\sigma]_{q'}]_p \xrightarrow[\beta^0]{j} u[P\sigma[(t|_q\sigma) \downarrow]_{q'}]_p$, and we are done. ◀

The case of a local peak $s \xleftarrow[p]{i} u \xrightarrow[q]{j} t$, where the higher-order step with $i : L \rightarrow R$ applies above another step with $j : G \rightarrow D$, a situation called (LAP \mathcal{R} a), is shown at Figure 1 (left). Its proof requires an important preliminary result:

► **Lemma 14 (Preservation).** *Let $u \xrightarrow[p]{i:L \rightarrow R} v$ and $q \in \mathcal{Pos}(u)$ such that $q \geq_P p \cdot F_L$ and $u(q) \in \mathcal{F}$. Then $\underline{u}_q = u[Z(\bar{y})]_q \xrightarrow[p]{i} w$ for some w , and $v = w\bar{u}^q = w\{Z \mapsto \lambda \bar{y}.u|_q\}$.*

Proof. By definition of splitting, let t be the term $\underline{u}_q = u[Z(\bar{y})]_q$ and τ be the substitution $\bar{u}^q = \{Z \mapsto \lambda \bar{y}.u|_q\}$, where \bar{y} is the vector of variables bound in u above the position q . Then, $u = t\tau$. Further, since $u|_q$ is normal and \mathcal{F} -headed, τ is preserving.

Since $q \geq_{\mathcal{P}} p \cdot F_L$, then $q = p \cdot o \cdot q'$, where $o \in F_L$. Let then $L|_o = ((X \bar{x})\bar{z})$ for some meta-variable X and variables \bar{x} and \bar{z} bound above in L .

By definition of higher-order rewriting, $u|_p =_{\beta^0} L\gamma$ for some normal substitution γ , hence $L\gamma \xrightarrow[\beta^0]{\geq_{\mathcal{P}} F_L} u|_p$ by Lemma 6. By definition of a substitution, $\gamma(X) = \lambda\bar{x}.M$, and by the previous property, $((X \bar{x})\bar{z})\gamma \xrightarrow[\beta^0]{} u|_{p \cdot o}$. Since $u|_{p \cdot o \cdot q'}$ is normal and \mathcal{F} -headed, these β^0 -steps must result in a term of which $u|_q$ is a subterm. Therefore, there exists a position q'' in M such that $M|_{q''} = u|_q$ (up to variable renaming).

Let now θ be the substitution identical to γ except for the meta-variable X for which $\gamma(X) = \lambda\bar{x}.M$ and $\theta(X_i) = \lambda\bar{x}.M[Z(\bar{y})]_{q''}$, where \bar{y} is the vector of variables bound above q in u . It follows that $\gamma = \theta\tau$. Since L is linear, $L\theta \xrightarrow[\beta^0]{} u|_p[Z(\bar{y})]_{o \cdot q''}$, a normal term, hence $u|_p[Z(\bar{y})]_{o \cdot q''} \xrightarrow[\beta^0]{P} R\theta$. By monotonicity, $t = u[Z(\bar{y})]_q = u[u|_p[Z(\bar{y})]_{o \cdot q''}]_p \xrightarrow[\beta^0]{P} w = u[R\theta]_p$. Further, $v = R\gamma = R\theta\tau = w\tau$ and we are done. \blacktriangleleft

► **Lemma 15 (LAPRa).** *Let $i : L \rightarrow R \in \mathcal{R}$, $j \in \mathcal{R} \cup \beta$, u be a term, and $p, q \in \mathcal{P}os(u)$ such that $q \geq_{\mathcal{P}} p \cdot F_L$ and $s \xleftarrow[\beta^0]{\frac{P}{i}} u \xrightarrow[\beta^0]{\frac{Q}{j}} t$. Then, $s \xleftarrow[\beta^0]{\frac{Q_R}{j}} r = r \downarrow \xrightarrow[\beta^0]{\geq_{\mathcal{P}} P} \xleftarrow[\beta^0]{\frac{P}{i}} \xleftarrow[\beta^0]{\frac{P}{j}} t$ for some set Q_R of parallel positions of s such that $Q_R \geq_{\mathcal{P}} p$.*

Proof. By assumption, $q = p \cdot o \cdot p'$, where $o \in F_L$, hence $L|_o = X(\bar{x})$. Let now $O \subseteq F_L$ be the set of positions whose pre-redex is $X(\bar{x})$, up to the renaming of variables in \bar{x} . Let finally $Q = p \cdot O \cdot p'$, and note that $Q = q$ in case L is linear.

Splitting u at q yields $u = v\sigma$, where $v = \underline{u}_q$ and $\sigma = \bar{u}^q = \{Z \mapsto \lambda\bar{y}.u|_q\}$ is preserving since it must be normal because $q \geq_{\mathcal{P}} p \cdot F_L$ and $u|_q$ is \mathcal{F} -headed. By assumption, $\sigma(Z) = \lambda\bar{y}.u|_q \xrightarrow[\beta^0]{\frac{P}{j}} \lambda\bar{y}.t|_q$. Let τ be σ with the exception $\tau(Z) = \lambda\bar{y}.t|_q$. Then $\sigma \xrightarrow[\beta^0]{} \tau$ and $t = v\tau$.

By Lemma 14, $v \xrightarrow[\beta^0]{P} w$ for some w such that $s = w\sigma$. By Lemma 9, $v\sigma \xrightarrow[\beta^0]{} v\tau = t$. Let Q_R be the set of parallel positions of $Z(\bar{y})$ in w . Then Q_R is a set of parallel positions of $w\sigma$. By Lemma 9, $w\sigma \xrightarrow[\beta^0]{\frac{Q_R}{j}} w\tau$. By lemma 10, $(v\tau) \downarrow \xrightarrow[\beta^0]{\geq_{\mathcal{P}} P} \xrightarrow[\beta^0]{\frac{P}{i}} \xrightarrow[\beta^0]{\geq_{\mathcal{P}} P} (w\tau) \downarrow \xrightarrow[\beta^0]{\geq_{\mathcal{P}} P}$. The result follows. \blacktriangleleft

Patterns being β -normal, note that the assumptions $q \geq_{\mathcal{P}} p$ and $q \geq_{\mathcal{P}} p \cdot F_L$ are equivalent.

A last, easy property that we shall use is that β^0 -steps commute with other steps.

3.6 Critical peaks

Critical peaks are obtained by unifying left-hand sides of rules at subterms, resulting in overlapping peaks which are minimal w.r.t. instantiation. Our definitions and their properties live in an untyped world, hence need not to coincide with those used in a typed world.

► **Definition 16.** Let $i : L \rightarrow R$ and $j : G \rightarrow D$ be two rules in \mathcal{R} and $o \in \mathcal{F}Pos(L)$ such that the equation $L|_o =_{\beta^0} G$ has a most general solution σ .

Then, the peak $R\sigma \xleftarrow[\beta^0]{\frac{\Lambda}{i}} L\sigma \downarrow \xrightarrow[\beta^0]{\frac{o}{j}} (L\sigma) \downarrow [D\sigma]_o$ is called a *critical peak* of j onto i at position o . Its associated *critical pair* is $\langle (R\sigma) \downarrow, (L\sigma[D\sigma]_o) \downarrow \rangle$.

Note first that this definition makes sense: since $o \in \mathcal{F}Pos(L)$, then $o <_{\mathcal{P}} F_L$, and therefore, $o \in \mathcal{F}Pos((L\sigma) \downarrow)$ by Lemma 6. The same lemma shows that σ must actually satisfy the equation $L|_o \xrightarrow[\beta^0]{} \xleftarrow[\beta^0]{} G$. The existence of most general solutions in our context follows therefore by basic narrowing with β^0 [12]. Termination of basic narrowing is straightforward here, since narrowing steps reduce the total size of the pre-redexes. Further, since left-hand sides are linear, a deterministic search is complete, hence the result, if any, is unique.

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Using standard techniques, we then get the analog of Nipkow's critical pair lemma developed for the case of simply typed higher-order rewrite rules:

► **Lemma 17** (Critical pair lemma). *Assume $s \xleftarrow{i}^p u \xrightarrow{j}^q t$ is an overlapping peak of $j : G \rightarrow D$ onto $i : L \rightarrow R$ at position $o \in \mathcal{FPos}(L)$ such that $p = q \cdot o$. Then, there is a critical peak $s' \xleftarrow{i}^{\Lambda} u' \xrightarrow{j}^o t'$ and a normal substitution θ such that $s'\theta \xrightarrow{\beta^0} s|_p$ and $t'\theta \xrightarrow{\beta^0} t|_p$.*

Proof. By definition of higher-order rewriting, there exists some normal substitution γ such that $L\gamma \xrightarrow{\beta^0}^{\geq_P F_L} u|_p$, $G\gamma \xrightarrow{\beta^0}^{\geq_P F_G} u|_q$, $s|_p = R\gamma$ and $t|_q = D\gamma$. Since $o \in \mathcal{FPos}(L)$, then $o <_P F_L$, and since $L\gamma \xrightarrow{\beta^0}^{\geq_P F_L} u|_p$, then $u_p[]_o = L\gamma[]_o$, hence $t|_p = u|_p[t|_q]_o = (L\gamma[D\gamma]_o) = L[D]_o\gamma$.

Since $o \in \mathcal{FPos}(L)$, $(L\gamma)|_o = L|_o\gamma$, hence $L|_o\gamma \xrightarrow{\beta^0} \xleftarrow{\beta^0} G\gamma$. Therefore, γ is a higher-order unifier of the equation $L|_o = G$. Let σ be the most general higher-order unifier of the equation $L|_o = G$. Then, there exist a normal substitution θ such that $\sigma\theta \xrightarrow{\beta^0} \gamma$, using the fact that γ is normal. It follows that $u[R\sigma\theta]_p \xrightarrow{\beta^0} s$ and $u[L[D]_o\sigma\theta] \xrightarrow{\beta^0} t$. Since $s' = R\sigma$ and $t' = L[D]_o\sigma$, we get the result. ◀

3.7 Confluence in $\lambda\mathcal{F}$

We can now address the problem of confluence of a higher-order rewrite theory $\lambda\mathcal{F}$. The road map will always be the same. Step 1: define the rewrite relation \longrightarrow to be proved confluent; step 2: prove that the confluence of this relation implies the confluence of $\lambda\mathcal{F}$, more precisely: $\xrightarrow{\lambda\mathcal{F}} \subseteq \xrightarrow{\longrightarrow}$ (completeness) and $\xrightarrow{\longrightarrow} \subseteq \xrightarrow{\lambda\mathcal{F}}$ (soundness); step 3: define labels for each rewrite step ; step 4: define what are the critical peaks for that relation and give the critical pair lemma ; step 5: show that decreasing diagrams remain decreasing under context application and substitution instance ; and last step 6: show that all local peaks are decreasing with respect to the labeling (stability). Trivial steps will be omitted. The last step is the most complex one, and requires specific assumptions that include the existence of decreasing diagrams for all local peaks of the rewrite relation generated by the rules.

4 Terminating left-linear theories

We assume given a set \mathcal{Rll} of left-linear rewrite rules, and will consider the case where the relation generated by \mathcal{Rll} , more precisely by $\beta^0 \cup \mathcal{Rll}$, is terminating.

We now need to recall (a variation of) the notion of orthogonal reductions introduced by Tait to show the confluence of the λ -calculus (we need it here for the same purpose).

First, we define the product of sets of positions:

► **Definition 18.** Given a linear term u , a substitution σ , a set of parallel positions $P \subseteq \mathcal{FPos}(u)$, and a family Q of sets of positions indexed by $\mathcal{Var}(u)$, such that $Q_x \subseteq \mathcal{FPos}(x\sigma)$ for each variable $x \in \mathcal{Var}(u)$, we define the *orthogonal product* $P \otimes_u Q$ as the set of so-called *orthogonal* positions in $\mathcal{FPos}(u\sigma)$ defined as $P \otimes_u Q = P \cup \{o \cdot Q_x : u|_o = x \in \mathcal{Var}(u|_P)\}$.

Note that Q is a set of sets, rather than a set, and that a position $p \in \mathcal{FPos}(u\sigma)$ can always be seen (in many ways) as a singleton set of orthogonal positions in $\mathcal{FPos}(u\sigma)$.

► **Definition 19** (Orthogonal reductions). Let s, t be terms such that $u = s\sigma$ and $v = t\tau$ for some linear term u , term v and substitutions σ, τ , and $O \subseteq \mathcal{FPos}(u\sigma)$ such that $O = P \otimes_u Q$.

Orthogonal rewriting is the relation on terms such that $u \xrightarrow[\beta]{O} v$ provided $s \xrightarrow[\beta]{P} t$ and $\sigma \xrightarrow[\beta]{Q} \tau$, where the extension of $\xrightarrow[\beta]{}$ to substitutions is as expected: $\forall x \in \mathcal{Pos}(\sigma)(x\sigma \xrightarrow[\beta]{Q_x} x\tau)$.

We shall need several properties of orthogonal β -reductions. Firstly, orthogonal reductions are monotonic and stable. Being monotonic, they satisfy the commutation lemma with any other monotonic rewrite relation. Further, they also satisfy (LAP), whether above or below. Unlike the “above case”, the “below case” listed first follows easily from Lemma (LAPRa):

► **Lemma 20** (LAPOb). *Let $s \xrightarrow[\mathcal{R}ll]{Q} u \xrightarrow[\beta]{P} t$, where $P \geq_P Q$. Then,*
 $s \xrightarrow[\beta]{O} \xrightarrow[\beta^0]{\geq_P Q} r = r \downarrow \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\mathcal{R}ll]{Q} \xrightarrow[\beta^0]{\geq_P Q} t$ for some set O of positions in s such that $O >_P Q$.

► **Lemma 21** (LAPOba). *Let $s \xrightarrow[\beta]{P} u \xrightarrow[\mathcal{R}ll]{q} t$, where $q >_P P$. Then,*
 $s \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\mathcal{R}ll]{Q} \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\beta]{P} t$ for some set Q of parallel positions in s such that $Q >_P P$.

► **Proof.** By induction on the pair $\langle |P|, |U| \rangle$, we prove a more general result for which $u \xrightarrow[\mathcal{R}ll]{O} t$, where $O >_P P$ and all redexes in $u|_O$ are identical up to the renaming of their bound variables. If $\Lambda \notin P$, then $u = f(\bar{u})$, we can then apply the induction hypothesis to \bar{u} , and conclude by monotonicity and grouping together the appropriate steps. Otherwise, $u = (\lambda x.M N) \xrightarrow[\beta]{\Lambda} u_1 \xrightarrow[\beta]{P'} s$. We then write $O = 1^2 \cdot O_1 \cup 2 \cdot O_2$, where O_1, O_2 are (possibly empty) sets of parallel positions of M, N respectively, hence $M \xrightarrow[\mathcal{R}ll]{O_1} M'$ and $N \xrightarrow[\mathcal{R}ll]{O_2} N'$ and $t = (\lambda x.M' N') \xrightarrow[\beta]{} M'\{x \mapsto N'\} = t_1$. There are now two cases:

1. $O_2 \neq \emptyset$. Since all redexes in $u|_O$ are identical, we can assume without loss of generality that $x \notin \text{Var}(M|_{O_1})$. It follows that these identical $\mathcal{R}ll$ -redexes $u|_O$ occur now in $M\{x \mapsto N\}$ at the set of parallel positions $O' = O_1 \cup \{o' \cdot o : M|_{o'} = x \text{ and } o \in O_1\}$, and $M\{x \mapsto N\} \xrightarrow[\mathcal{R}ll]{O'} M'\{x \mapsto N'\}$.

We have got $s \xrightarrow[\beta]{P'} u_1 \xrightarrow[\mathcal{R}ll]{O'} t_1$, a linear ancestor peak satisfying our assumptions. By

induction hypothesis, $s \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\mathcal{R}ll]{Q} \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\beta^0]{\geq_P Q} v \xrightarrow[\beta]{P'} t$ for some set Q of parallel positions in s such that $Q >_P P'$. By definition of orthogonal reductions, $t \xrightarrow[\beta]{P} v$ and we are done.

2. $O_2 = \emptyset$, hence $N' = N$. This case is a bit more complex because the variable x may now occur below the $\mathcal{R}ll$ -redexes in M . Its substitution will therefore give, after β^0 -normalization, a set of identical redexes different from the previous ones, but at the set of positions O_1 in $M\{x \mapsto N\}$. By Lemma (LAP β a), $u_1 \xrightarrow[\beta^0]{\geq_P Q_1} u' \xrightarrow[\mathcal{R}ll]{O_1} t' \xrightarrow[\beta^0]{\geq_P Q_1} \xrightarrow[\beta^0]{\geq_P Q_1} t_1$.

Positions in P' are above those in O_1 . By commutation of $\xrightarrow[\beta^0]{\geq_P Q_1}$ with $\xrightarrow[\beta]{P'}$, we get

$s \xrightarrow[\beta^0]{\geq_P Q} s' \xrightarrow[\beta]{P'} v$. By induction hypothesis applied to the peak $s' \xrightarrow[\beta]{P'} u' \xrightarrow[\mathcal{R}ll]{O_1} t'$, we get

$s' \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\mathcal{R}ll]{Q} \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\beta^0]{\geq_P Q} v \xrightarrow[\beta]{P'} t'$. By commutation with β^0 , $v \xrightarrow[\beta^0]{\geq_P Q} \xrightarrow[\beta^0]{\geq_P Q} w \xrightarrow[\beta]{P} t_1$, hence

$t \xrightarrow[\beta]{} w$ by definition of orthogonal reductions. We conclude by confluence of β^0 . ◀

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Rewriting is a relation between two terms generated by a set of rewrite rules. Here, the terms are split into a term part and a substitution applied to that term. It turns out that different *splittings* of u into a term s and a substitution σ , and of the set of position O into the respective sets P and Q yield the same term v . We will not need this property, but a few others originating from the residuals' theory of the λ -calculus [3].

We are now ready to proceed with the first step.

Step 1: let $\longrightarrow = \xrightarrow{\beta^0} \cup \xrightarrow{\mathcal{R}ll} \cup \xrightarrow{\beta} \otimes \xrightarrow{\beta}$.

Redundancy does not harm, of course, but does ease the analysis of local peaks.

Step 2: Soundness and completeness are straightforward properties here.

Step 3: labels are pairs: $\langle 1, u \rangle$ for $u \xrightarrow{\beta^0} v$; $\langle 2, u \rangle$ for $u \xrightarrow{\mathcal{R}ll} v$; and $\langle 3, \perp \rangle$ for $u \xrightarrow{\beta} v$, \perp being a don't care constant. Labels are compared lexicographically, the first argument in the order on natural numbers, the second in the order $\xrightarrow{\beta^0 \cup \mathcal{R}ll}$, \perp being chosen minimal.

Step 4: critical pairs are higher-order critical pairs of rules in $\mathcal{R}ll$. An easy property is:

► **Lemma 22.** *Assume that $s \xrightarrow{\beta^0 \cup \mathcal{R}ll} t$. Then, $s \downarrow \xrightarrow{\beta^0 \cup \mathcal{R}ll} t \downarrow$.*

Proof. We show that $s \downarrow \xrightarrow{\beta^0 \cup \mathcal{R}ll} t \downarrow$ by induction on the length of the derivation, using Lemma 10 and confluence of β^0 . ◀

Step 5: this is trivial here since labels are decreasing along reductions with $\beta^0 \cup \xrightarrow{\mathcal{R}ll}$.

Step 6: Let $(\mathcal{R}ll)$ be the assumption that $\mathcal{R}ll \cup \beta^0$ is terminating.

► **Theorem 23.** *Assuming $(\mathcal{R}ll)$, then $\lambda\mathcal{F}$ is confluent if all critical pairs of $\mathcal{R}ll$ are joinable with rules of $\beta^0 \cup \mathcal{R}ll$.*

► **Proof.** Let $s \xleftarrow{P} u \xrightarrow{Q} t$ be an arbitrary local peak, where P, Q are possibly singleton sets of orthogonal positions.

1. First, rewrite steps of monotonic relations always commute when $P \# Q$, yielding a DD. We are now left with all peaks for which $\neg(P \# Q)$.
2. $s \xleftarrow{\otimes} u \xrightarrow{\otimes} t$. Orthogonal β -reductions are known to be strongly confluent, hence $s \xrightarrow{\otimes} v \xleftarrow{\otimes} t$ for some P', Q', v , a DD.
3. $s \xleftarrow{\beta^0} u \xrightarrow{\beta^0} t$. Then, by confluence of β^0 , $s \xrightarrow{\beta^0} v \xleftarrow{\beta^0} t$ for some v , a DD.
4. $s \xleftarrow{\beta^0} u \xrightarrow{\otimes} t$. Then, either $s \xrightarrow{\otimes} v \xleftarrow{\beta^0} t$ for some v if the β^0 -redex is also a β -redex, or both steps commute otherwise. The obtained diagrams are particular DDs.
5. $s \xleftarrow{i} u \xrightarrow{j} t$ with $i : L \rightarrow R \in \mathcal{R}ll$, $j \in \mathcal{R}rl$ and $q \geq_{\mathcal{P}} p \cdot F_L$. Lemma (LAPRa) yields a decreasing diagram without facing steps because labels decrease along reductions.
6. $s \xleftarrow{i} u \xrightarrow{j} t$ with $i : L \rightarrow R \in \mathcal{R}ll$, $j : G \rightarrow D \in \mathcal{R}rl$, $q \in p \cdot o$ and $o \in \mathcal{FPos}(F_L)$. By Lemma 17, there is a critical peak obtained by overlapping G onto L at position o . By assumption, this peak is joinable with rules of $\beta^0 \cup \mathcal{R}ll$, hence the pair s, t is joinable by the monotonicity lemma 9 and Lemma 22. Note that there are no facing steps here, since labels decrease strictly along $(\mathcal{R}ll \cup \beta^0)$ -reductions.
7. $s \xleftarrow{\beta^0} u \xrightarrow{\mathcal{R}ll} t$. Since $u|_q$ must be normal, $q >_{\mathcal{P}} p$. Then, both steps commute, giving a DD.

8. $s \xleftarrow[\beta]{P} u \xrightarrow[i]{q} t$, where $i : L \rightarrow R \in \mathcal{R}ll$. Since $u(q) \in \mathcal{F}$, then $q \notin P$, hence $P = P_1 \cup P_2 \cup P_3$,

where $P_1 = \{o \in P : o \# q\}$, $P_2 = \{o \in P : o \geq_{\mathcal{P}} q\}$ and $P_3 = \{o \in P : o <_{\mathcal{P}} q\}$. Note that $P_1 \# (P_2 \cup P_3)$ and $P_2 = q \cdot P'_2$. The proof of this case is shown at Figure 2.

Let $v = u[X(\bar{x})]_q$ and $\sigma = \{X \mapsto \lambda \bar{x}. u|_q\}$, where \bar{x} is the list of variables bound above q in u . Then, $u = v\sigma$, $P_1 \cup P'_2 \subseteq \mathcal{FPos}(\sigma(X))$ and $P_3 \subseteq \mathcal{FPos}(v)$. By linearizing the orthogonal rewrite from u to s , we can decompose the local peak as follows: $v_2\tau \xleftarrow[\beta]{Q \cdot P_3} v_1\tau \xleftarrow[\beta]{P_2} v_1\sigma \xleftarrow[\beta]{P_1} u = v\sigma \xrightarrow[i]{q} v\theta = t$, where $\theta = \{X \mapsto \lambda \bar{x}. t|_q\}$. Note that, since $P_1 \# q$, then q and P_3 denote positions in v_1 .

By commutation, $v_1\sigma \xrightarrow[i]{q} v_1\theta \xleftarrow[\beta]{P_1} v\theta$. Hence $v_1\sigma \xrightarrow[\beta^0]{\geq_{\mathcal{P}} q} v_1\tau' \xrightarrow[i]{q} v_1\tau'' \xrightarrow[\beta^0]{\geq_{\mathcal{P}} q} v_1\gamma \xleftarrow[\beta]{P'_2} v_1\theta$

for some $P'_2 \triangleright_{\mathcal{P}} q$ by (LAPOb). We then get $v_2\tau \xrightarrow[\beta^0]{\geq_{\mathcal{P}} Q} w_2 \xleftarrow[\beta]{P_3} v_1\tau'$ by commutation of β^0 with other rewrites. By (LAPOa), $w_2 \xrightarrow[i]{Q} w_1 \xleftarrow[\beta^0]{P_3} v_1\tau'$. By commutation again,

$w_1 \xrightarrow[\beta^0]{P_3} v_2\gamma \xleftarrow[\beta]{P_3} v_1\tau' = w$. Since $P'_2 \geq_{\mathcal{P}} q$, then $P' = p_1 \cup p''_2 \cup P_3$ is a set of

orthogonal positions, hence $t \xrightarrow[\beta]{P'} w$, and we have got a DD for this last local peak.

By Lemma 12, all cases have been considered, we are therefore done.

Note that the last case in the proof is actually a generalization of (LAPOa) and (LAPOb) to an arbitrary local peak between β - and $\mathcal{R}ll$ -rewrites, which we could have singled out.

► **Example 24.** We consider Plotkin's and Power's theory of global states for a single location [19]. It is described by two types, Val for values and A for states, a unary operation lk for looking up a state, a binary operation ud for updating a state, and five higher-order rules which satisfy our format:

$$lk : (Val \rightarrow A) \rightarrow A \quad | \quad ud : Val, A \rightarrow A$$

$lk(\lambda v.t)$ looks up the state, binds its value to v , and continues with t while $ud(v,t)$ updates the state to v , and continues with t . We give their types for a better understanding:

$$\begin{array}{ll} (ll) & lk(\lambda w.lk(\lambda v.X[v,w])) \rightarrow lk(\lambda v.X[v,v]) & (ll) \\ (lu) & lk(\lambda v.ud(v,X[v])) \rightarrow lk(\lambda v.X[v]) \quad | \quad lk(\lambda v.X) \rightarrow X & (l) \\ (ul) & ud(V,lk(\lambda v.X[v])) \rightarrow ud(V,X[V]) \quad | \quad ud(V,ud(W,X)) \rightarrow ud(W,X) & (uu) \end{array}$$

This typed higher-order theory was studied by Hamana, who was indeed interested in its confluence investigated with his Haskell-based analysis tool SOL [10]. Our presentation is a simplification of Hamana's, whose one rule was actually superfluous. SOL shows the confluence of the example for simple types with prenex polymorphism. We show below that it is confluent for any type discipline for which it is terminating. To this end, we need to show first that it is terminating (in conjunction with β^0), and that the critical pairs are joinable. In the absence of $\beta^{\neq 0}$, first-order termination techniques can do. We are left with verifying the joinability of critical pairs, these computations are presented inside individual boxes. In the upper middle of each box appear two rules whose superposition is inside braces. The upper rule is in displayed in red, the lower one in blue. Next comes the unifier, then the colored right-hand sides, then the reduced right-hand sides, and finally the joinability verification itself. Colored rule names label the arrows. All critical pairs are considered: note that no rule can overlap l , because its variable X has arity 0.

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$\begin{array}{c} \text{\color{red} } \downarrow^u \\ lk(\lambda v.X_1[v, v])\sigma \\ \parallel \\ lk(\lambda v.lk(\lambda z.X_2[z, v])) \end{array}$	$\begin{array}{c} lk(\lambda w. \left\{ \begin{array}{l} \text{\color{red} } lk(\lambda v.X_1[v, w]) \\ \text{\color{blue} } lk(\lambda v.lk(\lambda z.X_2[z, v])) \end{array} \right\}) \\ \sigma = \{X_1 \mapsto \lambda v \lambda w.lk(\lambda z.X_2[z, v])\} \\ \xrightarrow{u} \quad lk(\lambda z.X_2[z, z]) \quad \xleftarrow{l} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^u \\ lk(\lambda w.lk(\lambda z.X_2[z, z]))\sigma \\ \parallel \\ lk(\lambda w.lk(\lambda z.X_2[z, z])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^u \\ lk(\lambda v.X_1[v, v])\sigma \\ \parallel \\ lk(\lambda v.ud(v, X_2[v])) \end{array}$	$\begin{array}{c} lk(\lambda w. \left\{ \begin{array}{l} \text{\color{red} } lk(\lambda v.X_1[v, w]) \\ \text{\color{blue} } lk(\lambda v.ud(v, X_2[v])) \end{array} \right\}) \\ \sigma = \{X_1 \mapsto \lambda v \lambda w.ud(v, X_2[v])\} \\ \xrightarrow{lu} \quad lk(\lambda v.X_2[v]) \quad \xleftarrow{l} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{lu} \\ lk(\lambda w.lk(\lambda v.X_2[v]))\sigma \\ \parallel \\ lk(\lambda w.lk(\lambda v.X_2[v])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{lu} \\ lk(\lambda w.X_1[w])\sigma \\ \parallel \\ lk(\lambda w.lk(\lambda v.X_2[v])) \end{array}$	$\begin{array}{c} lk(\lambda w. \left\{ \begin{array}{l} \text{\color{red} } ud(w, X_1[w]) \\ \text{\color{blue} } ud(V, lk(\lambda v.X_2[v])) \end{array} \right\}) \\ \sigma = \{V \mapsto w, X_1 \mapsto \lambda w.lk(\lambda v.X_2[v])\} \\ \xrightarrow{l} \quad lk(\lambda v.X_2[v]) =_{\alpha} lk(\lambda w.X_2[w]) \quad \xleftarrow{lu} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{ul} \\ lk(\lambda w.ud(V, X_2[V]))\sigma \\ \parallel \\ lk(\lambda w.ud(w, X_2[w])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{lu} \\ lk(\lambda w.X[w])\sigma \\ \parallel \\ lk(\lambda w.lk(\lambda w.ud(Z, T))) \end{array}$	$\begin{array}{c} lk(\lambda w. \left\{ \begin{array}{l} \text{\color{red} } ud(w, X[w]) \\ \text{\color{blue} } ud(Y, ud(Z, T)) \end{array} \right\}) \\ \sigma = \{Y \mapsto w, X \mapsto \lambda w.ud(Z, T)\} \\ \xrightarrow{u} \quad lk(\lambda w.ud(Z, T)) \quad = \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{uu} \\ lk(\lambda w.ud(Z, T))\sigma \\ \parallel \\ lk(\lambda w.ud(Z, T)) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{ul} \\ ud(V, X_1[V])\sigma \\ \parallel \\ ud(V, ud(V, X_2[V])) \end{array}$	$\begin{array}{c} ud(V, \left\{ \begin{array}{l} \text{\color{red} } lk(\lambda v.X_1[v]) \\ \text{\color{blue} } lk(\lambda v.ud(v, X_2[v])) \end{array} \right\}) \\ \sigma = \{X_1 \mapsto \lambda v.ud(v, X_2[v])\} \\ \xrightarrow{uu} \quad ud(V, X_2[V]) \quad \xleftarrow{ul} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{lu} \\ ud(V, lk(\lambda v.X_2[v]))\sigma \\ \parallel \\ ud(V, lk(\lambda v.X_2[v])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{ul} \\ ud(V, X_1[V])\sigma \\ \parallel \\ ud(V, lk(\lambda w.X_2[w, V])) \end{array}$	$\begin{array}{c} ud(V, \left\{ \begin{array}{l} \text{\color{red} } lk(\lambda v.X_1[v]) \\ \text{\color{blue} } lk(\lambda v.lk(\lambda w.X_2[w, v])) \end{array} \right\}) \\ \sigma = \{X_1 \mapsto \lambda v.lk(\lambda w.X_2[w, v])\} \\ \xrightarrow{ul} \quad ud(V, X_2[V, V]) \quad \xleftarrow{ul} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^u \\ ud(V, lk(\lambda w.X_2[w, w]))\sigma \\ \parallel \\ ud(V, lk(\lambda w.X_2[w, w])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{uu} \\ ud[W, X_1]\sigma \\ \parallel \\ (ud(W, lk(\lambda w.X_2[w]))) \end{array}$	$\begin{array}{c} ud(V, \left\{ \begin{array}{l} \text{\color{red} } ud(W, X_1) \\ \text{\color{blue} } ud(W, lk(\lambda w.X_2[W])) \end{array} \right\}) \\ \sigma = \{X_1 \mapsto lk(\lambda w.X_2[W])\} \\ \xrightarrow{ul} \quad ud(W, X_2[W]) \quad \xleftarrow{uu} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{ul} \\ ud(V, ud(W, X_2[W]))\sigma \\ \parallel \\ ud(V, ud(W, X_2[W])) \end{array}$
$\begin{array}{c} \text{\color{red} } \downarrow^{uu} \\ ud(Y, Z)\sigma \\ \parallel \\ ud(Y, ud(Z', T)) \end{array}$	$\begin{array}{c} ud(V, \left\{ \begin{array}{l} \text{\color{red} } ud(Y, Z) \\ \text{\color{blue} } ud(Y, ud(Z', T)) \end{array} \right\}) \\ \sigma = \{Z \mapsto ud(Z', T)\} \\ \xrightarrow{uu} \quad ud(Z', T) \quad \xleftarrow{uu} \end{array}$	$\begin{array}{c} \text{\color{red} } \downarrow^{uu} \\ ud(V, ud(Z', T))\sigma \\ \parallel \\ ud(V, ud(Z', T)) \end{array}$

5 Sufficiently complete left-linear theories

In this section, we generalize Felgenhauer's theorem [6] to higher-order theories, called *sufficiently complete*, whose confluence does not depend upon the use of $\beta^{\neq 0}$ -steps to join $\mathcal{R}ll$ -convertible terms. No termination assumption is necessary here, as is the case in [6].

Step 1: We replace rewriting with $\mathcal{R}ll$ by rewriting at parallel positions with some rule in $\mathcal{R}ll$, that is $\bigcup_i \xrightarrow{i}$ written $\xrightarrow{\mathcal{R}ll}$ for short. Therefore $\rightarrow = \xrightarrow{\beta^0} \cup \xrightarrow{\mathcal{R}ll} \cup \xrightarrow{\beta}$.

Step 2: Soundness and completeness are straightforward properties again here.

Step 3: The label of a rewrite step $s \xrightarrow{\mathcal{R}ll} t$ becomes $\langle 2, i \rangle$, where i is the rule's index. Other labels remain the same. Labels are compared lexicographically, the order on the second components ($>_{\text{Nat}}$ for $\mathcal{R}ll$ and $\xrightarrow{\beta^0}$ for β^0 -steps) depending upon the first component.

Step 4: Having slightly changed the rewrite relation has an important impact since we now need parallel critical pairs. Their definition is easily obtained from Definition 16 by overlapping several copies of a left-hand side of rule inside the left-hand side of another rule at a set of parallel positions. The critical peak lemma generalizes as well without difficulty.

Step 5: We need to show that decreasing diagrams are stable under context and substitution instance, which is not as straightforward as it was in the previous case. To this end, we consider a specific form of decreasing diagram:

► **Definition 25.** A rewrite sequence issuing from a term s is *normal*, in short NRS, if it is of the form $s \xrightarrow{\beta^0} s \downarrow$ or $s \xrightarrow{\beta^0} s \downarrow \xrightarrow{\geq \mathcal{P}P} s \downarrow \xrightarrow{\geq \mathcal{P}P} \frac{p}{i \in \mathcal{R}ll} t \xrightarrow{\mathcal{R}ll \cup \beta^0} u$, where $t \xrightarrow{\mathcal{R}ll \cup \beta^0} u$ is an NRS. We denote by $NDS(i, j)$ the set of normal rewrite sequences which are decreasing w.r.t. (i, j) .

A rewrite sequence in $NDS(i, j)$ issuing from s protects the positions in a set Q of parallel positions of s if it is of the form $s \xrightarrow{\triangleleft i, j} u$ (no facing step) or $s \left(\frac{\geq \mathcal{P}P}{\beta^0} \frac{p \# Q}{k \triangleleft i} \right)^* \xrightarrow{\frac{P \# Q}{j}} \xrightarrow{\triangleleft i, j} u$.

Normal rewrite sequences restrict the use of β^0 -steps to those which are strictly necessary to apply every $\mathcal{R}ll$ -step in the sequence before to reach some normal term. Note that normal sequences always end up in a normal term. In case there is a set Q of protected positions, there must be a final normalization phase at all positions which are either below, or above Q .

► **Definition 26.** The decreasing diagram of a parallel critical peak $s \xleftarrow{\Lambda} u \xrightarrow{O} t$ is *normal*, in short NDD, if it is a pair of normal decreasing rewrite sequences such that the one issuing from s (which may contain the *lower* facing step) protects the positions in Q , where $Q = \{q \in \text{Pos}(R) : R|_q = (X[\bar{t}]\bar{y}) \text{ where } X \notin \text{MVar}(L|_O) \text{ and } \bar{y} \in \mathcal{X}\}$.

Protection restricts the set of rewrite positions that precede the facing step of the sequence to those that cannot interfere with the rewrites below meta-variables of R that are not, as meta-variables of L , in the scope of O . This adapts to higher-order rewriting the so-called Toyama's Variable Condition (TVC) introduced long ago in a first-order setting by Toyama. In particular, if all meta-variables of L are below O , then $Q = \emptyset$ and (TVC) is automatically satisfied. Example 31 shows that all critical pairs of Example 24, whose decreasing diagrams have lower facing steps, satisfy this sufficient condition.

Our goal now is to prove the stability of these notions of sequences under substitution:

► **Lemma 27.** Let $s \xrightarrow[k \in \mathcal{R}ll]{p} \xrightarrow{\triangleright} t$ be an NRS, and γ a normal substitution. Then, there exists an NRS $s\gamma \xrightarrow[k]{p} \xrightarrow{\triangleright} t\gamma \downarrow$.

Proof. By induction on the definition of NRSs, using Lemma 10 and confluence of β^0 . ◀

This very strong property, saying that the instance of a normal decreasing sequence is again a normal decreasing sequence whose higher-order steps are indeed the instances of the previous ones at the same positions, is crucial, since it implies readily that protection of a set of positions is preserved under substitution. Therefore,

► **Corollary 28.** *Normal decreasing diagrams are closed under substitution instance.*

We can further exploit the preservation property of sequences in NDS:

► **Lemma 29 (Permutation).** *Let $w \left(\begin{smallmatrix} \geq_{\beta^0} \mathcal{P} o \\ o \# Q \\ k < i \end{smallmatrix} \right)^* \xrightarrow{j} P \# Q w'''$ be a NRS protecting the set of positions Q and $s \xrightarrow{j} Q t \xrightarrow{\geq_{\beta^0} \mathcal{P} P} t \downarrow \geq_{\mathcal{P} P} = w$ for some $p \geq_{\mathcal{P}} P \cup Q$.*

Then there exists a rewrite sequence $s \xrightarrow{\geq_{\beta^0} \mathcal{P} P} \xrightarrow{j} P \cup Q w'''$

Proof. We write \downarrow instead of $\downarrow \geq_{\mathcal{P} P}$. First, by Lemma 10 and commutation of β^0 , we have $s \downarrow \xrightarrow{j} Q \xrightarrow{\geq_{\beta^0} \mathcal{P} Q} t \downarrow$. Then we can commute the steps $s \downarrow \xrightarrow{j} Q \xrightarrow{\geq_{\beta^0} \mathcal{P} Q} t \downarrow \left(\begin{smallmatrix} \geq_{\beta^0} \mathcal{P} o \\ o \# Q \\ k < i \end{smallmatrix} \right)^*$ into $s \downarrow \left(\begin{smallmatrix} \geq_{\beta^0} \mathcal{P} o \\ o \# Q \\ k < i \end{smallmatrix} \right)^* \xrightarrow{j} Q \xrightarrow{\geq_{\beta^0} \mathcal{P} Q}$ and again commute $\xrightarrow{\geq_{\beta^0} \mathcal{P} Q} \xrightarrow{j} P \# Q$ into $\xrightarrow{j} P \# Q \xrightarrow{\geq_{\beta^0} \mathcal{P} Q}$ since they occur at uncomparable positions. For the same reason, we can merge $\xrightarrow{j} Q \xrightarrow{j} P \# Q$ into $\xrightarrow{j} P \cup Q$.

We therefore get the rewrite sequence: $s \xrightarrow{\geq_{\beta^0} \mathcal{P} P} s \downarrow \left(\begin{smallmatrix} \geq_{\beta^0} \mathcal{P} o \\ o \\ k < i \end{smallmatrix} \right)^* \xrightarrow{j} P \cup Q \xrightarrow{\geq_{\beta^0} \mathcal{P} Q} w'''$. ◀

Step 6: We are left with the last step:

► **Theorem 30.** *$\lambda\mathcal{F}$ is confluent if all parallel critical peaks of $\mathcal{R}ll$ have NDDs.*

► **Proof.** Let $s \xleftarrow{P} u \xrightarrow{Q} t$ be an arbitrary local peak, where P, Q are possibly singleton sets of orthogonal positions. The proof is by induction on the size of u .

1. Assume that $\Lambda \notin P \cup Q$, and let $\{u_k\}_k, \{s_k\}_k$ and $\{t_k\}_k$ be the immediate subterms of u, s and t , respectively. Let also $P = \bigcup_k k \cdot P_k$ and $Q = \bigcup_k k \cdot Q_k$. Then, $s_k \xleftarrow{P_k} u_k \xrightarrow{Q_k} t_k$. By induction hypothesis, these local peaks enjoy a DD of the form $s_i \longrightarrow v_i \longleftarrow t_i$.

We now show that putting these DDs together results in a DD $s \longrightarrow v \longleftarrow t$. To this end, we show that the decreasing reductions from u_i to s_i can be put together and yield a decreasing reduction from u to s . Let $u_i \longrightarrow v_i \longrightarrow w_i \longrightarrow s_i$ be the decreasing reduction from u_i to s_i , where $v_i \longrightarrow w_i$ is a possibly missing facing step, in which case $v_i = s_i$. Then, the reduction $s = F(s_1, \dots, s_n) \longrightarrow F(v_1, \dots, v_n) \longrightarrow F(w_1, \dots, w_n) \longrightarrow s$ is a decreasing reduction from u to s . The same technique applied to the pair (u, t) yields a decreasing diagram for the starting local peak.

We are now left with all peaks for which $\Lambda \in (P \# Q)$. W.l.o.g, we assume that $\Lambda \in P$. Cases whose proof remains the same as in the proof of Theorem 23 are omitted.

5. $s \xleftarrow{i} \Lambda u \xrightarrow{j} Q t$ with $i : L \rightarrow R \in \mathcal{R}ll$ and $j : G \rightarrow D \in \mathcal{R}rl$. This case is depicted at Figure 4. We first split Q into disjoint subsets $Q_1 = \{q \in Q : q \geq_{\mathcal{P}} F_L\}$ and $Q_2 = \{q \in Q : q <_{\mathcal{P}} F_L\}$. Let O_1 and O_2 be the subsets of F_L such that $O_1 = \{o \in F_L : Q_1 \geq_{\mathcal{P}} o\}$, and $O_2 = \{o \in F_L : o >_{\mathcal{P}} Q_2\}$. Since Q_1 and Q_2 are disjoint, so are O_1 and O_2 . Let now P_1 be the set of positions of the corresponding meta-variables in R . Since $O_1 \# Q_2$, positions in P_1 are protected in the critical peak defined by the overlap at Q_2 .

By (LAPRa), $s \xrightarrow[\beta^0]{Q'_1} w \xrightarrow[\beta^0]{Q'_1} s' \xrightarrow[\beta^0]{\Lambda} u' \xrightarrow[\beta^0]{\Lambda} v$, where $Q'_1 \geq_{\mathcal{P}} P_1$ and w is normal below Q'_1 . The β^0 -steps from v to u' occur at positions which are parallel to those in Q_2 , hence, by commutation, $u' \xrightarrow[\beta^0]{Q_2} t' \xrightarrow[\beta^0]{\Lambda} t$. The obtained local peak $s' \xrightarrow[\beta^0]{\Lambda} u' \xrightarrow[\beta^0]{Q_2} t'$ is critical by definition of Q_2 . Therefore, by assumption and Lemma 28, there exist an NDD from s', t' such that $s' \xrightarrow[\triangleleft i]{\beta^0} s'' \xrightarrow[\triangleleft i, j]{\beta^0} s''$ belongs to $NDS(i, j)$ that protects Q'_1 , hence the rewrites from s' to s'' occur at positions disjoint from P_1 , hence disjoint from $Q'_1 \geq_{\mathcal{P}} P_1$. By commutation of β^0 with other rewrites, $w \xrightarrow[\triangleleft i]{\beta^0} w''' \xrightarrow[\triangleleft i, j]{\beta^0} w' \xrightarrow[\beta^0]{\Lambda} s''$. An easy property of β^0 ensures that the derivation $w \xrightarrow[\triangleleft i]{\beta^0} w'''$ still occurs at positions disjoint from Q'_1 . Therefore, by Lemma 29, we can rearrange the derivation from s to w''' so that it belongs to $DS(i, j)$. Since the derivation from t to w belongs to $DS(j, i)$, we are done.

6. $s \xrightarrow[\beta^0]{\Lambda} u \xrightarrow[\beta^0]{Q} t$. Since $u|_Q$ must be normal, both steps commute, giving a DD.
7. $s \xrightarrow[\beta]{P} u \xrightarrow[\beta]{Q} t$, where $i : L \rightarrow R \in \mathcal{R}ll$. The proof is then obtained from the corresponding case in the proof of Theorem 23 by replacing $u|_q$ by $u|_Q$.

Example 24 can actually also serve to illustrate this theorem:

► **Example 31.** We now show the confluence of the theory of global states for a unique location without using the termination assumption. To this end, we need to show that all parallel critical peaks have a decreasing diagram, and that it satisfies (TVC). The form of the left-hand sides of rules ensures that no true parallel overlap is possible, hence we get the same set of critical peaks as in Example 24. Let us then note that (TVC) is trivial for the first four, since the meta-variables of all critical peaks always occur below the overlap position, hence $Q = \emptyset$. The next two have no lower facing step, hence all positions are protected. For the last two critical peaks, $Q = \emptyset$ since the only meta-variable V of uu which is not in the scope of the overlap position does not occur in the right-hand side of uu .

We now need to check that the diagrams obtained at Example 24 are decreasing. Referring to them by their order in the list given there, we define the order on rules that ensures decreasingness: diagrams 1,7 and 8 are decreasing because they are reduced to their facing steps, hence are normal; the diagrams from number 2 to number 6 need to satisfy the following successive ordering constraints:

$$l \triangleleft \{ll, lu\}; l \triangleleft \{lu, ul\}; ll \triangleleft \{lu, uu\}; uu \triangleleft \{ul, lu\}; ul \triangleleft ll\};$$

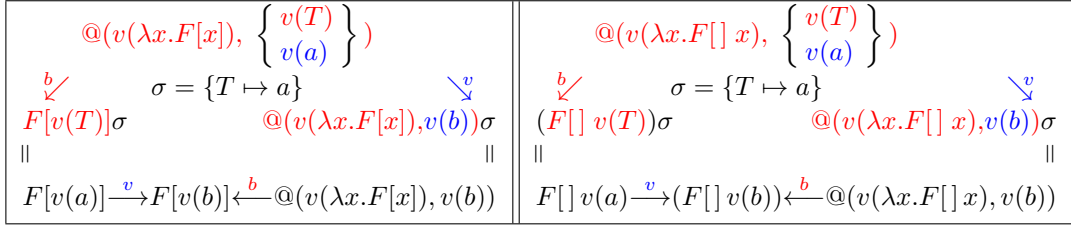
One order among others that satisfies the above constraints is $l \triangleleft ul \triangleleft ll \triangleleft uu \triangleleft lu$.

Our last example highlights the subtleties of confluence criteria in a higher order rewriting context and exemplifies the interaction between meta-variables arity and protection:

► **Example 32.** We illustrate here the tension between protected positions and arities of meta-variables. Let $\mathcal{F} = \{\@, v, a, b\}$, where $\@ : V \rightarrow V \rightarrow T$, $v : (V \rightarrow T) \rightarrow V$, $a : V \rightarrow T$ and $b : V \rightarrow T$. We consider two sets of rules, depending upon the arity of meta-variable F :

$$\begin{aligned} \mathcal{R}_1 &= \{ (b) \quad \@ (v(\lambda x.F[x]), v(T)) \rightarrow F[v(T)] \quad , \quad v(a) \rightarrow v(b) \quad (v) \} \\ \mathcal{R}_2 &= \{ (b) \quad \@ (v(\lambda x.F[] x), v(T)) \rightarrow F[] v(T) \quad , \quad v(a) \rightarrow v(b) \quad (v) \} \end{aligned}$$

In both systems, the term $\@ (v(\lambda x.\@(x, x)), v(\lambda x.\@(x, x)))$ reduces to itself. Relying on the second technique we thus need to check one critical pair for each system:



In the left diagram, the reduct $F[v(a)]$ must be protected at $Q = \{\Lambda\}$, since F is not below the overlap position. The obtained diagram is not an NDD, we cannot conclude confluence. In the right diagram, $Q = \{1\}$, the obtained diagram is an NDD, hence confluence is concluded.

6 Conclusion

Confluence of first-order rewrite rules is well understood, in both the terminating and non-terminating cases. Confluence of left-linear higher-order rules on simply-typed λ -terms is well understood too [15]. This is true as well of confluence of first-order rules in presence of β -reductions for any type discipline for which β is terminating [5]. Finally, positive confluence results in presence of both dependent types and higher-order rewrite rules are claimed in [1].

In this paper (together with [1]), we have described under which conditions on critical pairs is confluence in the untyped λ -calculus preserved by a set $\mathcal{R}ll$ of left-linear rewrite rules:

1. If $\mathcal{R}ll$ preserves termination of β^0 and the critical pairs of $\mathcal{R}ll$ are joinable in $\mathcal{R}ll \cup \beta^0$;
2. If the parallel critical pairs of $\mathcal{R}ll$ are joinable in $\mathcal{R}ll \cup \beta^0$ by a pair of decreasing rewrites with respect to rule labeling that satisfies Toyama's Variable Condition;
3. If the nested critical pairs of $\mathcal{R}ll$ are joinable in $\mathcal{R}ll \cup \beta$ by a pair of decreasing rewrites with respect to rule labeling [1, 8].

A non-trivial example has been given that shows the usefulness of the first two categories of theories. One may however wonder whether there are interesting examples of theories that are not sufficiently complete, that is, that require the use of $\beta^{\neq 0}$ -steps to join their critical pairs. One such theory is described in [7].

An important question is whether our present results can be combined, so as to use a divide and conquer approach to analyze a complex left-linear theory. Using van Oostrom decreasing diagrams technique is crucial in this respect, provided the rewrite relations used for the confluence proof and their labellings can be combined. This is why we used exactly the same relations with $\xrightarrow{\beta^0}$ and $\xrightarrow{\beta}$ in the proofs of our theorems –we could have done otherwise. Combining the results obtained for left-linear systems under different assumptions is therefore possible, this will require computing new critical pairs and parallel critical pairs between the two rewrite systems and checking that they are decreasing. Lack of space does not permit us doing it here.

Further results are on the way that deal with right-linear rules. In particular, we can show that all Klop's counter-examples preserve confluence on the set of β -strongly-normalizing terms. The case of rules which are semi-linear is of uttermost importance to us because encodings of complex type theories in $\lambda IIMod$ are not purely left-linear. We will therefore have to combine our results for left-linear systems with those for right-linear ones, but this other combination raises new technical difficulties since the techniques used for left-linear and right-linear systems, although both based on van Oostrom's decreasing diagrams, happen to be quite different.

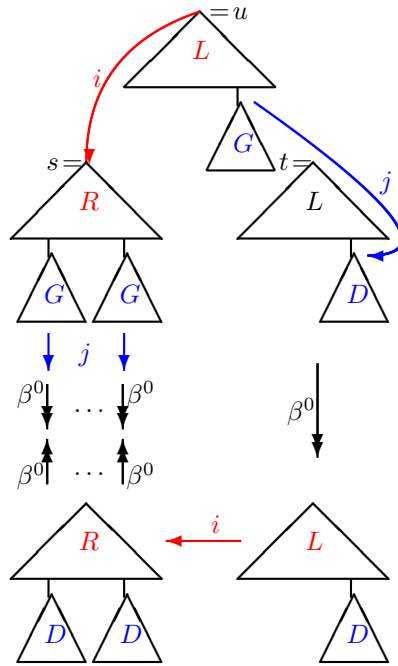
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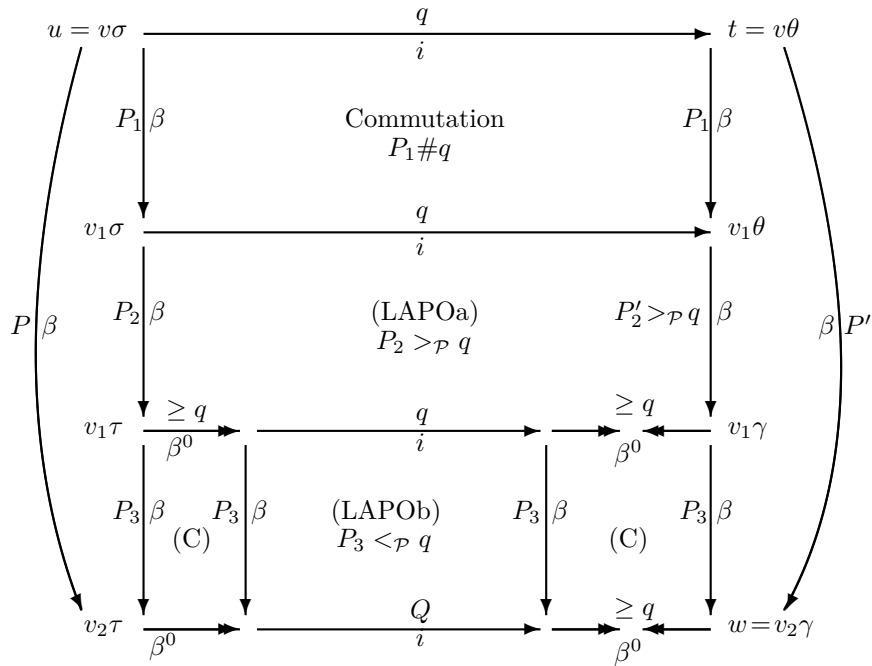
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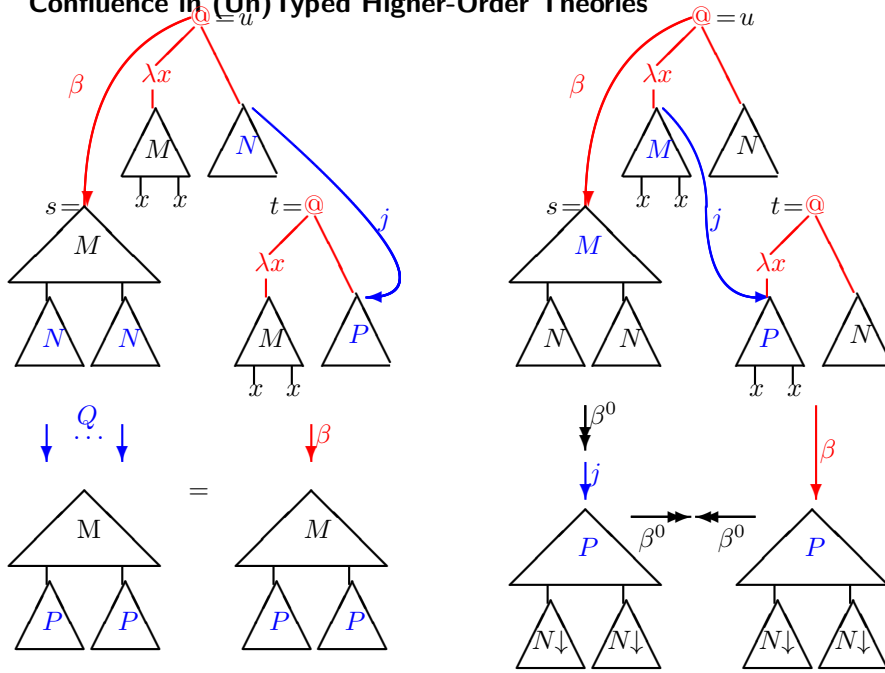


■ **Figure 1** Ancestor peaks in rewrite theories. L, G stand for terms rewriting to R, D , using a red rule in $\mathcal{R} \cup \beta$ and a blue rule in $\mathcal{R} \cup \beta$.

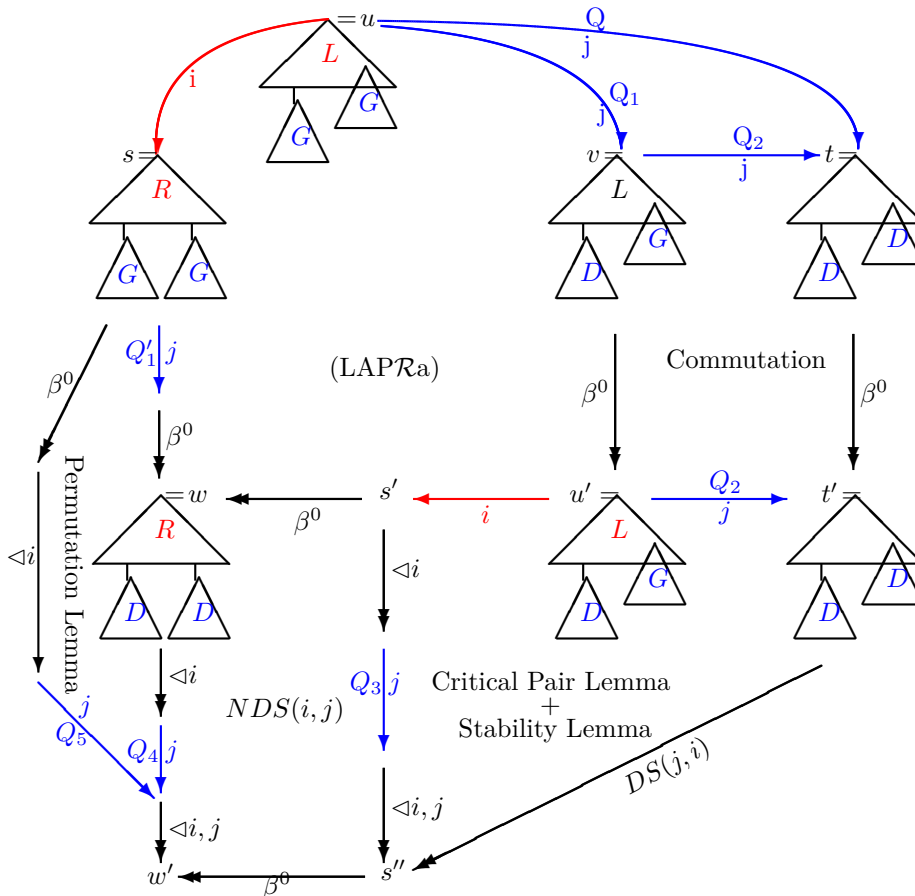


■ **Figure 2** Construction of a decreasing diagram for heterogeneous local peaks.

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■ **Figure 3** Ancestor peaks in rewrite theories with β above and \mathcal{R} below



■ **Figure 4** Construction of a decreasing diagram for \mathcal{R} -local peaks. Our pictures are simplified by identifying rule instances with their normal forms.