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FRACTIONAL CHROMATIC NUMBER, MAXIMUM DEGREE AND GIRTH

FRANÇOIS PIROT AND JEAN-SÉBASTIEN SERENI

Abstract. We prove new lower bounds on the independence ratio of graphs of maximum degree $\Delta \in \{3,4,5\}$ and girth $g \in \{6,\dots,12\}$, notably $1/3$ when $(\Delta, g) = (4,10)$ and $2/7$ when $(\Delta, g) = (5,8)$. We establish a general upper bound on the fractional chromatic number of triangle-free graphs, which implies that deduced from the fractional version of Reed's bound for triangle-free graphs and improves it as soon as $\Delta \geq 17$, matching the best asymptotic upper bound known for off-diagonal Ramsey numbers. In particular, the fractional chromatic number of a triangle-free graph of maximum degree Δ is less than 9.916 if $\Delta = 17$, less than 22.17 if $\Delta = 50$ and less than 249.06 if $\Delta = 1000$. Focusing on smaller values of Δ, we also demonstrate that every graph of girth at least 7 and maximum degree Δ has fractional chromatic number at most $\min_{k\in\mathbb{N}} \frac{2\Delta+2^{k-3}+k}{k}$. In particular, the fractional chromatic number of a graph of girth 7 and maximum degree Δ is at most $\frac{2\Delta+9}{5}$ when $\Delta \in [3,8]$, at most $\frac{\Delta+7}{3}$ when $\Delta \in [8,20]$, at most $\frac{2\Delta+23}{7}$ when $\Delta \in [20,48]$, and at most $\frac{\Delta}{4}+5$ when $\Delta \in [48,112]$.

1. Introduction

Independent sets in graphs are fundamental objects, at the heart of several problems and notions such as graph colouring. Of particular interest is the order $\alpha(G)$ of a largest stable set in a graph G, which often is divided by the number of vertices of G: this is the *independence ratio* of G. One can then consider a hereditary version of the inverse of the independence ratio, called the *Hall ratio*, which is defined as

$$\rho(G) \coloneqq \max\left\{\frac{|V(H)|}{\alpha(H)} \,:\, H \subseteq G\right\}.$$

Since a k-colouring of a graph is a partition of the vertex set into k stable sets, it follows that the inverse of the independence ratio of a graph is a lower bound on its chromatic number, and by heredity of the chromatic number, so is the Hall ratio. For instance, the 4-colour theorem thus implies that every planar graph has Hall ratio at most 4. Interestingly enough, no one seems to know how to prove this last statement, sometimes called the "Erdős-Vizing conjecture", without using the 4-colour theorem — or a proof of a similar nature and length.

The Hall ratio of a graph has often been studied in relation with the *girth*, which is the length of a smallest cycle in the graph. A first result in this direction is the celebrated introduction of the so-called "deletion method" in graph theory by Erdős, who used it to demonstrate the existence of graphs with arbitrarily large girth and chromatic number. The latter is actually established by proving that the Hall ratio of the graph is arbitrarily large. As a large girth is not strong enough a requirement to imply a constant upper bound on the chromatic number, a way to pursue this line of research is to express the upper bound in terms of the maximum degree $\Delta(G)$ of the graph G considered. This also applies to the Hall ratio.

Letting $\mathrm{girth}(G)$ stand for the girth of the graph G, that is, the length of a shortest cycle in G if G is not a forest and $+\infty$ otherwise, we define $\rho(d,g)$ to be the supremum of the Hall ratios among all graphs of maximum degree at most d and girth at least g. We also let $\rho(d,\infty)$ be the limit as $g \to \infty$ of $\rho(d,g)$ — note that if we fix d then $\rho(d,g)$ is a non-increasing function of g. In

symbols,

$$\rho(d,g) \coloneqq \sup\left\{\frac{|V(G)|}{\alpha(G)} \,:\, G \text{ graph with } \Delta(G) \leq d \text{ and } \operatorname{girth}(G) \geq g\right\},$$
$$\rho(d,\infty) \coloneqq \lim_{g\to\infty} \rho(d,g).$$

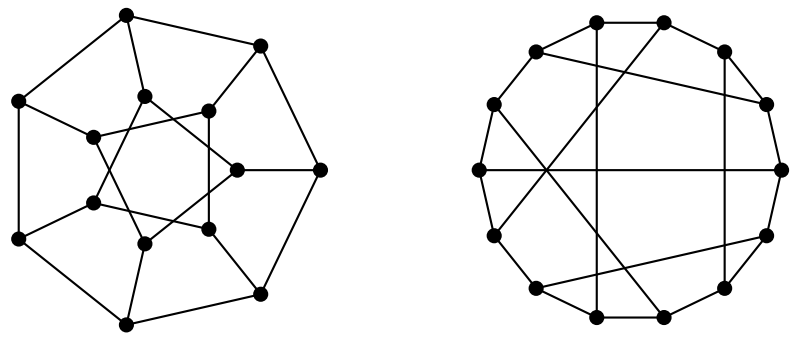


FIGURE 1. The two cubic triangle-free connected graphs with Hall ratio $\frac{14}{5}$.

In 1979, Staton [17] established that $\rho(d,4) \leq \frac{5d-1}{5}$, in particular implying that $\rho(3,4) \leq \frac{14}{5}$. The two graphs depicted in Figure 1, called the graphs of Fajtlowicz and of Locke, have fourteen vertices each, girth 5, and no stable set of order 6. It follows that $\rho(3,4) = \frac{14}{5} = \rho(3,5)$. It is known that the graphs of Fajtlowicz and of Locke are the only two cubic triangle-free and connected graphs with Hall ratio $\frac{14}{5}$. This follows from a result of Fraughnaugh and Locke [9] for graphs with more than 14 vertices completed by an exhaustive computer check on graphs with at most 14 vertices performed by Bajnok and Brinkmann [1].

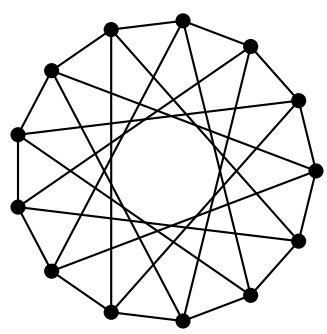


FIGURE 2. The only known 4-regular triangle-free connected graph of Hall ratio $\frac{13}{4}$.

In 1983, Jones [10] reached the next step by establishing that $\rho(4,4) = \frac{13}{4}$. Only one connected graph is known to attain this value: it has 13 vertices and is represented in Figure 2. The value of $\rho(d,4)$ when $d \geq 5$ is still unknown; the best general upper bound is due to Shearer [16]. He also provides an upper bound for $\rho(d,6)$ as a consequence of a stronger result on graphs with no cycle of length 3 or 5.

Theorem 1 (Shearer [16])**.** *For every non-negative integer d, set*

$$f(d) \coloneqq \begin{cases} 1 & \text{if } d = 0, \\ \frac{1+(d^2-d)f(d-1)}{\frac{d^2+1}{2}} & \text{if } d \geq 1. \end{cases}$$

If G is a triangle-free graph on n vertices with degree sequence $d_1, \ldots, d_n$, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i).$$

Theorem 2 (Shearer [16])**.** *For every non-negative integer d, set*

$$f(d) := \begin{cases} 0 & \text{if } d = 0, \\ \frac{4}{7} & \text{if } d = 1, \\ \frac{1+(d^2-d)f(d-1)}{d^2+1} & \text{if } d \geq 2. \end{cases}$$

If G is a graph on n vertices with degree sequence $d_1, \ldots, d_n$ and with no 3-cycle and no 5-cycle, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i) - \frac{n_{11}}{7},$$

where n_{11} is the number of pairs of adjacent vertices of degree 1 in G.

Theorems 1 and 2 allow us to compute upper bounds on $\rho(d, 4)$ and on $\rho(d, 6)$ for small values of d, as indicated in Table 1. When $d \geq 5$, these bounds are the best known ones.

d	upper bound of $\rho(d,4)$		upper bound on $\rho(d,6)$	
2	$\frac{5}{2}$	$= 2.5$	$\frac{7}{3}$	≈ 2.33333
3	$\frac{50}{17}$	≈ 2.94118	$\frac{14}{5}$	$= 2.8$
4	$\frac{425}{127}$	≈ 3.34646	$\frac{119}{37}$	≈ 3.21622
5	$\frac{2210}{593}$	≈ 3.72681	$\frac{3094}{859}$	≈ 3.60186
6	$\frac{8177}{2000}$	≈ 4.0885	$\frac{57239}{14432}$	≈ 3.96612
7	$\frac{408850}{92177}$	≈ 4.43549	$\frac{408850}{94769}$	≈ 4.31417
8	$\frac{13287625}{2785381}$	≈ 4.77049	$\frac{13287625}{2857957}$	≈ 4.64934
9	$\frac{1089585250}{213835057}$	≈ 5.09545	$\frac{1089585250}{219060529}$	≈ 4.9739
10	$\frac{11004811025}{2033474038}$	≈ 5.41183	$\frac{11004811025}{2080503286}$	≈ 5.28949

TABLE 1. Upper bounds on $\rho(d, 4)$ and $\rho(d, 6)$ for $d \leq 10$ derived from Theorems 1 and 2.

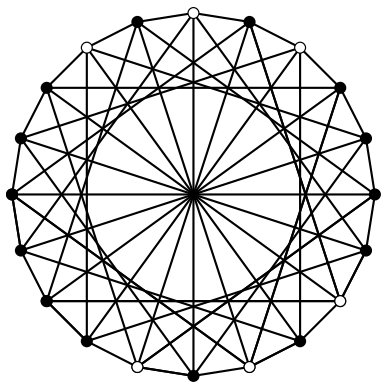
We are not aware of any non trivial lower bounds on $\rho(5, 4)$ and $\rho(6, 4)$. Figure 3 show graphs illustrating that $\rho(5, 4) \geq \frac{10}{3} \approx 3.33333$ and $\rho(6, 4) \geq \frac{29}{8} = 3.625$. These two graphs are circulant graphs, which are Cayley graphs over $\mathbb{Z}_n$.

The value of $\rho(3, g)$ has also been studied when g goes to infinity. Kardoš, Král' and Volec [11] proved the existence of an integer g_0 such that $\rho(3, g_0) \leq 2.2978$. More strongly, their upper bound holds for the fractional chromatic number of every (sub)cubic graph of girth at least g_0. In the other direction, Bollobás [4] proved a general lower bound on $\rho(d, g)$.

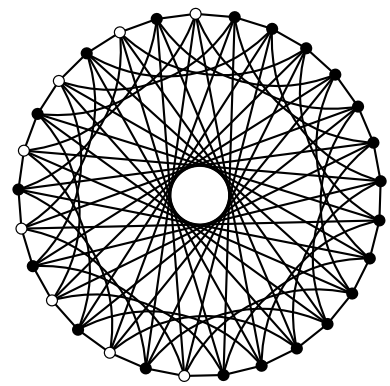
Theorem 3 (Bollobás, 1981)**.** *Let $\Delta \geq 3$. Let α be a real number in $(0, 1)$ such that*

$$\alpha(\Delta \ln 2 - \ln(\alpha)) + (2 - \alpha)(\Delta - 1) \ln(2 - \alpha) + (\alpha - 1)\Delta \ln(1 - \alpha) < 2(\Delta - 1) \ln 2.$$

For every positive integer g, there exists a Δ-regular graph with girth at least g and Hall ratio more than $2/\alpha$.



(A) A 5-regular triangle-free (vertex-transitive) graph with Hall ratio $\frac{10}{3}$. It is the Cayley graph over $\mathbb{Z}_{20}$ with generating set $\{\pm 1, \pm 6, \pm 10\}$. There is no stable set of order 7, and the white vertices form a stable set of order 6.



(B) A 6-regular triangle-free (vertex-transitive) graph with Hall ratio $\frac{29}{8}$. It is the Cayley graph over $\mathbb{Z}_{29}$ with generating set $\{\pm 1, \pm 5, \pm 13\}$. There is no stable set of order 9, and the white vertices form a stable set of order 8.

FIGURE 3. Two possibly extremal regular triangle-free graphs for the Hall ratio.

d	lower bound on $\rho(d, \infty)$
2	2
3	2.17835
4	2.3775
5	2.57278
6	2.76222
7	2.94606
8	3.1249
9	3.29931
10	3.46981
d	$d/(2 \ln d)$

TABLE 2. Lower bounds on $\rho(d, \infty)$ implied by Theorem 3.

Theorem 3 allows us to compute lower bounds on $\rho(d, \infty)$ for any value of d, the smaller ones being represented in Table 2. All these values can be generalised into a looser but asymptotically equivalent general lower bound of $d/(2 \ln d)$ [4, Corollary 3].

The fractional chromatic number $\chi_f(G)$ of a graph G is a refinement of the chromatic number. It is the fractional solution to a linear program the integer solution of which is the chromatic number. Let G be a given graph; we define $\mathscr{S}_{\max}(G)$ to be the set of all maximal stable sets of G and $\mathscr{S}_{\alpha}(G)$ to be the set of all maximum stable sets of G. Then $\chi_f(G)$ is the solution of the following linear program.

$$\min \sum_{S \in \mathscr{S}_{\max}(G)} w_S$$

$$\text{such that} \begin{cases} w_S \in [0,1] & \text{for each } S \in \mathscr{S}_{\max} \\ \displaystyle\sum_{\substack{S \in \mathscr{S}_{\max} \\ v \in S}} w_S \geq 1 & \text{for each } v \in V(G). \end{cases}$$

A *fractional colouring of weight w of G* is any instance within the domain of the above linear program such that $\sum w_S = w$. You can note that a k-colouring of G is a special case of a fractional colouring of weight k of G, where $w_S = 1$ if S is a monochromatic class of the k-colouring, and $w_S = 0$ otherwise. Note also that if G is a clique, then any fractional colouring of G is of weight at least $|V(G)|$. This allows us to write the following inequalities

$$\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \Delta(G) + 1,$$

where $\omega(G)$ is the maximum order of a clique in G, and $\Delta(G)$ is the maximum degree of G. Equality holds between $\omega(G)$ and $\chi(G)$, and so in particular between $\omega(G)$ and $\chi_f(G)$, when G is a perfect graph. Those are the graphs that contain no odd hole nor odd antihole, as was conjectured by Berge [2] in 1961, and proved by Chudnovsky *et al.* [6] in 2006. On the other side, the characterisation of the graphs G for which equality holds between $\chi(G)$ and $\Delta(G)+1$ was established by Brooks [5] in 1941, and those graphs are cliques and odd cycles. Since $\chi_f(C_{2k+1}) = \frac{k}{2k+1}$, the only graphs G such that $\chi_f(G) = \Delta(G) + 1$ are cliques. Moreover, equality holds between the Hall ratio of G and its fractional chromatic number for example when G is vertex transitive.

Recently, Molloy [14] proved the best known extremal upper bounds for the chromatic number of graphs of given clique number and maximum degree.

Theorem 4 (Molloy, 2019)**.** *Let G be a graph of maximum degree Δ.*

- *If G is triangle-free, then for every $\varepsilon > 0$, there exists Δ_ε such that, assuming that $\Delta \geq \Delta_\varepsilon$,*
$$\chi(G) \leq (1+\varepsilon)\frac{\Delta}{\ln \Delta}.$$
- *If G has clique number $\omega(G) > 2$, then*
$$\chi(G) \leq 200\omega(G)\frac{\Delta \ln\ln \Delta}{\ln \Delta}.$$

The first bound is sharp up to a multiplicative factor in a strong sense, since as shown by Bollobás [4, Corollaries 3 and 4] for all integers g and $\Delta \geq 3$ there exists a graph with maximum degree Δ, girth at least g and chromatic number at least $\frac{\Delta}{2\ln \Delta}$.

There remains however a substantial range of degrees not concerned by the bound for triangle-free graphs given by Theorem 4, namely when Δ is smaller than Δ_ε, which is larger than $20^{2/\varepsilon}$. To this date, the best known general upper bound in terms of clique number and maximum degree for the *fractional* chromatic number[1] is due to Molloy and Reed [15, Theorem 21.7, p. 244].

Theorem 5 (Molloy and Reed, 2002)**.** *For every graph G,*

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

If one considers a convex combination of the clique number and the maximum degree plus one for an upper bound on the (fractional) chromatic number of a graph, then because the chromatic number of a graph never exceeds its maximum degree plus one, the aim is to maximise the coefficient in front of the clique number. The convex combination provided by Theorem 5 (which is conjectured to hold, after taking the ceiling, also for the chromatic number), is best possible. Indeed, for every positive integer k the graph $G_k \coloneqq C_5 \boxtimes K_k$ is such that $\omega(G_k) = 2k, \Delta(G_k) = 3k-1, \chi_f(G_k) = \frac{5k}{2} = \frac{\omega(G_k)+\Delta(G_k)+1}{2}$.

[1]For the chromatic number, the reader is referred to a nice theorem of Kostochka [13], which for instance implies that every graph with maximum degree at most 5 and girth at least 35 has chromatic number at most 4 (Corollary 2 in *loc. cit.*). The general upper bound on the chromatic number guaranteed by Kostochka's theorem is never less than the floor of half the maximum degree plus two.

A local form of Theorem 5 exists: it was first devised by McDiarmid (unpublished) and appears as an exercise in Molloy and Reed's book [15]. A published version is found in the thesis of Andrew King [12, Theorem 2.10, p. 12].

Theorem 6 (McDiarmid, unpublished)**.** *Let G be a graph, and set $f_G(v) := \frac{\omega_G(v)+\deg_G(v)+1}{2}$ for every $v \in V(G)$, where $\omega_G(v)$ is the order of a largest clique in G containing v. Then*

$$\chi_f(G) \leq \max\{f_G(v) \,:\, v \in V(G)\}.$$

In Subsection 3.1, we slightly strengthen the local property of Theorem 6 as a way to illustrate the arguments used later on.

Our first contribution is to establish a (non-explicit) formula for an upper bound on the fractional chromatic number of triangle-free graphs depending on their maximum degree Δ. The upper bound which can be effectively computed from this formula improves on the one which can be derived from Theorem 6 as soon as $\Delta \geq 17$.

$\Delta(G)$	k	λ	upper bound on $\chi_f(G)$
$1\ldots16$	2	∞	$\frac{\Delta(G)+3}{2}$
17	3	3.41613	9.91552
18	3	3.50195	10.3075
19	3	3.58603	10.6981
20	3	3.66847	11.0875
50	4	2.04455	22.1644
100	5	1.48418	38.0697
200	6	1.24061	66.151
500	8	0.915598	139.842
1000	10	0.734978	249.058

TABLE 3. Upper bounds on $\chi_f(G)$ when G is triangle-free.

Theorem 7. *For every triangle-free graph G of maximum degree Δ,*

$$\chi_f(G) \leq 1 + \min_{k\in\mathbb{N}} \inf_{\lambda>0} \frac{(1+\lambda)^k + \lambda(1+\lambda)\Delta}{\lambda(1+k\lambda)}.$$

Theorem 7 lets us derive the upper bounds for the fractional chromatic number of triangle-free graphs in Table 3.

Remark 1. When we use the couple $(k,\lambda) = (2,\infty)$, Theorem 7 implies the fractional Reed bound of Theorem 5. On the other hand, when we use the couple $(k,\lambda) = (\lfloor \ln\Delta(\ln\Delta - 2\ln\ln\Delta)\rfloor, 1/\ln\Delta)$, we obtain the bound

$$\chi_f(G) \leq 1 + \left(1 + \frac{2}{\ln\Delta}\right)\frac{\Delta}{\ln\Delta - 2\ln\ln\Delta},$$

assuming that $\Delta \geq 3$, and hence that $k\lambda \geq \ln\Delta - 2\ln\ln\Delta - 1$. This in particular implies the fractional version of the triangle-free bound of Theorem 4. So Theorem 7 yields a smooth transition for the fractional chromatic number of triangle-free graphs from Reed's bound to Molloy's bound, as Δ increases.

In order to obtain upper bounds smaller than that of Theorem 6 for smaller values of the maximum degree, we needed to consider graphs of higher girth. Our second contribution is to establish good upper bounds for the fractional chromatic number of graphs of girth 7. Moreover, these bounds have the same local property as those of Theorem 6.

Theorem 8. *Let* $f(x) := 1 + \min_{k\in\mathbb{N}} \frac{2x+2^{k-3}}{k}$. *If* G *is a graph of girth at least* 7*, then* G *admits a fractional colouring such that for every induced subgraph* H *of* G*, the restriction of* c *to* H *has weight at most* $f\left(\max\left\{\deg_G(v) : v \in V(H)\right\}\right)$*. In particular,*

$$\chi_f(G) \leq f(\Delta(G)).$$

Remark 2. In Theorem 8, if $x \geq 3$ then the minimum of the function $k \to \frac{2x+2^{k-3}}{k}$ (over $\mathbb{N}$) is attained for $k = [4 + \log_2 x - \log_2 \log_2 x]$. So if $x \geq 3$, then $f(x) = (2\ln 2 + o(1))x/\ln x$, which is off by a multiplicative factor $2\ln 2$ from the asymptotic value for triangle-free graphs which can be derived from Theorem 7. The turning point happens when the maximum degree is approximately $3{\cdot}10^6$. We also note that for every non-negative integer x, the minimum of the function $k \to \frac{2x+2^{k-3}}{k}$ (over $\mathbb{N}$) is attained at an integer greater than 3.

Finally, we provide improved upper bounds on the Hall ratio of graphs of maximum degree in $\{3, 4, 5\}$ and girth in $\{6, \ldots, 12\}$. In particular, these are upper bounds on the fractional chromatic number of vertex-transitive graphs in these classes. These upper bounds are obtained *via* a systematic computer-assisted method.

Theorem 9. *The values presented in Table 4 are upper bounds on* $\rho(d, g)$ *for* $d \in \{3, 4, 5\}$ *and* $g \in \{6, \ldots, 12\}$.

d \ g	6	7	8	9	10	11	12
3	$30/11 \approx 2.727272$	$30/11$	2.625224	2.604167	2.557176	2.539132	2.510378
4	$41/13 \approx 3.153846$	$41/13$	3.038497	3.017382	3		
5	$69/19 \approx 3.631579$	3.6	3.5				

TABLE 4. Upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, \ldots, 12\}$.

The bounds provided by Theorem 9 when $d \in \{3, 4\}$ and $g = 7$ are the same as those for $g = 6$. It seems that this could be a general phenomenon. We therefore offer the following conjecture, implicitly revealing that we expect our method to produce an upper bound of 2.5 on $\rho(3, 13)$.

Conjecture 1. *The values presented in Table 5 are upper bounds on* $\rho(d, g)$ *for* $d \in \{3, 4, 5\}$ *and* $g \in \{6, 8, 10, 12\}$.

d \ g	6	8	10	12
3		2.604167	2.539132	2.5
4		3.017382	3	
5	3.6	3.5		

TABLE 5. Conjectured upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, \ldots, 12\}$.

Notation. We introduce some notation before establishing a few technical lemmas, from which we will prove Theorems 8 and 9. If v is a vertex of a graph G and r a non-negative integer, then $N_G^r(v)$ is the set of all vertices of G at distance exactly r from v in G, while $N_G^r[v]$ is $\bigcup_{j=0}^r N_G^j(v)$. If u

is also a vertex of G, we write $\mathrm{dist}_G(u,v)$ for the distance in G between u and v. Further, if R is a subset of vertices of G, then we write $N_G(R)$ for the set of vertices that are not in R and have a neighbour in R, while $N_G[R]$ is $N_G(R) \cup R$. We will omit the graph subscript when there is no ambiguity, and sometimes write $N_X(v)$ instead of $N(v) \cap X$, for any subset of vertices $X \subseteq V(G)$. The set of all stable sets of G is $\mathscr{S}(G)$, while $\mathscr{S}_{\max}(G)$ is the set of all maximal stable sets of G and $\mathscr{S}_\alpha(G)$ is the set of all maximum stable sets of G. If w is a mapping from $\mathscr{S}(G)$ to $\mathbb{R}$ then for every vertex $v \in V(G)$ we set

$$w[v] \coloneqq \sum_{\substack{S \in \mathscr{S}(G) \\ v \in S}} w(S).$$

Further, if X is a collection of maximal stable sets of G, then $w(X) \coloneqq \sum_{S \in X} w(S)$. If S in a stable set of a graph G, a vertex v is *covered* by S if v belongs to S or has a neighbour in S. A vertex that is not covered by S is *uncovered* (by S). If G is a graph rooted at a vertex v, then for every positive integer d, the set of all vertices at distance d from v in G is a *layer* of G.

2. Technical lemmas

In this section we present the tools needed for the proofs of the main theorems.

2.1. **Greedy fractional colouring algorithm.** Our results on fractional colouring are obtained using a greedy algorithm analysed in a recent work involving the first author [7]. This algorithm is a generalisation of an algorithm first described in the book of Molloy and Reed [15, p. 245] for the uniform distribution over maximum stable sets. The setting here is, for each induced subgraph H of the graph we wish to fractionally colour, a probability distribution over the stable sets of H. We shall use only distributions over maximal stable sets.

Lemma 1 (de Joannis de Verclos *et al.*, 2018)**.** *Let G be a graph given with well-chosen parameters α_v, β_v for every vertex $v \in V(G)$. For every induced subgraph H of G, let $\mathbf{S}_H$ be a random stable set of H drawn according to a given probability distribution, and assume that*

$$\alpha_v \mathbb{P}\left[v \in \mathbf{S}_H\right] + \beta_v \mathbb{E}\left[|N(v) \cap \mathbf{S}_H|\right] \geq 1,$$

for every vertex $v \in V(H)$. Then the greedy fractional algorithm *defined by Algorithm 1 produces a fractional colouring w of G such that the restriction of w to any subgraph H of G is a fractional colouring of H of weight at most $\max_{v \in V(H)} \alpha_v + \beta_v \deg_G(v)$. In particular,*

$$\chi_f(G) \leq \max_{v \in V(G)} \alpha_v + \beta_v \deg_G(v).$$

Algorithm 1 The greedy fractional algorithm

```
for I ∈ 𝒮_max(G) do
    w(I) ← 0
end for
H ← G
while |V(H)| > 0 do
    ι ← min { min_{v∈V(H)} (1 − w[v]) / ℙ[v ∈ S_H], min_{v∈V(H)} (α_v + β_v deg_G(v)) − w(𝒮_max(G)) }
    for S ∈ 𝒮_max(H) do
        w(S) ← w(S) + ℙ[S_H = S] ι
    end for
end while
```

2.2. **Hard-core model.** In the setting of Lemma 1, we need a probability distribution over the stable sets of a given graph H. For instance, Molloy and Reed used the uniform distribution over the maximum stable sets of H, and obtained the fractional Reed bound as a result. As we will show in Section 3.1, this bound is best possible when restricting to the maximum stable sets, even for trees. Therefore, we need to include the non-maximum stable sets with non-zero probability in order to hope for improved bounds. Moreover, in order to perform a local analysis of the possible random outcomes, we need our probability distribution to have a good relative independence between the random outcomes in a local part of the graph, and the ones outside this part.

The probability distribution that we are going to use as a setting of Lemma 1 is the hard-core distribution over the stable sets of a graph, which has the Spatial Markov Property. Given a family $\mathscr{S}$ of stable sets of a graph H, and a positive real λ, a random stable set $\mathbf{S}$ drawn according to the hard-core distribution at fugacity λ over $\mathscr{S}$ is such that

$$\mathbb{P}[\mathbf{S} = S] = \frac{\lambda^{|S|}}{\sum\limits_{T \in \mathscr{S}} \lambda^{|T|}},$$

for every $S \in \mathscr{S}$.

Along this work, we consider two possible families $\mathscr{S}$ of stable sets of H, the first one being the whole set $\mathscr{S}(H)$ of stable sets of H. Note that when $\mathscr{S} = \mathscr{S}(H)$, and $\lambda \to \infty$, the hard-core distribution converges towards the uniform distribution over the maximum stable sets of H.

Lemma 2 (Spatial Markov Property)**.** *Given a graph H, and a real $\lambda > 0$, let $\mathbf{S}$ be drawn according to the hard-core distribution at fugacity λ over the stable sets $\mathscr{S}(H)$ of H. Let $X \subseteq V(H)$ be any given subset of vertices, and R any possible outcome of $\mathbf{S} \setminus X$. Then, conditioned on the fact that $\mathbf{S} \setminus X = R$, the random stable set $\mathbf{S} \cap X$ follows the hard-core distribution at fugacity λ over the stable sets of $H[X \setminus N(R)]$.*

The proof of this result is standard and follows from a simple consideration of the marginal probabilities. Things are more complicated with our second choice for $\mathscr{S}$, that is the set $\mathscr{S}_{\max}(H)$ of maximal stable sets of H. Indeed, in this setting, one has to make sure that the local outcome of the stable set is compatible with the fact that the global outcome of the stable set is maximal, *i.e.* there remains no uncovered vertices in H. This adds a new level of dependency, and we need the extra assumption (1) to be able to handle it. For two disjoint subsets of vertices X and U of a graph G, we define $P^2_X(U)$ to be the set of vertices $x \in X$ such that there exists a path $ux'x$ of length 2 with $u \in U$ and $x' \in X$.

Lemma 3 (Spatial Markov Property for maximal stable sets)**.** *Given a graph H, and a real $\lambda > 0$, let $\mathbf{S}$ be drawn according to the hard-core distribution at fugacity λ over the maximal stable sets $\mathscr{S}_{\max}(H)$ of H. Let $X \subseteq V(H)$ be any given subset of vertices, R any possible outcome of $\mathbf{S} \setminus X$, and $U \coloneqq (V(H) \setminus X) \setminus N[R]$ the set of vertices outside of X that are uncovered (by R). Moreover, we assume that*

$$|N(v) \cap X| \leq 1, \tag{1}$$

for any vertex $v \notin X$. Then, conditioned on the fact that $\mathbf{S} \setminus X = R$, the random stable set $\mathbf{S} \cap X$ follows the hard-core distribution at fugacity λ over the maximal stable sets of

$$H[X \setminus (N(R) \cup P^2_X(U)].$$

Proof. Set $W \coloneqq X \setminus (N(R) \cup P^2_X(U))$. First let S_X be any possible realisation of $\mathbf{S} \cap X$, conditioned on the fact that $\mathbf{S} \setminus X = R$. We prove that $S_X \in \mathscr{S}_{\max}(H[W])$. To this end, we begin by showing that $N_X(U) \subseteq S_X$. By the definitions of R and U, every vertex in U must be adjacent to a vertex in X, and hence for each $u \in U$ there exists a unique vertex v_u in X that is adjacent to U. It follows that $N_X(U)$ is contained in S_X. This in particular implies that no vertex in $P^2_X(U)$ can

belong to S_X, and hence $S_X \subseteq W$. We note for later that we just established that $N_X(U)$ is a set of isolated vertices of $H[W]$ (that is, these vertices belong to W and have no neighbour in $H[W]$).

Next we observe that S_X is maximal in $H[W]$. Indeed, let $w \in W \setminus S_X$. Because $S_X \cup R$ is a maximal independent set of H, there exists $v \in S_X \cup R$ that is adjacent to w in H. Since $W \subseteq X \setminus N(R)$ by definition, we deduce that $v \in S_X$ and hence S_X is maximal in $H[W]$.

Second, given any set $S_X \in \mathscr{S}_{\max}(H[W])$, the set $S_X \cup R$ is a valid realisation of $\mathbf{S}$. Indeed, S_X and R are stable sets, and so is their union as $S_X \cap N(R) = \varnothing$. To prove that $S_X \cup R$ is maximal in H, it suffices to show that every vertex x in $U \cup (X \setminus W)$ has a neighbour in S_X. As reported earlier, $N_X(U)$ is contained in W and forms a set of isolated vertices in $H[W]$. Therefore, $N_X(U)$ is contained in every maximal independent of $H[W]$, and hence in S_X. Since every vertex in U has a neighbour in X, it therefore only remains to deal with case where $x \in X \setminus W$. Then $x \in N(R) \cup P_X^2(U)$, and hence x has a neighbour in $R \cup N_X(U)$, which is contained in $S_X \cup R$.

In conclusion, the set of realisations of $\mathbf{S} \cap X$ is exactly $\mathscr{S}_{\max}(H[W])$, and each such realisation S_X has a probability proportional to $\lambda^{|S_X|+|R|}$, and hence proportional to $\lambda^{|S_X|}$ since R is fixed. This finishes the proof. □

2.3. **Independence ratio.** We state two lemmas which can be proved in similar ways. We only present the proof of the second one, the argument for the first lemma being very close but a little simpler.

Lemma 4. *Let r be a positive integer and G be a d-regular graph on n vertices. Let $\alpha_0, \dots, \alpha_r$ be real numbers such that $\sum_{i=1}^r \alpha_i (d-1)^{i-1} \geq 0$. Assume that there exists a probability distribution p on $\mathscr{S}_{\max}(G)$ such that*

$$\forall v \in V(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(v)\right] \geq 1, \tag{2}$$

where $\mathbf{X}_i(v)$ is the random variable counting the number of paths of length i between v and a vertex belonging to a random stable set $\mathbf{S}$ chosen following p. Then

$$\frac{n}{\alpha(G)} \leq \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1}. \tag{3}$$

Lemma 5. *Let r be a positive integer and G be a d-regular graph on n vertices. Let $\alpha_0, \dots, \alpha_r$ be real numbers such that $\sum_{i=0}^r \alpha_i (d-1)^{i} \geq 0$. Assume that there exists a probability distribution p on $\mathscr{S}_{\max}(G)$ such that*

$$\forall e \in E(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(e)\right] \geq 1, \tag{4}$$

where $\mathbf{X}_i(e)$ is the random variable counting the number of paths of length $i+1$ starting with e and ending at a vertex belonging to a random stable set $\mathbf{S}$ chosen following p. Then

$$\frac{n}{\alpha(G)} \leq \sum_{i=0}^{r} 2\alpha_i (d-1)^{i}. \tag{5}$$

Proof. Given an integer $i \in \{0, \dots, r\}$ and an edge e of G, the contribution of an arbitrary vertex $v \in \mathbf{S}$ to $\mathbf{X}_i(e)$ is the number of paths of length $i+1$ starting at v and ending with e. It follows that the total contribution of any vertex $v \in \mathbf{S}$ to $\sum_{e \in E(G)} \mathbf{X}_i(e)$ is the number of paths of G with length $i+1$ that start at v, which is $d(d-1)^i$ since G is a d-regular graph. Consequently,

$$\mathbb{E}\left[\sum_{e \in E(G)} \mathbf{X}_i(e)\right] = \sum_{v \in V(G)} \mathbb{P}\left[v \in \mathbf{S}\right] d(d-1)^i.$$

We now sum (4) over all edges of G.

$$\sum_{e\in E(G)}\sum_{i=0}^{r}\alpha_i\mathbb{E}\left[\mathbf{X}_i(e)\right] \geq |E(G)| = \frac{nd}{2}$$

$$\sum_{i=0}^{r}\alpha_i\sum_{e\in E(G)}\mathbb{E}\left[\mathbf{X}_i(e)\right] \geq \frac{nd}{2}$$

$$\sum_{i=0}^{r}\alpha_i\sum_{v\in V(G)}\mathbb{P}\left[v\in\mathbf{S}\right]d(d-1)^i \geq \frac{nd}{2}$$

$$\sum_{i=0}^{r}2\alpha_i\mathbb{E}\left[|\mathbf{S}|\right](d-1)^i \geq n$$

$$\sum_{i=0}^{r}2\alpha_i(d-1)^i \geq \frac{n}{\alpha(G)}$$

□

The next lemma allows us to generalise Lemmas 4 and 5 to non-regular graphs. To this end, we use a standard argument coupled with the existence of specific vertex-transitive type-1 regular graphs with any given degree and girth. These are provided by a construction of Exoo and Jajcay [8] in the proof of their Theorem 19, which is a direct generalisation of a construction for cubic graphs designed by Biggs [3, Theorem 6.2]. We slighty reformulate their theorem, the mentioned edge-colouring and transitivity property following simply from the fact that the graph constructed is a Cayley graph obtained from a generating set consisting only of involutions. Given a graph G endowed with an edge-colouring c, an automorphism f of G is *c-preserving* if $c(\{f(u),f(v)\}) = c(u,v)$ for each edge $\{u,v\}$ of G. The graph G is *c-transitive* if for every pair (u,v) of vertices of G there exists a c-preserving automorphism f of G such that $f(u)=v$.

Theorem 10 (Exoo & Jajcay, 2013)**.** *For every integers d and g both at least 3, there exists a d-regular graph H with girth at least g along with a proper edge-colouring c using d colours such that H is c-transitive.*

Lemma 6. *From any graph G of maximum degree d and girth g, we can construct a d-regular graph $\varphi(G)$ of girth g whose vertex set can be partitioned into induced copies of G, and such that any vertex $v\in G$ can be sent to any of its copies through an automorphism.*

Proof. Set $k := \sum_{v\in G}(d-\deg(v))$. Let G' be the supergraph of G obtained by adding k vertices $v'_1,\dots,v'_k$ each of degree 1, such that all other vertices have degree d. We let e'_i be the edge of G' incident to v'_i, for each $i\in\{1,\dots,k\}$. By Theorem 10, there exists a k-regular graph H of girth at least g together with a proper edge-colouring c using k colours, such that H is c-transitive. Let $n(H)$ be the number of vertices of H and write $V(H)=\{1,\dots,n(H)\}$.

We construct $\varphi(G)$ by starting from the disjoint union of $n(H)$ copies $G_1,\dots,G_{n(H)}$ of G. For each edge $e=\{i,j\}\in E(H)$, letting u_e be the vertex of G incident to the edge $e'_{c(e)}$ in G', we add an edge between the copy of u_e in G_i and that in G_j.

Any cycle in $\varphi(G)$ either is a cycle in G, and hence has length at least g, or contains all the edges of a cycle in H, and hence has length at least g. It follows that $\varphi(G)$ has girth g.

The last statement follows directly from the fact that H is c-transitive. □

Corollary 1. *Let d and g be integers greater than two. If there exists a constant $B=B(d,g)$ such that every d-regular graph H with girth g has independence ratio at least B, then every graph G with maximum degree d and girth g also has independence ratio at least B. In particular, if Lemma 4 or*

Lemma 5 can be applied to the class of d-regular graphs of girth g, then the conclusion also holds for the class of graphs with maximum degree d and girth g, that is, for $\rho(d, g)$.

Proof. Let G be a graph with maximum degree d and girth g on n vertices. Let $\varphi(G)$ be the graph provided by Lemma 6. In particular, $|V(\varphi(G))| = kn$ where k is the number of induced copies of G partitioning $V(\varphi(G))$. By assumptions, $\varphi(G)$ contains a stable set I of order at least $B \cdot kn$. Letting I_i be the set of vertices of the i-th copy of G contained in I, by the pigeon-hole principle there exists $i \in \{1, \dots, k\}$ such that $|I_i| \geq B \cdot n$, and hence G has independence ratio at least B. □

3. Fractional colourings

3.1. **A local version of Reed's bound.** For the sake of illustration, we begin by showing how Lemma 1 can be used to prove Theorem 5. We actually establish a slight strengthening of Theorem 6, the local form of Theorem 5. The argument relies on the relation (A) below [12, Lemma 2.11], which is a local version of the relation (21.10) appearing in Molloy and Reed's book [15]. The short argument, however, stays the same and we provide it here only for explanatory purposes, since it is the inspiration for the argument used in the proof of Theorem 8.

Proposition 1. *Let G be a graph, and set $f_G(v) := \frac{\omega_G(v)+\deg_G(v)+1}{2}$ for every $v \in V(G)$, where $\omega_G(v)$ is the order of a largest clique in G containing v. Then G admits a fractional colouring c such that the restriction of c to any induced subgraph H of G has weight at most $\max_{v \in V(H)} f_G(v)$. In particular,*

$$\chi_f(G) \leq \max \{ f_G(v) \, : \, v \in V(G) \} .$$

Proof. We demonstrate the statement by applying Lemma 1. To this end, we use the uniform distribution on maximum stable sets, which corresponds to the hard-core distribution at fugacity $\lambda = \infty$.

(A). *For every induced subgraph H of G, let $\mathbf{S}_H$ be a maximum stable set of H, drawn uniformly at random. Then for every vertex $v \in V(H)$,*

$$\frac{\omega(v)+1}{2} \mathbb{P}\left[v \in \mathbf{S}_H\right] + \frac{1}{2} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H|\right] \geq 1.$$

The conclusion then follows by applying Lemma 1, with $r = 1$, $\alpha_v = \frac{1}{2} \cdot (\omega(v) + 1)$ and $\beta_v = \frac{1}{2}$ for every vertex $v \in V(G)$.

It remains to establish (A). We let R be any possible outcome of $\mathbf{S}_H \setminus N[v]$, and $W = N[v] \setminus N(R)$. We condition on the random event E_R that $\mathbf{S}_H \setminus N[v] = R$, and the Spatial Markov Property of the uniform distribution over the maximum stable sets of H ensures that $\mathbf{S}_H \cap W$ is a uniform random maximum stable set of $H[W]$. There are two cases.

(i) If W is a clique of size $k \leq \omega(v)$, then exactly one vertex from W belongs to $\mathbf{S}_H$, and every vertex in W has equal probability $1/k$ to be in $\mathbf{S}_H$. So, in this case,

$$\frac{\omega(v)+1}{2} \mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] + \frac{1}{2} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] = \frac{\omega(v)+1}{2k} + \frac{k-1}{2k} \geq 1.$$

(ii) If W is not a clique, then $|W \setminus \{v\} \cap \mathbf{S}_H| \geq 2$ and $v \notin \mathbf{S}_H$, since $\mathbf{S}_H$ is a maximum stable set. So, in this case,

$$\frac{\omega(v)+1}{2} \mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] + \frac{1}{2} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] \geq \frac{1}{2} \times 2 = 1$$

The validity of (A) follows by summing over all possible realisations R of $\mathbf{S}_H \setminus N[v]$. □

We finish by noting that the bound provided by Theorem 6 is best possible over the class of unicyclic triangle-free graphs if one uses the fractional greedy colouring of Lemma 1 together with any probability distribution on the *maximum* stable sets of the graph.

Lemma 7. *If the probability distribution used in Lemma 1 gives positive probability only to maximum stable sets, then the greedy fractional colouring algorithm can return a fractional colouring of weight up to $\frac{d+3}{2}$ in general for graphs of degree d, should they be acyclic when d is odd, or have a unique cycle (of length 5) when d is even.*

Proof. We prove the statement by induction on the positive integer d.

- If $d = 1$, then let G_1 consist only of an edge. The algorithm returns a fractional colouring of G_1 of weight 2.
- If $d = 2$, then let G_2 be the cycle of length 5. The algorithm returns a fractional colouring of G_2 of weight $\frac{5}{2}$.
- If $d > 2$, then let G_d be obtained from G_{d-2} by adding two neighbours of degree 1 to every vertex. This creates no new cycles, so G_d is acyclic when d is odd, and contains a unique cycle, which is of length 5, when d is even.

  For every $d \geq 3$, the graph G_d contains a unique maximum stable set, namely $S_0 := V(G_d) \setminus V(G_{d-2})$. After the first step of the algorithm applied to G_d, all the vertices in S_0 have weight 1, and we are left with the graph G_{d-2} where every vertex has weight 0. By the induction hypothesis, the total weight of the fractional colouring returned by the algorithm is therefore $1 + \frac{(d-2)+3}{2} = \frac{d+3}{2}$.

□

3.2. **Triangle-free graphs.** Using a similar approach, it is possible to obtain improved bounds for the fractional chromatic number of a given triangle-free graph G, if we apply Lemma 1 with a stable set $\mathbf{S}_H$ drawn according to the hard-core distribution at fugacity λ over the set $\mathscr{S}(H)$ of all stable sets of any induced subgraph H of G, for a carefully chosen $\lambda > 0$.

(B). *For every subgraph H of G, every vertex $v \in V(H)$, and every integer $k \geq 1$,*

$$\left(1 + \frac{(1+\lambda)^k}{\lambda(1+k\lambda)}\right) \mathbb{P}\left[v \in \mathbf{S}_H\right] + \frac{1+\lambda}{1+k\lambda} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H|\right] \geq 1.$$

Note that when $\omega = 2$, we deduce (A) from (B) by taking $k = 2$ and letting λ go to infinity.

Proof. We let R be any possible realisation of $\mathbf{S}_H \setminus N[v]$. By the Spatial Markov Property of the hard-core distribution, if we condition on the event E_R that $\mathbf{S}_H \setminus N[v] = R$ and write $W := N[v] \setminus N(R)$, then $\mathbf{S}_H \cap N[v]$ follows the hard-core distribution at fugacity λ over $\mathscr{S}(H[W])$. Since G (and therefore also H) is triangle-free, then $H[W]$ is a star $K_{1,d}$, for some integer $d \in \{0, \ldots, \Delta(G)\}$. An analysis of the hard-core distribution over the stable sets of a star yields that

(i) $\mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] = \dfrac{\lambda}{\lambda + (1+\lambda)^d}$, and

(ii) $\mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] = \dfrac{d\lambda(1+\lambda)^{d-1}}{\lambda + (1+\lambda)^d}$.

For some positive real numbers α and β, we let

$$g(x) := \alpha \frac{\lambda}{\lambda + (1+\lambda)^x} + \beta \frac{x\lambda(1+\lambda)^{x-1}}{\lambda + (1+\lambda)^x}.$$

We observe that g is a convex function, and therefore its minimum over the (non-negative) reals is reached at its unique critical point x^* such that $g'(x^*) = 0$ (if it exists). Moreover, if there are numbers y and $z > y$ such that $g(y) = g(z)$, then Rolle's theorem ensures that $x^* \in (y, z)$, and $g(x) \geq g(y)$ for every $x \notin (y, z)$.

Let k be a positive integer. We now fix

$$\alpha := 1 + \frac{(1+\lambda)^k}{\lambda(1+k\lambda)} \quad \text{and} \quad \beta = \frac{1+\lambda}{1+k\lambda}.$$

One can easily check that with these values for α and β, it holds that

$$g(k-1) = 1 \quad \text{and} \quad g(k) = 1.$$

We conclude that $g(d) \geq 1$ for every non-negative integer d, which means that

$$\left(1 + \frac{(1+\lambda)^k}{\lambda(1+k\lambda)}\right) \mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] + \frac{1+\lambda}{1+k\lambda} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] \geq 1, \tag{6}$$

for any possible realisation R of $\mathbf{S}_H \setminus N[v]$. The conclusion follows again by a convex combination of (6) over all the possible values of R. □

3.3. **A stronger bound for graphs of girth 7.** Let G be a graph of girth (at least) 7 and H an induced subgraph of G. We wish to apply Lemma 1 with a stable set $\mathbf{S}_H$ draw according to the hard-core distribution at fugacity λ over the set $\mathscr{S}_{\max}(H)$ of all *maximal* stable sets of H, for the specific value $\lambda = 4$. We now establish the following assertion.

(C). *For every given vertex $v \in V(H)$ and every integer $k \geq 4$,*

$$\frac{2^{k-3}+k}{k} \mathbb{P}\left[v \in \mathbf{S}_H\right] + \frac{2}{k} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H|\right] \geq 1.$$

Proof. Let R be any possible realisation of $\mathbf{S}_H \setminus N^2[v]$. We are going to condition on the random event E_R that $\mathbf{S}_H \setminus N^2[v] = R$. Let $U := (V(H) \setminus N^2[v]) \setminus N[R]$ be the set of vertices at distance more than 2 from v that are uncovered (by R), and set $W := N^2[v] \setminus (N(R) \cup P^2_{N^2[V]}(U))$. (It follows from the definitions that $v \in W$.) Because the girth of G is greater than 6, no vertex outside of $N^2[v]$ has more than one neighbour in $N^2[v]$. Then Lemma 3 ensures that $\mathbf{S}_H \cap W$ follows the hard-core distribution at fugacity $\lambda = 4$ over the maximal stable sets of $H[W]$.

In general, $H[W]$ is not connected, but it is sufficient for us to assume that it is, by considering the connected component of $H[W]$ containing v.

We let W_i be the set of vertices in W at distance i from v in $H[W]$, for $i \in \{0,1,2\}$, and $W_{1,j}$ be the subset of vertices of W_1 with j neighbours in W_2. We set $x_j := |W_{1,j}|$. Thus $|W_1| = \sum_{j=0}^{d-1} x_j$ and $|W_2| = \sum_{j=1}^{d-1} j x_j$, since G has girth greater than 4.

Note that $W_1 \in \mathscr{S}_{\max}(H[W])$ and that $\mathbb{P}\left[\mathbf{S}_H \cap W = W_1\right]$ is proportional to $\lambda^{\sum_{j=0}^{d-1} x_j}$. In order to ease the following computations and verifications, we compute a weight $w(S)$ for each stable set $S \in \mathscr{S}_{\max}(H[W])$ that is proportional to $\mathbb{P}\left[\mathbf{S}_H \cap W = S\right]$, such that $w(W_1) = 1$.

There is exactly one maximal stable set S_0 that contains v, namely $S_0 := \{v\} \cup W_2$, of normalised weight $w_0 := \omega(S_0) = \lambda^{1+\sum_{j\geq 0}(j-1)x_j}$. Every other maximal stable set $S \in \mathscr{S}_{\max}(H[W]) \setminus \{S_0, W_1\}$ contains $W_{1,0}$. In addition, for every vertex $u \in W_1 \setminus W_{1,0}$, the set S either contains u or it contains all the neighbours of u in W_2. Therefore, it follows that if $x_0 > 0$, then the sum of the weights of these other stable sets is

$$T := \sum_{i_1 \leq x_1, \ldots, i_{d-1} \leq x_{d-1}} \prod_{j=1}^{d-1} \binom{x_j}{i_j} \left(\lambda^{j-1}\right)^{i_j} = \prod_{j=1}^{d-1} \left(1 + \lambda^{j-1}\right)^{x_j}.$$

If $x_0 = 0$, then the sum of their weights is $T - \frac{w_0}{\lambda}$, since there is no stable set containing W_2 in whole and not v in this case.

We let $D \coloneqq w_0 + T$ if $x_0 > 0$, and $D \coloneqq T + w_0\left(1 - \frac{1}{\lambda}\right)$ otherwise. It follows that

$$\mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] = \frac{w_0}{D} \qquad \text{and}$$

$$\mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] = \frac{T}{D}\left(x_0 + \sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right).$$

There remains to check that, when $\lambda = 4$, it holds that

$$\frac{2^{k-3}+k}{k}\mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] + \frac{2}{k}\mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] \geq 1.$$

This translates to

$$\left(2^{k-3} + \frac{k}{\lambda}\right) \cdot w_0 \geq T\left(k - 2\sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 = 0, \text{ and to}$$

$$2^{k-3} w_0 \geq T\left(k - 2x_0 - 2\sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 \neq 0.$$

Notice that the condition is verified if $x_0 \geq k/2$, so we assume from now on that $x_0 < k/2$. We use the two following facts.

Fact 1: For every positive integer j, the function $\lambda \mapsto \left(1 + \frac{1}{\lambda^{j-1}}\right)^{1+\lambda^{j-1}}$ is non increasing on $(0, +\infty)$, and in particular always bounded from above by $\frac{3125}{1024}$ when $\lambda \geq 4$ and $j \geq 2$, and by $\left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{1+\lambda^{j_0-1}}$ when $\lambda \geq 1$ and $j \geq j_0$.

Fact 2: For all real numbers y_0, A and B with $A > 1$ and $B > 0$, the maximum of the function $f\colon y \mapsto A^y(B-2y)$ on the domain $[y_0, +\infty)$ is $f(y_0)$ when $B/2 - 1/\ln A \leq y_0$, and $\frac{2A^{B/2}}{e \ln A}$ otherwise.

Let us discriminate on the possible values for x_0, noting that $w_0 \geq \lambda^{1-x_0}$.

(i) When $x_0 = 0$, it suffices to show that

$$2^{k-3}\lambda + k \geq \prod_{j=1}^{d-1}\left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right). \tag{7}$$

(ii) When $1 \leq x_0 < k/2$, it suffices to show that

$$2^{k-3}\lambda^{1-x_0} \geq \prod_{j=1}^{d-1}\left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k - 2x_0 - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right). \tag{8}$$

Recall that, according to the definition, each value x_j is a non-negative integer. Note that the right side of inequality (7) and that of inequality (8) are both at most 0 if $x_1 \geq k - 2x_0$; so we may assume that $x_1 \in [k - 2x_0 - 1]$. Let us now fix $\lambda = 4$, and prove the stronger statement that the right side of inequality (8), which we call $R8$, is always at most 2^{k-2x_0-1}. This implies both (7) and (8). Since the inequality is verified if $R8 \leq 0$, we assume now on that $R8$ is positive. We define $y_j \coloneqq \dfrac{x_j}{1+\lambda^{j-1}}$, for every $j \in \{1, \ldots, d-1\}$.

- If $x_1 = k - 2x_0 - 1$, then

$$\begin{aligned}
R8 &= 2^{k-2x_0-1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(1 - 2\sum_{j=2}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{k-2x_0-1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j} \left(1 - 2\sum_{j=2}^{d-1} y_j\right) \\
&\le 2^{k-2x_0-1} \cdot \prod_{j=2}^{d-1} \left(\frac{3125}{1024}\right)^{y_j} \left(1 - 2\sum_{j=2}^{d-1} y_j\right) && \text{by Fact 1 because } R8 > 0 \\
&= 2^{k-2x_0-1} \cdot \left(\frac{3125}{1024}\right)^{y} (1-2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1} y_j \\
&\le 2^{k-2x_0-1} \cdot \max_{y \in \mathbb{R}^+} \left(\frac{3125}{1024}\right)^{y} (1-2y) && \text{because } R8 > 0 \\
&\le 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}$$

- If $x_1 = k - 2x_0 - 2$, then

$$R8 = 2^{k-2x_0-2} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(2 - 2\sum_{j=2}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right).$$

  If $x_j = 0$ for every $j \in \{2, \dots, d-1\}$, then $R8 \le 2^{k-2x_0-1}$. Let us now assume otherwise, and set $j_0 \coloneqq \min\{j \ge 2 : x_j > 0\}$. In particular $x_{j_0} \ge 1$ and $y_{j_0} \ge \frac{1}{1+\lambda^{j_0-1}}$. Then

$$\begin{aligned}
R8 &= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(2 - 2\sum_{j=j_0}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) \\
&\le 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) && \text{by Fact 1 because } R8 > 0 \\
&= 2^{k-2x_0-2} \cdot \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y} (2-2y) && \text{where } y \coloneqq \sum_{j=j_0}^{d-1} y_j \ge \frac{1}{1+\lambda^{j_0-1}} \\
&\le 2^{k-2x_0-2} \cdot \max_{y \ge \frac{1}{1+\lambda^{j_0-1}}} \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{(1+\lambda^{j_0-1})y} (2-2y) && \text{because } R8 > 0 \\
&\le 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}$$

- If $x_1 \leq k - 2x_0 - 3$, then

$$\begin{aligned}
R8 &= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) \\
&\leq 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(\frac{3125}{1024}\right)^{y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) && \text{by Fact 1 because } R8 > 0 \\
&= 2^{x_1} \cdot \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1} y_j \\
&\leq 2^{x_1} \cdot \max_{y \in \mathbb{R}} \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) && \text{because } R8 > 0 \\
&\leq 2^{x_1} \frac{2\left(\frac{3125}{1024}\right)^{\frac{k-2x_0-x_1}{2}}}{e \ln\left(\frac{3125}{1024}\right)} && \text{by Fact 2} \\
&\leq 2^{k-2x_0-1} && \text{as } (k - 2x_0 - 1)/2 > 1.
\end{aligned}$$

This finishes to establish that

$$\frac{2^{k-3} + k}{k} \mathbb{P}\left[v \in \mathbf{S}_H \mid E_R\right] + \frac{2}{k} \mathbb{E}\left[|N(v) \cap \mathbf{S}_H| \mid E_R\right] \geq 1. \tag{9}$$

The conclusion follows by the convex combination of (9) over all possible values of R. □

We set $\lambda \coloneqq 4$, and apply Lemma 1 with

$$\alpha_v = 1 + \frac{2^{k(v)-3}}{k(v)} \quad \text{and} \quad \beta_v = \frac{2}{k(v)}$$

for every vertex $v \in V(G)$, where $k(v)$ is chosen such that $\frac{2\deg(v)+2^{k-3}}{k}$ is minimised when $k = k(v)$, and is always at least 4 since $\deg(v)$ is a non-negative integer. This ends the proof of Theorem 8.

4. Bounds on the Hall ratio

We focus on establishing upper bounds on the Hall ratios of graphs with bounded maximum degree and girth. These bounds are obtained by using the uniform distribution on $\mathscr{S}_\alpha(G)$, for G in the considered class of graphs, into Lemma 4 or Lemma 5.

4.1. Structural analysis of a neighbourhood.

We start by introducing some terminology.

Definition 1.

(1) A *pattern of depth* r is any graph P given with a root vertex v such that

$$\forall u \in V(G), \quad \mathrm{dist}_G(u, v) \leq r.$$

The *layer at depth* i of P is the set of vertices at distance i from its root vertex v.

(2) A pattern P of depth r and root v is *d-regular* if all its vertices have degree exactly d, except maybe in the two deepest layers where the vertices have degree at most d.

Definition 2. Let P be a pattern with depth r and root v. Let $\mathbf{S}$ be a uniform random maximum stable of P. We define $e_i(P) \coloneqq \mathbb{E}\left[|\mathbf{S} \cap N_P^i(v)|\right]$ for each $i \in \{0, \ldots, r\}$.

(1) The *constraint* associated to P is the pair $c(P) = (\mathbf{e}(P), n(P))$, where $\mathbf{e}(P) = (e_i(P))_{i=0}^r \in (\mathbb{Q}^+)^{r+1}$, and $n(P) \in \mathbb{N}$ is the *cardinality* of the constraint, which is the number of maximum stable sets of P. Most of the time, we only need to know the value of $\mathbf{e}(P)$, in which case we characterise the constraint $c(P) = (\mathbf{e}(P), n(P))$ only by $\mathbf{e}(P)$. The value of $n(P)$ is only needed for a technical reason, in order to be able to compute constraints inductively.

(2) Given two constraints $\mathbf{e}, \mathbf{e}' \in (\mathbb{Q}^+)^{r+1}$, we say that $\mathbf{e}$ is *weaker* than $\mathbf{e}'$ if, for any vector $\boldsymbol{\alpha} \in (\mathbb{Q}^+)^{r+1}$ it holds that
$$\boldsymbol{\alpha}^\top \mathbf{e}' \geq 1 \quad \Longrightarrow \quad \boldsymbol{\alpha}^\top \mathbf{e} \geq 1.$$
If the above condition holds only for all vectors $\boldsymbol{\alpha} \in (\mathbb{Q}^+)^{r+1}$ with non-increasing coordinates, then we say that $\mathbf{e}$ is *relatively weaker* than $\mathbf{e}'$.

Note that $\mathbf{e}$ is weaker than $\mathbf{e}'$ if and only if
$$\forall i \in \{0, \dots, r\}, \quad e_i \geq e'_i,$$
and $\mathbf{e}$ is relatively weaker than $\mathbf{e}'$ if and only if
$$\forall i \in \{0, \dots, r\}, \quad \sum_{j=0}^{i} e_j \geq \sum_{j=0}^{i} e'_j.$$

Remark 3. Let P be a pattern such that one of its vertices u is adjacent with some vertices $u_1, \dots, u_k$ of degree 1 in the next layer, where $k \geq 2$. Then every maximum stable set of P contains $\{u_1, \dots u_k\}$ and not u. Consequently, $\mathbf{e}(P)$ is weaker than $\mathbf{e}(P \setminus \{u_3, \dots, u_k\})$ since, letting i be the distance between u_1 and the root of P, one has
$$e_j(P) = \begin{cases} e_j(P \setminus \{u_3, \dots, u_k\}) & \text{if } j \neq i, \text{ and} \\ e_i(P \setminus \{u_3, \dots, u_k\}) + (k-2) & \text{if j=i.} \end{cases}$$

4.2. **Tree-like patterns.**

4.2.1. *Rooting at a vertex.* Fix an integer $r \geq 2$. Let G be a d-regular graph of girth at least $2r+2$, and let $\mathbf{S}$ be a uniform random maximum stable set of G. For any fixed vertex v, we let R be any possible realisation of $\mathbf{S} \setminus N^r[v]$, and $W := N^r[v] \setminus N(R)$. By the Spatial Markov Property of the uniform distribution over the maximum stable sets of G, the random stable set $\mathbf{S} \cap N^r[v]$ follows the uniform distribution over the maximum stable sets of $G[W]$. Now, observe that $G[W]$ is a d-regular pattern of depth r with root vertex v, and since G has girth at least $2r+2$, this is a tree. Let $\mathcal{T}_r(d)$ be the set of acyclic d-regular patterns of depth r.

Let us define $\mathbf{X}_i(v) := \mathbf{S} \cap N^i(v)$, for every $i \in \{0, \dots, r\}$. We seek parameters $(\alpha_i)_{i=0}^r$ such that the inequality $\sum_{i=0}^r \alpha_i \mathbb{E}\left[|\mathbf{X}_i(v)|\right] \geq 1$ is satisfied regardless of the choice of v. To this end, it is enough to pick the rational numbers α_i in such a way that the inequality is satisfied in any tree $T \in \mathcal{T}_r(d)$, when v is the root vertex. In a more formal way, given any $T \in \mathcal{T}_r(d)$, the vector $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r)$ must be compatible with the constraint $\mathbf{e}(T)$, that is, $\boldsymbol{\alpha}^\top \mathbf{e}(T) \geq 1$ for each $T \in \mathcal{T}_r(d)$.

An application of Lemma 4 then lets us conclude that the desired bound is the solution to the following linear program.

$$\frac{|G|}{\alpha(G)} \leq \min \quad \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1} \tag{10}$$
$$\text{such that} \begin{cases} \forall T \in \mathcal{T}_r(d), & \displaystyle\sum_{i=0}^{r} \alpha_i e_i(T) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

The end of the proof is made by computer generation of $\mathcal{T}_r(d)$, in order to generate the desired linear program, which is then solved again by computer computation. For the sake of illustration, we give a complete human proof of the case where $r = 2$ and $d = 3$. There are 10 trees in $\mathcal{T}_2(3)$. One can easily compute the constraint $(e_0(T), e_1(T), e_2(T))$ for each $T \in \mathcal{T}_2(3)$; they are depicted in Figure 4. Note that constraints $\mathbf{e}_8$, $\mathbf{e}_9$ and $\mathbf{e}_{10}$ are weaker than constraint $\mathbf{e}_7$, so we may disregard these constraints in the linear program to solve. Note also that constraint $\mathbf{e}_0$ is relatively weaker than constraint $\mathbf{e}_1$, and so may be disregarded as well, provided that the solution of the linear program is attained by a vector $\boldsymbol{\alpha}$ with non-increasing coordinates, which will have to be checked. The linear program to solve is therefore the following.

$$\begin{array}{ll} \text{minimise} & \alpha_0 + 3\alpha_1 + 6\alpha_2 \\ \text{such that} & \begin{cases} 5/2 \cdot \alpha_1 + 1/2 \cdot \alpha_2 & \geq 1 \\ 2\alpha_1 + 2\alpha_2 & \geq 1 \\ 1/5 \cdot \alpha_0 + 8/5 \cdot \alpha_1 + 6/5 \cdot \alpha_2 & \geq 1 \\ 1/3 \cdot \alpha_0 + \alpha_1 + 8/3 \cdot \alpha_2 & \geq 1 \\ 1/2 \cdot \alpha_0 + 1/2 \cdot \alpha_1 + 4\alpha_2 & \geq 1 \\ \alpha_0 + 3\alpha_2 & \geq 1 \\ \alpha_0, \alpha_1, \alpha_2 \geq 0. \end{cases} \end{array}$$

The solution of this linear program is $\frac{85}{31} \approx 2.741935$, attained by $\boldsymbol{\alpha} = \left(\frac{19}{31}, \frac{14}{31}, \frac{4}{31}\right)$, which indeed has non-increasing coordinates. This is an upper bound on $\rho(3, 6)$, though we prove a stronger one through a more involved computation in Section 4.3.2.

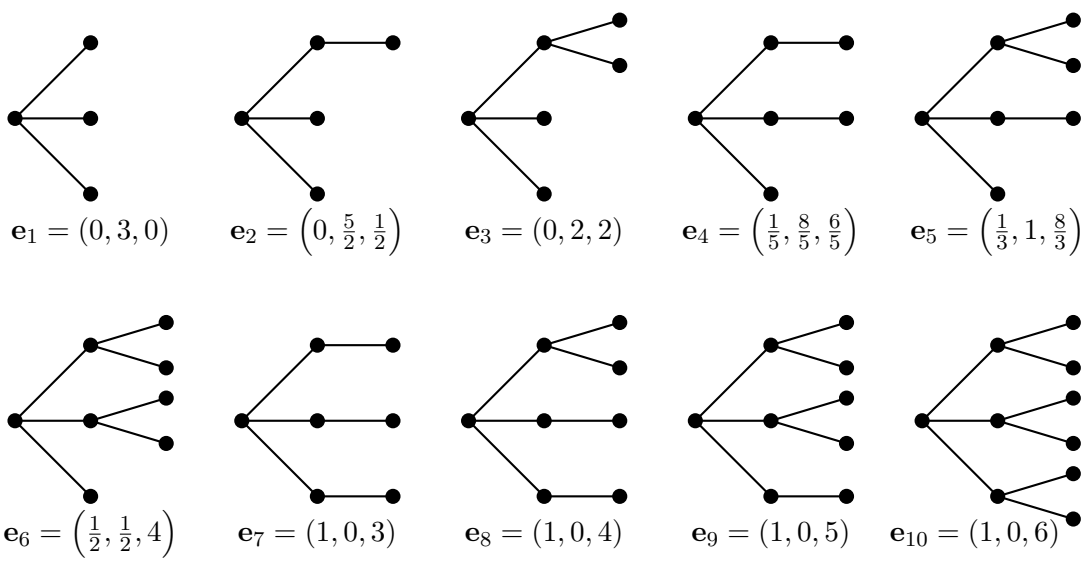


FIGURE 4. An enumeration of $\mathbf{e}(T)$ for all trees $T \in \mathcal{T}_2(3)$.

To compute $\mathbf{e}(T)$ for each $T \in \mathcal{T}_r(d)$, one can enumerate all the maximum stable sets of T and average the size of their intersection with each layer of T. For general graphs, there might be no better way of doing so, however the case of $\mathcal{T}_r(d)$ can be treated inductively by a standard approach: we distinguish between the maximum stable sets that contain the root and those that do not. We introduce the following notation; to this end, we slightly extend our notion of "constraint" to any

pair $(\mathbf{e}, n)$ where $\mathbf{e} \in (\mathbb{Q}^+)^{r+1}$ and n is a non-negative integer; the constraints we shall use have a combinatorial interpretation with respect to some pattern.

Definition 3. Let $c = (\mathbf{e}, n)$ and $c' = (\mathbf{e}', n')$ be two constraints.

(1) The operation $\vee$ on c and c' returns the constraint

$$c \vee c' \coloneqq \begin{cases} \left(\dfrac{n}{n+n'}\mathbf{e} + \dfrac{n'}{n+n'}\mathbf{e}', n+n'\right) & \text{if } \|\mathbf{e}\|_1 = \|\mathbf{e}'\|_1, \\ c & \text{if } \|\mathbf{e}\|_1 > \|\mathbf{e}'\|_1, \\ c' & \text{if } \|\mathbf{e}\|_1 < \|\mathbf{e}'\|_1. \end{cases}$$

(2) The operation $\oplus$ on c and c' returns the constraint $c \oplus c' \coloneqq (\mathbf{e} + \mathbf{e}', n \cdot n')$.

For a given tree $T \in \mathcal{T}_r(d)$ with root v, let $c_1(T)$ be the constraint associated to T where v is forced (that is, we restrict to the maximum stable sets that contain v when computing the constraint $c_1(T)$), and let $c_0(T)$ be the constraint associated to T where v is forbidden. It readily follows from Definition 3 that

$$c(T) = c_0(T) \vee c_1(T).$$

If $(T_i)_{i=1}^d$ are the subtrees of T rooted at the children of the root v (some of which might be empty), then

$$c_0(T) = ((0, \mathbf{e}), n) \qquad \text{where } (\mathbf{e}, n) = \bigoplus_{i=1}^{d} c(T_i), \quad \text{and}$$

$$c_1(T) = ((1, \mathbf{e}), n) \qquad \text{where } (\mathbf{e}, n) = \bigoplus_{i=1}^{d} c_0(T_i).$$

We thus obtain an inductive way of computing $\mathbf{e}(T)$ by using the following initial values.

$$c_0(\varnothing) \coloneqq ((0), 1) \qquad c_1(\varnothing) \coloneqq ((0), 0)$$
$$c_0(\{v\}) \coloneqq ((0), 1) \qquad c_1(\{v\}) \coloneqq ((1), 1).$$

Using this inductive way to enumerate the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}_r(d)$, the following statement is obtained by computer calculus.

Lemma 8. *The solution to the linear program* (10) *is*

$$\mathcal{T}_3(3): \quad \frac{5849}{2228} \approx 2.625224 \qquad \textit{with } \boldsymbol{\alpha} = \left(\frac{953}{2228}, \frac{162}{557}, \frac{81}{557}, \frac{21}{557}\right),$$

$$\mathcal{T}_4(3): \quad \frac{2098873192}{820777797} \approx 2.557176 \qquad \textit{with } \boldsymbol{\alpha} = \left(\frac{225822361}{820777797}, \frac{18575757}{91197533}, \frac{10597368}{91197533}, \frac{5054976}{91197533}, \frac{1172732}{91197533}\right),$$

$$\mathcal{T}_5(3): \quad \frac{29727802051155412}{11841961450578397} \approx 2.510378 \quad \textit{with } \boldsymbol{\alpha} = \left(\frac{3027359065168972}{11841961450578397}, \frac{2216425114872980}{11841961450578397}, \frac{2224040336719575}{23683922901156794}, \frac{2026654050681425}{47367845802313588}, \frac{403660478424775}{23683922901156794}, \frac{51149140376400}{11841961450578397}\right),$$

$$\mathcal{T}_3(4): \quad \frac{7083927}{2331392} \approx 3.038497 \qquad \textit{with } \boldsymbol{\alpha} = \left(\frac{123345}{333056}, \frac{68295}{291424}, \frac{12283}{145712}, \frac{2911}{145712}\right),$$

$\mathcal{T}_4(4)$: $\quad 3 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{7}{43}, \frac{6}{43}, \frac{19}{258}, \frac{7}{258}, \frac{1}{258}\right)$,

$\mathcal{T}_2(5)$: $\quad \frac{69}{19} \approx 3.631579 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{37}{57}, \frac{6}{19}, \frac{4}{57}\right)$,

$\mathcal{T}_3(5)$: $\quad \frac{7}{2} = 3.5 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{77}{282}, \frac{25}{141}, \frac{17}{282}, \frac{2}{141}\right)$.

4.2.2. *Rooting in an edge.* Definition 1 can be extended to a pattern with a root-edge instead of a root-vertex. The distance in a pattern P between a vertex w and an edge uv is defined to be $\min\{\mathrm{dist}_P(w,u), \mathrm{dist}_P(w,v)\}$. The depth of a pattern P rooted in an edge e is then the largest distance between e and a vertex in P. It is possible to follow the same analysis as in Section 4.2.1 with edge-rooted patterns: in order for the edge-rooted pattern of depth r to always be a tree, the graph G must have girth at least $2r+3$. Let $\mathcal{T}'_r(d)$ be the set of acyclic edge-rooted d-regular patterns of depth r. By Lemma 5, the linear program to solve is now the following.

$$\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^{r} \alpha_i (d-1)^i \tag{11}$$

$$\text{such that} \begin{cases} \forall T \in \mathcal{T}'_r(d), & \sum_{i=0}^{r} \alpha_i e_i(T) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

For a given tree $T \in \mathcal{T}'_r(d)$ rooted in $e = uv$, it is possible to compute $\mathbf{e}(T)$ using the constraints associated to vertex-rooted trees. If T_u and T_v are the subtrees of T respectively rooted at u and at v, then it readily follows from Definition 3 that

$$c(T) = \Big(c_0(T_u) \oplus c_0(T_v)\Big) \vee \Big(c_0(T_u) \oplus c_1(T_v)\Big) \vee \Big(c_1(T_u) \oplus c_0(T_v)\Big). \tag{12}$$

Following the enumeration of the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}'_r(d)$ described earlier, the next statement is obtained by computer calculus.

Lemma 9. *The solution to the linear program* (11) *is*

$\mathcal{T}'_2(3)$: $\quad \frac{30}{11} \approx 2.72727 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{1}{2}, \frac{13}{44}, \frac{3}{44}\right)$,

$\mathcal{T}'_3(3)$: $\quad \frac{125}{48} \approx 2.604167 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{11}{32}, \frac{5}{24}, \frac{3}{32}, \frac{1}{48}\right)$,

$\mathcal{T}'_4(3)$: $\quad \frac{14147193}{5571665} \approx 2.539132 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{98057}{506515}, \frac{159348}{1114333}, \frac{3688469}{44573320}, \frac{1752117}{44573320}, \frac{402569}{44573320}\right)$,

$\mathcal{T}'_2(4)$: $\quad \frac{41}{13} \approx 3.153846 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{11}{26}, \frac{3}{13}, \frac{2}{39}\right)$,

$\mathcal{T}'_3(4)$: $\quad \frac{127937}{42400} \approx 3.017382 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{5539}{16960}, \frac{1737}{10600}, \frac{257}{5300}, \frac{399}{42400}\right)$,

$\mathcal{T}'_2(5)$: $\quad \frac{18}{5} = 3.6 \qquad$ *with* $\boldsymbol{\alpha} = \left(\frac{17}{45}, \frac{8}{45}, \frac{2}{45}\right)$.

The bounds obtained in Lemma 9 are valid for graphs of girth at least $2r+3$. It turns out that the same bounds, with the same $\boldsymbol{\alpha}$, remain valid for graphs of girth $2r+2 = 6$, when $r = 2$ and $d \in \{3, 4\}$. We were not able to check this for higher values of r or d, but we propose the following conjecture which would explain and generalise this phenomenon.

Conjecture 2. *Let P be a d-regular edge-rooted pattern of depth r and of girth $2r+2$. Then the constraint $\mathbf{e}(P)$ is weaker than some convex combination of constraints $\mathbf{e}(T)$ with $T \in \mathcal{T}'_r(d)$. More formally, there exist $T_1, \dots, T_m \in \mathcal{T}'_r(d)$ and $\lambda_1, \dots, \lambda_m \in [0,1]$ with $\sum_{i=1}^m \lambda_i = 1$ such that for any $\boldsymbol{\alpha} \in (\mathbb{Q}^+)^{r+1}$,*

$$\alpha^\top \left(\sum_{i=1}^m \lambda_i \mathbf{e}(T_i) \right) \geq 1 \quad \Longrightarrow \quad \alpha^\top \mathbf{e}(P) \geq 1.$$

4.3. **More complicated patterns.**

4.3.1. *Rooting at a vertex.* Let us fix a depth $r \geq 2$. Let G be a d-regular graph of girth $g \leq 2r+1$. We repeat the same analysis as in Section 4.2.1: we end up having to find a vector $\boldsymbol{\alpha} \in \mathbb{Q}^{r+1}$ compatible with all the constraints generated by vertex-rooted d-regular patterns of depth r and girth g. Letting $\mathcal{P}_r(d,g)$ be the set of such patterns, we thus want that

$$\forall P \in \mathcal{P}_r(d,g), \quad \boldsymbol{\alpha}^\top \mathbf{e}(P) \geq 1.$$

In this setting, we could do no better than performing an exhaustive enumeration of every possible pattern $P \in \mathcal{P}_r(d,g)$, and computing the associated constraint $\mathbf{e}(P)$ through an exhaustive enumeration of $\mathscr{S}_\alpha(P)$. The complexity of such a process grows fast, and we considered only depth $r \leq 2$ and degree $d \leq 4$. Since the largest value of the Hall ratio over the class of 3-regular graphs of girth 4 or 5 is known to be $\frac{14}{5} = 2.8$, and the one of 4-regular graphs of girth 4 is known to be $\frac{13}{4} = 3.25$, the only open value in these settings is for the class of 4-regular graphs of girth 5. Unfortunately, this method is not powerful enough to prove an upper bound lower than $\frac{13}{4}$, the obtained bound for $\mathcal{P}_2(4,5)$ being $\frac{82}{25} = 3.28$. It is more interesting to root the patterns in an edge.

4.3.2. *Rooting in an edge.* Similarly, we define $\mathcal{P}'_r(d,g)$ to be the set of edge-rooted d-regular patterns of girth g. For fixed r and g, we seek for the solution of the following linear program.

$$\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^r \alpha_i (d-1)^i \tag{13}$$

$$\text{such that} \begin{cases} \forall P \in \mathcal{P}'_r(d,g), & \displaystyle\sum_{i=0}^r \alpha_i e_i(P) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

Again, our computations were limited to the cases where $r \leq 2$ and $d \leq 4$. However, we managed to prove improved bounds for girth 6 when $d \in \{3,4\}$, which seems to support Conjecture 2.

Lemma 10. *The solution to the linear program* (13) *is*

$$\mathcal{P}'_2(3,6): \qquad \frac{30}{11} \approx 2.72727 \qquad \textit{with } \boldsymbol{\alpha} = \left(\frac{1}{2}, \frac{13}{44}, \frac{3}{44}\right),$$

$$\mathcal{P}'_2(4,6): \qquad \frac{41}{13} \approx 3.153846 \qquad \textit{with } \boldsymbol{\alpha} = \left(\frac{11}{26}, \frac{3}{13}, \frac{2}{39}\right).$$

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