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CHARACTERIZATION OF THE REDUCED PERIPHERAL SYSTEM OF LINKS

BENJAMIN AUDOUX AND JEAN-BAPTISTE MEILHAN

ABSTRACT. The reduced peripheral system was introduced by Milnor in the fifties for the study of links up to link-homotopy, *i.e.* up to isotopies and crossing changes within each link component. However, for four or more components, this invariant does not yield a complete link-homotopy invariant. This paper provides two characterizations of links having the same reduced peripheral system: a diagrammatical one, in terms of link diagrams, seen as welded diagrams up to self-virtualization, and a topological one, in terms of ribbon solid tori in 4-space up to ribbon link-homotopy.

INTRODUCTION

A celebrated consequence of Waldhausen's theorem on Haken 3-manifolds [15] is that the fundamental group of the complement, endowed with a peripheral system, forms a complete invariant of links in the 3-sphere up to ambient isotopy. The peripheral system is the data, for each component of a link L in S^3 , of a pair of elements $\{m_i, l_i\}$ of $\pi_1(S^3 \setminus L)$ —a meridian and a preferred longitude—that generates the fundamental group of the corresponding boundary component of $S^3 \setminus L$. Although rather intractable in practice, the peripheral system is nonetheless a fundamental link invariant, and it is natural to expect that some weaker equivalence relations than ambient isotopy could be classified by an appropriate adaptation of the peripheral system. During the fifties, this has been the strategy of J. Milnor in his attempt to classify links up to *link-homotopy* [12], that is up to homotopy deformations during which distinct connected components remain disjoint at all time. In order to address this link-homotopy classification problem, Milnor introduced the *reduced peripheral system*. Roughly speaking, the *reduced fundamental group* $R\pi_1(L^c)$ of a link L is the largest quotient of the fundamental group of the complement where any generator commutes with any of its conjugates; if $\{\mu_i, \lambda_i\}_i$ is a peripheral system for L , with image $\{m_i, l_i\}_i$ under the projection onto $R\pi_1(L^c)$, then a *reduced peripheral system* of L is $\{m_i, l_i N_i\}_i$, where N_i is the normal subgroup of $R\pi_1(L^c)$ generated by m_i .

The reduced peripheral system, however, only yields a complete link-homotopy invariant for links with at most 3 components. The 4-component case was tackled by J. Levine [10] only 40 years later, using a smaller normal subgroup for defining the reduced longitudes. As a matter of fact, there exists a pair of 4-component links, exhibited by J.R. Hughes, with equivalent reduced peripheral systems but which are link-homotopically distinct [7]. It seems still unknown whether Levine's peripheral system classifies links up to link-homotopy. In fact, this classification was achieved by N. Habegger and X.S. Lin by a rather different approach, which relies on representing links as the closure of string links [6].

The goal of the present paper is to provide the following two characterizations of the information captured by Milnor's reduced peripheral system.

Main Theorem. *Let L and L' be two oriented links in the 3-sphere. The following are equivalent:*

- i. *L and L' have equivalent reduced peripheral systems;*
- ii. *L and L' are sv-equivalent, as welded links;*
- iii. *$\text{Spun}^\bullet(L)$ and $\text{Spun}^\bullet(L')$ are ribbon link-homotopic, as ribbon immersed solid tori.*

Here, *welded links* are generalized link diagrams, where we allow for virtual crossings in addition to the usual crossings, regarded up to an extended set of Reidemeister moves. This is a sensible generalization in the sense that classical links inject into welded links, and that the fundamental group and (reduced) peripheral system naturally extends to this larger class of objects. Part ii then gives a diagrammatic characterization of the reduced peripheral systems of links, by regarding them as welded links via their diagrams, up to *sv-equivalence*, which is the equivalence relation generated by the replacement of a classical crossing involving two strands of a same component by a virtual one; this stresses in particular the fact that the sv-equivalence

is a refinement of link-homotopy for classical links, see Remark 1.6. This characterization actually follows from a more general result, which classifies all welded links up to sv -equivalence, see Theorem 2.1. Also, it follows that the non link-homotopic 4-component links exhibited by Hugues in [7] are sv -equivalent; this is made explicit in Appendix A.

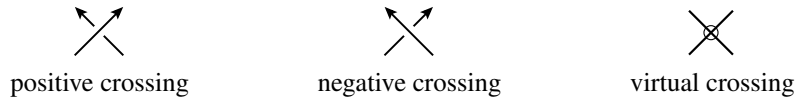
Part iii gives a topological characterization, in terms of 4-dimensional topology. A classical construction dating back to Artin [1] produces a knotted surface in 4-space from a link in 3-space by spinning it around some plane. By spinning as well projection rays from the link to the plane, this can be extended to a map Spun^\bullet producing *ribbon-immersed solid tori*, i.e. solid tori in 4-space intersecting along only ribbon singularities. The *ribbon link-homotopy* for such objects is a notion of link-homotopy within the realm of ribbon-immersed solid tori, which allows for the removal/insertion of such ribbon singularities inside a same connected component. The reduced peripheral system for links hence appears in this way as an intrinsically 4-dimensional invariant, rather than a 3-dimensional one.¹ As above, this characterization is obtained as a consequence of a more general result, characterizing the reduced peripheral system of welded links in terms of 4-dimensional topology; see Theorem 3.6. It is thus noteworthy that our purely topological characterization $i \Leftrightarrow iii$ for classical links is actually obtained as an application of virtual/welded knot theory.

1. THE REDUCED PERIPHERAL SYSTEM OF CLASSICAL AND WELDED LINKS

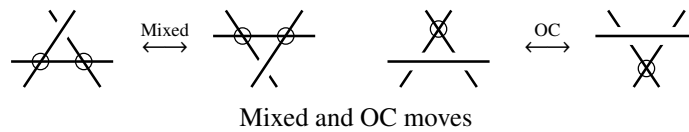
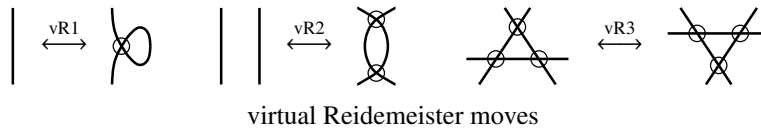
1.1. **Welded links.** In this section, we review the theory of welded links and Gauss diagrams.

Definition 1.1.

- An n -component *welded diagram* is a planar immersion of n ordered and oriented circles, whose singular set is a finite number of transverse double points, each double point being labelled either as a *positive or negative (classical) crossing*, or as a *virtual crossing*:



- We denote by $w\mathcal{L}_n$ the set of n -component welded diagrams up to the following *welded moves*:



and classical Reidemeister moves R1, R2 and R3, which are the three usual moves of classical knot theory. Elements of $w\mathcal{L}_n$ are called *welded links*.

A welded diagram with no virtual crossing is called *classical*. It is well-known [8, 5] that this set-theoretical inclusion induces an injection of the set \mathcal{L}_n of n -component classical link diagrams up to classical Reidemeister moves, into $w\mathcal{L}_n$; as pointed out in Remark 1.10, this follows from the fact that the peripheral system is a complete link invariant.

An alternative approach to welded links, which is often more tractable in practice, is through the notion of Gauss diagrams.

Definition 1.2. An n -component *Gauss diagram* is an abstract collection of n ordered and oriented circles, together with disjoint signed arrows whose endpoints are pairwise disjoint points of these circles. For each

¹One might expect for this 4-dimensional incarnation to be in terms of knotted surfaces, we explain in Section 3.3 why this is not the case.

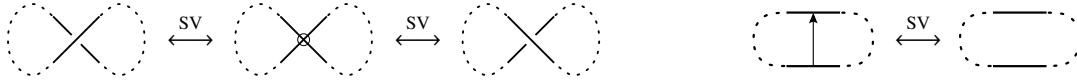
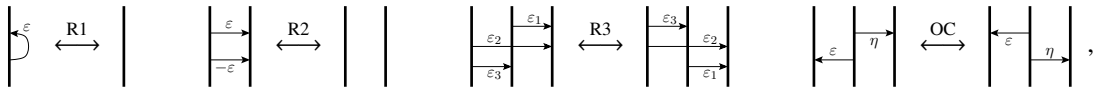


FIGURE 1. Self-virtualization move, at the welded and Gauss diagram level

arrow, the two endpoints are called *head* and *tail*, with the obvious convention that the arrow orientation goes from the tail to the head.

To a welded diagram corresponds a unique Gauss diagram, given by joining the two preimages of each classical crossing by an arrow, oriented from the overpassing to the underpassing strand and labelled by the crossing sign.

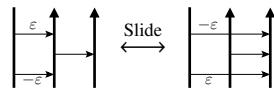
Definition 1.3. Two Gauss diagrams are *welded equivalent* if they are related by a sequence of the following *welded moves*:



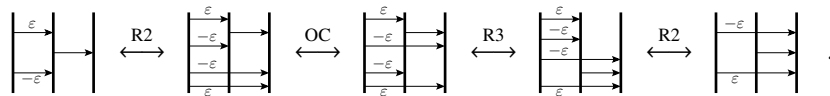
where move R3 requires the additional sign condition that $\varepsilon_2\varepsilon_3 = \tau_2\tau_3$, where $\tau_i = 1$ if the i^{th} strand (from left to right) is oriented upwards, and -1 otherwise.

As the notation suggests, these four moves are just the Gauss diagram analogues, using the above correspondance, of the three classical Reidemeister moves and the OC move for welded diagrams (the Gauss diagram versions of the virtual Reidemeister and Mixed moves being trivial). As a matter of fact, it is well-known that welded equivalence classes of Gauss diagrams are in one-to-one correspondence with welded links; see for example [4, Sec. 4.5].

Remark 1.4. We will make use the following *Slide* move



which is a consequence of the other moves, as follows:



Note that this is a Gauss diagram analogue of the Slide move on arrow diagrams [11].

In a welded diagram, a *self-crossing* is a crossing where both preimages belong to the same component.

Definition 1.5. A *self-virtualization* is a local move SV, illustrated in Figure 1, which replaces a classical self-crossing by a virtual one. The *sv-equivalence* is the equivalence relation on welded diagrams generated by self-virtualizations. We denote by $w\mathcal{L}_n^{\text{sv}}$ the quotient of $w\mathcal{L}_n$ under this relation.

At the Gauss diagram level, a self-crossing is represented by a *self-arrow*, that is an arrow whose endpoints lie on the same component, and a self-virtualization move simply erases a self-arrow, see Figure 1.

Remark 1.6. The link-homotopy relation for classical links, as defined by Milnor, is generated by the self-crossing change, *i.e.* the local move that exchanges the relative position of two strands of a same component. As the left-hand side of Figure 1 suggests, a self-crossing change can be realized by two self-virtualizations, and the sv-equivalence is thus a refinement of the link-homotopy relation for classical links.

1.2. Welded link groups and peripheral systems. Let L be a welded diagram.

Definition 1.7.

- The *arcs* of L are the maximal pieces of L which do not underpass any classical crossing. An arc is hence either a whole component or a piece of strand which starts and ends at some (possibly the same) crossings; it might pass through some virtual crossings and overpass some classical ones.
- The *group of L* , denoted by $G(L)$, is defined by a Wirtinger-type presentation, where each arc yields a generator, and each classical crossing yields a relation, as follows:

$$\begin{array}{c} \nearrow \gamma \\ \alpha \quad \searrow \beta \end{array} \rightsquigarrow \alpha^{-1}\beta\alpha\gamma^{-1} \qquad \begin{array}{c} \nearrow \gamma \\ \alpha \quad \searrow \beta \end{array} \rightsquigarrow \alpha\beta\alpha^{-1}\gamma^{-1}$$

Since virtual crossings do not produce extra generator or relation, it is clear that virtual Reidemeister moves and Mixed moves preserve the group presentation. It is also easily checked that the isomorphism class of this group is invariant under classical Reidemeister and OC moves, and is thus an invariant of welded links [8, 14]. If L is a diagram of a classical link \mathcal{L} , then $G(L)$ is merely the fundamental group of the complement of an open tubular neighborhood of \mathcal{L} in S^3 ; in this case, an arc corresponds to the topological meridian which positively enlaces it. By analogy, arcs of welded diagrams can be seen as some combinatorial meridians, and in what follows, we will often blur the distinction between arcs/meridians of L and the corresponding generators of $G(L)$. We will also regularly, and sometimes implicitly, make use of the simple fact that two meridians of a same components are always conjugate.

Definition 1.8.

- A *basing* of L is a choice of one meridian for each component of L .
- For each i , the *i th preferred longitude* of L with respect to the basing $\{\mu_1, \dots, \mu_n\}$ is the element $\lambda_i \in G(L)$ obtained as follows: when running along the i th component of L , starting at the arc labelled by μ_i and following the orientation, write ω^ε when passing under an arc labelled by ω at a classical crossing of sign ε , and finally write μ_i^{-k} , where k is the algebraic number of classical self-crossings in the i th component.
- A *peripheral system* for L is the group $G(L)$ endowed with the choice of a basing and the data, for each i , of the i th preferred longitude.

When L is a classical link, a basing is the choice of a topological meridian for each component, and the i th preferred longitude represents a parallel copy of the i th component having linking number zero with it. Hence the above definitions naturally generalize the usual notion of peripheral system of links.

Two peripheral systems $(G, \{(\mu_i, \lambda_i)\}_i)$ and $(G, \{(\mu'_i, \lambda'_i)\}_i)$ are *conjugate* if, for each i , there exists $\omega_i \in G$ such that $\mu'_i = \omega_i^{-1}\mu_i\omega_i$ and $\lambda'_i = \omega_i^{-1}\lambda_i\omega_i$. Two peripheral systems $(G, \{(\mu_i, \lambda_i)\}_i)$ and $(G', \{(\mu'_i, \lambda'_i)\}_i)$ are *equivalent* if there exists an isomorphism $\psi : G' \rightarrow G$ such that $(G, \{(\mu_i, \lambda_i)\}_i)$ and $(G, \{(\psi(\mu'_i), \psi(\lambda'_i))\}_i)$ are conjugate.

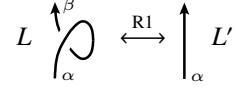
The following is well-known, see for example [9, Prop.6].

Lemma 1.9. *Up to conjugation, peripheral systems are well-defined for welded diagrams and yield, up to equivalence, a well-defined invariant of welded links.*

Proof. Suppose that $\{(\mu_i, \lambda_i)\}_i$ is a peripheral system of the welded diagram L , and let μ'_i be another choice of meridian for the i th component, yielding hence another preferred i th longitude λ'_i . Then $\mu'_i = \omega_i^{-1}\mu_i\omega_i$ for some $\omega_i \in G(L)$, and by definition $\lambda'_i = \omega_i^{-1}(\lambda_i\mu_i^k)\omega_i\mu_i^{-k}$. But substituting μ'_i for $\omega_i^{-1}\mu_i\omega_i$ in λ'_i then gives $\lambda'_i = \omega_i^{-1}\lambda_i\mu_i^k\omega_i\omega_i^{-1}\mu_i^{-k}\omega_i = \omega_i^{-1}\lambda_i\omega_i$. This proves that the peripheral system of L is uniquely determined up to conjugation.

Using this fact, it is then an easy exercise to check that equivalence classes of peripheral systems are well-defined for welded links, *i.e.* that it is invariant under welded and classical Reidemeister moves. More precisely, by an appropriate choice of basing, one can check that each classical Reidemeister move induces an isomorphism of the groups of the diagrams which preserves each preferred longitude; the argument is even simpler for welded Reidemeister moves.

As an elementary illustration, let us consider two welded diagrams L and L' which differ by an R1 move as shown on the right. The generators α and β of $G(L)$ shown in the figure satisfy $\alpha = \beta$, so $G(L)$ and $G(L')$ are clearly isomorphic. Pick $\beta \in G(L)$, resp. $\alpha \in G(L')$, as meridian for the depicted component of L , resp. L' .



Then the corresponding preferred longitude of L is of the form $\omega\alpha\alpha^{-k}$ for some $\omega \in G(L)$ and some $k \in \mathbb{Z}$, while the corresponding preferred longitude of L' reads $\omega\alpha^{-k+1}$, since L' contains one less positive self-crossing. Hence the above isomorphism from $G(L)$ to $G(L')$ preserves the peripheral system. \square

Remark 1.10. The peripheral system of classical links is a complete invariant [15]. Since this invariant extends to welded links, in the sense that the above-defined invariant coincides with the usual peripheral system for classical links, this shows that classical links inject into welded links [8, 5].

1.3. Reduced group and reduced peripheral system. As before, let us consider a welded diagram L .

Definition 1.11. For a group G given with a finite generating set X , the *reduced group of G* , denoted by RG , is the quotient of G by its normal subgroup generated by all elements $[\zeta, \omega^{-1}\zeta\omega]$, where $\zeta \in X$ and $\omega \in G$. In particular, we define the *reduced group of L* as the reduced group $RG(L)$ of $G(L)$ with respect to its Wirtinger generators.

Note that $RG(L)$ is the largest quotient of $G(L)$ where any meridian commutes with any of its conjugates. Since any two meridians of a same component are conjugate elements, it can also be defined as the quotient of $G(L)$ by the normal subgroup generated by the elements $[\mu_i, \omega^{-1}\mu_i\omega]$ for all $\omega \in G(L)$, where $\{\mu_i\}_i$ is a fixed basing for L .

Convention. In the rest of this paper, we shall use greek letters with tilda for elements in the group of a welded diagram, and use the same letters, but without the tilda, to denote the corresponding elements in the reduced group. In particular, we respectively denote by μ_i and λ_i the images in $RG(L)$ of any meridian $\tilde{\mu}_i$ and longitude $\tilde{\lambda}_i$ in $G(L)$.

Lemma 1.12. *The reduced fundamental group $RG(L)$ has the following presentation*

$$RG(L) = \langle \mu_1, \dots, \mu_n \mid [\mu_i, \lambda_i], [\mu_i, \omega^{-1}\mu_i\omega], \text{ for all } i \text{ and for all } \omega \in F(\mu_i) \rangle,$$

where $F(\mu_i)$ denotes the free group on $\{\mu_i\}_i$.

Proof. The argument follows closely the proof of [13, Thm. 4], where Milnor gives, in the case of a classical link L , a presentation for the quotients of $G(L)$ by its lower central series subgroups $\Gamma_k G(L)$; we recall that the lower central series of a group G is inductively defined by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. As a matter of fact, Milnor's argument being purely algebraic, it applies verbatim to the case of welded links,² and requires only minor adjustments to be adapted to reduced link groups. Yet, Milnor's proof being quite insightful, giving in particular an algorithm expressing the reduced longitudes as a word in the reduced free group, we give here a complete proof.

We start with the Wirtinger presentation $G(L) = \langle \tilde{\mu}_{i,j} \mid R_{i,j} \rangle$, where the generators $\tilde{\mu}_{i,j}$ and relations $R_{i,j}$ are taken for $1 \leq i \leq n$ and $1 \leq j \leq k_i$, with k_i the number of arcs in the i th component, and where the relations are of the form $R_{i,j} = \tilde{\mu}_{i,j+1}^{-1} \cdot \tilde{\nu}_{i,j}^{-1} \cdot \tilde{\mu}_{i,j} \cdot \tilde{\nu}_{i,j}$ for some elements $\tilde{\nu}_{i,j}$ (here, the j label is taken modulo k_i for each i). We stress that we pick here a diagram L such that the i th preferred longitude is given by $\tilde{\lambda}_i = \prod_{m=1}^{k_i} \tilde{\nu}_{i,m}$; this is always possible up to R1 moves.

We first rewrite this presentation as $G(L) = \langle \tilde{\mu}_{i,j} \mid \tilde{\sigma}_{i,j} \rangle$, where the relations $\tilde{\sigma}_{i,j}$; $1 \leq i \leq n$, $1 \leq j \leq k_i$ are given (taking again the j label modulo k_i) by $\tilde{\sigma}_{i,j} = \tilde{\mu}_{i,j+1}^{-1} \cdot \tilde{\rho}_{i,j}^{-1} \cdot \tilde{\mu}_{i,j} \cdot \tilde{\rho}_{i,j}$ with $\tilde{\rho}_{i,j} = \prod_{m=1}^j \tilde{\nu}_{i,m}$. Removing tildas since we are dealing now with the reduced quotient, this gives the following presentation for the reduced group of L :

$$RG(L) = \langle \mu_{i,j} \mid \sigma_{i,j}, [\mu_{i,j}, \omega^{-1}\mu_{i,j}\omega], \text{ for all } i, j \text{ and all } \omega \in F(\mu_{i,j}) \rangle,$$

where $F(\mu_{i,j})$ denotes the free group on $\{\mu_{i,j}\}_{i,j}$, hence the following presentation for its nilpotent quotient

$$RG(L)/\Gamma_k RG(L) = \langle \mu_{i,j} \mid \Gamma_k F(\mu_{i,j}), \sigma_{i,j}, [\mu_{i,j}, \omega^{-1}\mu_{i,j}\omega], \text{ for all } i, j \text{ and all } \omega \in F(\mu_{i,j}) \rangle.$$

²This means in particular that the nilpotent quotient $G(L)/\Gamma_k G(L)$ has a presentation $\langle \mu_1, \dots, \mu_n \mid \Gamma_k F(\mu_i), [\mu_i, \lambda_i] \text{ for all } i \rangle$.

The main idea is then to consider a family of homomorphisms $\eta_k : F(\mu_{i,j}) \rightarrow F$ ($k \geq 1$), where F denotes the free group on $\{\mu_{i,1}\}_i$, defined inductively by

$$\eta_1(\mu_{i,j}) = \mu_{i,1} \text{ for all } i, j, \quad \text{and} \quad \eta_{k+1}(\mu_{i,1}) = \mu_{i,1}, \eta_{k+1}(\mu_{i,j+1}) = \eta_k(\rho_{i,j}^{-1} \cdot \mu_{i,1} \cdot \rho_{i,j}).$$

The proof then mainly relies on two facts, which are both shown by induction on k :³

$$(4_k) : \eta_k(\mu_{i,j}) \equiv \mu_{i,j} \pmod{\Gamma_k F(\mu_{i,j}) \cdot S} \quad \text{and} \quad (6_k) : \eta_{k+1}(\mu_{i,j}) \equiv \eta_k(\mu_{i,j}) \pmod{\Gamma_k F},$$

where S is the normal subgroup of $F(\mu_{i,j})$ generated by all relations $\sigma_{i,j}$.

First, it is easily verified that $\eta_k(\Gamma_k F(\mu_{i,j})) = \Gamma_k F$. Fact (4_k) tells us that relations $\eta_k(\mu_{i,j})\mu_{i,j}^{-1}$ are implied by the relations $\sigma_{i,j}$ and $\Gamma_k F(\mu_{i,j})$ in the above presentation of $RG(L)/\Gamma_k RG(L)$, and can therefore be used to reduce it to

$$RG(L)/\Gamma_k RG(L) = \langle \mu_{i,1} \mid \Gamma_k F, \eta_k(\sigma_{i,j}), \eta_k([\mu_{i,j}, \omega^{-1}\mu_{i,j}\omega]), \text{ for all } i, j \text{ and all } \omega \in F(\mu_{i,j}) \rangle.$$

Now, Fact (6_k) implies that, for all $1 \leq j < k_i$,

$$\eta_k(\sigma_{i,j}) = \eta_k(\mu_{i,j+1})^{-1} \cdot \eta_k(\rho_{i,j}^{-1} \cdot \mu_{i,1} \cdot \rho_{i,j}) \equiv \eta_{k+1}(\mu_{i,j+1})^{-1} \cdot \eta_k(\rho_{i,j}^{-1} \cdot \mu_{i,1} \cdot \rho_{i,j}) = 1 \pmod{\Gamma_k F},$$

and in the case $j = k_i$ we have $\eta_k(\sigma_{i,k_i}) = \eta_k(\mu_{i,1}^{-1} \cdot \lambda_i^{-1} \cdot \mu_{i,1} \cdot \lambda_i)$. Similarly, by Fact (6_k) we have

$$\eta_k(\mu_{i,j+1}) \equiv \eta_{k+1}(\mu_{i,j+1}) = \eta_k(\rho_{i,j})^{-1} \cdot \mu_{i,1} \cdot \eta_k(\rho_{i,j}) \pmod{\Gamma_k F},$$

which shows that all relations $\eta_k([\mu_{i,j}, \omega^{-1}\mu_{i,j}\omega])$ with $j > 1$ are consequences of those for $j = 1$. Finally, it is easily verified that $\Gamma_k F$ is equivalent to $\Gamma_k RF$ modulo those latter relations. This further reduces the presentation to

$$RG(L)/\Gamma_k RG(L) = \langle \mu_i \mid \Gamma_k RF, [\mu_i, \lambda_i^{(k)}], [\mu_i, \omega^{-1}\mu_i\omega], \text{ for all } i \text{ and all } \omega \in F(\mu_i) \rangle,$$

where we set $\mu_i = \mu_{i,1}$ and $\lambda_i^{(k)} = \eta_k(\lambda_i)$. But $RG(L)$ and RF are nilpotent groups of class at most n by [6, Lem. 1.3], so taking $k \geq n + 1$ gives the desired presentation for $RG(L)$. \square

Definition 1.13. The *reduced peripheral system* for L is the data

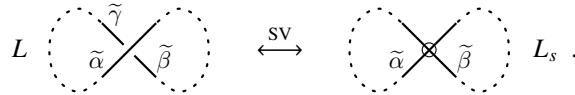
$$(RG(L), \{(\mu_i, \lambda_i N_i)\}_i),$$

associated to a peripheral system $(G(L), \{(\tilde{\mu}_i, \tilde{\lambda}_i)\}_i)$ where, for each i , $\lambda_i N_i$ denotes the coset of λ_i with respect to N_i , the normal subgroup generated by the i th reduced meridian μ_i . Two reduced peripheral systems are *conjugate* if they come from conjugate peripheral systems; and they are *equivalent* if there is a group isomorphism sending one to a conjugate of the other.

As explained in the introduction, Milnor introduced the reduced peripheral system for classical links, and showed that it is a link-homotopy invariant. We have the following generalization.

Lemma 1.14. *Up to equivalence, the reduced peripheral system is a well-defined invariant of welded links up to sv-equivalence.*

Proof. Since equivalent peripheral systems obviously yield equivalent reduced peripheral systems, it suffices to prove the invariance under a single SV move. Pick a self-crossing s of some welded diagram L , and denote by L_s the diagram obtained by replacing s by a virtual crossing:



Consider the three generators $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ of $G(L)$ involved in s , as shown above. Since the meridians $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ all belong to the same component, there are $\tilde{\omega}, \tilde{\zeta} \in G(L)$ such that $\tilde{\beta} = \tilde{\omega}^{-1}\tilde{\alpha}\tilde{\omega}$ and $\tilde{\alpha} = \tilde{\zeta}^{-1}\tilde{\gamma}\tilde{\zeta}$. For L , the Wirtinger relation at s is $\tilde{\gamma} = \tilde{\alpha}^{-1}\tilde{\beta}\tilde{\alpha}$; hence, we have that $\gamma = \alpha^{-1}\omega^{-1}\alpha\omega\alpha \equiv \omega^{-1}\alpha\omega = \beta$ holds in $RG(L)$, which shows that $RG(L)$ is isomorphic to $RG(L_s)$.

It remains to show that this isomorphism preserves the reduced peripheral system. Pick $\tilde{\alpha} \in G(L)$ as meridian $\tilde{\mu}$ for the component of L containing s ; the corresponding preferred longitude is given by $\tilde{\lambda} = \tilde{\omega}\tilde{\alpha}\tilde{\zeta}\tilde{\alpha}^{-k}$ for some integer k . Take the meridian $\tilde{\mu}_s$ of the corresponding component of L_s to be represented by

³Here we follow Milnor's numbering, and refer to [13, pp. 290] for the proof.

$\tilde{\alpha}$ again, so that the preferred longitude is given by $\tilde{\lambda}_s = \tilde{\omega}\tilde{\zeta}\tilde{\alpha}^{-k+1}$. Then the isomorphism $RG(L) \rightarrow RG(L_s)$ maps μ to μ_s , and the equality

$$\lambda = \omega\alpha\zeta\alpha^{-k} \equiv \omega[\zeta, \alpha]\alpha\zeta\alpha^{-k} = \omega\zeta\alpha\zeta^{-1}\alpha^{-1}\alpha\zeta\alpha^{-k} = \omega\zeta\alpha^{-k+1} \pmod{N}$$

shows that $\lambda.N$ is mapped to $\lambda_s.N_s$, where $N \subset RG(L)$ and $N_s \subset RG(L_s)$ denote the normal subgroups generated by α . This handles one version of the SV move, but the other one is strictly similar. \square

2. DIAGRAMMATIC CHARACTERIZATION OF THE REDUCED PERIPHERAL SYSTEM

This section is devoted to the proof of the following result, which readily implies the equivalence i \Leftrightarrow ii in our main theorem, and which more generally classifies welded links up to sv-equivalence.

Theorem 2.1. *Two welded links are sv-equivalent if and only if they have isomorphic reduced peripheral systems.*

For convenience, we will adopt here the Gauss diagram point of view, fix a number $n \in \mathbb{N}^*$ of components and set $F(\mu)$ the free group over elements μ_1, \dots, μ_n .

Proof. The ‘‘only if’’ part of Theorem 2.1 was proved in Lemma 1.14, so we only have to prove here the converse implication. There will be two steps in the proof. We will first prove it for a special case of, so-called sorted, Gauss diagrams, and then prove that any Gauss diagram is sv-equivalent to a sorted one.

We say that a Gauss diagram is *sorted* if each circle C_i splits into two arcs, the t and the h -arc, containing respectively all tails and all heads. Note that, thanks to the OC move, the relative position of the tails need not be specified, and that t and h -arcs are naturally oriented by the orientation of the circles; such a sorted Gauss diagram D can be hence described by n words $\Lambda_D := \left\{ \prod_{r=1}^{k_i} (s_{j_r}^i)^{\varepsilon_{j_r}^i} \right\}_i$ in the alphabet $\{1^{\pm 1}, \dots, n^{\pm 1}\}$, the word $(s_{j_r}^i)^{\varepsilon_{j_r}^i}$ meaning that the r th head met on C_i when running along its oriented h -arc is connected by an $\varepsilon_{j_r}^i$ -signed arrow to one of the tails on $C_{s_{j_r}^i}$. According to Lemma 1.12, and using t -arcs as basing, $RG(D)$ has then the following presentation:

$$\left\langle \mu_1, \dots, \mu_n \mid [\mu_i, \prod_{r=1}^{k_i} \mu_{s_{j_r}^i}^{\varepsilon_{j_r}^i}], [\mu_i, \omega^{-1}\mu_i\omega], \text{ for all } i \text{ and for all } \omega \in F(\mu) \right\rangle,$$

and, for each i , the i th reduced longitude is $\prod_{r=1}^{k_i} \mu_{s_{j_r}^i}^{\varepsilon_{j_r}^i}$.

Now consider another sorted Gauss diagram D' which has an equivalent reduced peripheral system. We first modify D' so that the isomorphism between $RG(D')$ and $RG(D)$ identifies the basings of D and D' . For this purpose, we introduce the following TaH (tail across head) move:

$$\text{TaH} : \begin{array}{c} \uparrow \quad \uparrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \varepsilon \end{array} \xrightarrow{\text{R2}} \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \varepsilon \quad \varepsilon \quad \varepsilon \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \varepsilon \quad \varepsilon \quad \varepsilon \end{array} \xrightarrow{\text{Slide}} \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \varepsilon \quad \varepsilon \quad \varepsilon \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \varepsilon \quad \varepsilon \quad \varepsilon \end{array}.$$

Now, we observe that, for any i and j , using an R2 move to add two parallel arrows with tails on the j th t -arc and heads on the starting extremity of the i th t -arc, and then pulling one of these two heads along the i th t -arc using the TaH move, provides an equivalent Gauss diagram which is still sorted and whose associated basing is obtained from the former one by conjugating the i th meridian (and hence the i th longitude) by the j th meridian or its inverse, depending on the sign of the pulled arrow. This can be used repeatedly to realize any conjugation on the reduced peripheral system of D' .

That being done, the i th longitude λ'_i of D' is now sent to the i th longitude λ_i of D for every i . According to the above presentation for $RG(D)$, this means that, as words in the μ_j 's, λ'_i and λ_i differ by a sequence of the following moves:

- i. $\mu_j^{\pm 1}\mu_j^{\mp 1} \leftrightarrow 1$;
- ii. $\omega\mu_j^{\pm 1} \rightarrow \mu_j^{\pm 1}\omega$ where ω is of the form $\zeta^{-1}\mu_j^{\pm 1}\zeta$ with $\zeta \in F(\mu)$;
- iii. $\mu_j^{\pm 1} \rightarrow \omega^{-1}\mu_j^{\pm 1}\omega$ where ω is any representative in $F(\mu)$ of the element λ_j .

But each of these moves can be realized by modifying D' . Relations i correspond indeed to R2 moves. Relations ii can be handled exactly as in the proof of [4, Lem. 4.26]. For relations iii, consider first the particular representative ω_0 of λ_j extracted out of D' by reading in order the letters, or their inverses for (-1) -signed arrows, given by the arrow heads on the j th h -arc. Now, at the level of the diagram D' , the term

$\mu_j^{\pm 1}$ in relation iii corresponds to an arrow a whose tail sits on the j th circle; moving this tail along the whole circle component, following the orientation, does conjugate $\mu_j^{\pm 1}$ by ω_0 . Indeed, using TaH moves, the tail of a will cross every head on its way at the cost of conjugating the head of a with the desired arrows. For other representatives of λ_j , we note that they differ from ω_0 by a sequence of the following moves:

- i'. $\mu_k^{\pm 1} \mu_k^{\mp 1} \leftrightarrow 1$;
- ii'. $\zeta^{-1} \mu_k^{\pm 1} \zeta \rightarrow \mu_k^{\pm 1}$ where ζ is any element in $F(\mu)$.

Before sliding the tail of a over the circle component, D' should hence be modified so that the slide operation does conjugate by the right word in $F(\mu)$. Again, relations i' can be realized using R2 moves. For relation ii', use the Slide move to remove by pairs the arrows of ζ and ζ^{-1} away from the circle. Then move the tail of a along this loop, and perform the relations i' and ii' backwards.

At this point, the longitudes of D and D' correspond to the same words in $F(m)$, meaning that $\Lambda_D = \Lambda_{D'}$, and hence that $D = D'$. We have proved so far that any two sorted Gauss diagrams having same reduced peripheral system describe the same welded link, so it only remains to observe that any Gauss diagram is sv-equivalent to a sorted one. The strategy to achieve this is to sort, one by one, every circle in order by removing first all self-arrows using SV moves and then gathering all heads together using TaH moves. Since there is no self-arrow left, the two extra arrows appearing in the latter moves won't have endpoints on the considered circle, and as their heads, resp. tails, are close to the already existing arrow head, resp. tail, this won't unsort already sorted components. This complete the proof. \square

3. A TOPOLOGICAL CHARACTERIZATION OF THE REDUCED PERIPHERAL SYSTEM

Welded links are closely related to *ribbon knotted tori* and *ribbon solid tori* in S^4 , and the characterization of classical links having same reduced peripheral systems given by Theorem 2.1 can be recasted in terms of 4-dimensional topology.

3.1. The enhanced Spun map. Given a classical link $L \subset \mathbb{R}^3$, a well-known procedure to construct ribbon knotted tori in 4-space is to take the *Spun* of L : consider a plane \mathcal{P} which is disjoint from a 3-ball containing L , and spin L around \mathcal{P} inside $\mathbb{R}^4 \supset \mathbb{R}^3$. The result is a union of knotted tori, which we denote by $\text{Spun}(L)$. If the projection $D(L, \mathcal{P})$ of L onto the plane \mathcal{P} is regular, then spinning as well the orthogonal projection rays from L to \mathcal{P} provides immersed solid tori whose boundary is $\text{Spun}(L)$ and whose singularities are so-called *ribbon disks*, corresponding to the crossings of $D(L, \mathcal{P})$. Of course, this *ribbon filling* depends on the choice of plane \mathcal{P} , and more precisely on the diagram $D(L, \mathcal{P})$, which may be changed by some sequence of Reidemeister moves. But for each Reidemeister move, there is an associated singular diagram, that is a singular plane \mathcal{P}_s , and spinning L around \mathcal{P}_s provides some singular ribbon filling which can be infinitesimally desingularized into the spun of one or the other side of the Reidemeister move. This leads to the following definition, which settles a notion of (singular) ribbon solid tori.

Definition 3.1. Let $\varphi : M \rightarrow S^4$ be an immersed 3-dimensional manifold. Let D be a connected component of the singular set of $\varphi(M)$ contained in an open 4-ball $B \subset S^4$. We say that D is a *ribbon singularity*:

- of type 0 if $\varphi^{-1}(B)$ is the disjoint union $B_1 \sqcup B_2$ of two 3-balls and there is a local system of coordinates for $B \cong \mathbb{R}^4$ such that
$$\begin{cases} \varphi(B_1) = \left\{ (t, r \cos(s), r \sin(s), 0) \mid t, s \in \mathbb{R}, r \in [0, 2] \right\} \\ \varphi(B_2) = \left\{ (0, r \cos(s), r \sin(s), t) \mid t, s \in \mathbb{R}, r \in [0, 1] \right\} \end{cases};$$
- of type 2 if $\varphi^{-1}(B)$ is the disjoint union $B_1 \sqcup B_2$ of two 3-balls and there is a local system of coordinates for $B \cong \mathbb{R}^4$ such that
$$\begin{cases} \varphi(B_1) = \left\{ (t, r \cos(s), r \sin(s), t^2) \mid t, s \in \mathbb{R}, r \in [0, 2] \right\} \\ \varphi(B_2) = \left\{ (t, r \cos(s), r \sin(s), -t^2) \mid t, s \in \mathbb{R}, r \in [0, 1] \right\} \end{cases};$$
- of type 3 if $\varphi^{-1}(B)$ is the disjoint union $B_1 \sqcup B_2 \sqcup B_3$ of three 3-balls and there is a local system of coordinates for $B \cong \mathbb{R}^4$ such that
$$\begin{cases} \varphi(B_1) = \left\{ (t, r \cos(s), r \sin(s), 0) \mid t, s \in \mathbb{R}, r \in [0, 2] \right\} \\ \varphi(B_2) = \left\{ (t, r \cos(s), r \sin(s), t) \mid t, s \in \mathbb{R}, r \in [0, 1] \right\} \\ \varphi(B_3) = \left\{ (t, r \cos(s), r \sin(s), -t) \mid t, s \in \mathbb{R}, r \in [0, \frac{1}{2}] \right\} \end{cases};$$

- of type SV if $\varphi^{-1}(B)$ is the disjoint union $B_1 \sqcup B_2$ of two 3–balls, B_1 and B_2 belongs to the same connected component of M , and there is a local system of coordinates for $B \cong \mathbb{R}^4$ such that

$$\begin{cases} \varphi(B_1) = \left\{ (r, t, s, 0) \mid t, s \in \mathbb{R}, r \in \mathbb{R}_- \right\} \\ \varphi(B_2) = \left\{ (0, r \cos(s), r \sin(s), t) \mid t, s \in \mathbb{R}, r \in [0, 1], r \right\} \end{cases} .$$

Remark 3.2. In all four cases, the ribbon singularity D corresponds to the disk $\left\{ (0, r \cos(s), r \sin(s), 0) \mid s \in \mathbb{R}, r \in [0, 1] \right\}$. Type 0 corresponds to two solid tubes, one being smaller than the other, intersecting transversally; these are the usual ribbon singularities. Type 2 corresponds to two solid tubes, one being smaller than the other, intersecting tangentially; these occur when spinning a link around a plane on which the link projects with two tangential strands. Type 3 corresponds to three solid tubes of increasing width, intersecting simultaneously and transversally; these occur when spinning a link around a plane on which the link projects with a triple point. Type SV differs from type 0 in that one preimage of the singular disk lies on the boundary of M instead of its interior, and in that the two preimages belong to the same solid torus; these occur when performing the link-homotopy which pushes at once a usual ribbon (self) singularity through the boundary of M . Note that a type 1 seems to be missing here, which would correspond to spinning a link around a plane on which the link projects with a cusp, but this does not introduce any new kind of ribbon singularity.

Definition 3.3. *Ribbon solid tori* are immersed solid tori in S^4 whose singular locus is made of ribbon singularities of type 0. *Generalized ribbon solid tori* are immersed solid tori in S^4 whose singular locus is made of ribbon singularities of type 0, 2 and 3. *Self-singular ribbon solid tori* are immersed solid tori in S^4 whose singular locus is made of ribbon singularities of type 0 and SV.

We say that two (generalized) ribbon solid tori are *equivalent* if there is a path among generalized ribbon solid tori connecting them, and we say that they are *ribbon link-homotopic* if there is path among generalized and self-singular ribbon solid tori connecting them.

Adding the spun of projection rays in the above definition of the Spun map provides a well-defined map Spun^\bullet from classical links to generalized ribbon soli tori. The following result is the topological characterization of the reduced peripheral system given by the equivalence $i \Leftrightarrow iii$ in our main theorem.

Theorem 3.4. *Two classical links L and L' have isomorphic reduced peripheral systems if and only if $\text{Spun}^\bullet(L)$ and $\text{Spun}^\bullet(L')$ are ribbon link-homotopic.*

The proof is given in the next section. As for the diagrammatic characterization given in Section 2, this will follow from a more general result, Theorem 3.6, characterizing the reduced peripheral system of welded links in terms of 4–dimensional topology.

3.2. The enhanced Tube map. In this section, we prove Theorem 3.4, using the so-called Tube map. Recall from [14] that Satoh’s generalization of Yajima’s Tube map is defined from welded links to ribbon knotted 2-tori, and that for any welded link L , $\text{Tube}(L)$ actually comes with a canonical ribbon filling. In order to fully record this ribbon filling in the Tube map, and to connect with the Spun^\bullet map, we are led to the following notion.

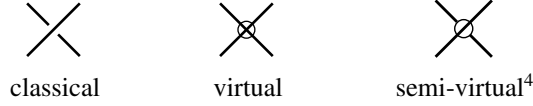
Definition 3.5. We define *generalized welded diagrams* as diagrams with cups and the following kind of crossings:



Then classical Reidemeister moves are replaced by a path of diagrams going through the corresponding cusp, tangential ou triple point. Other welded moves are still locally allowed.

³Semi-virtual crossings were already introduced in [5] in connection with finite type invariants of virtual knots.

We define *self-singular welded diagrams* as diagrams with the following kind of crossings, where the two strands involved in a semi-virtual crossing belong to a same component:



Self-virtualization is defined for generalized welded diagrams as the equivalence relation generated by the local moves turning a semi-virtual crossing into either a classical crossing, or a virtual one as follows:



Following [3, Sec. 3.2], one can then define a map

$$\text{Tube}^\bullet : \frac{\{\text{generalized welded diagrams}\}}{\text{self-virtualization}} \rightarrow \frac{\{\text{generalized ribbon solid tori}\}}{\text{ribbon link-homotopy}}$$

which, respectively, associates ribbon singularities of type 0, 2, 3 and SV to classical, tangential, triple and semi-virtual crossings and connects these various singularities by pairwise disjoint 3-balls, as prescribed by the welded diagram. It is then a straightforward adaptation of [3, Prop. 3.7] to prove that Tube^\bullet is one-to-one.

As a direct corollary of Theorem 2.1, we obtain the following alternative characterization of the reduced peripheral system, which holds for all welded links.

Theorem 3.6. *Two welded links L_1 and L_2 have isomorphic reduced peripheral systems if and only if $\text{Tube}^\bullet(L_1)$ and $\text{Tube}^\bullet(L_2)$ are ribbon link-homotopic.*

Theorem 3.4 follows from this results. Indeed, as essentially pointed out by Satoh in [14], it is clear from the definition of the Spun^\bullet map that, starting with a diagram D of a classical link L , the ribbon solid tori $\text{Spun}^\bullet(L)$ consists of ribbon singularities which are connected by 3-balls as combinatorially prescribed by D : $\text{Spun}^\bullet(L)$ and $\text{Tube}^\bullet(L)$ are hence equivalent.

3.3. Link-homotopy of ribbon surfaces in 4-space. The original versions of the Spun and Tube maps produce ribbon 2-tori, which are just the boundary of some ribbon solid tori, rather than 3-dimensional objects. Obviously, any ribbon link-homotopy between two ribbon solid tori induces a usual link-homotopy between their boundaries. Building on this remark, it follows that:

Proposition 3.7. *If two classical links L_1 and L_2 have isomorphic reduced peripheral systems, then $\text{Spun}(L_1)$ and $\text{Spun}(L_2)$ are link-homotopic.*

It is hence tempting to hope for the converse to hold true: this would give a topological characterization of the reduced peripheral system in terms of spun surfaces up to link-homotopy. However, this is not the case. There is indeed a known global move on welded links, related to the torus eversion in S^4 , under which the Spun map is invariant, and this move transforms every classical link into its *reversed image*, which is the mirror image with reversed orientation (see [16] or [2, Prop. 2.7]). Furthermore, it can be checked that the (reduced) peripheral system of a reversed image is given from the initial one by just inverting the longitudes. It follows easily from these two observations that, for instance, the positive and negative Hopf links have non-equivalent reduced peripheral systems, whereas their spuns are isotopic, hence link-homotopic. As a consequence, keeping track of the ribbon filling is mandatory to preserve (reduced) peripheral systems, and Theorem 3.4 is in this sense optimal.

APPENDIX A. HUGUES' COUNTEREXAMPLE

As mentioned in the introduction, the fact that the reduced peripheral system of classical links is not a complete link-homotopy invariant was made explicit by J. Hugues in [7]. There, a pair of 4-component links is given, which have isomorphic reduced peripheral systems but are not link-homotopic; this latter fact is

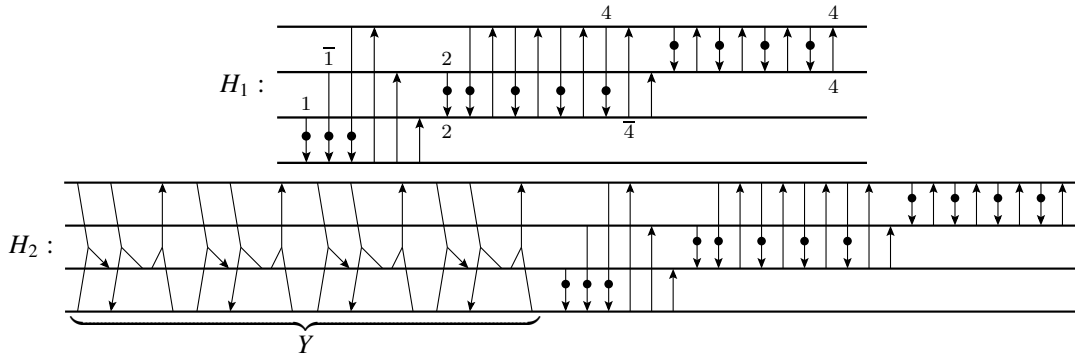
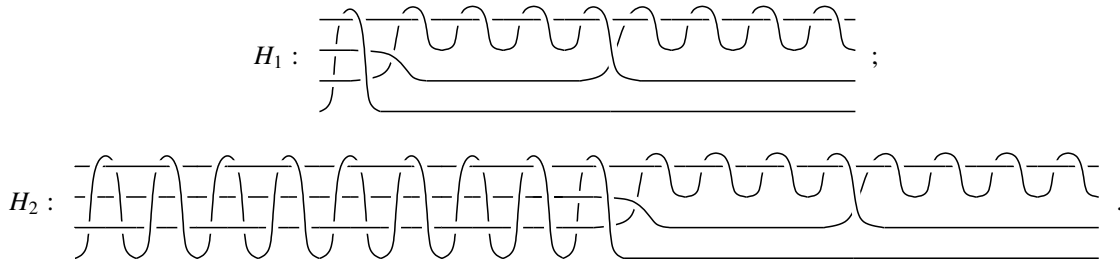


FIGURE 2. Arrow presentations of Hughes' links H_1 and H_2

proved using Levine's refinement of Milnor invariants developed in [10]. These two links, H_1 and H_2 , are given by the closures of the following pure braids, oriented from left to right:

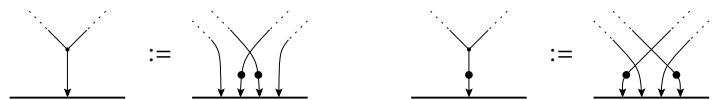


Our main theorem implies that, although not link-homotopic, the links H_1 and H_2 are sv-equivalent. This fact, however, is rather difficult to verify by hand, and we outline in this appendix the method that we used for this verification.

We make use of the theory of *arrow calculus* developed in [11], which is in some sense a 'higher order Gauss diagram' theory. We only give here a quick overview of this theory, and refer to [11] for precisions and further details. Roughly speaking, a w-arrow for a diagram L is an oriented interval, possibly decorated by a dot, immersed in the plane so that the endpoints lie on L ; one can perform *surgery* on L along this w-arrow to obtain a new diagram as follows:

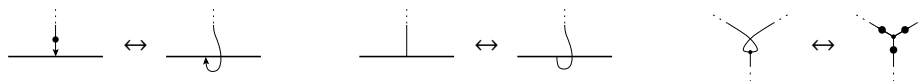


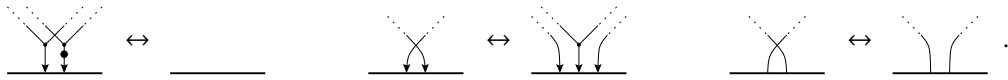
More generally, one defines w-trees, which are oriented univalent trees defined recursively by the rules:



There, the dotted parts represent 'parallel' subtrees, see [11, Conv. 5.1]. Any welded diagram can be represented as a diagram without any crossing, but with w-trees. For example, the links H_1 and H_2 can be represented in this way as the closures of the diagrams given in Figure 2 (ignoring the integer labels). Note that the presentation for H_2 only differs from that for H_1 by the addition of a union Y of Y-shaped w-trees.

It is shown in [11, Sections 4–5] that two dots on a same edge do cancel, and that the following moves can be performed on w-trees:





Moreover, up to sv -equivalence, it is shown in [11, Section 9] that so-called *repeated* w -trees having at least two endpoints on a same connected component can be removed, and that the following moves can also be performed:



Then, one can start with the diagram for H_1 given in Figure 2, pick some w -arrow endpoint, and slide it all around the component it is attached to, in either direction: by the above moves, this will create Y -shaped w -trees, which can in turn be ‘gathered’ at the cost of higher order w -trees. By performing the appropriate sequence of slides and cancelling inverse pairs and repeated w -trees, one can create the union Y of w -trees realizing the presentation for H_2 , thus showing that the two links H_1 and H_2 are indeed sv -equivalent. Such an appropriate sequence of slides is indicated in the upper part of Figure 2: there, an integer label k (resp. \bar{k}) near an arrow end indicates k full turn in the left (resp. right) direction.

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