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A lower bound on the size of an absorbing set in an arc-coloured tournament

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Abstract

Bousquet, Lochet and Thomassé recently gave an elegant proof that for any integer n , there is a least integer $f(n)$ such that any tournament whose arcs are coloured with n colours contains a subset of vertices S of size $f(n)$ with the property that any vertex not in S admits a monochromatic path to some vertex of S . In this note we provide a lower bound on the value $f(n)$.

A *directed graph* or *digraph* D is a pair (V, A) where V is a set called the *vertex set* of D and A is a subset of V^2 called the *arc set* of D . When (u, v) is in A , we say there is an arc from u to v . A *tournament* is a digraph for which there is exactly one arc between each pair of vertices (in only one direction).

Given a colouring of the arcs of a digraph, we say that a vertex x is *absorbed* by a vertex y if there is a monochromatic path from x to y . A subset S of V is called an *absorbing set* if any vertex in $V \setminus S$ is absorbed by some vertex in S .

In the early eighties, Sands, Sauer and Woodrow [7] suggested the following problem (also attributed to Erdős in the same paper):

Problem (Sands, Sauer and Woodrow [7]). *For each n , is there a (least) positive integer $f(n)$ so that every finite tournament whose arcs are coloured with n colours contains an absorbing set S of size $f(n)$?*

This problem has been investigated in weaker settings by forbidding some structures (see [5], [3]), or by making stronger claims on the nature of the tournament (see [6]). Hahn, Ille and Woodrow [4] also approached the infinite case.

Recently, Pálvölgyi and Gyárfás [6] have shown that a positive answer to this problem would imply a new proof of a former result from Bárány and Lehel [1] stating that any set X of points in \mathbb{R}^d can be covered by $f(d)$ X -boxes (each

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box is defined by two points in X). In 2017, Bousquet Lochet and Thomassé [2] gave a positive answer to the problem of Sands, Sauer and Woodrow.

Theorem 1 (Bousquet, Lochet and Thomassé [2]). *Function f is well defined and $f(n) = O(\ln(n) \cdot n^{n+2})$.*

In this note, we provide a lower bound on the value of $f(n)$.

Theorem 2. *For any integer n , let*

$$p = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

There is a tournament arc-coloured with n colours, such that no set of size less than p absorbs the rest of the tournament, and thus $f(n) \geq p$.

Proof. Let n be an integer and let \mathcal{P} be the family of all subsets of $\{1, \dots, n-1\}$ of size exactly $\lfloor \frac{n-1}{2} \rfloor$. Note that \mathcal{P} has size p .

For any integer m , let $V(m)$ be the set $\{1, \dots, m\} \times \mathcal{P}$ and let $\mathcal{T}(m)$ be the probability space consisting of arc-coloured tournaments on $V(m)$ where the orientation of each arc is fairly determined (probability $1/2$ for each orientation), and the colour of an arc from (i, P) to (j, P') is n if $P = P'$, and picked randomly in $P' \setminus P$ otherwise. Note that if P and P' are distinct, then $P' \setminus P$ can not be empty since both sets have same size.

For any set P in \mathcal{P} , we shall denote by $B(P)$ the set of vertices having P as second coordinate. We may call this set, the *bag of P* . A key observation is that the arcs coming in a bag, the arcs leaving a bag and the arcs contained in a bag share no common colour. As a consequence, a monochromatic path is either contained in a bag or of length 1.

Let us find an upper bound on the probability that such a tournament is absorbed by a set of size strictly less than p . Let S be a subset of $V(m)$ of size $p-1$. There must exist a set P in \mathcal{P} such that S does not hit $B(P)$. For S to be absorbing, each vertex in $B(P)$ must have an outgoing arc to some vertex of S . Let x be a vertex in $B(P)$,

$$Pr(x \text{ is absorbed by } S) = 1 - \left(\frac{1}{2}\right)^{p-1}.$$

The events of being absorbed by S are pairwise independent for elements of $B(P)$. Then,

$$Pr(B(P) \text{ is absorbed by } S) = \left(1 - \left(\frac{1}{2}\right)^{p-1}\right)^m.$$

This is an upper bound on the probability for S to absorb the whole tournament. We may sum this for every possible choice of S .

$$Pr(\text{some } S \text{ is absorbing}) \leq \binom{mp}{p-1} \left(1 - \left(\frac{1}{2}\right)^{p-1}\right)^m.$$

Finally, by using the classic inequalities $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and $(1+x)^n \leq e^{nx}$, we obtain

$$Pr(\text{some } S \text{ is absorbing}) \leq \left(\frac{emp}{p-1}\right)^{p-1} e^{-m(\frac{1}{2})^{p-1}}.$$

When m tends to infinity, this last quantity tends to 0. So that, for m large enough, there is a tournament which is not absorbed by $p-1$ vertices. \square

Remark. By Stirling's approximation we derive that the bound obtained in Theorem 2 is of order $\frac{2^n}{\sqrt{n}}$.

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References

- [1] I. Bárány and J. Lehel, Covering with Euclidean Boxes, *European Journal of Combinatorics*, vol. 8, pp. 13–19, 1987.
- [2] N. Bousquet, W. Lochet and S. Thomassé, *A proof of the Erdős-Sands-Sauer-Woodrow conjecture*, arXiv:1703.08123v1, 2017.
- [3] H. Galeana-Sánchez and R. Rojas-Monroy, A counterexample to a conjecture on edge coloured tournaments, *Discrete Mathematics*, vol. 282, pp. 275–276, 2004.
- [4] G. Hahn, P. Ille and R. Woodrow, Absorbing sets in arc-coloured tournaments, *Discrete Mathematics*, vol. 283(1–3), pp. 93–99, 2004.
- [5] S. Minggang, On monochromatic paths in m -coloured tournaments, *Journal of Combinatorial Theory, Series B*, vol. 45, pp. 108–111, 1988.
- [6] D. Pálvölgyi and A. Gyárfás, Domination in transitive colorings of tournaments, *Journal of Combinatorial Theory, Series B*, vol. 107, pp. 1–11, 2014.
- [7] B. Sands, N. Sauer and R. Woodrow, On monochromatic paths in edge-colored digraphs, *Journal of Combinatorial Theory, Series B*, vol. 33, pp. 271–275, 1982.