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1 **OPTIMAL CONTROL OF A POPULATION DYNAMICS MODEL**  
2 **WITH MISSING BIRTH RATE\***

3 CYRILLE KENNE<sup>†</sup>, GÜNTER LEUGERING<sup>‡</sup>, AND GISÈLE MOPHOU<sup>§</sup>

4 **Abstract.** We consider a model of population dynamics with age dependence and spatial  
5 structure but unknown birth rate. Using the notion of Low-regret [1], we prove that we can bring  
6 the state of the system to a desired state by acting on the system via a localized distributed control.  
7 We provide the optimality systems that characterize the Low-regret control. Moreover, using an  
8 appropriate Hilbert space, we prove that the family of Low-regret controls tends to a so-called No-  
9 regret control, which we, in turn, characterize.

10 **Key words.** Population dynamics, incomplete data, optimal control, No-regret control, Low-  
11 regret control, Euler-Lagrange formula.

12 **AMS subject classifications.** 49J20, 92D25, 93C41

13 **1. Introduction.** In the modeling of the dynamics of some invasive species  
14 governed by diffusive systems with age dependency (for instance invasive plants in  
15 bounded domains), it may not be possible to have sufficient observations to obtain a  
16 good approximation of the birth function or/and the mortality function [2]. In order  
17 to avoid the disappearance of the other species it seems natural to control those inva-  
18 sive species. More precisely, let  $\Omega \subset \mathbb{R}^3$  be a bounded domain in which the invasive  
19 species live. We denote by  $\Gamma$  the boundary of the domain and we assume that it is  
20 of class  $C^2$ . For the time  $T > 0$  and the life expectancy of an individual  $A > 0$ , we  
21 set  $U = (0, T) \times (0, A)$ ,  $Q = U \times \Omega$ ,  $\Sigma = U \times \Gamma$ ,  $Q_A = (0, A) \times \Omega$ ,  $Q_T = (0, T) \times \Omega$   
22 and  $Q_\omega = U \times \omega$ , where  $\omega$  is a non-empty open subset of  $\Omega$ . Then we consider a  
23 model describing the dynamics of an invasive species with age dependence and spatial  
24 structure:

$$25 \quad (1.1) \quad \begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y = f + v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot, \cdot) = y^0 & \text{in } Q_A, \\ y(\cdot, 0, \cdot) = \int_0^A g(a)y(t, a, x) da & \text{in } Q_T, \end{cases}$$

26 where

- 27 •  $y = y(t, a, x)$  is the distribution of individuals of age  $a \geq 0$ , at time  $t \geq 0$  and  
28 location  $x \in \Omega$ .
- 29 • The recruitment  $f \in L^2(Q)$  is a positive periodic function.
- 30 • The control  $v \in L^2(Q)$  which corresponds to the removal of the individuals in  
31 a sub-domain  $\omega$  of  $\Omega$  and  $\chi_\omega$  denote the characteristic function of the control  
32 set  $\omega$ .

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- The mortality rate  $\mu = \mu(a) \geq 0$  is a known increasing positive function which is continuous on  $[0, A]$ , whereas the fertility rate  $g = g(a) \in L^\infty(0, A)$  is unknown and positive.

We assume as in [3], that:

$$(H_1) \quad \lim_{a \rightarrow A} \int_0^a \mu(s) ds = +\infty,$$

which means that each individual in the population dies before age  $A$ . For more literature on the population dynamics model and the signification of assumption  $(H_1)$ , we refer to [3, 4, 5, 6, 7, 8, 9] and the reference therein.

*Remark 1.1.* Set

$$(1.2) \quad W(T, A) = \left\{ \rho \in L^2(U; H_0^1(\Omega)); \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2(U; H^{-1}(\Omega)) \right\}.$$

Then we have (see [10]) that

$$(1.3) \quad W(T, A) \subset \mathcal{C}([0, T], L^2(Q_A)) \text{ and } W(T, A) \subset \mathcal{C}([0, A], L^2(Q_T)).$$

Under the assumption on the data, (1.1) has a solution  $y(v, g) = y(t, a, x; v, g)$  in  $W(T, A)$ . We define the cost function

$$(1.4) \quad J(v, g) = \|y(v, g) - z_d\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q_\omega)}^2,$$

where  $z_d \in L^2(Q)$  and  $N > 0$  are given.

In this paper, we don't want to determine the unknown fertility rate  $g$ . Actually, we want to bring the distribution of individuals to a desired distribution  $z_d$  by acting on the system via a control  $v$ . In other words, we are interested in solving the following optimization problem:

$$\inf_{v \in L^2(Q_\omega)} \sup_{g \in L^2(0, A)} J(v, g).$$

But observing that we could have  $\sup_{g \in L^2(0, A)} J(v, g) = +\infty$ , we consider the optimization problem:

$$(1.5) \quad \inf_{v \in L^2(Q_\omega)} \sup_{g \in L^2(0, A)} (J(v, g) - J(0, g)).$$

Problem (1.5) is called the No-regret control problem. The notions of No-regret control and Low-regret control were introduced by J. -L. Lions [1] in order to control a phenomenon described by a parabolic equation with missing initial condition. Let us recall that one obtains the Low-regret control problem by relaxing the No-regret control one. See (3.23), (3.24), (3.25) for the relaxation used in this article. By means of the Legendre-Fenchel transform, we prove that the Low-regret control problem is equivalent to a classical optimal control problem. The most difficult task is to prove that this family of controls (called Low-regret controls) converges towards the No-regret control. Also in [11], J. -L. Lions proved that these notion can be used in the framework of decomposition methods. In [12], O. Nakoulima et al. applied this notion to linear evolution equations with incomplete data and they proved that the Low-regret controls converges to the No-regret control for which they obtained a singular optimality system. B. Jacob et al. [13] generalized the notion of No-regret control to population dynamics with incomplete initial data with a distributed control

66 acting on the whole domain. They proved the existence and uniqueness of the No-  
 67 regret control and gave a singular optimality system that characterizes this control. In  
 68 the nonlinear case, this notion was considered by O. Nakoulima et al. [14] to control  
 69 on the whole domain a nonlinear system with incomplete data. Observing on the one  
 70 hand that the No-regret control is typically not easy to characterize and, on the other  
 71 hand that the Low-regret cost function may not be convex, they proved by adapting  
 72 this cost to a No-regret control that the adapted Low-regret control converge towards  
 73 this No-regret control characterized by a singular optimality system. In [15], J. Vélin  
 74 studied systems governed by quasilinear equations with unknown boundary condition  
 75 and a control acting on the whole domain. After established some regularity results  
 76 for the control-to-state and control-perturbation applications and its derivatives, he  
 77 proved by proceeding as in [14] that the adapted Low-regret control converge towards a  
 78 No-regret control characterized gain by a singular optimality. Note that in the above  
 79 papers, the convergence of the Low-regret control towards the No-regret control is  
 80 obtained by controlling on the whole domain.

81 In this paper, we use the notion of No-regret and Low-regret to control a model  
 82 describing the dynamics of population with age dependence and spatial structure  
 83 with missing birth rate by acting on a part of the domain. Observing that with an  
 84 unknown the birth rate, the control problem considered is now non-linear, we start by  
 85 proving some regularity results. Then we prove the existence of a No-regret control.  
 86 We then regularize the No-regret control problem to a Low-regret control problem  
 87 ((3.23),(3.24),(3.25)). We introduce an appropriate Hilbert space to obtain estimates  
 88 on the states satisfying the optimality systems and by that characterize the Low-regret  
 89 control. Then we prove that the adapted Low-regret control converges towards a No-  
 90 regret control and establish a singular optimality system that, in turn, characterizes  
 91 this no-regret control.

92 The rest of this paper is structured as follows. In section 2, we give some regularity  
 93 results. We study the Low-regret and no-regret control and their characterizations in  
 94 section 3. A conclusion is given in section 4.

95 **2. Preliminary results.** In order to solve the optimization problem (3.2), we  
 96 need some preliminary results.

97 In what follows, we adopt the following notation

$$98 \quad (2.1) \quad \begin{cases} L &= \frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \Delta + \mu I, \\ L^* &= -\frac{\partial}{\partial t} - \frac{\partial}{\partial a} - \Delta + \mu I, \end{cases}$$

99 where  $I$  is the identity operator.

100 **PROPOSITION 2.1.** *Let  $y = y(v, g)$  be a solution of (1.1). Then the application*  
 101  *$(v, g) \mapsto y(v, g)$  is a continuous function from  $L^2(Q_\omega) \times L^2(0, A)$  onto  $L^2(U, H_0^1(\Omega))$ .*

102 *Proof.* Let  $(v_0, g_0) \in L^2(Q_\omega) \times L^2(0, A)$ . We show that  $(v, g) \mapsto y(v, g)$  is contin-  
 103 uous at  $(v_0, g_0)$ .

104 Set  $\bar{y} = y(v, g) - y(v_0, g_0)$ , then  $\bar{y}$  is solution to the problem

$$105 \quad (2.2) \quad \begin{cases} L\bar{y} &= v\chi_{Q_\omega} - v_0\chi_{Q_\omega} & \text{in } Q, \\ \bar{y} &= 0 & \text{on } \Sigma, \\ \bar{y}(0, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \bar{y}(\cdot, 0, \cdot) &= \eta & \text{in } Q_T, \end{cases}$$

where for  $(t, x) \in Q_T$ ,

$$\eta(t, x) = \int_0^A [g(a)y(t, a, x; v, g) - g_0(a)y(t, a, x; v_0, g_0)] da.$$

106 If we set  $z = e^{-rt}\bar{y}$  with  $r > 0$ , then we obtain that  $z$  is solution to the problem

$$107 \quad (2.3) \quad \begin{cases} Lz + rz & = e^{-rt}(v\chi_{Q_\omega} - v_0\chi_{Q_\omega}) & \text{in } Q, \\ z & = 0 & \text{on } \Sigma, \\ z(0, \cdot, \cdot) & = 0 & \text{in } Q_A, \\ z(\cdot, 0, \cdot) & = e^{-rt}\eta & \text{in } Q_T. \end{cases}$$

Multiplying the first equation of system (2.3) by  $z$  and integrating by parts over  $Q$ , we get

$$\begin{aligned} \int_{Q_\omega} e^{-rt}(v - v_0)z \, dxdt \, da &= \frac{1}{2} \|z(T, \cdot, \cdot)\|_{L^2(Q_A)}^2 - \frac{1}{2} \|z(0, \cdot, \cdot)\|_{L^2(Q_A)}^2 \\ &+ \frac{1}{2} \|z(\cdot, A, \cdot)\|_{L^2(Q_T)}^2 - \frac{1}{2} \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 \\ &+ \|\nabla z\|_{L^2(Q)}^2 + \int_Q (r + \mu)z^2. \end{aligned}$$

108 From this we deduce that

$$109 \quad (2.4) \quad \|\nabla z\|_{L^2(Q)}^2 + r\|z\|_{L^2(Q)}^2 \leq \frac{1}{2} \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \frac{1}{2} \|v - v_0\|_{L^2(Q_\omega)}^2 + \frac{1}{2} \|z\|_{L^2(Q)}^2,$$

110 because  $\mu \geq 0$ . On the other hand, observing that for  $(t, x) \in Q_T$ ,

$$\begin{aligned} 111 \quad z(t, 0, x) &= e^{-rt} \int_0^A [g(a)y(t, a, x, v, g) - g_0(a)y(t, a, x, v_0, g_0)] da \\ 112 \quad &= e^{-rt} \int_0^A [(g(a) - g_0(a))y(t, a, x, v, g)] da + \int_0^A g_0(a)z da, \end{aligned}$$

113 we obtain

$$114 \quad \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 \leq 2\|g - g_0\|_{L^2(0,A)}^2 \|y\|_{L^2(Q)}^2 + 2\|g_0\|_{L^2(0,A)}^2 \|z\|_{L^2(Q)}^2.$$

Thus, (2.4) gives

$$\begin{aligned} \|\nabla z\|_{L^2(Q)}^2 + \left(r - \|g_0\|_{L^2(0,A)}^2 - \frac{1}{2}\right) \|z\|_{L^2(Q)}^2 &\leq \|g - g_0\|_{L^2(0,A)}^2 \|y\|_{L^2(Q)}^2 \\ &+ \frac{1}{2} \|v - v_0\|_{L^2(Q_\omega)}^2. \end{aligned}$$

115 Choosing  $r$  such that  $r > \|g_0\|_{L^2(0,A)}^2 + \frac{1}{2}$ , we have

$$116 \quad \|z\|_{L^2(U; H_0^1(\Omega))}^2 \leq \|g - g_0\|_{L^2(0,A)}^2 \|y\|_{L^2(Q)}^2 + \frac{1}{2} \|v - v_0\|_{L^2(Q_\omega)}^2.$$

117 From this we deduce

$$118 \quad \|z\|_{L^2(U; H_0^1(\Omega))} \leq \|g - g_0\|_{L^2(0,A)} \|y\|_{L^2(Q)} + \frac{\sqrt{2}}{2} \|v - v_0\|_{L^2(Q_\omega)}.$$

119 Therefore,

$$120 \quad \|\bar{y}\|_{L^2(U; H_0^1(\Omega))} \leq e^{rT} \|g - g_0\|_{L^2(0,A)} \|y\|_{L^2(Q)} + \frac{\sqrt{2}}{2} e^{rT} \|v - v_0\|_{L^2(Q_\omega)}.$$

121 As  $(v, g) \rightarrow (v_0, g_0)$ , we have  $\bar{y} \rightarrow 0$  strongly in  $L^2(U; H_0^1(\Omega))$ . Hence  $y(v, g) \rightarrow$   
 122  $y(v_0, g_0)$  strongly in  $L^2(U; H_0^1(\Omega))$  as  $(v, g) \rightarrow (v_0, g_0)$ .  $\square$

123 **PROPOSITION 2.2.** *Let  $\lambda > 0$ . Let  $g, h \in L^2(0, A)$  and  $v, w \in L^2(Q_\omega)$ . Let also*  
 124  *$y = y(v, g)$  be a solution of (1.1). Set  $\bar{y}_\lambda = \frac{y(v + \lambda w, g + \lambda h) - y(v, g)}{\lambda}$ . Then  $(\bar{y}_\lambda)$*   
 125 *converges strongly in  $L^2(U; H_0^1(\Omega))$  as  $\lambda \rightarrow 0$  to a function  $\bar{y}$  which is solution of*

$$126 \quad (2.5) \quad \begin{cases} L\bar{y} = w\chi_{Q_\omega} & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{y}(\cdot, 0, \cdot) = \int_0^A g(a)\bar{y} da + \int_0^A h(a)y(t, a, x; v, g) da & \text{in } Q_T. \end{cases}$$

127 *Proof.*  $\bar{y}_\lambda$  is a solution to the problem

$$128 \quad \begin{cases} L\bar{y}_\lambda = w\chi_{Q_\omega} & \text{in } Q, \\ \bar{y}_\lambda = 0 & \text{on } \Sigma, \\ \bar{y}_\lambda(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{y}_\lambda(\cdot, 0, \cdot) = \int_0^A g(a)\bar{y}_\lambda da + \int_0^A h(a)y(t, a, x; v + \lambda w, g + \lambda h) da & \text{in } Q_T. \end{cases}$$

129 Define  $y_\lambda = \bar{y}_\lambda - \bar{y}$ , where  $\bar{y}$  is a solution to (2.5). Then  $y_\lambda$  is a solution to

$$130 \quad (2.6) \quad \begin{cases} Ly_\lambda = 0 & \text{in } Q, \\ y_\lambda = 0 & \text{on } \Sigma, \\ y_\lambda(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ y_\lambda(\cdot, 0, \cdot) = \int_0^A g(a)y_\lambda da + \eta_1 & \text{in } Q_T, \end{cases}$$

where for  $(t, x) \in Q_T$ ,

$$\eta_1(t, x) = \int_0^A h(a) [y(t, a, x; v + \lambda w, g + \lambda h) - y(t, a, x; v, g)] da.$$

131 We set  $z_\lambda = e^{-rt}y_\lambda$  with  $r > 0$ . Then we obtain that  $z_\lambda$  is a solution to the problem

$$132 \quad (2.7) \quad \begin{cases} Lz_\lambda + rz_\lambda = 0 & \text{in } Q, \\ z_\lambda = 0 & \text{on } \Sigma, \\ z_\lambda(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ z_\lambda(\cdot, 0, \cdot) = \int_0^A g(a)z_\lambda da + e^{-rt}\eta_1 & \text{in } Q_T. \end{cases}$$

133 Multiplying the first equation of system (2.7) by  $z_\lambda$  and integrating by parts over  $Q$ ,  
 134 then using the fact that  $\mu \geq 0$ , we obtain

$$135 \quad (2.8) \quad \|\nabla z_\lambda\|_{L^2(Q)}^2 + r\|z_\lambda\|_{L^2(Q)}^2 \leq \frac{1}{2}\|z_\lambda(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2.$$

Since

$$\begin{aligned} \|z_\lambda(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 &\leq 2\|g\|_{L^2(0,A)}^2 \|z_\lambda\|_{L^2(Q)}^2 \\ &\quad + 2\|h\|_{L^2(0,A)}^2 \|y(v + \lambda w, g + \lambda h) - y(v, g)\|_{L^2(U; H_0^1(\Omega))}^2, \end{aligned}$$

it follows from (2.8) that

$$\begin{aligned} & \|\nabla z_\lambda\|_{L^2(Q)}^2 + (r - \|g\|_{L^2(0,A)}^2) \|z_\lambda\|_{L^2(Q)}^2 \leq \\ & \|h\|_{L^2(0,A)}^2 \|y(v + \lambda w, g + \lambda h) - y(v, g)\|_{L^2(U; H_0^1(\Omega))}^2. \end{aligned}$$

136 Choosing  $r$  such that  $r > \|g\|_{L^2(0,A)}^2$ , we deduce

$$137 \quad (2.9) \quad \|z_\lambda\|_{L^2(U; H_0^1(\Omega))} \leq \|h\|_{L^2(0,A)} \|y(v + \lambda w, g + \lambda h) - y(v, g)\|_{L^2(U; H_0^1(\Omega))}.$$

138 Hence,

$$139 \quad (2.10) \quad \|y_\lambda\|_{L^2(U; H_0^1(\Omega))} \leq e^{rT} \|h\|_{L^2(0,A)} \|y(v + \lambda w, g + \lambda h) - y(v, g)\|_{L^2(Q)}.$$

140 Passing to the limit in this latter identity when  $\lambda \rightarrow 0$  and using the fact that the func-  
141 tion  $(v, g) \mapsto y(v, g)$  is continuous, it follows that  $y_\lambda \rightarrow 0$  strongly in  $L^2(U; H_0^1(\Omega))$ .  
142 This means that  $(\bar{y}_\lambda)$  converges to  $\bar{y}$  strongly in  $L^2(U; H_0^1(\Omega))$  as  $\lambda \rightarrow 0$ .  $\square$

143 **PROPOSITION 2.3.** *The mapping*

$$144 \quad \frac{\partial y}{\partial g}(\cdot, g): L^2(Q_\omega) \rightarrow \mathcal{L}(L^2(0, A); L^2(U; H_0^1(\Omega)))$$

$$145 \quad v \mapsto \frac{\partial y}{\partial g}(v, g),$$

146 *is continuous.*

147 *Proof.* From Proposition 2.2, we have that  $\bar{y}(h) = \frac{\partial y}{\partial g}(v, g)(h)$  is a solution to

$$149 \quad \begin{cases} L\bar{y}(h) = 0 & \text{in } Q, \\ \bar{y}(h) = 0 & \text{on } \Sigma, \\ \bar{y}(h)(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{y}(h)(\cdot, 0, \cdot) = \int_0^A g(a)\bar{y}(h) da + \int_0^A h(a)y(t, a, x; v, g) da & \text{in } Q_T. \end{cases}$$

150 Let  $v_1, v_2 \in L^2(Q_\omega)$ . Set  $\bar{y}_1(h) = \frac{\partial y}{\partial g}(v_1, g)(h)$ ,  $\bar{y}_2(h) = \frac{\partial y}{\partial g}(v_2, g)(h)$  and take  $\bar{z}(h) =$   
151  $e^{-rt}(\bar{y}_1(h) - \bar{y}_2(h))$ ,  $r > 0$ . It then follows that  $\bar{z}(h)$  is a solution to problem

$$152 \quad (2.11) \quad \begin{cases} L\bar{z}(h) + r\bar{z}(h) = 0 & \text{in } Q, \\ \bar{z}(h) = 0 & \text{on } \Sigma, \\ \bar{z}(h)(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{z}(h)(\cdot, 0, \cdot) = \int_0^A g(a)\bar{z}(h) da + e^{-rt}\eta_4 & \text{in } Q_T, \end{cases}$$

153 where for  $(t, x) \in Q_T$ ,

$$154 \quad (2.12) \quad \eta_4(t, x) = \int_0^A h(a)(y(t, a, x; v_1, g) - y(t, a, x; v_2, g)) da.$$

155 Multiplying the first equation of system (2.11) by  $\bar{z}(h)$  and integrating by parts over  
156  $Q$ , we obtain

$$157 \quad (2.13) \quad \begin{aligned} & \frac{1}{2}\|\bar{z}(h)(T, \cdot, \cdot)\|_{L^2(Q_A)}^2 + \frac{1}{2}\|\bar{z}(h)(\cdot, A, \cdot)\|_{L^2(Q_T)}^2 - \frac{1}{2}\|\bar{z}(h)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 \\ & + \|\nabla \bar{z}(h)\|_{L^2(Q)}^2 + \int_Q (r + \mu)\bar{z}(h) = 0. \end{aligned}$$

Observing

$$\begin{aligned} \|\bar{z}(h)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 &\leq 2\|g\|_{L^2(0,A)}^2 \|\bar{z}(h)\|_{L^2(Q)}^2 \\ &\quad + \|h\|_{L^2(0,A)}^2 \|y(v_1, g) - y(v_2, g)\|_{L^2(U; H_0^1(\Omega))}^2 \end{aligned}$$

158 and choosing in (2.13)  $r$  such that  $r > \|g\|_{L^2(0,A)}^2$ , we deduce

$$159 \quad (2.14) \quad \|\bar{z}(h)\|_{L^2(U; H_0^1(\Omega))} \leq \|h\|_{L^2(0,A)} \|y(v_1, g) - y(v_2, g)\|_{L^2(U; H_0^1(\Omega))}.$$

160 Therefore,

$$161 \quad (2.15) \quad \|\bar{y}_1(h) - \bar{y}_2(h)\|_{L^2(U; H_0^1(\Omega))} \leq e^{rT} \|h\|_{L^2(0,A)} \|y(v_1, g) - y(v_2, g)\|_{L^2(U; H_0^1(\Omega))},$$

from which we deduce

$$\begin{aligned} \|\bar{y}_1 - \bar{y}_2\| &= \sup_{h \in L^2(0,A), \|h\| \leq 1} \|\bar{y}_1(h) - \bar{y}_2(h)\|_{L^2(U; H_0^1(\Omega))} \\ &\leq e^{rT} \|y(v_1, g) - y(v_2, g)\|_{L^2(U; H_0^1(\Omega))}, \end{aligned}$$

where  $\|\cdot\|$  stands for the norm in  $\mathcal{L}(L^2(0, A); L^2(U; H_0^1(\Omega)))$ . This leads us to

$$\left\| \left\| \frac{\partial y}{\partial g}(v_1, g) - \frac{\partial y}{\partial g}(v_2, g) \right\| \right\| \leq e^{rT} \|y(v_1, g) - y(v_2, g)\|_{L^2(U; H_0^1(\Omega))}.$$

162 Passing to the limit in this latter inequality when  $v_1 \rightarrow v_2$  while using Proposition  
163 2.2, we obtain that  $\frac{\partial y}{\partial g}(v_1, g)$  converges to  $\frac{\partial y}{\partial g}(v_2, g)$  in  $\mathcal{L}(L^2(0, A); L^2(U; H_0^1(\Omega)))$ .  $\square$

164 **3. Resolution of the optimization problem (1.5).** In this section, we are  
165 concerned with the optimization problem (1.5). As the Low-regret and No-regret  
166 notion introduced by Lions [1] uses the decomposition of the solution of (1.1) on the  
167 form  $y(v, g) = y(v, 0) + \varphi(g)$  where  $y(v, 0)$  is solution of (1.1) with  $g = 0$  and  $\varphi(g)$   
168 a function depending of  $g$ , this decomposition is no longer valid because the map  
169  $g \mapsto y(v, g)$  from  $L^2(0, A)$  to  $L^2(U; H_0^1(\Omega))$  is non-linear. Thus using the regularity  
170 results of  $y$  proven in Proposition 2.1 and Proposition 2.2, we replace the cost function  
171 defined in (1.4) by its linearized form with respect to  $g$ . We thus consider as in [17]  
172 the new cost-function

$$173 \quad (3.1) \quad J_1(v, g) = J(v, 0) + \frac{\partial J}{\partial g}(v, 0)(g).$$

174 Then, we consider the following new optimization problem:

$$175 \quad (3.2) \quad \inf_{v \in L^2(Q_\omega)} \sup_{g \in L^2(0,A)} (J_1(v, g) - J_1(0, g)).$$

176 Let  $y(v, 0) \in L^2(U; H_0^1(\Omega))$  be the solution of

$$177 \quad (3.3) \quad \begin{cases} Ly(v, 0) = f + v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot, \cdot) = y^0 & \text{in } Q_A, \\ y(\cdot, 0, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

178 Then we have the following result.

179 PROPOSITION 3.1. For any  $(v, g) \in L^2(Q_\omega) \times L^2(0, A)$ , the following equality  
180 holds:

$$181 \quad (3.4) \quad J_1(v, g) = J(v, 0) + 2 \int_Q \left( \frac{\partial y}{\partial g}(v, 0)(g) \right) (y(v, 0) - z_d) dt da dx,$$

where  $J$  is the cost function defined in (1.4) and

$$\frac{\partial J}{\partial g}(v, 0)(g) = \lim_{t \rightarrow 0} \frac{J(v, tg) - J(v, 0)}{t}.$$

*Proof.* Observing on the one hand

$$\begin{aligned} J(v, tg) &= \|y(v, tg) - z_d\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q_\omega)}^2 \\ &= J(v, 0) + \|y(v, tg) - y(v, 0)\|_{L^2(Q)}^2 \\ &\quad + 2 \int_Q (y(v, tg) - y(v, 0))(y(v, 0) - z_d) dt da dx, \end{aligned}$$

and on the other hand

$$\frac{\partial J}{\partial g}(v, 0)(g) = \lim_{t \rightarrow 0} \frac{J(v, tg) - J(v, 0)}{t},$$

using Proposition 2.2, we obtain that

$$\frac{\partial J}{\partial g}(v, 0)(g) = 2 \int_Q \left( \frac{\partial y}{\partial g}(v, 0)(g) \right) (y(v, 0) - z_d) dt da dx.$$

$$182 \quad \text{So, } J_1(v, g) = J(v, 0) + 2 \int_Q \left( \frac{\partial y}{\partial g}(v, 0)(g) \right) (y(v, 0) - z_d) dt da dx. \quad \square$$

183 PROPOSITION 3.2. For any  $(v, g) \in L^2(Q_\omega) \times L^2(0, A)$ , we have

$$184 \quad (3.5) \quad J_1(v, g) - J_1(0, g) = J(v, 0) - J(0, 0) + 2 \int_0^A S(a; v) g(a) da,$$

185 where for any  $a \in (0, A)$ ,

$$186 \quad (3.6) \quad S(a; v) = \int_{Q_T} [y(t, a, x; v, 0) \xi(v)(t, 0, x) - y(t, a, x; 0, 0) \xi(0)(t, 0, x)] dt dx$$

187 with  $\xi(v)$ , a solution to

$$188 \quad (3.7) \quad \begin{cases} L^* \xi(v) = y(v, 0) - z_d & \text{in } Q, \\ \xi(v) = 0 & \text{on } \Sigma, \\ \xi(v)(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \xi(v)(\cdot, A, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

189 *Proof.* In view of (3.4), we have

$$\begin{aligned} 190 \quad (3.8) \quad J_1(v, g) - J_1(0, g) &= J(v, 0) - J(0, 0) \\ &\quad + 2 \int_Q \left( \frac{\partial y}{\partial g}(v, 0)(g) \right) (y(v, 0) - z_d) dt da dx \\ &\quad - \int_Q \left( \frac{\partial y}{\partial g}(0, 0)(g) \right) (y(0, 0) - z_d) dt da dx. \end{aligned}$$

191 From Proposition 2.2, we have that  $\bar{y}(g) = \frac{\partial y}{\partial g}(v, 0)(g)$  is the solution to

$$192 \quad (3.9) \quad \begin{cases} L\bar{y}(g) &= 0 & \text{in } Q, \\ \bar{y}(g) &= 0 & \text{on } \Sigma, \\ \bar{y}(g)(0, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \bar{y}(g)(\cdot, 0, \cdot) &= \int_0^A g(a)y(t, a, x; v, 0) da & \text{in } Q_T. \end{cases}$$

So, if we multiply the first equation of (3.9) by  $\xi(v)$  and integrate by parts over  $Q$ , we get

$$- \int_Q g(a)y(t, a, x; v, 0)\xi(v)(t, 0, x) dt da dx + \int_Q \bar{y}(g)(y(v, 0) - z_d) dt da dx = 0,$$

193 which can be rewritten as

$$194 \quad (3.10) \quad \begin{aligned} & \int_Q \left( \frac{\partial y}{\partial g}(v, 0)(g) \right) (y(v, 0) - z_d) dt da dx = \\ & \int_Q g(a)y(t, a, x; v, 0)\xi(v)(t, 0, x) dt da dx. \end{aligned}$$

195 We also have

$$196 \quad (3.11) \quad \begin{aligned} & \int_Q \left( \frac{\partial y}{\partial g}(0, 0)(g) \right) (y(0, 0) - z_d) dt da dx = \\ & \int_Q g(a)y(t, a, x; 0, 0)\xi(0)(t, 0, x) dt da dx. \end{aligned}$$

197 Using (3.8), (3.10) and (3.11), it follows that

$$198 \quad (3.12) \quad \begin{aligned} & J_1(v, g) - J_1(0, g) = J(v, 0) - J(0, 0) + \\ & 2 \int_0^A g(a) \int_{Q_T} [y(t, a, x; v, 0)\xi(v)(t, 0, x) - y(t, a, x; 0, 0)\xi(0)(t, 0, x)] dadtdx \\ & = J(v, 0) - J(0, 0) + 2 \int_0^A S(a; v) g(a) da. \end{aligned}$$

199

□

200 LEMMA 3.3. Let  $\xi(v)$  be the solution of problem (3.7). Then the application  $v \mapsto$   
201  $\xi(v)$  is continuous from  $L^2(Q_\omega)$  onto  $L^2(U; H_0^1(\Omega))$ .

202 Proof. Let  $v_1, v_2 \in L^2(Q_\omega)$ , and define  $\bar{\xi} = \xi(v_1) - \xi(v_2)$ . Then  $\bar{\xi}$  is the solution  
203 to problem

$$204 \quad (3.13) \quad \begin{cases} L^*\bar{\xi} &= y(v_1, 0) - y(v_2, 0) & \text{in } Q, \\ \bar{\xi} &= 0 & \text{on } \Sigma, \\ \bar{\xi}(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ \bar{\xi}(\cdot, A, \cdot) &= 0 & \text{in } Q_T. \end{cases}$$

205 By setting  $z = e^{-rt}\bar{\xi}$ , it follows that  $z$  solves

$$206 \quad (3.14) \quad \begin{cases} L^*z + rz &= (y(v_1, 0) - y(v_2, 0))e^{-rt} & \text{in } Q, \\ z &= 0 & \text{on } \Sigma, \\ z(T, \cdot, \cdot) &= 0 & \text{in } Q_A, \\ z(\cdot, A, \cdot) &= 0 & \text{in } Q_T. \end{cases}$$

If we multiply the first equation of system (3.14) by  $z$  and integrating by parts over  $Q$ , we get

$$\begin{aligned} & \frac{1}{2} \|z(0, \cdot, \cdot)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \|\nabla z\|_{L^2(Q)}^2 + \int_Q (r + \mu) z^2 dt da dx = \\ & \int_Q (y(v_1, 0) - y(v_2, 0)) z e^{-rt} dt da dx. \end{aligned}$$

It then follows

$$\frac{1}{2} \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \|\nabla z\|_{L^2(Q)}^2 + r \|z\|_{L^2(Q)}^2 \leq \frac{1}{2} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)}^2 + \frac{1}{2} \|z\|_{L^2(Q)}^2.$$

Taking  $r = \frac{1}{2}$  in this latter identity yields

$$\frac{1}{2} \|z(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \frac{1}{2} \|z\|_{L^2(U; H_0^1(\Omega))}^2 \leq \frac{1}{2} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)}^2.$$

207 Thus,

$$208 \quad (3.15) \quad \|\bar{\xi}(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \|\bar{\xi}\|_{L^2(U; H_0^1(\Omega))}^2 \leq e^T \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)},$$

from which we deduce

$$\|\bar{\xi}\|_{L^2(U; H_0^1(\Omega))} \leq e^{T/2} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)}.$$

209 Using Proposition 2.1, while passing to limit in this latter inequality when  $v_1 \rightarrow v_2$ ,  
210 we obtain that  $\bar{\xi} \rightarrow 0$  strongly in  $L^2(U; H_0^1(\Omega))$ . This means that  $\xi(v_1) \rightarrow \xi(v_2)$   
211 strongly in  $L^2(U; H_0^1(\Omega))$  as  $v_1 \rightarrow v_2$ .  $\square$

212 **PROPOSITION 3.4.** *Let  $S(\cdot; v)$  be the function defined in (3.6). Then the map*  
213  *$v \mapsto S(\cdot; v)$  is continuous form  $L^2(Q_\omega)$  onto  $L^2(0, A)$ .*

*Proof.* Let  $v_1$  and  $v_2$ . Then in view of (3.6),

$$\begin{aligned} S(a; v_1) - S(a; v_2) &= \int_{Q_T} (y(t, a, x; v_1, 0) - y(t, a, x; v_2, 0)) \xi(v_1)(t, 0, x) dt dx \\ &\quad - \int_{Q_T} y(t, a, x; v_2, 0) (\xi(v_2)(t, 0, x) - \xi(v_1)(t, 0, x)) dt dx. \end{aligned}$$

Using the Cauchy Schwarz inequality, we have

$$\begin{aligned} |S(a; v_1) - S(a; v_2)| &\leq \|y(\cdot, a, \cdot; v_1, 0) - y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q_T)} \|\xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)} \\ &\quad + \|y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q_T)} \|\xi(v_2)(\cdot, 0, \cdot) - \xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)}. \end{aligned}$$

Observing on the one hand that  $\xi(v_2) - \xi(v_1)$  is solution of (3.13), and, on the other hand that, in view of (3.15),

$$\|\xi(v_2)(\cdot, 0, \cdot) - \xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 \leq e^{T/2} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)},$$

we have

$$\begin{aligned} |S(a; v_1) - S(a; v_2)| &\leq \|y(\cdot, a, \cdot; v_1, 0) - y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q_T)} \|\xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)} \\ &\quad + e^{T/2} \|y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q_T)} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)}. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_0^A |S(a; v_1) - S(a; v_2)|^2 da \leq \\ & 2\|y(\cdot, a, \cdot; v_1, 0) - y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q)}^2 \|\xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)}^2 + \\ & 2e^T \|y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q)}^2 \|y(v_1, 0) - y(v_2, 0)\|_{L^2(Q)}^2. \end{aligned}$$

It then follows from Poincaré inequality,

$$\begin{aligned} & \|S(a; v_1) - S(a; v_2)\|_{L^2(0, A)} \leq \\ & C(\Omega) \|y(\cdot, a, \cdot; v_1, 0) - y(\cdot, a, \cdot; v_2, 0)\|_{L^2(U; H_0^1(\Omega))} \|\xi(v_1)(\cdot, 0, \cdot)\|_{L^2(Q_T)} + \\ & C(\Omega) e^{T/2} \|y(\cdot, a, \cdot; v_2, 0)\|_{L^2(Q)} \|y(v_1, 0) - y(v_2, 0)\|_{L^2(U; H_0^1(\Omega))}, \end{aligned}$$

214 where  $C(\Omega) > 0$  is a constant depending on  $\Omega$ . In view of Proposition 2.1, it follows  
215 that  $S(\cdot, v_1) \rightarrow S(\cdot, v_2)$  as  $v_1 \rightarrow v_2$ .  $\square$

216 The following Lemma will be useful to prove the existence of the No-regret and Low-  
217 regret controls.

LEMMA 3.5. *Let  $S(\cdot, v)$  be defined as in (3.6) for any  $a \in L^2(0, A)$ . For any  $\gamma > 0$ , we consider the sequences  $y^\gamma = y(t, a, x; u^\gamma, 0)$  and  $\xi(u^\gamma)$ , respectively, solutions of (3.3) and (3.7) with  $v = u^\gamma$ . Assume that there exists  $C > 0$  independent of  $\gamma$  such that*

$$\|S(\cdot, u^\gamma)\|_{L^2(0, A)} < C.$$

218 Assume also that  $\hat{u} \in L^2(Q_\omega)$ ,  $\hat{\xi}(\cdot, 0, \cdot) \in L^2(Q_T)$  and  $\hat{y} = y(t, a, x; \hat{u}, 0) \in L^2(U; H_0^1(\Omega))$   $\blacksquare$   
219 solution of (3.3), such that

$$\begin{aligned} 220 \quad (3.16a) \quad & u^\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(U \times \omega), \\ 221 \quad (3.16b) \quad & y^\gamma \rightharpoonup \hat{y} = y(t, a, x; \hat{u}, 0) \text{ weakly in } L^2(U, H_0^1(\Omega)), \\ 222 \quad (3.16c) \quad & \xi(u^\gamma)(\cdot, 0, \cdot) \rightharpoonup \hat{\xi}(\cdot, 0, \cdot) \text{ weakly in } L^2(Q_T). \end{aligned}$$

Then we have

$$S(a; u^\gamma) \rightharpoonup S(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

224 *Proof.* Let  $\mathcal{D}((0, A))$  be the set of  $C^\infty$  function with compact support on  $(0, A)$ .  
225 Set for any  $\phi \in \mathcal{D}((0, A))$

$$226 \quad (3.17) \quad z^\gamma(t, x) = \int_0^A y(t, a, x; u^\gamma, 0) \phi(a) da, \quad (t, x) \in Q_T.$$

Then, in view of (3.16b), there exists a constant  $C > 0$  independent of  $\gamma$  such that

$$\|z^\gamma\|_{L^2(Q_T)} \leq \|y^\gamma\|_{L^2(Q)} \|\phi\|_{L^2(0, A)} \leq C.$$

227 Consequently, there exists  $z \in L^2(Q_T)$  such that

$$228 \quad (3.18) \quad z^\gamma \rightharpoonup z \text{ weakly in } L^2(Q_T).$$

Moreover, using (3.17) we have that,

$$\int_{Q_T} z^\gamma(t, x) \psi(t, x) dx dt = \int_{Q_T} \int_0^A y(t, a, x; u^\gamma, 0) \phi(a) \psi(t, x) dx dt da, \quad \forall \psi \in \mathcal{D}(Q_T),$$

which passing to the limit when  $\gamma \rightarrow 0$  while using (3.16b) gives

$$\lim_{\gamma \rightarrow 0} \int_{Q_T} z^\gamma(t, x) z(t, x) dx dt = \int_{Q_T} \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) \psi(t, x) dx dt da \quad \forall \psi \in \mathcal{D}(Q_T).$$

This means that

$$z^\gamma \rightharpoonup \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da \text{ weakly in } \mathcal{D}'(Q_T).$$

229 It follows from (3.18) and the uniqueness of the limit that,

$$230 \quad (3.19) \quad z(t, x) = \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da, \quad (t, x) \in Q_T.$$

Because  $y^\gamma = y(t, a, x; u^\gamma, 0)$  solves (3.3) with  $v = u^\gamma$ , we have that  $z^\gamma$  solves

$$\begin{cases} \frac{\partial z^\gamma}{\partial t} - \Delta z^\gamma = k^\gamma & \text{in } Q_T, \\ z^\gamma = 0 & \text{on } (0, T) \times \Gamma, \\ z^\gamma(0) = \int_0^A y^0(a, x) \phi(a) da & \text{in } \Omega, \end{cases}$$

where

$$k^\gamma(t, x) = \int_0^A (f + u^\gamma \chi_\omega) \phi da - \int_0^A \mu(a) y^\gamma \phi da - \int_0^A \frac{\partial y^\gamma}{\partial a} \phi da.$$

Consequently, in view of (3.16a) and (3.16b), we have there exists a positive constant  $C$  independent of  $\gamma$  such that

$$\begin{aligned} \|k^\gamma\|_{L^2(Q_T)} &\leq \left( 2\|f\|_{L^2(Q)}^2 + 2\|u^\gamma\|_{L^2(Q_\omega)}^2 + \|\mu\|_{L^\infty(0,A)}^2 \|y^\gamma\|_{L^2(Q)}^2 \right)^{1/2} \|\phi\|_{L^2(0,A)} + \\ \|y^\gamma\|_{L^2(Q)} \left\| \frac{\partial \phi}{\partial a} \right\|_{L^2(0,A)} &\leq C. \end{aligned}$$

It then follows that there is  $C > 0$ , independent of  $\gamma$ , such that

$$\begin{cases} \|z^\gamma\|_{L^2((0,T);H_0^1(\Omega))} &\leq C, \\ \left\| \frac{\partial z^\gamma}{\partial t} \right\|_{L^2((0,T);H^{-1}(\Omega))} &\leq C. \end{cases}$$

231 Therefore, it follows from Aubin-Lions's Lemma that

$$232 \quad (3.20) \quad z^\gamma \rightarrow z \text{ strongly in } L^2(Q_T),$$

where

$$z(t, x) = \int_0^A y(t, a, x; \hat{u}, 0) \phi(a) da, \quad (t, x) \in Q_T$$

233 because of (3.19).

Now in view of (3.6)

$$S(a; u^\gamma) = \int_{Q_T} [y(t, a, x; u^\gamma, 0) \xi(u^\gamma)(t, 0, x) - y(t, a, x; 0, 0) \xi(0)(t, 0, x)] dt dx.$$

Therefore for any  $\phi \in \mathcal{D}(0, A)$ ,

$$\begin{aligned}
 \int_0^A S(a; u^\gamma) \phi(a) da &= \int_{Q_T} \int_0^A (y(t, a, x; u^\gamma, 0) \phi(a) da) \xi(u^\gamma)(t, 0, x) dt da dx \\
 &- \int_Q y(t, a, x; 0, 0) \xi(0)(t, 0, x) \phi(a) dt da dx \\
 &= \int_{Q_T} z^\gamma(t, x) \xi(u^\gamma)(t, 0, x) dt da dx \\
 &- \int_Q y(t, a, x; 0, 0) \xi(0)(t, 0, x) \phi(a) dt da dx
 \end{aligned}$$

Passing this latter identity to the limit while using (3.20), (3.19) and (3.16c), we obtain

$$\begin{aligned}
 \int_0^A S(a; u^\gamma) \phi(a) da &\rightarrow \int_{Q_T} z(t, x) \xi(\hat{u})(t, 0, x) dt da dx \\
 &- \int_Q y(t, a, x; 0, 0) \xi(0)(t, 0, x) \phi(a) dt da dx \\
 &= \int_{Q_T} \int_0^A (y(t, a, x; \hat{u}, 0) \phi(a) da) \xi(\hat{u})(t, 0, x) dt da dx \\
 &- \int_Q y(t, a, x; 0, 0) \xi(0)(t, 0, x) \phi(a) dt da dx \quad \forall \phi \in \mathcal{D}(0, A),
 \end{aligned}$$

which in view of (3.6), proves

$$S(a; u^\gamma) \rightharpoonup S(a; \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

234

□

235 From now on, we denote by  $\mathcal{D}(\Theta)$  the set of  $\mathcal{C}^\infty$  function with compact support on  $\Theta$   
 236 and by  $\mathcal{D}'(\Theta)$ , its dual.

237 **3.1. Existence of No-regret control and Low-regret control.** In view of  
 238 (3.5), the optimization problem (3.2) is equivalent to the following problem:

$$239 \quad (3.21) \quad \inf_{v \in L^2(Q_\omega)} \sup_{g \in L^2(0, A)} [J(v, 0) - J(0, 0) + 2 \int_0^A S(a; v) g(a) da].$$

240 As  $\int_0^A S(a; v) g(a) da$  is either equal to 0 or  $+\infty$ , we look for the control  $v$  in the set,

$$241 \quad (3.22) \quad \mathcal{M} = \left\{ v \in L^2(Q_\omega); \int_0^A S(a; v) g(a) da = 0, \quad \forall g \in L^2(0, A) \right\}.$$

242 Note that the set  $\mathcal{M}$  is strongly closed in  $L^2(Q_\omega)$ . Now, observing on the one hand  
 243 that the application  $v \mapsto J(v, 0) - J(0, 0)$  is coercive on  $L^2(Q_\omega)$ , bounded below by  
 244  $-J(0, 0)$ , and continuous because of Proposition 2.1, and on the other hand that the  
 245 application  $v \mapsto S(\cdot; v)$  is continuous on  $L^2(Q_\omega)$ , using minimizing sequences and  
 246 Lemma 3.5, we prove that there exists a No-regret control  $\tilde{u}$  in  $\mathcal{M}$  satisfying (3.21).  
 247 We thus have proved the following result.

248 **LEMMA 3.6.** *There exists a solution  $\tilde{u}$  of (3.21) in  $\mathcal{M}$ .*

249 As such a control  $\tilde{u}$  is not easy to characterize, we consider for any  $\gamma > 0$ , the relaxed  
 250 optimization problem, which we refer to as the *Low-regret-control problem*:

$$251 \quad (3.23) \quad \inf_{v \in L^2(Q_\omega)} \sup_{g \in L^2(0,A)} \left[ J(v, 0) - J(0, 0) + 2 \int_0^A S(a; v)g(a)da - \gamma \|g\|_{L^2(0,A)}^2 \right].$$

Observing that

$$\sup_{g \in L^2(0,A)} \left[ J(v, 0) - J(0, 0) + 2 \int_0^A S(a; v)g(a)da - \gamma \|g\|_{L^2(0,A)}^2 \right] = \\ J(v, 0) - J(0, 0) + 2\gamma \sup_{g \in L^2(0,A)} \left[ \int_0^A \frac{S(a; v)}{\gamma} g(a)da - \frac{1}{2} \|g\|_{L^2(0,A)}^2 \right],$$

using Fenchel-Legendre transform (see [16]), we obtain that,

$$2\gamma \sup_{g \in L^2(0,A)} \left[ \int_0^A \frac{S(a; v)}{\gamma} g(a)da - \frac{1}{2} \|g\|_{L^2(0,A)}^2 \right] = \frac{1}{\gamma} \|S(\cdot; v)\|_{L^2(0,A)}^2$$

252 and (3.23) is equivalent to

$$253 \quad (3.24) \quad \inf_{v \in L^2(Q_\omega)} \mathcal{J}^\gamma(v),$$

254 with

$$255 \quad (3.25) \quad \mathcal{J}^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(\cdot; v)\|_{L^2(0,A)}^2.$$

256

257 **PROPOSITION 3.7.** *Let  $\gamma > 0$ . Then there exists at least in  $L^2(Q_\omega)$  a Low-regret*  
 258 *control  $u_\gamma$  solution of problem (3.24).*

259 *Proof.* We have  $\mathcal{J}^\gamma(v) \geq -J(0, 0)$  and  $\mathcal{J}^\gamma(0) = 0$ . Using minimizing sequences,  
 260 **Proposition 2.1**, **Proposition 3.4** and **Lemma 3.5**, we prove as for **Lemma 3.6** that  
 261 problem (3.24) has at least one solution  $u_\gamma \in L^2(Q_\omega)$ .  $\square$

262 **Remark 3.8.** The uniqueness of  $u_\gamma \in L^2(Q_\omega)$ , solution of (3.24) is not guaranteed  
 263 because the application  $v \mapsto S(\cdot; v)$  from  $L^2(Q_\omega)$  to  $L^2(0, A)$  is not necessarily strictly  
 264 convex. Consequently, we are not sure that control  $u_\gamma$  will converge to a No-regret  
 265 control  $\tilde{u} \in \mathcal{M}$ . So, in order to have a Low-regret control which will converge in  $\mathcal{M}$ ,  
 266 we adapt the cost function  $\mathcal{J}^\gamma$  to a No-regret control  $\tilde{u}$ .

267 **3.2. Existence of the adapted low-regret control.** Let  $\tilde{u}$  be a No-regret  
 268 optimal control. For any  $\gamma > 0$ , we define the adapted cost function  $\tilde{\mathcal{J}}^\gamma$  by:

$$269 \quad (3.26) \quad v \mapsto \tilde{\mathcal{J}}^\gamma(v) = J(v, 0) - J(0, 0) + \|v - \tilde{u}\|_{L^2(Q_\omega)}^2 + \frac{1}{\gamma} \|S(\cdot; v)\|_{L^2(0,A)}^2.$$

270 Then, we consider the following optimal control problem:

$$271 \quad (3.27) \quad \inf_{v \in L^2(Q_\omega)} \tilde{\mathcal{J}}^\gamma(v).$$

272

273 PROPOSITION 3.9. Let  $\gamma > 0$ . Then problem (3.27) has at least a solution  $\tilde{u}_\gamma$  in  
 274  $L^2(Q_\omega)$ .

275 *Proof.* One proceeds as for the proof of Proposition 3.7 using the fact that  $v \mapsto$   
 276  $\tilde{\mathcal{J}}^\gamma(v)$  is continuous on  $L^2(Q_\omega)$  (thanks to Proposition 2.1 and Proposition 3.4) and  
 277 the fact that  $\lim_{\|v\|_{L^2(Q_\omega)} \rightarrow +\infty} \mathcal{J}^\gamma(v) = +\infty$ .  $\square$

278 PROPOSITION 3.10. Let  $\tilde{u}_\gamma \in L^2(Q_\omega)$  be a solution of (3.27). Then there exist  
 279  $\tilde{p}_\gamma = p(\tilde{u}_\gamma) \in L^2(U; H_0^1(\Omega))$  and  $\tilde{q}_\gamma = q(\tilde{u}_\gamma) \in L^2(U; H_0^1(\Omega))$  such that  $\{\tilde{y}_\gamma, \tilde{\xi}_\gamma, \tilde{p}_\gamma, \tilde{q}_\gamma\}$   
 280 is a solution of the systems:

$$281 \quad (3.28) \quad \begin{cases} L\tilde{y}_\gamma = f + \tilde{u}_\gamma \chi_{Q_\omega} & \text{in } Q, \\ \tilde{y}_\gamma = 0 & \text{on } \Sigma, \\ \tilde{y}_\gamma(0, \cdot, \cdot) = y^0 & \text{in } Q_A, \\ \tilde{y}_\gamma(\cdot, 0, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

$$283 \quad (3.29) \quad \begin{cases} L^*\tilde{\xi}_\gamma = \tilde{y}_\gamma - z_d & \text{in } Q, \\ \tilde{\xi}_\gamma = 0 & \text{on } \Sigma, \\ \tilde{\xi}_\gamma(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{\xi}_\gamma(\cdot, A, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

$$285 \quad (3.30) \quad \begin{cases} L\tilde{p}_\gamma = 0 & \text{in } Q, \\ \tilde{p}_\gamma = 0 & \text{on } \Sigma, \\ \tilde{p}_\gamma(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{p}_\gamma(\cdot, 0, \cdot) = \frac{1}{\sqrt{\gamma}} \int_0^A y(t, a, x; \tilde{u}_\gamma, 0) S(a; \tilde{u}_\gamma) da & \text{in } Q_T, \end{cases}$$

$$287 \quad (3.31) \quad \begin{cases} L^*\tilde{q}_\gamma = y(\tilde{u}_\gamma, 0) - z_d + \varrho^\gamma & \text{in } Q, \\ \tilde{q}_\gamma = 0 & \text{on } \Sigma, \\ \tilde{q}_\gamma(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{q}_\gamma(\cdot, A, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

288 and

$$289 \quad (3.32) \quad (N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma = 0 \quad \text{in } Q_\omega,$$

290 where  $\varrho^\gamma = \frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma + \frac{1}{\gamma}S(a; \tilde{u}_\gamma)\xi(\tilde{u}_\gamma)(t, 0, x)$ ,  $\tilde{y}_\gamma = y(\tilde{u}_\gamma, 0)$  and  $\tilde{\xi}_\gamma = \xi(\tilde{u}_\gamma)$ .

*Proof.* We write the Euler-Lagrange optimality condition that characterizes  $\tilde{u}_\gamma$ :

$$\lim_{\lambda \rightarrow 0} \frac{\tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma + \lambda w) - \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma)}{\lambda} = 0, \quad \forall w \in L^2(Q_\omega).$$

291 Using Proposition 2.2 and Proposition 3.2, we obtain after some calculations  
 (3.33)

$$292 \quad \begin{aligned} 0 &= \int_Q \left( \frac{\partial y}{\partial v}(\tilde{u}_\gamma, 0)(w) \right) \left( y(\tilde{u}_\gamma, 0) - z_d + \frac{1}{\gamma} \xi(\tilde{u}_\gamma)(\cdot, 0, \cdot) S(\cdot; \tilde{u}_\gamma) \right) dt da dx \\ &+ \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx + \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx \\ &+ \frac{1}{\gamma} \int_Q \frac{\partial \xi}{\partial v}(\tilde{u}_\gamma)(w)(\cdot, 0, \cdot) y(\tilde{u}_\gamma, 0) S(\cdot; \tilde{u}_\gamma) dt da dx, \quad \forall w \in L^2(Q_\omega), \end{aligned}$$

293 where  $\bar{y}(w) = \frac{\partial y}{\partial v}(\tilde{u}_\gamma, 0)(w)$  and  $\bar{\xi} = \frac{\partial \xi}{\partial v}(\tilde{u}_\gamma)(w)$  are respectively solutions to

$$294 \quad (3.34) \quad \begin{cases} L\bar{y}(w) = w & \text{in } Q, \\ \bar{y}(w) = 0 & \text{on } \Sigma, \\ \bar{y}(w)(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{y}(w)(\cdot, 0, \cdot) = 0 & \text{in } Q_T \end{cases}$$

295 and

$$296 \quad (3.35) \quad \begin{cases} L^*\bar{\xi} = \bar{y}(w) & \text{in } Q, \\ \bar{\xi} = 0 & \text{on } \Sigma, \\ \bar{\xi}(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \bar{\xi}(\cdot, A, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

297 To interpret (3.33), we use the adjoint states  $\tilde{q}_\gamma$  and  $\tilde{p}_\gamma$  solutions of (3.31) and (3.30)  
298 respectively.

299 So if we multiply the first term of (3.34) by a function  $\tilde{q}_\gamma$  and the first equation  
300 of (3.35) by a function  $\frac{1}{\sqrt{\gamma}}p_\gamma$ , then integrate by parts over  $Q$ , we, respectively, obtain

$$301 \quad (3.36) \quad \int_Q \bar{y}(w) \left( y(\tilde{u}_\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}S(a; \tilde{u}_\gamma)\xi(\tilde{u}_\gamma)(t, 0, x) \right) dt da dx = \\ \int_{Q_\omega} w \tilde{q}_\gamma dt da dx,$$

302 and

$$303 \quad (3.37) \quad \frac{1}{\gamma} \int_Q \bar{\xi}(\cdot, 0, \cdot) y(t, a, x; \tilde{u}_\gamma, 0) S(a; \tilde{u}_\gamma) dt da dx = \frac{1}{\sqrt{\gamma}} \int_Q \bar{y}(w) p_\gamma dt da dx.$$

304 Combining (3.36), (3.37) and (3.33), we have

$$305 \quad \int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma) w dt da dx = 0, \quad \forall w \in L^2(Q_\omega),$$

306 which implies that

$$307 \quad (3.38) \quad (N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma = 0 \text{ in } Q_\omega. \quad \square$$

308 **PROPOSITION 3.11.** *Let  $\tilde{u}_\gamma \in L^2(Q_\omega)$  be a solution of (3.27). Let also  $\tilde{y}_\gamma, \tilde{\xi}_\gamma, \tilde{p}_\gamma$   
309 and  $\tilde{q}_\gamma$  be such that (3.28)-(3.32) hold true. Then we have following estimations:*

$$310 \quad (3.39) \quad \|\tilde{u}_\gamma\|_{L^2(Q_\omega)} \leq C(N, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$311 \quad (3.40) \quad \frac{1}{\sqrt{\gamma}} \|S(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$312 \quad (3.41) \quad \|S(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)} \leq \sqrt{\gamma} C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$313 \quad (3.42) \quad \|\tilde{y}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$314 \quad (3.43) \quad \|\tilde{\xi}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$315 \quad (3.44) \quad \|\tilde{\xi}_\gamma(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$316 \quad (3.45) \quad \|\tilde{p}_\gamma(\cdot, 0, \cdot)\|_{L^2(Q_T)} \leq C(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$317 \quad (3.46) \quad \|\tilde{p}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

$$318 \quad (3.47) \quad \|\tilde{q}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}),$$

319 where from now on,  $C(X)$  to denote a positive constant whose value varies from a  
 320 line to another but depends on  $X$ .

321 *Proof.* We proceed in three steps.

322 **Step 1.** We prove the estimations (3.39)-(3.43).

323 As  $\tilde{u}_\gamma$  is solution of (3.27), we can write

$$324 \quad (3.48) \quad \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(0) = \|\tilde{u}\|_{L^2(Q_\omega)}.$$

It then follows from the definition of  $\tilde{\mathcal{J}}^\gamma$  and  $J$  given respectively by (3.26) and (1.4) that,

$$\begin{aligned} & \|\tilde{y}_\gamma - z_d\|_{L^2(Q)}^2 + N\|\tilde{u}_\gamma\|_{L^2(Q_\omega)}^2 + \|\tilde{u}_\gamma - \tilde{u}\|_{L^2(Q)}^2 + \frac{1}{\gamma}\|S(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)}^2 \leq \\ & \|\tilde{u}\|_{L^2(Q_\omega)}^2 + \|y(0,0) - z_d\|_{L^2(Q)}^2 = C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right). \end{aligned}$$

325 Hence we deduce (3.39), (3.40), (3.41) and

$$326 \quad (3.49) \quad \|\tilde{y}_\gamma - z_d\|_{L^2(Q)} \leq C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right).$$

Observing  $\tilde{y}_\gamma$  and  $\tilde{\xi}_\gamma$  are respectively solution of (3.28) and (3.29), proceeding as for  $\bar{y}$  in pages 3-4, we obtain that

$$\|\tilde{y}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq \frac{1}{\sqrt{2}}e^{2T}\left(\|y_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\tilde{u}_\gamma\|_{L^2(Q_\omega)}\right)$$

and

$$\|\tilde{\xi}_\gamma(\cdot, 0, \cdot)\|_{L^2(Q_T)} + \|\tilde{\xi}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq \frac{\sqrt{2}}{2}e^{2T}\|\tilde{y}_\gamma - z_d\|_{L^2(Q)},$$

327 from which we, respectively, deduce (3.42), (3.43) and (3.44) because of (3.39) and  
 328 (3.49).

329 **Step 2.** We prove the estimations (3.45) and (3.46).

To prove (3.45), we observe that

$$\begin{aligned} & \left| \frac{1}{\sqrt{\gamma}} \int_0^A y(t, a, x; \tilde{u}_\gamma, 0) S(a; \tilde{u}_\gamma) da \right| \leq \\ & \frac{1}{\sqrt{\gamma}} \|S(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)} \left( \int_0^A y(t, a, x; \tilde{u}_\gamma, 0)^2 da \right)^{1/2}. \end{aligned}$$

So using (3.40) and (3.49), we deduce

$$\begin{aligned} \int_{Q_T} \left| \frac{1}{\sqrt{\gamma}} \int_0^A y(t, a, x; \tilde{u}_\gamma, 0) S(a; \tilde{u}_\gamma) da \right|^2 dt dx & \leq \frac{1}{\gamma} \|S(\cdot; \tilde{u}_\gamma)\|_{L^2(0,A)}^2 \|\tilde{y}_\gamma\|_{L^2(Q)}^2 \\ & \leq C, \end{aligned}$$

where  $C = C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right) > 0$ . This means

$$\|p_\gamma(\cdot, 0, \cdot)\|_{L^2(Q)} \leq C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right).$$

Since  $p_\gamma$  is solution of (3.30), proceeding as for  $\bar{y}$  in pages 3-4 while using (3.45), we obtain

$$\|\tilde{p}_\gamma\|_{L^2(U; H_0^1(\Omega))} \leq \frac{1}{\sqrt{2}}e^{2T}\left(\|y_0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|\tilde{u}_\gamma\|_{L^2(Q_\omega)}\right).$$

330 **Step 3.** We prove (3.47).

331 We observe that  $\tilde{q}_\gamma$ , solution of (3.31), can be decomposed as  $\tilde{q}_\gamma = \tilde{q}_\gamma^1 + \tilde{q}_\gamma^2$ , where  
 332  $\tilde{q}_\gamma^1$  is solution to

$$333 \quad (3.50) \quad \begin{cases} L^* \tilde{q}_\gamma^1 = \tilde{y}_\gamma - z_d & \text{in } Q, \\ \tilde{q}_\gamma^1 = 0 & \text{on } \Sigma, \\ \tilde{q}_\gamma^1(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{q}_\gamma^1(\cdot, A, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

334 and  $\tilde{q}_\gamma^2$  is solution to

$$335 \quad (3.51) \quad \begin{cases} L^* \tilde{q}_\gamma^2 = \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} S(a; \tilde{u}_\gamma) \xi(\tilde{u}_\gamma)(t, 0, x) & \text{in } Q, \\ \tilde{q}_\gamma^2 = 0 & \text{on } \Sigma, \\ \tilde{q}_\gamma^2(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{q}_\gamma^2(\cdot, A, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

336 Proceeding as for  $\bar{y}$  in pages 3-4, while using (3.49), we obtain

$$337 \quad (3.52) \quad \|\tilde{q}_\gamma^1\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}).$$

338 Combining (3.33) and (3.37), we obtain

$$339 \quad (3.53) \quad \begin{aligned} 0 &= \int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d) dt da dx \\ &+ \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx + \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx \\ &+ \int_Q \bar{y}(w) \left( \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma) \right) dt da dx, \forall w \in L^2(Q_\omega). \end{aligned}$$

340 Set

$$341 \quad (3.54) \quad \mathcal{E} = \{ \bar{y}(w), \quad w \in L^2(Q_\omega) \}.$$

342 Then  $\mathcal{E} \subset L^2(Q)$ . We define on  $\mathcal{E} \times \mathcal{E}$  the inner product:

$$343 \quad (3.55) \quad \langle \bar{y}(v), \bar{y}(w) \rangle_{\mathcal{E}} = \int_{Q_\omega} v w dt da dx + \int_Q \bar{y}(v) \bar{y}(w) dt da dx, \forall \bar{y}(v), \bar{y}(w) \in \mathcal{E}.$$

344 Then  $\mathcal{E}$  endowed with the norm

$$345 \quad (3.56) \quad \|\bar{y}(w)\|_{\mathcal{E}}^2 = \|w\|_{L^2(Q_\omega)}^2 + \|\bar{y}(w)\|_{L^2(Q)}^2, \forall \bar{y}(w) \in \mathcal{E}$$

is a Hilbert space. We set

$$T_\gamma(\tilde{u}_\gamma) = \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma).$$

346 Then, in view of (3.53), we have for any  $w \in L^2(Q_\omega)$ ,

$$347 \quad (3.57) \quad \begin{aligned} \int_Q T_\gamma(\tilde{u}_\gamma) \bar{y}(w) dt da dx &= - \int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d) dt da dx \\ &- \int_{Q_\omega} N \tilde{u}_\gamma w dt da dx - \int_{Q_\omega} (\tilde{u}_\gamma - \tilde{u}) w dt da dx. \end{aligned}$$

Using the Cauchy Schwarz inequality, we have

$$\begin{aligned} & \left| - \int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d) dt da dx - \int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u}) w dt da dx \right| \leq \\ & \|\tilde{y}_\gamma - z_d\|_{L^2(Q)} \|\bar{y}(w)\|_{L^2(Q)} + (N+1) \|\tilde{u}_\gamma\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)} + \\ & \|\tilde{u}\|_{L^2(Q_\omega)} \|w\|_{L^2(Q_\omega)}. \end{aligned}$$

Therefore, using (3.49) and (3.39),

$$\begin{aligned} & \left| - \int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d) dt da dx - \int_{Q_\omega} ((N+1)\tilde{u}_\gamma - \tilde{u}) w dt da dx \right| \leq \\ & \left( \|\tilde{y}_\gamma - z_d\|_{L^2(Q)}^2 + [(N+1)\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} + \|\tilde{u}\|_{L^2(Q_\omega)}]^2 \right)^{1/2} \|\bar{y}(w)\|_{\mathcal{E}} \leq \\ & C \|\bar{y}(w)\|_{\mathcal{E}}, \end{aligned}$$

where  $C = C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}) > 0$ . It then follows from (3.57)

$$\left| \int_Q T_\gamma(\tilde{u}_\gamma) \bar{y}(w) dt da dx \right| \leq C \|\bar{y}(w)\|_{\mathcal{E}}.$$

Consequently,

$$\|T_\gamma(\tilde{u}_\gamma)\|_{\mathcal{E}'} = \left\| \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma) \right\|_{\mathcal{E}'} \leq C.$$

348 In particular,

$$349 \quad (3.58) \quad \left\| \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma) \right\|_{L^2(Q)} \leq C,$$

350 where  $C = C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}) > 0$ .

351 Now, proceeding as for  $\bar{y}$  in pages 3-4, while using (3.58), we obtain that

$$352 \quad (3.59) \quad \|\tilde{q}_\gamma^2\|_{L^2(U; H_0^1(\Omega))} \leq C(N, T, \|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}).$$

353 Finally from (3.52) and (3.59) we deduce (3.47).  $\square$

### 354 3.3. Characterization of the No-regret control.

355 PROPOSITION 3.12. *The adapted Low-regret optimal control  $\tilde{u}_\gamma$  converges in  $L^2(Q_\omega)$  to the No-regret control  $\tilde{u} \in \mathcal{M}$ .*

357 *Proof.* In view of (3.39)-(3.44), there exists a subsequence of  $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{\xi}_\gamma, S(\cdot, \tilde{u}_\gamma))$   
 358 still denoted by  $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{\xi}_\gamma, S(\cdot, \tilde{u}_\gamma))$  and  $\hat{u} \in L^2(Q_\omega)$ ,  $\tilde{y} \in L^2(U, H_0^1(\Omega))$ ,  $\tilde{\xi} \in L^2(U, H_0^1(\Omega))$ ,  
 359  $\alpha \in L^2(0, A)$ ,  $\tau \in L^2(Q_T)$  such that

$$360 \quad (3.60) \quad \tilde{u}_\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(Q_\omega),$$

$$361 \quad (3.61) \quad \frac{1}{\sqrt{\gamma}} S(\cdot, \tilde{u}_\gamma) \rightharpoonup \alpha \text{ weakly in } L^2(0, A),$$

$$362 \quad (3.62) \quad S(\cdot, \tilde{u}_\gamma) \rightarrow 0 \text{ strongly in } L^2(0, A),$$

$$363 \quad (3.63) \quad \tilde{y}_\gamma \rightharpoonup \tilde{y} \text{ weakly in } L^2(U; H_0^1(\Omega)),$$

$$364 \quad (3.64) \quad \tilde{\xi}_\gamma \rightharpoonup \tilde{\xi} \text{ weakly in } L^2(U; H_0^1(\Omega)),$$

$$365 \quad (3.65) \quad \tilde{\xi}_\gamma(\cdot, 0, \cdot) \rightharpoonup \tau \text{ weakly in } L^2(Q_T).$$

If we multiply the first equation (3.28) by  $\phi \in \mathcal{D}(Q)$  and the first equation in (3.29) by  $\psi \in \mathcal{D}(Q)$  and integrate by parts over  $Q$ , we have

$$\int_Q \tilde{y}_\gamma L^* \phi \, dt \, da \, dx = \int_Q (f + \tilde{u}_\gamma \chi_\omega) \phi \, dt \, da \, dx$$

and

$$\int_Q \tilde{\xi}_\gamma L \psi \, dt \, da \, dx = \int_Q (\tilde{y}_\gamma - z_d) \psi \, dt \, da \, dx.$$

Passing in these two latter identities to the limit, while using (3.60), (3.63) and (3.64), we obtain

$$\int_Q \tilde{y} L^* \phi \, dt \, da \, dx = \int_Q (f + \hat{u} \chi_\omega) \phi \, dt \, da \, dx$$

and

$$\int_Q \tilde{\xi} L \psi \, dt \, da \, dx = \int_Q (\tilde{y} - z_d) \psi \, dt \, da \, dx,$$

which after an integration by parts over  $Q$  give

$$\int_Q L \tilde{y} \phi \, dt \, da \, dx = \int_Q (f + \hat{u} \chi_\omega) \phi \, dt \, da \, dx, \quad \forall \phi \in \mathcal{D}(Q)$$

and

$$\int_Q L^* \tilde{\xi} \psi \, dt \, da \, dx = \int_Q (\tilde{y} - z_d) \psi \, dt \, da \, dx, \quad \forall \psi \in \mathcal{D}(Q),$$

366 respectively. Hence, we can deduce

$$367 \quad (3.66) \quad L \tilde{y} = f + \hat{u} \chi_\omega \quad \text{in } Q$$

368 and

$$369 \quad (3.67) \quad L^* \tilde{\xi} = \tilde{y} - z_d \quad \text{in } Q.$$

370 Note that  $\tilde{y}, \tilde{\xi} \in L^2(U, H_0^1(\Omega))$ . This implies that  $\tilde{y}(t, a)|_\Gamma$  and  $\tilde{\xi}(t, a)|_\Gamma$  exist and  
 371 belong to  $L^2(\Gamma)$  for almost every  $(t, a) \in U$ . On the other hand from (3.66), (3.67)  
 372 and the expression of the operator  $L$  and  $L^*$  given by (2.1), we have  $\tilde{y}, \tilde{\xi} \in W(T, A)$ .  
 373 It follows from Remark 1.1 that  $(\tilde{y}(0, \cdot, \cdot), \tilde{\xi}(T, \cdot, \cdot))$  exists and belongs  $(L^2(Q_A))^2$  and  
 374  $(\tilde{y}(\cdot, 0, \cdot), \tilde{\xi}(\cdot, A, \cdot), \tilde{\xi}(\cdot, 0, \cdot))$  exists and belongs  $(L^2(Q_T))^2$ .

Now, if we multiply the first equation (3.28) by  $\phi \in \mathcal{C}^\infty(\bar{Q})$  such that  $\phi = 0$  on  $\Sigma$ ,  $\phi(\cdot, A, \cdot) = 0$  in  $Q_T$  and  $\phi(T, \cdot, \cdot) = 0$  in  $Q_A$  and the first equation in (3.29) by  $\psi \in \mathcal{C}^\infty(\bar{Q})$  such that  $\psi = 0$  on  $\Sigma$  and  $\psi(0, \cdot, \cdot) = 0$  in  $Q_A$  and integrate by parts over  $Q$ , we respectively have that

$$- \int_{Q_A} y^0 \phi(0, a, x) \, da \, dx + \int_Q \tilde{y}_\gamma L^* \phi \, da \, dx = \int_Q (f + \tilde{u}_\gamma \chi_\omega) \phi \, dt \, da \, dx$$

and

$$\int_{Q_A} \tilde{\xi}_\gamma(t, 0, x) \psi(0, a, x) \, dt \, dx + \int_Q \tilde{\xi}_\gamma L \psi \, da \, dx = \int_Q (\tilde{y}_\gamma - z_d) \psi \, dt \, da \, dx.$$

Passing these two latter identities to the limit while using (3.60), (3.63), (3.64) and (3.65), we obtain

$$-\int_{Q_A} y^0 \phi(0, a, x) da dx + \int_Q \tilde{y} L^* \phi dt da dx = \int_Q (f + \hat{u} \chi_\omega) \phi dt da dx,$$

$$\forall \phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \phi|_\Gamma = 0, \phi(\cdot, A, \cdot)|_{Q_T} = 0, \phi(T, \cdot, \cdot)|_{Q_A} = 0,$$

and

$$\int_{Q_A} \tau \psi(0, a, x) dt dx + \int_Q \tilde{\xi} L \psi dt da dx = \int_Q (\tilde{y} - z_d) \psi dt da dx,$$

$$\forall \psi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \psi|_\Gamma = 0, \psi(0, \cdot, \cdot)|_{Q_A} = 0,$$

respectively, which after an integration by parts over  $Q$  give

$$\int_Q (f + \hat{u} \chi_\omega) \phi dt da dx = - \int_{Q_A} (y^0 - \tilde{y}(0, a, x)) \phi(0, a, x) da dx +$$

$$\int_{Q_T} \tilde{y}(t, 0, x) \phi(t, 0, x) dt dx - \int_\Sigma \tilde{y} \frac{\partial \phi}{\partial \nu} dt da dx + \int_Q L \tilde{y} \phi dt da dx,$$

$$\forall \phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \phi|_\Gamma = 0, \phi(\cdot, A, \cdot)|_{Q_T} = 0, \phi(T, \cdot, \cdot)|_{Q_A} = 0,$$

and

$$\int_Q (\tilde{y} - z_d) \psi dt da dx = \int_{Q_A} (\tau - \tilde{\xi}(t, 0, x)) \psi(0, a, x) dt dx +$$

$$\int_Q L^* \tilde{\xi} \psi dt da dx + \int_{Q_A} \tilde{\xi}(T, a, x) \psi(T, a, x) da dx +$$

$$\int_{Q_T} \tilde{\xi}(t, A, x) \psi(t, A, x) dt dx + \int_\Sigma \tilde{\xi} \frac{\partial \psi}{\partial \nu} dt da dx,$$

$$\forall \psi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \psi|_\Gamma = 0, \psi(0, \cdot, \cdot)|_{Q_A} = 0.$$

375 Using (3.66) and (3.67), we deduce from these latter identities that,

$$0 = - \int_{Q_A} (y^0 - \tilde{y}(0, a, x)) \phi(0, a, x) da dx +$$

$$376 \quad (3.68) \quad \int_{Q_T} \tilde{y}(t, 0, x) \phi(t, 0, x) dt dx - \int_\Sigma \tilde{y} \frac{\partial \phi}{\partial \nu} dt da dx,$$

$$\forall \phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \phi|_\Gamma = 0, \phi(\cdot, A, \cdot)|_{Q_T} = 0, \phi(T, \cdot, \cdot)|_{Q_A} = 0,$$

377 and

$$0 = \int_{Q_A} (\tau - \tilde{\xi}(t, 0, x)) \psi(0, a, x) dt dx +$$

$$378 \quad (3.69) \quad \int_{Q_A} \tilde{\xi}(T, a, x) \psi(T, a, x) da dx +$$

$$\int_{Q_T} \tilde{\xi}(t, A, x) \psi(t, A, x) dt dx + \int_\Sigma \tilde{\xi} \frac{\partial \psi}{\partial \nu} dt da dx,$$

$$\forall \psi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \psi|_\Gamma = 0, \psi(0, \cdot, \cdot)|_{Q_A} = 0.$$

If we successively take in (3.68) and (3.69),

$$\phi(\cdot, 0, \cdot)|_{Q_T} = 0 \text{ and } \frac{\partial \phi}{\partial \nu}|_\Gamma = 0,$$

$$\psi(\cdot, 0, \cdot)|_{Q_T} = \psi(\cdot, A, \cdot)|_{Q_T} = 0 \text{ and } \frac{\partial \psi}{\partial \nu}|_\Gamma = 0,$$

$$\begin{aligned} \frac{\partial \phi}{\partial \nu} \Big|_{\Gamma} &= 0, \\ \psi(\cdot, A, \cdot) \Big|_{Q_T} &= 0 \text{ and } \frac{\partial \psi}{\partial \nu} \Big|_{\Gamma} = 0, \end{aligned}$$

then in (3.69),

$$\frac{\partial \psi}{\partial \nu} \Big|_{\Gamma} = 0,$$

we successively obtain

$$(3.70) \quad \tilde{y}(0, \cdot, \cdot) = y^0 \text{ in } Q_A,$$

$$(3.71) \quad \tilde{\xi}(T, \cdot, \cdot) = 0 \text{ in } Q_A,$$

382

$$(3.72) \quad \tilde{y}(\cdot, 0, \cdot) = 0 \text{ in } Q_T,$$

$$(3.73) \quad \tilde{\xi}(\cdot, A, \cdot) = 0 \text{ in } Q_T,$$

385 then

$$(3.74) \quad \tilde{y} = 0 \text{ on } \Sigma,$$

$$(3.75) \quad \tilde{\xi}(\cdot, 0, \cdot) = \tau \text{ in } Q_T,$$

388 and finally,

$$(3.76) \quad \tilde{\xi} = 0 \text{ on } \Sigma.$$

Now, using (3.60), (3.63), (3.65), (3.75) and (3.41), we have from Lemma 3.5 that

$$S(\cdot, u^\gamma) \rightharpoonup S(\cdot, \hat{u}) \text{ weakly in } \mathcal{D}'(0, A).$$

Hence, using (3.62) and the uniqueness of the limit that

$$S(\cdot, \tilde{u}_\gamma) \rightarrow S(\cdot, \hat{u}) = 0 \text{ strongly in } L^2(0, A).$$

Consequently,

$$\int_0^A S(a; \tilde{u}_\gamma) g(a) da \rightarrow \int_0^A S(a; \hat{u}) g(a) da = 0.$$

390 Thus  $\hat{u} \in \mathcal{M}$  and we also have  $\|S(\cdot; \hat{u})\|_{L^2(0, A)} = 0$ . Since  $\tilde{u}$  is a No-regret control and  
391  $\hat{u} \in \mathcal{M}$ , it follows from (3.21) that

$$(3.77) \quad J(\tilde{u}, 0) - J(0, 0) \leq J(\hat{u}, 0) - J(0, 0),$$

393 Observing that  $\tilde{u}_\gamma$  solves the problem  $\inf_{v \in L^2(Q_\omega)} \tilde{\mathcal{J}}^\gamma(v)$ , we have

$$(3.78) \quad \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}) = J(\tilde{u}, 0) - J(0, 0),$$

which, in view of the definition of  $\tilde{\mathcal{J}}^\gamma$  given by (3.26), implies that

$$J(\tilde{u}_\gamma, 0) - J(0, 0) + \|\tilde{u}_\gamma - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq \tilde{\mathcal{J}}^\gamma(\tilde{u}) = J(\tilde{u}, 0) - J(0, 0).$$

395 Using the convexity and lower semi-continuity of  $J$  on  $L^2(Q_\omega)$ , (3.60) and (3.63), we  
396 obtain

$$(3.79) \quad J(\hat{u}, 0) - J(0, 0) + \|\hat{u} - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq \liminf_{\gamma \rightarrow 0} \tilde{\mathcal{J}}^\gamma(\tilde{u}_\gamma) \leq J(\tilde{u}, 0) - J(0, 0),$$

which combining with (3.77) gives

$$\|\hat{u} - \tilde{u}\|_{L^2(Q_\omega)}^2 \leq 0.$$

398 Hence,

$$399 \quad (3.80) \quad \hat{u} = \tilde{u} \text{ in } Q_\omega.$$

400 Thus the adapted Low-regret controls converge in  $L^2(Q_\omega)$  to the No-regret control.  
 401 Moreover from (3.80), (3.66), (3.70), (3.72) and (3.74), it follows that  $\tilde{y} = y(\tilde{u}, 0) \in$   
 402  $L^2(U; H_0^1(\Omega))$  unique solution of

$$403 \quad (3.81) \quad \begin{cases} L\tilde{y} = f + \tilde{u}\chi_{Q_\omega} & \text{in } Q, \\ \tilde{y} = 0 & \text{on } \Sigma, \\ \tilde{y}(0, \cdot, \cdot) = y^0 & \text{in } Q_A, \\ \tilde{y}(\cdot, 0, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

404 Similarly, from (3.67), (3.71), (3.73) and (3.76), we infer that  $\tilde{\xi} = \xi(\tilde{u})$  is the unique  
 405 solution of

$$406 \quad (3.82) \quad \begin{cases} L^*\tilde{\xi} = \tilde{y} - z_d & \text{in } Q, \\ \tilde{\xi} = 0 & \text{on } \Sigma, \\ \tilde{\xi}(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{\xi}(\cdot, A, \cdot) = 0 & \text{in } Q_T. \end{cases}$$

407

□

408 PROPOSITION 3.13. *The No-regret control  $\tilde{u} \in \mathcal{M}$  is characterized by the func-*  
 409 *tions  $\tilde{u}$ ,  $\tilde{y}$ ,  $\tilde{\xi}$ ,  $\tilde{p}$  and  $\tilde{q}$  which are unique solutions of the optimality system:*

$$410 \quad (3.83) \quad \begin{cases} L\tilde{y} = f + \tilde{u}\chi_{Q_\omega} & \text{in } Q, \\ \tilde{y} = 0 & \text{on } \Sigma, \\ \tilde{y}(0, \cdot, \cdot) = y^0 & \text{in } Q_A, \\ \tilde{y}(\cdot, 0, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

411

$$412 \quad (3.84) \quad \begin{cases} L^*\tilde{\xi} = \tilde{y} - z_d & \text{in } Q, \\ \tilde{\xi} = 0 & \text{on } \Sigma, \\ \tilde{\xi}(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{\xi}(\cdot, A, \cdot) = 0 & \text{in } Q_T, \end{cases}$$

413

$$414 \quad (3.85) \quad \begin{cases} L\tilde{p} = 0 & \text{in } Q, \\ \tilde{p} = 0 & \text{on } \Sigma, \\ \tilde{p}(0, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{p}(\cdot, 0, \cdot) = \lambda_1 & \text{in } Q_T, \end{cases}$$

415

$$416 \quad (3.86) \quad \begin{cases} L^*\tilde{q} = \tilde{y} - z_d + \lambda_2 & \text{in } Q, \\ \tilde{q} = 0 & \text{on } \Sigma, \\ \tilde{q}(T, \cdot, \cdot) = 0 & \text{in } Q_A, \\ \tilde{q}(\cdot, A, \cdot) = 0 & \text{in } Q_T \end{cases}$$

417 and

$$418 \quad (3.87) \quad N\tilde{u} + \tilde{q} = 0,$$

where

$$\begin{aligned} \lambda_1 &= \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{\gamma}} \int_0^A y_\gamma(t, a, x, \tilde{u}_\gamma) S(a; \tilde{u}_\gamma) da, \\ \lambda_2 &= \lim_{\gamma \rightarrow 0} \left( \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma) \right). \end{aligned}$$

419 *Proof.* We have already proved (3.83) and (3.84) (see Page 23).

420 From (3.45), (3.58), (3.46) and (3.47), we have there exist  $\lambda_1 \in L^2(Q_T)$ ,  $\lambda_2 \in$   
421  $L^2(Q)$ ,  $\tilde{p} \in L^2(U; H_0^1(\Omega))$  and  $\tilde{q} \in L^2(U; H_0^1(\Omega))$  such that

$$422 \quad (3.88) \quad \tilde{p}_\gamma(\cdot, 0, \cdot) \rightharpoonup \lambda_1 \text{ in } L^2(Q_T),$$

$$423 \quad (3.89) \quad \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma} \tilde{\xi}_\gamma(0) S(a; \tilde{u}_\gamma) \rightharpoonup \lambda_2 \text{ in } L^2(Q),$$

$$424 \quad (3.90) \quad \tilde{p}_\gamma \rightharpoonup \tilde{p} \text{ in } L^2(U; H_0^1(\Omega)),$$

$$425 \quad (3.91) \quad \tilde{q}_\gamma \rightharpoonup \tilde{q} \text{ in } L^2(U; H_0^1(\Omega)).$$

426 Then, proceeding as for  $\tilde{y}_\gamma$  and  $\tilde{\xi}_\gamma$  in Pages 19-22 while using (3.89)-(3.91), we prove  
427 (3.85) and (3.86). To obtain (3.87), we pass to the limit in (3.38) while using (3.60),  
428 (3.80) and (3.91).  $\square$

429 **4. Conclusions.** We used the notion of No-regret and Low-regret to control a  
430 model describing the dynamics of population with age dependence and spatial struc-  
431 ture with missing birth rate. In contrast to some works on the topic which need the  
432 control to act on the whole domain to obtain the convergence of the optimality system  
433 that characterizes the Low-regret control towards the singular optimality characteriz-  
434 ing the No-regret control, our control acts on a part of the domain. We then introduce  
435 an appropriate Hilbert space and apply the Aubin-Lions Lemma to an appropriate  
436 auxiliary problem to obtain the convergence of an adapted Low-regret control towards  
437 a No-regret control that we characterize.

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