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# POSITIVE SOLUTIONS FOR LARGE RANDOM LINEAR SYSTEMS

PIERRE BIZEUL AND JAMAL NAJIM

ABSTRACT. Consider a large linear system where  $A_n$  is an  $n \times n$  matrix with independent real standard Gaussian entries,  $\mathbf{1}_n$  is an  $n \times 1$  vector of ones and with unknown the  $n \times 1$  vector  $\mathbf{x}_n$  satisfying

$$\mathbf{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \mathbf{x}_n .$$

We investigate the (componentwise) positivity of the solution  $\mathbf{x}_n$  depending on the scaling factor  $\alpha_n$  as the dimension  $n$  goes to infinity. We prove that there is a sharp phase transition at the threshold  $\alpha_n^* = \sqrt{2 \log n}$ : below the threshold ( $\alpha_n \ll \sqrt{2 \log n}$ ),  $\mathbf{x}_n$  has negative components with probability tending to 1 while above ( $\alpha_n \gg \sqrt{2 \log n}$ ), all the vector's components are eventually positive with probability tending to 1. At the critical scaling  $\alpha_n^*$ , we provide a heuristics to evaluate the probability that  $\mathbf{x}_n$  is positive.

Such linear systems arise as solutions at equilibrium of large Lotka-Volterra (LV) systems of differential equations, widely used to describe large biological communities with interactions. In the domain of positivity of  $\mathbf{x}_n$  (a property known as *feasibility* in theoretical ecology), our results provide a stability criterion for such LV systems for which  $\mathbf{x}_n$  is the solution at equilibrium.

## 1. INTRODUCTION

Denote by  $A_n$  an  $n \times n$  matrix with independent Gaussian  $\mathcal{N}(0, 1)$  entries and by  $\alpha_n$  a positive sequence. We are interested in the componentwise positivity of the  $n \times 1$  vector  $\mathbf{x}_n$ , solution of the linear system

$$(1.1) \quad \mathbf{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \mathbf{x}_n ,$$

where  $\mathbf{1}_n$  is the  $n \times 1$  vector with components 1.

It is well-known since Geman [7] that the limsup of the spectral radius of  $\frac{A_n}{\sqrt{n}}$  is almost surely (a.s.)  $\leq 1$ , so that matrix  $\left( I_n - \frac{A_n}{\alpha_n \sqrt{n}} \right)$  is eventually invertible as long as  $\alpha_n \gg 1$ . In this case, vector  $\mathbf{x}_n = (x_k)_{k \in [n]}$ , where we denote by  $[n] = \{1, \dots, n\}$ , is

$$\mathbf{x}_n = \left( I_n - \frac{A_n}{\alpha_n \sqrt{n}} \right)^{-1} \mathbf{1}_n \quad \text{with} \quad x_k = \mathbf{e}_k^* \left( I_n - \frac{A_n}{\alpha_n \sqrt{n}} \right)^{-1} \mathbf{1}_n ,$$

where  $\mathbf{e}_k$  is the  $n \times 1$  canonical vector and  $B^*$  is the transconjugate of  $B$  (or simply its transpose if  $B$  is real).

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The positivity of the  $x_k$ 's is a key issue in the study of Large Lotka-Volterra (LV) systems, widely used in mathematical biology and ecology to model populations with interactions.

Consider for instance a given foodweb and denote by  $\mathbf{x}_n(t) = (x_k(t))_{k \in [n]}$  the vector of abundances of the various species within the foodweb at time  $t$ . A standard way to connect the various abundances is via a LV system of equations

$$(1.2) \quad \frac{dx_k(t)}{dt} = x_k(t) \left( 1 - x_k(t) + \frac{1}{\alpha_n \sqrt{n}} \sum_{\ell \in [n]} A_{k\ell} x_\ell(t) \right) \quad \text{for } k \in [n],$$

where the interactions  $(A_{k\ell})$  can be modeled as random in the absence of any prior information. Here, the  $A_{k\ell}$ 's are assumed to be i.i.d.  $\mathcal{N}(0, 1)$ . At the equilibrium  $\frac{d\mathbf{x}_n}{dt} = 0$ , the abundance vector  $\mathbf{x}_n$  is a solution of (1.1) and a key issue is the existence of a *feasible* solution, that is a solution  $\mathbf{x}_n$  where all the  $x_k$ 's are positive. Dougoud et al. [5] based on Geman et al. [8] proved that a feasible solution is very unlikely to exist if  $\alpha_n \equiv \alpha$  is a constant. In fact, the CLT proved in [8] asserts that for any fixed number  $M$  of components

$$(x_k - 1)_{k \in [M]} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z \sim \mathcal{N}(0, \sigma_\alpha^2 I_M),$$

where  $\xrightarrow{\mathcal{D}}$  (resp.  $\xrightarrow{\mathcal{P}}$ ) stands for the convergence in distribution (resp. in probability) and where  $\sigma_\alpha^2 = \mathcal{O}(1)$ . As an important consequence, vectors  $\mathbf{x}_n$  with positive components will become extremely rare since

$$\mathbb{P}\{x_k > 0, k \in [M]\} \xrightarrow[n \rightarrow \infty]{} \left( \int_{-\frac{1}{\sigma_\alpha}}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \right)^M \Rightarrow \mathbb{P}\{x_k > 0, k \in [n]\} \xrightarrow[n \rightarrow \infty]{} 0.$$

In this article, we consider a growing scaling factor  $\alpha_n \rightarrow \infty$  and study the positivity of  $\mathbf{x}_n$ 's components in relation with  $\alpha_n$ .

We find that there exists a critical threshold  $\alpha_n^* = \sqrt{2 \log n}$  below which feasible solutions hardly exist and above which feasible solutions are more and more likely to exist. More precisely, we prove the following:

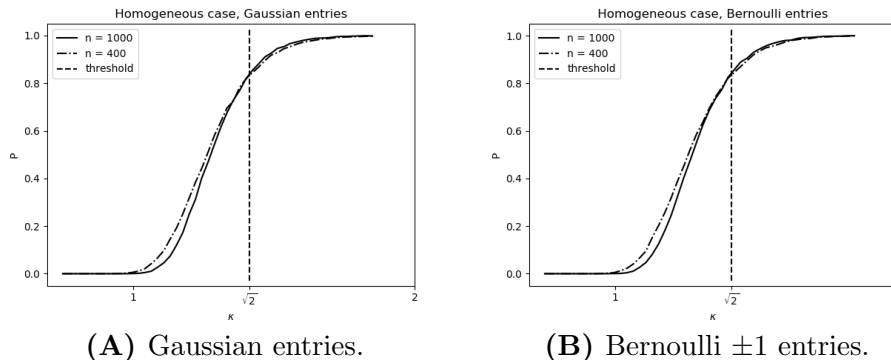
**Theorem 1.1** (Feasibility). *Let  $\alpha_n \xrightarrow[n \rightarrow \infty]{} \infty$  and denote by  $\alpha_n^* = \sqrt{2 \log n}$ . Let  $\mathbf{x}_n = (x_k)_{k \in [n]}$  be the solution of (1.1).*

- (1) *If  $\exists \varepsilon > 0$  such that  $\alpha_n \leq (1 - \varepsilon)\alpha_n^*$  then  $\mathbb{P}\{\min_{k \in [n]} x_k > 0\} \xrightarrow[n \rightarrow \infty]{} 0$ ,*
- (2) *If  $\exists \varepsilon > 0$  such that  $\alpha_n \geq (1 + \varepsilon)\alpha_n^*$  then  $\mathbb{P}\{\min_{k \in [n]} x_k > 0\} \xrightarrow[n \rightarrow \infty]{} 1$ .*

We illustrate the transition toward feasibility In Figure 1.

*Remark 1.2* (Beyond the Gaussian case). Proof of Theorem 1.1 is based on an analysis of the order of magnitude of the extreme values of the  $x_k$ 's, which heavily relies on sub-Gaussianness of Lipschitz functionals with Gaussian entries. This property remains true if the  $A_{ij}$ 's satisfies a logarithmic sobolev inequality - details are provided in Section 4.3. The case of discrete entries remains open although simulations (see Figure 1(B)) indicate that a similar phase transition occurs.

*Remark 1.3.* Notice that  $\frac{1}{\alpha_n^*}$  goes to zero extremely slowly, as shown in Table 1. For modeling purposes, the threshold  $\sigma_n^* := \frac{1}{\alpha_n^*}$  acts as an  $n$ -dependent upper bound of the standard deviation of the entries of  $(\alpha_n^{-1} A_n)$ , under which feasibility occurs.



(A) Gaussian entries.

 (B) Bernoulli  $\pm 1$  entries.

**Figure 1.** Transition toward feasibility. We consider different values of  $n$ , respectively 1000 (dashed line), 4000 (solid line). For each  $n$  and each  $\kappa$  on the  $x$ -axis, we simulate 10000  $n \times n$  matrices  $A_n$  and compute the solution  $\mathbf{x}_n$  of (1.1) at the scaling  $\alpha_n(\kappa) = \kappa\sqrt{\log(n)}$ . Each curve represents the proportion of feasible solutions  $\mathbf{x}_n$  obtained for 10000 simulations. The dotted vertical line corresponds to the critical scaling  $\alpha_n^* = \sqrt{2\log(n)}$  for  $\kappa = \sqrt{2}$ .

To complement the picture, we provide the following heuristics at the critical scaling  $\alpha_n^* = \sqrt{2\log n}$ :

$$(1.3) \quad \mathbb{P} \left\{ \min_{k \in [n]} x_k > 0 \right\} \approx 1 - \sqrt{\frac{e}{4\pi \log n}} + \frac{e}{8\pi \log n} \quad \text{as } n \rightarrow \infty.$$

**Table 1.** The quantity  $\frac{1}{\alpha_n^*} = \frac{1}{\sqrt{2\log n}}$  vanishes extremely slowly..

$n$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$\frac{1}{\alpha_n^*}$	0.33	0.27	0.23	0.21	0.19

Aside from the question of feasibility arises the question of *stability*: for a large complex system, that is a system of coupled differential equations describing the time evolution of the abundances of the various species of a given foodweb, how likely a perturbation of the solution  $\mathbf{x}$  will return to the equilibrium? Gardner and Ashby [6] considered stability issues of complex systems connected at random. Based on the circular law for large matrices with i.i.d. entries, May [14] provided a complexity/stability criterion and motivated the systematic use of large random matrix theory in the study of foodwebs, see for instance Allesina et al. [1]. Recently, Stone [15] and Gibbs *et al.* [9] revisited the relation between feasibility and stability.

We complement the result of Theorem 1.1 by addressing the question of stability in the context of a LV system (1.2) and prove that under the second condition of the theorem feasibility and stability occur simultaneously.

Recall that the solution at equilibrium  $\mathbf{x}_n$  is stable if the Jacobian matrix  $\mathcal{J}$  of the Lotka-Volterra system evaluated at  $\mathbf{x}_n$ , that is

$$(1.4) \quad \mathcal{J}(\mathbf{x}_n) = \text{diag}(\mathbf{x}_n) \left( -I_n + \frac{A_n}{\alpha_n \sqrt{n}} \right)$$

has all its eigenvalues with negative real part.

**Corollary 1.4** (Stability). *Let  $\mathbf{x}_n = (x_k)_{k \in [n]}$  be the solution of (1.1). Assume that  $\ell^+ := \limsup_{n \rightarrow \infty} \frac{\sqrt{2 \log n}}{\alpha_n} < 1$ . Denote by  $\mathcal{S}_n$  the spectrum of  $\mathcal{J}(\mathbf{x}_n)$ . Then*

$$(1.5) \quad \max_{\lambda \in \mathcal{S}_n} \min_{k \in [n]} |\lambda + x_k| \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad \text{and} \quad \max_{\lambda \in \mathcal{S}_n} \text{Re}(\lambda) \leq -(1 - \ell^+) + o_P(1).$$

Proof of Corollary 1.4 relies on standard perturbation results from linear algebra and on Theorem 1.1.

**Organization of the paper.** Theorem 1.1 is proved in Section 2, Corollary 1.4 in Section 3. In Section 4, elements supporting heuristics (1.3) are provided, together with extensions to non-homogeneous systems (where vector  $\mathbf{1}_n$  in (1.1) is replaced by a generic deterministic vector  $\mathbf{r}_n$ ) and non-Gaussian entries.

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## 2. POSITIVE SOLUTIONS: PROOF OF THEOREM 1.1

We will use the following notations for the various norms at stake: if  $\mathbf{v}$  is a vector then  $\|\mathbf{v}\|$  stands for its euclidian norm; if  $A$  is a matrix then  $\|A\|$  stands for its spectral norm and  $\|A\|_F = \sqrt{\sum_{ij} |A_{ij}|^2}$  for its Frobenius norm. Let  $\varphi$  be a function from  $\Sigma = \mathbb{R}$  or  $\mathbb{C}$  to  $\mathbb{C}$  then  $\|\varphi\|_\infty = \sup_{x \in \Sigma} |\varphi(x)|$ .

**2.1. Some preparation and strategy of the proof.** Denote by  $Q_n = \left( I_n - \frac{A_n}{\alpha_n \sqrt{n}} \right)^{-1}$  the resolvent and by  $s(B)$  the largest singular value of a given matrix  $B$ . Then it is well known that almost surely  $s_n := s(n^{-1/2} A_n) \xrightarrow[n \rightarrow \infty]{} 2$  (see for instance [3, Chapter 5]) hence  $s\left(\frac{1}{\alpha_n \sqrt{n}} A_n\right) \xrightarrow[n \rightarrow \infty]{} 0$ . In particular, the solution

$$\mathbf{x}_n = (x_k)_{k \in [n]} = \left( I_n - \frac{A_n}{\alpha_n \sqrt{n}} \right)^{-1} \mathbf{1}_n = Q_n \mathbf{1}_n,$$

with  $I_n$  the  $n \times n$  identity, is uniquely defined almost surely for all  $n$  large. In order to study the minimum of  $\mathbf{x}_n$ 's components, we partially unfold the above resolvent (in the sequel, we will simply denote  $A, \alpha, \mathbf{1}, Q$  instead of  $A_n, \alpha_n, \mathbf{1}_n, Q_n$ ) and write:

$$\begin{aligned} x_k &= \mathbf{e}_k^* \mathbf{x} = \mathbf{e}_k^* Q \mathbf{1} = \sum_{\ell=0}^{\infty} \mathbf{e}_k^* \left( \frac{A}{\alpha \sqrt{n}} \right)^\ell \mathbf{1}, \\ (2.1) &= 1 + \frac{1}{\alpha} \mathbf{e}_k^* \left( n^{-1/2} A \right) \mathbf{1} + \frac{1}{\alpha^2} \mathbf{e}_k^* \left( n^{-1/2} A \right)^2 Q \mathbf{1} = 1 + \frac{1}{\alpha} Z_k + \frac{1}{\alpha^2} R_k, \end{aligned}$$

where

$$(2.2) \quad Z_k = \mathbf{e}_k^* \left( n^{-1/2} A \right) \mathbf{1} = \frac{1}{\sqrt{n}} \sum_{i=1}^n A_{ki} \quad \text{and} \quad R_k = \mathbf{e}_k^* \left( n^{-1/2} A \right)^2 Q \mathbf{1}.$$

Notice that the  $Z_k$ 's are i.i.d.  $\mathcal{N}(0, 1)$ .

*Extreme values of Gaussian random variables.* Consider the sequence  $(Z_k)$  of standard Gaussian i.i.d. random variables, recall that  $\alpha_n^* = \sqrt{2 \log n}$  and let

$$(2.3) \quad M_n = \max_{k \in [n]} Z_k, \quad \check{M}_n = \min_{k \in [n]} Z_k \quad \text{and} \quad \beta_n^* = \alpha_n^* - \frac{1}{2\alpha_n^*} \log(4\pi \log n).$$

Denote by  $G(x) = e^{-e^{-x}}$  the Gumbel cumulative distribution. Then the following results are standard, (i.e. [12, Theorem 1.5.3]): for all  $x \in \mathbb{R}$

$$(2.4) \quad \mathbb{P} \{ \alpha_n^* (M_n - \beta_n^*) \leq x \} \xrightarrow{n \rightarrow \infty} G(x), \quad \mathbb{P} \{ \alpha_n^* (\check{M}_n + \beta_n^*) \geq -x \} \xrightarrow{n \rightarrow \infty} G(x).$$

*Strategy of the proof.* Eq. (2.1) immediately yields

$$\begin{cases} \min_{k \in [n]} x_k & \geq 1 + \frac{1}{\alpha} \check{M} + \frac{1}{\alpha^2} \min_{k \in [n]} R_k, \\ \min_{k \in [n]} x_k & \leq 1 + \frac{1}{\alpha} \check{M} + \frac{1}{\alpha^2} \max_{k \in [n]} R_k. \end{cases}$$

We rewrite the first equation as

$$(2.5) \quad \min_{k \in [n]} x_k \geq 1 + \frac{\alpha_n^*}{\alpha_n} \left( \frac{\check{M} + \beta_n^*}{\alpha_n^*} - \frac{\beta_n^*}{\alpha_n^*} + \frac{\min_{k \in [n]} R_k}{\alpha_n^* \alpha_n} \right) \\ = 1 + \frac{\alpha_n^*}{\alpha_n} \left( -1 + o_P(1) + \frac{\min_{k \in [n]} R_k}{\alpha_n^* \alpha_n} \right),$$

where we have used the fact that  $(\alpha_n^*)^{-1}(\check{M} + \beta_n^*) = o_P(1)$ . Similarly,

$$\min_{k \in [n]} x_k \leq 1 + \frac{\alpha_n^*}{\alpha_n} \left( -1 + o_P(1) + \frac{\max_{k \in [n]} R_k}{\alpha_n^* \alpha_n} \right).$$

The theorem will then follow from the following lemma.

**Lemma 2.1.** *The following convergence holds*

$$\frac{\max_{k \in [n]} R_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad \text{and} \quad \frac{\min_{k \in [n]} R_k}{\alpha_n \sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Proof of Lemma 2.1 requires a careful analysis of the order of magnitude of the extreme values of the remaining term  $(R_k)_{k \in [n]}$ . It is postponed to Section 2.3.

**2.2. Lipschitz property and tightness of  $R_k(A)$ .** Let  $\eta \in (0, 1)$  and  $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$  be a smooth (infinitely differentiable) function with values

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [0, 2 + \eta] \\ 0 & \text{if } x \geq 3 \end{cases},$$

and strictly decreasing from 1 to zero as  $x$  goes from  $2 + \eta$  to 3. Notice in particular that  $\|\varphi'\|_\infty$  is finite. Recall that  $s_n = s(n^{-1/2} A)$  is the largest singular value of the normalized matrix  $n^{-1/2} A$  and denote by

$$\varphi_n := \varphi(s_n) = \varphi \left( s(n^{-1/2} A) \right).$$

Notice that  $\mathbb{P}\{\varphi_n < 1\} = \mathbb{P}\{s_n > 2 + \eta\} \xrightarrow{n \rightarrow \infty} 0$  (by the a.s. convergence of  $s_n$  to 2). Instead of working with  $R_k$  we introduce the truncated quantity:

$$(2.6) \quad \tilde{R}_k = \varphi_n R_k .$$

For a given  $n \times n$  matrix  $A$ , we may consider its  $2n \times 2n$  hermitized matrix  $\mathcal{H}(A)$  defined as  $\mathcal{H}(A) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ . Notice that the singular values of  $A$  together with their negatives are the eigenvalues of  $\mathcal{H}(A)$ .

**Lemma 2.2.** *Let  $\tilde{R}_k$  be given by (2.6), then the function  $A \mapsto \tilde{R}_k(A)$  is Lipschitz, i.e.*

$$(2.7) \quad \left| \tilde{R}_k(A) - \tilde{R}_k(B) \right| \leq K \|A - B\|_F ,$$

where  $\|A\|_F$  is the Frobenius norm and  $K$  is a constant independent from  $k$  and  $n$ .

*Proof.* Notice that  $\varphi(s_n) = 0$  and  $\varphi'(s_n) = 0$  for  $s_n \geq 3$ , which implies that one may consider the bound  $s_n \leq 3$  in the following computations, for  $\tilde{R}_k$  or its derivatives would be zero otherwise. Recall the definition of the resolvent  $Q = \left( I - \frac{A}{\alpha\sqrt{n}} \right)^{-1}$  then  $Q^{-1}Q = I$  which yields  $Q = I + \frac{A}{\alpha\sqrt{n}}Q$  from which we deduce that

$$(2.8) \quad \varphi_n \|Q\| \leq \varphi_n \left( 1 - \frac{1}{\alpha} \left\| n^{-\frac{1}{2}} A \right\| \right)^{-1} \leq \frac{1}{1 - 3\alpha^{-1}} \leq 3$$

for  $n$  large enough.

We first consider a matrix  $A$  such that  $\mathcal{H}(A)$  has simple spectrum (i.e. with  $2n$  distinct eigenvalues, each with multiplicity 1). We denote by  $\partial_{ij} = \frac{\partial}{\partial A_{ij}}$  and prove that the vector  $\nabla \tilde{R}_k(A) = \left( \partial_{ij} \tilde{R}_k(A), i, j \in [n] \right)$  satisfies

$$(2.9) \quad \|\nabla \tilde{R}_k(A)\| = \sqrt{\sum_{ij} \left| \partial_{ij} \tilde{R}_k(A) \right|^2} \leq K .$$

We may occasionally drop the dependence of  $\tilde{R}_k$  in  $A$ . We begin by computing

$$\begin{aligned} \partial_{ij} \tilde{R}_k &= \lim_{h \rightarrow 0} \frac{\tilde{R}_k(A + h \mathbf{e}_i \mathbf{e}_j^*) - \tilde{R}_k(A)}{h} , \\ &= (\partial_{ij} \varphi_n) R_k + \varphi_n \mathbf{e}_k^* \left( \partial_{ij} \left( n^{-\frac{1}{2}} A \right)^2 \right) Q \mathbf{1} + \varphi_n \mathbf{e}_k^* \left( n^{-\frac{1}{2}} A \right)^2 (\partial_{ij} Q) \mathbf{1} \\ &=: T_{1,ij} + T_{2,ij} + T_{3,ij} . \end{aligned}$$

Straightforward computations yield

$$(2.10) \quad \partial_{ij} \left( n^{-\frac{1}{2}} A \right)^2 = \frac{1}{n} (A \mathbf{e}_i \mathbf{e}_j^* + \mathbf{e}_i \mathbf{e}_j^* A) \quad \text{and} \quad \partial_{ij} Q = \frac{1}{\alpha\sqrt{n}} Q \mathbf{e}_i \mathbf{e}_j^* Q .$$

It remains to compute  $\partial_{ij} \varphi_n = \varphi'(s_n) \partial_{ij} s_n$ . Recall that  $\mathcal{H}(A)$  has a simple spectrum and notice that  $A \mapsto s_n(A)$  is differentiable. In fact, since  $s_n$  is simple, it is a simple root of the characteristic polynomial. In particular, it is not a root of its derivative and one can use the implicit function theorem to conclude its differentiability. Let  $\mathbf{u}$  and  $\mathbf{v}$  be respectively the left and right normalized singular vectors

associated to  $s(A)$ . Then

$$\mathcal{H}(A)\mathbf{w} = s(A)\mathbf{w} \quad \text{with} \quad \mathbf{w} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \quad \text{and} \quad \|\mathbf{w}\|^2 = 2,$$

moreover  $\mathbf{w}$  is (up to scaling) the unique eigenvector of  $s(A)$  since  $s(A)$  is simple by assumption. We now apply [10, Theorem 6.3.12] to compute  $s_n$ 's derivative:

(2.11)

$$\partial_{ij}s(A) = \frac{1}{\|\mathbf{w}\|^2} (\mathbf{u}^* \mathbf{e}_i \mathbf{e}_j^* \mathbf{v} + \mathbf{v}^* \mathbf{e}_j \mathbf{e}_i^* \mathbf{u}) = \mathbf{u}^* \mathbf{e}_i \mathbf{e}_j^* \mathbf{v} \quad \text{hence} \quad \partial_{ij}s_n = \frac{1}{\sqrt{n}} \mathbf{u}^* \mathbf{e}_i \mathbf{e}_j^* \mathbf{v}$$

(recall that all the considered vectors are real). We first handle the term  $T_{1,ij}$ .

$$\begin{aligned} \sum_{ij} |T_{1,ij}|^2 &= \sum_{ij} \left| \mathbf{u}^* \mathbf{e}_i \mathbf{e}_j^* \mathbf{v} \varphi'(s_n) \mathbf{e}_k^* \left( n^{-1/2} A \right)^2 Q \frac{\mathbf{1}}{\sqrt{n}} \right|^2, \\ &\leq 3^6 \|\varphi'\|_\infty^2 \sum_i |\mathbf{u}^* \mathbf{e}_i|^2 \sum_j |\mathbf{e}_j^* \mathbf{v}|^2 \leq 3^6 \|\varphi'\|_\infty^2. \end{aligned}$$

We now handle the term  $T_{2,ij}$ .

$$\begin{aligned} \sum_{ij} |T_{2,ij}|^2 &= \sum_{ij} \left| \varphi_n \mathbf{e}_k^* \left( \frac{A}{\sqrt{n}} \mathbf{e}_i \mathbf{e}_j^* + \mathbf{e}_i \mathbf{e}_j^* \frac{A}{\sqrt{n}} \right) Q \frac{\mathbf{1}}{\sqrt{n}} \right|^2 \\ &\leq 2\varphi_n^2 \sum_i \left| \mathbf{e}_k^* \frac{A}{\sqrt{n}} \mathbf{e}_i \right|^2 \sum_j \left| \mathbf{e}_j^* Q \frac{\mathbf{1}}{\sqrt{n}} \right|^2 + 2\varphi_n^2 \sum_i |\mathbf{e}_k^* \mathbf{e}_i|^2 \sum_j \left| \mathbf{e}_j^* \frac{A}{\sqrt{n}} Q \frac{\mathbf{1}}{\sqrt{n}} \right|^2, \\ &= 2\varphi_n^2 \left( \mathbf{e}_k^* \frac{A}{\sqrt{n}} \frac{A^*}{\sqrt{n}} \mathbf{e}_k \right) \left( \frac{\mathbf{1}^*}{\sqrt{n}} Q^* Q \frac{\mathbf{1}}{\sqrt{n}} \right) + 2\varphi_n^2 \left( \frac{\mathbf{1}^*}{\sqrt{n}} Q^* \frac{A^* A}{n} Q \frac{\mathbf{1}}{\sqrt{n}} \right) \leq 2^2 \times 3^4. \end{aligned}$$

The term  $T_{3,ij}$  can be handled similarly and one can prove  $\sum_{ij} |T_{3,ij}|^2 \leq 3^8$ . Gathering all these estimates, we finally obtain the desired bound:

$$\sqrt{\sum_{ij} \left| \partial_{ij} \tilde{R}_k \right|^2} \leq \sqrt{3 \sum_{ij} |T_{1,ij}|^2 + 3 \sum_{ij} |T_{2,ij}|^2 + 3 \sum_{ij} |T_{3,ij}|^2} \leq K,$$

where  $K$  neither depends on  $k$  nor on  $n$ .

Having proved a local estimate over  $\|\nabla \tilde{R}_k(A)\|$  for each matrix  $A$  such that  $\mathcal{H}(A)$  has simple spectrum, we now establish the Lipschitz estimate (2.7) for two such matrices  $A, B$ .

Let  $A, B$  such that  $\mathcal{H}(A)$  and  $\mathcal{H}(B)$  have simple spectrum and consider  $A_t = (1-t)A + tB$  for  $t \in [0, 1]$ . Notice first that the continuity of the eigenvalues implies that there exists  $\delta > 0$  sufficiently small such that  $\mathcal{H}(A_t)$  has a simple spectrum for  $t \leq \delta$  and  $t \geq 1 - \delta$ . To go beyond  $[0, \delta] \cup (1 - \delta, 1]$  and prove that  $\mathcal{H}(A_t)$  has simple spectrum for the entire interval  $[0, 1]$  except maybe for a finite number of points, we rely on the argument in Kato [11, Chapter 2.1] which states that apart from a finite number of  $t_\ell$ 's:  $t_0 = 0 < t_1 < \dots < t_L < t_{L+1} = 1$ , the number of eigenvalues of  $\mathcal{H}(A_t)$  remains constant for  $t \in [0, 1]$  and  $t \neq t_\ell, \ell \in [L]$ . Since  $\mathcal{H}(A_t)$  has simple spectrum for  $t \in [0, \delta] \cup (1 - \delta, 1]$ , it has simple spectrum for all  $t \notin \{t_\ell, \ell \in [L]\}$ .

We can now proceed:

$$\begin{aligned} \left| \tilde{R}_k(A_{t_1}) - \tilde{R}_k(A) \right| &= \left| \lim_{\tau \nearrow t_1} \int_0^\tau \frac{d}{dt} \tilde{R}_k(A_t) dt \right| = \left| \lim_{\tau \nearrow t_1} \int_0^\tau \nabla \tilde{R}_k(A_t) \circ \frac{d}{dt} A_t dt \right|, \\ &\leq \lim_{\tau \nearrow t_1} \int_0^\tau \|\nabla \tilde{R}_k(A_t)\| \times \|B - A\|_F dt \leq K t_1 \|B - A\|_F. \end{aligned}$$



By iterating this process, we obtain

$$\begin{aligned} \left| \tilde{R}_k(B) - \tilde{R}_k(A) \right| &\leq \sum_{\ell=1}^{L+1} \left| \tilde{R}_k(A_{t_\ell}) - \tilde{R}_k(A_{t_{\ell-1}}) \right| \\ &\leq \sum_{\ell=1}^{L+1} K(t_\ell - t_{\ell-1}) \|B - A\|_F = K \|B - A\|_F, \end{aligned}$$

hence the Lipschitz property along the segment  $[A, B]$ .

The general property follows by density of such matrices in the set of  $n \times n$  matrices and by continuity of  $A \mapsto \tilde{R}_k(A)$ . Let  $A, B$  be given and  $A_\varepsilon \rightarrow A$  and  $B_\varepsilon \rightarrow B$  be such that  $\mathcal{H}(A_\varepsilon)$  and  $\mathcal{H}(B_\varepsilon)$  have simple spectrum then:

$$\begin{aligned} \left| \tilde{R}_k(B) - \tilde{R}_k(A) \right| &\leq \left| \tilde{R}_k(B_\varepsilon) - \tilde{R}_k(B) \right| + K \|B_\varepsilon - B\| + \left| \tilde{R}_k(A_\varepsilon) - \tilde{R}_k(A) \right| \\ &\xrightarrow{\varepsilon \rightarrow 0} K \|B - A\|_F. \end{aligned}$$

Proof of Lemma 2.2 is completed.  $\square$

We now use concentration arguments to bound  $\mathbb{E} \max_{k \in [n]} (\tilde{R}_k - \mathbb{E} \tilde{R}_k)$ .

**Proposition 2.3.** *Let  $K$  be as in Lemma 2.2, then  $\mathbb{E} \max_{k \in [n]} (\tilde{R}_k - \mathbb{E} \tilde{R}_k) \leq K \sqrt{2 \log n}$ .*

*Proof.* By applying Tsirelson-Ibragimov-Sudakov inequality [4, Theorem 5.5] to  $\tilde{R}_k(A)$ , we obtain the following exponential estimate:

$$\mathbb{E} e^{\lambda(\tilde{R}_k(A) - \mathbb{E} \tilde{R}_k(A))} \leq e^{\frac{\lambda^2 K^2}{2}} \quad \forall \lambda \in \mathbb{R}.$$

We now estimate the expectation of the maximum (we drop the dependence in  $A$ ).

$$\begin{aligned} \exp \left( \lambda \mathbb{E} \max_{k \in [n]} (\tilde{R}_k - \mathbb{E} \tilde{R}_k) \right) &\leq \mathbb{E} \exp \left( \lambda \max_{k \in [n]} (\tilde{R}_k - \mathbb{E} \tilde{R}_k) \right) \\ &\leq \sum_{k=1}^n \mathbb{E} e^{\lambda(\tilde{R}_k - \mathbb{E} \tilde{R}_k)} \leq n e^{\frac{\lambda^2 K^2}{2}}. \end{aligned}$$

Hence for  $\lambda > 0$ ,

$$\mathbb{E} \max_{k \in [n]} (\tilde{R}_k - \mathbb{E} \tilde{R}_k) \leq \frac{\log n}{\lambda} + \frac{\lambda K^2}{2} =: \Phi(\lambda).$$

Optimizing in  $\lambda$ , we get  $\lambda^* = \frac{\sqrt{2 \log n}}{K}$  and  $\Phi(\lambda^*) = K \sqrt{2 \log n}$ , the desired estimate.  $\square$

**Proposition 2.4.** *We have  $\mathbb{E} \tilde{R}_k(A_n) = \mathcal{O}(1)$  uniformly in  $k \in [n]$ .*

*Proof.* By exchangeability, we have  $\mathbb{E} \tilde{R}_k = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{R}_i$  and

$$\left| \frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{R}_i \right| = \left| \frac{1}{n} \mathbb{E} \varphi_n \mathbf{1}^* \left( \frac{A}{\sqrt{n}} \right)^2 Q \mathbf{1} \right| \leq \left\| \frac{\mathbf{1}}{\sqrt{n}} \right\|^2 \mathbb{E} \left( \varphi_n \left\| \frac{A}{\sqrt{n}} \right\|^2 \|Q\| \right) = \mathcal{O}(1).$$

by (2.8). Proof of Proposition 2.4 is completed.  $\square$

We are now in position to prove Lemma 2.1.

**2.3. Proof of Lemma 2.1.** Since the  $\tilde{R}_i(A)$ 's are exchangeable,  $\mathbb{E}\tilde{R}_k(A) = \mathbb{E}\tilde{R}_1(A)$ . Notice that  $\max_{k \in [n]} \tilde{R}_k(A) - \tilde{R}_1(A)$  is nonnegative hence by Markov inequality,

$$\begin{aligned} \mathbb{P} \left\{ \frac{\max_{k \in [n]} \tilde{R}_k(A) - \tilde{R}_1(A)}{\alpha\sqrt{2 \log n}} \geq \varepsilon \right\} &\leq \frac{\mathbb{E} \left( \max_{k \in [n]} \tilde{R}_k(A) - \tilde{R}_1(A) \right)}{\varepsilon\alpha\sqrt{2 \log n}} \\ &= \frac{\mathbb{E} \left( \max_{k \in [n]} \left( \tilde{R}_k(A) - \mathbb{E}\tilde{R}_k(A) + \mathbb{E}\tilde{R}_1(A) \right) - \tilde{R}_1(A) \right)}{\varepsilon\alpha\sqrt{2 \log n}}, \\ &= \frac{\mathbb{E} \left( \max_{k \in [n]} \left( \tilde{R}_k(A) - \mathbb{E}\tilde{R}_k(A) \right) \right)}{\varepsilon\alpha\sqrt{2 \log n}} \leq \frac{K}{\varepsilon\alpha} \end{aligned}$$

by Proposition 2.3. This implies that

$$(2.12) \quad \frac{\max_{k \in [n]} \tilde{R}_k(A) - \tilde{R}_1(A)}{\alpha\sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

We now prove that

$$(2.13) \quad \tilde{R}_1(A) / \left( \alpha\sqrt{2 \log n} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

By Proposition 2.4,  $\mathbb{E}\tilde{R}_1(A) = \mathcal{O}(1)$  hence  $\mathbb{E}\tilde{R}_1(A) / (\alpha\sqrt{2 \log(n)}) \rightarrow 0$ . Applying Poincaré's inequality (cf. [4, Theorem 3.20] and its extension to Lipschitz functionals on p. 73) to the Lipschitz functional  $A \mapsto \tilde{R}_1(A)$  (cf. Lemma 2.2), we can bound  $\tilde{R}_1(A)$ 's variance by  $K^2$  and obtain

$$\mathbb{P} \left( \left| \frac{\tilde{R}_1(A) - \mathbb{E}\tilde{R}_1(A)}{\alpha\sqrt{2 \log n}} \right| > \delta \right) \leq \frac{\text{var}(\tilde{R}_1(A))}{2\delta^2\alpha^2 \log n} \leq \frac{K^2}{2\delta^2\alpha^2 \log n} \xrightarrow[n \rightarrow \infty]{} 0.$$

This and Proposition 2.4 yield (2.13). Combining (2.12) and (2.13) finally yields:

$$\frac{\max_{k \in [n]} \tilde{R}_k(A)}{\alpha\sqrt{2 \log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

In order to obtain the result for the untilded quantities, we write

$$\begin{aligned} \mathbb{P} \left\{ \left| \frac{\max_k R_k(A)}{\alpha\sqrt{2 \log n}} \right| > \varepsilon \right\} &\leq \mathbb{P} \left\{ \max_k R_k(A) \neq \max_k \tilde{R}_k(A) \right\} + \mathbb{P} \left\{ \left| \frac{\max_k \tilde{R}_k(A)}{\alpha\sqrt{2 \log n}} \right| > \frac{\varepsilon}{2} \right\}, \\ &= \mathbb{P}\{\varphi_n < 1\} + \mathbb{P} \left\{ \left| \frac{\max_k \tilde{R}_k(A)}{\alpha\sqrt{2 \log n}} \right| > \varepsilon/2 \right\} \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

One proves the second assertion similarly. This concludes the proof of Lemma 2.1.

### 3. STABILITY: PROOF OF COROLLARY 1.4

In order to study the stability of large Lotka-Volterra systems, we are led to study the matrix

$$\mathcal{J}(\mathbf{x}_n) = \text{diag}(\mathbf{x}_n) \left( -I_n + \frac{A_n}{\alpha_n\sqrt{n}} \right).$$

We first establish the following estimates

$$(3.1) \quad \begin{cases} \min_{k \in [n]} x_k \geq 1 - \ell^+ - o_P(1), \\ \max_{k \in [n]} x_k \leq 1 + \ell^+ + o_P(1). \end{cases}$$

The first estimate immediately follows from (2.5) together with Lemma 2.1. From  $x_k$ 's decomposition (2.1) we have

$$\begin{aligned} \max_{k \in [n]} x_k &\leq 1 + \frac{M_n}{\alpha_n} + \frac{\max_{k \in [n]} R_k}{\alpha_n^2} = 1 + \frac{\alpha_n^*}{\alpha_n} \left( \frac{M_n - \beta_n^*}{\alpha_n^*} + \frac{\beta_n^*}{\alpha_n^*} + \frac{\max_{k \in [n]} R_k}{\alpha_n^* \alpha_n} \right) \\ &\leq 1 + \ell^+ + o_P(1), \end{aligned}$$

where the last inequality follows from Lemma 2.1 and the fact that  $(\alpha_n^*)^{-1} (M_n - \beta_n^*) \xrightarrow{P} 0$ .

We now compare the spectra of matrices  $\mathcal{D}(\mathbf{x}_n) = -\text{diag}(\mathbf{x}_n)$  and  $\mathcal{J}(\mathbf{x}_n)$  by relying on Bauer and Fike's theorem [10, Theorem 6.3.2]: for every  $\lambda \in \mathcal{S}_n$ , there exists a component  $x_k$  of vector  $\mathbf{x}_n$  such that

$$\begin{aligned} |\lambda + x_k| &\leq \left\| \text{diag}(\mathbf{x}_n) \frac{A_n}{\alpha_n \sqrt{n}} \right\| \leq \frac{1}{\alpha_n} \|\text{diag}(\mathbf{x}_n)\| \left\| \frac{A_n}{\sqrt{n}} \right\| \\ &\stackrel{(a)}{\leq} \frac{1}{\alpha_n} (1 + \ell^+ + o_P(1)) (2 + o_P(1)) = o_P(1). \end{aligned}$$

where (a) follows from the second estimate in (3.1) and from the spectral norm estimate. Notice that the majorization above is uniform for  $\lambda \in \mathcal{S}_n$ . The first part of the corollary is proved. Finally,

$$\text{Re}(\lambda) + x_k \leq |\lambda + x_k| = o_P(1) \quad \Rightarrow \quad \text{Re}(\lambda) \leq -\min_{k \in [n]} x_k + o_P(1).$$

The estimate (1.5) finally follows from the first estimate in (3.1).

#### 4. HEURISTICS AT CRITICAL SCALING, NON-HOMOGENEOUS SYSTEMS AND NON-GAUSSIAN ENTRIES

**4.1. A heuristics at the critical scaling.** We provide here a heuristics to compute the probability that a solution  $\mathbf{x}_n$  is feasible at critical scaling  $\alpha_n^* = \sqrt{2 \log n}$ .

**Heuristics 4.1.** *The probability that a solution is feasible at the critical scaling  $\alpha_n^*$  is asymptotically given by*

$$(4.1) \quad \mathbb{P}(x_k > 0, k \in [n]) \approx 1 - \sqrt{\frac{e}{4\pi \log n}} + \frac{e}{8\pi \log n} =: H_1(n).$$

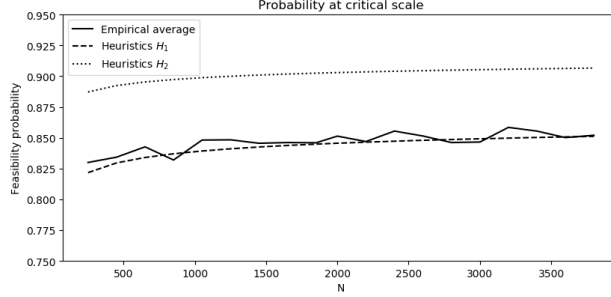
In Figure 2, we compare the heuristics with results from simulations.

*Arguments.* Consider

$$x_k = 1 + \mathbf{e}_k^* \frac{A_n}{\alpha_n^* \sqrt{n}} \mathbf{1}_n + \frac{R_k}{(\alpha_n^*)^2} = 1 + \frac{Z_k}{\alpha_n^*} + \frac{R_k}{(\alpha_n^*)^2} = 1 + \frac{1}{\alpha_n^*} \left( Z_k + \frac{R_k}{\alpha_n^*} \right).$$

Following Geman and Hwang [8, Lemma A.1], one could prove that  $Z_k$  and  $R_k$  are asymptotically independent centered Gaussian random variables, each with variance one. We thus approximate the quantity  $Z_k + \frac{R_k}{\alpha_n^*}$  by a Gaussian random variable with distribution  $\mathcal{N}\left(0, 1 + \frac{1}{(\alpha_n^*)^2}\right)$  and set  $x_k \approx 1 + \left(\frac{1}{\alpha_n^*} \sqrt{1 + \frac{1}{(\alpha_n^*)^2}}\right) U_k$  where the  $U_k$ 's are i.i.d.  $\mathcal{N}(0, 1)$ . Denote by  $\tilde{M}_n^U = \min_{k \in [n]} U_k$  then

$$\mathbb{P}(x_k > 0, k \in [n]) \approx \mathbb{P}\left(1 + \left(\frac{1}{\alpha_n^*} \sqrt{1 + \frac{1}{(\alpha_n^*)^2}}\right) \tilde{M}_n^U > 0\right).$$



**Figure 2.** Probability at critical scaling. The solid curve corresponds to the proportion of feasible solutions at critical scaling  $\alpha_n^*$  obtained for 10000 simulations (for  $n$  ranging from 50 to 3750 with a 200-increment) - notice the strong standard deviation. The dashed curve represents the heuristics  $H_1$  defined in (4.1). The dotted curve represents the heuristics  $H_2$  introduced in Remark 4.1. Notice the substantial discrepancy between  $H_1$  and  $H_2$ .

Recall that standard Gaussian extreme value convergence results yield

$$(4.2) \quad \mathbb{P} \{ \alpha_n^* (-\check{M}_n^U - \beta_n^*) < x \} = \mathbb{P} \{ \alpha_n^* (\check{M}_n^U + \beta_n^*) > -x \} \xrightarrow{n \rightarrow \infty} G(x) = e^{-e^{-x}},$$

where  $\beta_n^*$  is defined in (2.3). Denote by  $\Theta(\alpha) = \sqrt{1 + \alpha^{-2}}$  then

$$\mathbb{P} \left( 1 + \Theta(\alpha_n^*) \frac{\check{M}_n^U}{\alpha_n^*} > 0 \right) = \mathbb{P} \left( \alpha_n^* (\check{M}_n^U + \beta_n^*) > -\frac{(\alpha_n^*)^2}{\Theta(\alpha_n^*)} + \alpha_n^* \beta_n^* \right).$$

Notice that

$$-\frac{(\alpha_n^*)^2}{\Theta(\alpha_n^*)} + \alpha_n^* \beta_n^* = \frac{1}{2} - \frac{1}{2} \log(4\pi \log n) + \mathcal{O} \left( \frac{1}{(\alpha_n^*)^2} \right) = \frac{1}{2} + \log \frac{1}{\sqrt{2\pi} \alpha_n^*} + \mathcal{O} \left( \frac{1}{(\alpha_n^*)^2} \right).$$

Hence

$$(4.3) \quad \begin{aligned} \mathbb{P} \left( 1 + \Theta(\alpha_n^*) \frac{\check{M}_n^U}{\alpha_n^*} > 0 \right) &= \mathbb{P} \left( \alpha_n^* (\check{M}_n^U + \beta_n^*) > \frac{1}{2} + \log \frac{1}{\sqrt{2\pi} \alpha_n^*} + \mathcal{O} \left( \frac{1}{(\alpha_n^*)^2} \right) \right), \\ &\stackrel{(a)}{\approx} e^{-\exp \left( \frac{1}{2} + \log \frac{1}{\sqrt{2\pi} \alpha_n^*} + \mathcal{O} \left( \frac{1}{(\alpha_n^*)^2} \right) \right)} = e^{-\sqrt{\frac{e}{2\pi}} \frac{1}{\alpha_n^*} (1 + \mathcal{O}((\alpha_n^*)^{-2}))}, \\ &= 1 - \sqrt{\frac{e}{2\pi}} \frac{1}{\alpha_n^*} + \frac{1}{2} \frac{e}{2\pi} \frac{1}{(\alpha_n^*)^2} + \mathcal{O} \left( \frac{1}{(\alpha_n^*)^3} \right). \end{aligned}$$

We finally end up with the announced approximation

$$\mathbb{P}(x_k > 0, k \in [n]) \approx H_1(n) := 1 - \sqrt{\frac{e}{4\pi \log n}} + \frac{e}{8\pi \log n}.$$

*Remark 4.1.* A rougher approximation would have been to set  $x_k \approx 1 + \frac{Z_k}{\alpha_n^*}$  with  $Z_k \sim \mathcal{N}(0, 1)$  and to drop the next term  $\frac{R_k}{(\alpha_n^*)^2}$  in the heuristics but this would have resulted in the following approximation

$$\mathbb{P}(x_k > 0, k \in [n]) \approx 1 - (4\pi \log(n))^{-1/2} + (8\pi \log(n))^{-1} =: H_2(n),$$

which is worse than  $H_1(n)$ , as illustrated in Figure 2.

□

**4.2. Positivity for a non-homogeneous linear system.** By homogeneous, we refer to a LV system where the intrinsic growth rate of species  $i$  is equal to 1. If not, the system is non-homogeneous (NH). The results developed so far extend to a NH linear system where  $\mathbf{1}_n$  is replaced by a vector  $\mathbf{r}_n$  with slight modifications. In particular, we identify a regime where feasibility and stability occur simultaneously. Denote by  $\mathbf{r}_n = (r_k)$  a  $n \times 1$  deterministic vector with positive components and consider the linear system

$$(4.4) \quad \mathbf{x}_n = \mathbf{r}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \mathbf{x}_n.$$

Introduce the notations

$$r_{\min}(n) = \min_{k \in [n]} r_k, \quad r_{\max}(n) = \max_{k \in [n]} r_k \quad \text{and} \quad \sigma_{\mathbf{r}}(n) = \|\mathbf{r}/\sqrt{n}\| = \sqrt{n^{-1} \sum_{k \in [n]} r_k^2}.$$

Assume that there exist  $\rho_{\min}, \rho_{\max}$  independent from  $n$  such that eventually

$$0 < \rho_{\min} \leq r_{\min}(n) \leq \sigma_{\mathbf{r}}(n) \leq r_{\max}(n) \leq \rho_{\max} < \infty.$$

Then

**Theorem 4.2** (Feasibility - NH case). *Let  $\alpha_n \xrightarrow[n \rightarrow \infty]{} \infty$  and denote by  $\alpha_n^* = \sqrt{2 \log n}$ . Let  $\mathbf{x}_n = (x_k)_{k \in [n]}$  be the solution of (4.4).*

- (1) *If  $\exists \varepsilon > 0$  such that  $\alpha_n \leq (1 - \varepsilon) \frac{\alpha_n^* \sigma_{\mathbf{r}}(n)}{r_{\max}(n)}$  then  $\mathbb{P} \{ \min_{k \in [n]} x_k > 0 \} \xrightarrow[n \rightarrow \infty]{} 0$ .*
- (2) *If  $\exists \varepsilon > 0$  such that  $\alpha_n \geq (1 + \varepsilon) \frac{\alpha_n^* \sigma_{\mathbf{r}}(n)}{r_{\min}(n)}$  then  $\mathbb{P} \{ \min_{k \in [n]} x_k > 0 \} \xrightarrow[n \rightarrow \infty]{} 1$ .*

*Remark 4.3.* Contrary to the homogeneous system where there is a sharp transition at  $\alpha_n^* = \sqrt{2 \log(n)}$ , the situation is not as clean-cut here and there is a buffer zone

$$\alpha_n \in \left[ \frac{\sigma_{\mathbf{r}}(n)}{r_{\max}(n)} \sqrt{2 \log(n)}, \frac{\sigma_{\mathbf{r}}(n)}{r_{\min}(n)} \sqrt{2 \log(n)} \right]$$

in which the study of the feasibility is not clear.

This buffer zone is illustrated in Figure 3 where we simulate the transition toward feasibility for a non-homogeneous system (4.4) in the case where deterministic vector  $\mathbf{r}_n$  is equally distributed over  $[1, 3]$ , i.e.

$$(4.5) \quad r_n(i) = 1 + \frac{2i}{n}, \quad \sigma_{\mathbf{r}}(n) = \sqrt{\frac{1}{n} \sum_{i \in [n]} r_n^2(i)} \xrightarrow[n \rightarrow \infty]{} \sqrt{\int_0^1 (1 + 2x)^2 dx}$$

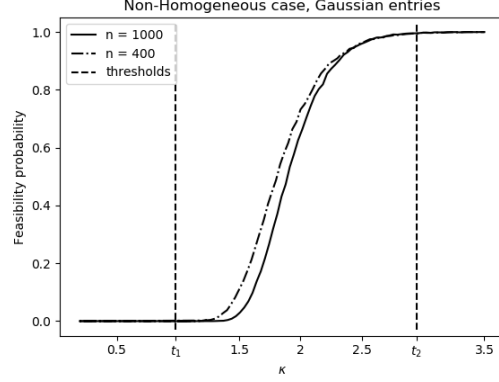
We introduce the quantities

$$(4.6) \quad t_1 = \lim_N \frac{\sqrt{2} \sigma_{\mathbf{r}}(N)}{r_{\max}} \simeq 0.98 \quad \text{and} \quad t_2 = \lim_N \frac{\sqrt{2} \sigma_{\mathbf{r}}(N)}{r_{\min}} \simeq 2.94.$$

As one may notice, the transition region is wider than in the homogeneous case.

*Elements of proof.* We have

$$x_k = \mathbf{e}_k^* Q \mathbf{r}_n = r_k + \frac{1}{\alpha} \frac{\sum_{i=1}^n r_i A_{ki}}{\sqrt{n}} + \frac{1}{\alpha^2} \mathbf{e}_k^* \left( \frac{A}{\sqrt{n}} \right) Q \mathbf{r}_n = r_k + \frac{\sigma_{\mathbf{r}}(n)}{\alpha} U_k + \frac{1}{\alpha^2} R_k^{(\mathbf{r})}$$



**Figure 3.** Transition toward feasibility for a NH system. The curves are obtained as for Figure 1 for  $\mathbf{r}_N$  defined in (4.5). The thresholds  $t_1$  and  $t_2$  are computed in (4.6)..

where the  $U_k$ 's are i.i.d.  $\mathcal{N}(0,1)$ . One can check by carefully reading the proof of Lemma 2.1 that the conclusions of the lemma apply to  $R_k^{(r)}$ . In particular, one may check that Proposition 2.4 holds uniformly in  $k \in [n]$  in the non-homogeneous case. Denote by  $\check{M} = \min_{k \in [n]} U_k$ , then

$$\begin{aligned} \min_{k \in [n]} x_k &\leq r_{\max}(n) + \frac{\sigma_{\mathbf{r}}(n)}{\alpha} \check{M} + \frac{\max_{k \in [n]} R_k^{(r)}}{\alpha^2}, \\ &\leq r_{\max}(n) + \frac{\sigma_{\mathbf{r}}(n) \alpha^*}{\alpha} \left( \frac{\check{M} + \beta^*}{\alpha^*} - \frac{\beta^*}{\alpha^*} + \frac{\max_{k \in [n]} R_k^{(r)}}{\sigma_{\mathbf{r}}(n) \alpha^* \alpha} \right), \\ &= r_{\max}(n) + \frac{\sigma_{\mathbf{r}}(n) \alpha^*}{\alpha} (-1 + o_P(1)). \end{aligned}$$

The first statement of the theorem follows. The second statement follows similarly, noticing that  $\min_{k \in [n]} x_k \geq r_{\min}(n) + \alpha^{-1} \sigma_{\mathbf{r}}(n) \check{M} + \alpha^{-2} \min_{k \in [n]} R_k^{(r)}$ . Proof of Theorem 4.2 is completed.  $\square$

A non homogeneous system (4.4) is associated to the following Lotka-Volterra system

$$\frac{dx_k(t)}{dt} = x_k(t) \left( r_k - x_k(t) + \frac{1}{\alpha_n \sqrt{n}} \sum_{\ell \in [n]} A_{k\ell} x_{\ell}(t) \right)$$

for  $k \in [n]$  whose jacobian at equilibrium is still given by (1.4).

**Theorem 4.4** (Stability - NH case). *Let  $\mathbf{x}_n = (x_k)_{k \in [n]}$  be the solution of (4.4) and assume that*

$$\ell^+ := \limsup_{n \rightarrow \infty} \frac{\alpha_n^* \sigma_{\mathbf{r}}(n)}{\alpha_n r_{\min}(n)} < 1.$$

*Denote by  $\mathcal{S}_n$  the spectrum of  $\mathcal{J}(\mathbf{x}_n)$ . Then for every  $\lambda \in \mathcal{S}_n$ ,*

$$\max_{\lambda \in \mathcal{S}_n} \min_{k \in [n]} |\lambda + x_k| \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad \text{and} \quad \max_{\lambda \in \mathcal{S}_n} \operatorname{Re} \lambda \leq -(1 - \ell^+) + o_P(1).$$

**4.3. Beyond the Gaussian case.** In this section, we extend the result to the class of random variables satisfying a Logarithmic Sobolev Inequality. We first recall standard facts that can be found in [13].

A random variable  $X$  on  $\mathbb{R}^n$  satisfies a Logarithmic Sobolev Inequality with constant  $\rho$  (denoted by  $X \in LSI(\rho)$ ) if for every function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  regular enough (for instance  $g^2$  is of finite entropy [13, Section 5.1] and  $g$  locally Lipschitz<sup>1</sup>):

$$\text{Ent}(g^2(X)) := \mathbb{E}(g^2(X) \log g^2(X)) - \mathbb{E}g^2(X) \log \mathbb{E}g^2(X) \leq \frac{2}{\rho} \mathbb{E}[\|\nabla g(X)\|^2].$$

We say that  $X$  satisfies a Poincaré inequality with constant  $c$  (denoted by  $X \in \text{Poinc}(c)$ ) if for every  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  locally Lipschitz:

$$\text{Var}(f(X)) \leq c \mathbb{E}[\|\nabla f(X)\|^2].$$

A well known fact is that Log-Sobolev inequalities are stronger than Poincaré's :

**Proposition 4.5.** *If  $X$  satisfies  $LSI(\rho)$  then it satisfies  $\text{Poinc}(\rho^{-1})$ .*

Another standard result is that  $LSI(\rho)$  implies Gaussian type concentration for Lipschitz functions:

**Theorem 4.6.** *If  $X \in LSI(\rho)$ , then for every function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$   $L$ -Lipschitz:*

$$\mathbb{E} \left[ e^{\lambda(f(X) - \mathbb{E}f(X))} \right] \leq e^{\lambda^2 L^2 / 2\rho}.$$

Important examples of random variables satisfying  $LSI(\rho)$  are strongly log-concave variables and couples of independent variables satisfying  $LSI$ :

**Proposition 4.7** ([13, Theorem 5.2]). *If  $X$  in  $\mathbb{R}^n$  is strongly log-concave, that is has a density  $e^{-V(x)} dx$  where  $V : \mathbb{R}^n \mapsto \mathbb{R}$  is  $C^2$  with  $\text{Hess}(V) \geq \rho I_n$  ( $\rho > 0$ ), then  $X$  satisfies  $LSI(\rho)$ .*

**Proposition 4.8** ([13, Corollary 5.7]). *If  $X \in LSI(\rho_X)$  and  $Y \in LSI(\rho_Y)$  are independent, then  $Z = (X, Y)$  satisfies  $LSI(\rho_Z)$ , where  $\rho_Z = \min(\rho_X, \rho_Y)$ .*

We finally mention that the uniform measure on an interval also satisfies  $LSI(\rho)$  for some  $\rho$  depending on the interval. Indeed it is a Lipschitz push-forward of the Gaussian distribution. In fact, let  $Z \sim \mathcal{N}(0, 1)$  and  $\Phi : \mathbb{R} \rightarrow (0, 1)$  its cumulative distribution function, then  $\Phi$  is  $\frac{1}{\sqrt{2\pi}}$ -Lipschitz and  $U = \Phi(Z) \sim \mathcal{U}(0, 1)$ , the uniform distribution satisfies  $LSI(\frac{1}{2\pi})$  by direct computations. In our case, the random variable  $A = (A_{ij})_{i,j \in [n]}$  is a random vector of  $\mathbb{R}^{n^2}$  whose entries are i.i.d. By virtue of Proposition 4.8, vector  $A$  satisfies  $LSI(\rho)$  as long as the r.v.  $A_{11}$  does.

We now go back to the extension of Theorem 1.1 for non-Gaussian entries. The Gaussianity of the entries is used at three crucial steps, and in each case, a milder  $LSI(\rho)$  assumption, for some  $\rho > 0$  is enough :

- (1) Gaussian entries immediately imply that the  $Z_k$ 's are independent standard Gaussian random variables, for which the study of the extrema is standard.

In the case where the entries are not Gaussian, the  $Z_k$ 's are no longer Gaussian but this issue can easily be circumvented since by the CLT the  $Z_k$ 's converge in distribution to a standard Gaussian. The extreme value study of such families of

<sup>1</sup>A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz if for all  $x \in \mathbb{R}^n$ , there exist  $\xi_x, L_x > 0$  such that  $y \in B(x, \xi_x) \Rightarrow |g(x) - g(y)| \leq L_x \|x - y\|$ .

$Z_k$ 's has been carried out in [2, Propositions 2 & 3]. In our case, it is easy to check that the  $LSI(\rho)$  condition ensures that we can apply [2, Proposition 3].

- (2) Gaussian concentration has been used to prove sub-Gaussianness of  $\tilde{R}_k(A)$ . The value of the constant being irrelevant to us, Theorem 4.6 yields the same conclusion.
- (3) In the proof of Lemma 2.1 the Gaussian Poincaré inequality is used to prove that  $\tilde{R}_1(A)/(\alpha\sqrt{2\log(n)})$  goes to zero in probability. A Poincaré inequality remains available by Proposition 4.5 for  $LSI(\rho)$  r.v.

Hence we can extend Theorem 1.1 as :

**Theorem 4.9.** *Assume that the entries  $A_{ij}$  are i.i.d. centered, with finite variance equal to 1 and satisfy  $LSI(\rho)$  for some  $\rho > 0$ , then the conclusions of Theorem 1.1 hold.*

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INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UMR 7586, SORBONNE UNIVERSITÉS, 4, PLACE JUSSIEU,, 75005 PARIS

*E-mail address:* pierre.bizeul@imj-prg.fr

LABORATOIRE D'INFORMATIQUE GASPARD MONGE, UMR 8049, CNRS & UNIVERSITÉ PARIS EST MARNE-LA-VALLÉE, 5, BOULEVARD DESCARTES,, CHAMPS SUR MARNE, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

*E-mail address:* najim@univ-mlv.fr