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Necessary conditions for tiling finitely generated amenable groups

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Abstract

We consider a set of necessary conditions which are efficient heuristics for deciding when a set of Wang tiles cannot tile a group.

Piantadosi [?] gave a necessary and sufficient condition for the existence of a valid tiling of any free group. This condition is actually necessary for the existence of a valid tiling for an arbitrary finitely generated group.

We then consider two other conditions: the first, also given by Piantadosi [?], is a necessary and sufficient condition to decide if a set of Wang tiles gives a strongly periodic tiling of the free group; the second, given by Chazottes et. al. [?], is a necessary condition to decide if a set of Wang tiles gives a tiling of \mathbb{Z}^2 .

We show that these last two conditions are equivalent. Joining and generalising approaches from both sides, we prove that they are necessary for having a valid tiling of any finitely generated amenable group, confirming a remark of Jeandel [?].

MCS classification : Primary 37B50; Secondary 37B10, 05B45.

Keywords : Symbolic dynamics, tilings, groups, periodicity, amenability, domino problem

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1 Introduction

\mathbb{Z}^2 -subshifts of finite type (SFT) are a set of colorings of the 2-dimensional lattice \mathbb{Z}^2 , or *tilings*, defined by a finite set of local restrictions. There are various equivalent ways to express the restrictions, such as the Wang tiles formalism introduced by Hao Wang [?]. This formalism was introduced to study the *domino problem*: given as input a set of restrictions (e.g. a set of Wang tiles), is there an algorithm that decides whether there is a tiling of \mathbb{Z}^2 that respects those restrictions?

R. Berger [?] showed that the domino problem is undecidable. The proof depends heavily on notions of periodicity and aperiodicity, more precisely on the existence of a set of Wang tiles that only tile \mathbb{Z}^2 in a strongly aperiodic manner. This is in stark contrast with the situation on \mathbb{Z} where the domino problem is decidable thanks to a graph representation [?].

There has been a recent interest in symbolic dynamics on more general contexts, such as where the lattice \mathbb{Z}^2 is replaced by the Cayley graph of an infinite, finitely generated group. Using again the existence of strongly aperiodic SFTs, the domino problem was shown to be undecidable, apart from \mathbb{Z}^d , on some semisimple Lie groups [?], the Baumslag-Solitar groups [?], the discrete Heisenberg group (announced, [?]), surface groups [?, ?], semidirect products on \mathbb{Z}^2 [?] or some direct products [?], polycyclic groups [?], some hyperbolic groups [?]. . . It is decidable on free groups [?] and conjectured to be decidable on every virtually free groups, which is proved for virtually nilpotent groups [?].

As a consequence, outside of free and virtually free groups, one can not expect to find simple necessary and sufficient conditions for admitting a valid tiling. However, heuristics can be very useful when making an exhaustive search for SFTs with desired properties; necessary conditions in particular allow fast rejection of most empty SFTs. For example, a transducer-based heuristic was used in the search for the smallest set of Wang tiles that yield a strongly aperiodic \mathbb{Z}^2 -SFT [?]. It is also of theoretical interest to understand how the group properties impact necessary conditions.

1.1 Statements of results

We first consider a necessary and sufficient condition introduced by Piantadosi for an SFT on the free group to admit a valid tiling [?]. It is well-known that an SFT on a finitely generated group can only admit a tiling if the “corresponding” SFT on the free group does, so this becomes a necessary condition on an arbitrary f.g. group (Corollary 4.3).

The next two stronger conditions were introduced by Piantadosi (to decide if an SFT admits a strongly periodic tiling of the free group) and by Chazottes-Gambaudo-Gautero [?] in a more general context of tiling the plane by polygons, but which is necessary for an SFT to admit a tiling of \mathbb{Z}^2 [?]. We prove that the two conditions are equivalent (Theorem 3.10), and that they form a necessary condition for an SFT to admit a valid tiling on any amenable group (Theorem 5.3), confirming a remark of Jeandel [?].

Finally, we provide for any non-free finitely generated group a counterexample that satisfies all conditions but does not provide a valid tiling.

2 Preliminaries

2.1 Symbolic dynamics on groups

In the whole article G is an infinite, finitely generated group with unit element 1_G . We write $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ where $\mathcal{S} = \{g_1, \dots, g_d\}$ is a finite set of generators and $\mathcal{R} = \{r_1, \dots, r_m\} \subset (S \cup S^{-1})^*$ is a finite set of relations. By convention $r \in \mathcal{R}$ means that $r = 1_G$.

For instance:

- the free group \mathbb{F}_d is the group on d generators with no relations;
- $\mathbb{Z}^2 = \langle \{g_1, g_2\} \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$.

Let \mathcal{A} be a finite set endowed with the discrete topology; denote its cardinality $\#\mathcal{A}$. Let $\mathcal{A}^G = \{(x_g)_{g \in G} \mid \forall g \in G : x_g \in \mathcal{A}\}$ be the set of all functions from G to \mathcal{A} endowed with the product topology. Given a finite subset $F \subset G$, an element $P \in \mathcal{A}^F$ is called a *pattern* and $F = \text{supp}(P)$ its *support*; the set of all patterns is denoted \mathcal{A}^* . We say that a pattern P *appears* in a configuration $x \in \mathcal{A}^G$ (and we write $P \sqsubset x$) if there exists $g \in G$ such that $x|_{gF} = P$.

\mathcal{A}^G is a compact space called the G -full shift. It is a symbolic dynamical system under the following G -action, called the G -shift:

$$\forall x \in \mathcal{A}^G, \forall h \in G, (\sigma_h(x_g))_{g \in G} = (x_{h^{-1}g})_{g \in G}$$

We call G -subshift a closed shift-invariant subset $Y \subset \mathcal{A}^G$. Given a set of forbidden patterns $\mathcal{F} \subset \mathcal{A}^*$, we can define the corresponding G -subshift:

$$Y = Y_{\mathcal{F}} = \{x \in \mathcal{A}^G \mid \forall P \sqsubset x : P \notin \mathcal{F}\}.$$

Every G -subshift can be defined in this way using a set of forbidden patterns. When a subshift can be defined by a *finite* set of forbidden patterns, we say it is a G -subshift of finite type (G -SFT). If furthermore the set of forbidden patterns can be chosen so that every pattern in \mathcal{F} has support of the form $\{1_G, g_i\}$ where $g_i \in \mathcal{S}$ for some set of generators \mathcal{S} , we say it is a G -nearest-neighbor subshift of finite type (G -NNSFT). Notice that this definition depends on the choice of \mathcal{S} which is usually clear in the context.

For example, If we consider $G = \mathbb{Z}$ with generator $+1$, $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11\}$ we obtain a \mathbb{Z} -NNSFT, the golden mean shift, a classical example in symbolic dynamics.

Definition 2.1 (Weakly & strongly aperiodic). For a configuration $x \in \mathcal{A}^G$, we define the orbit of the element x under the shift action as $\text{orb}_{\sigma}(x) = \{\sigma_g(x) \mid g \in G\}$ and the set of element on G that fixes the configuration x by $\text{stab}_{\sigma}(x) = \{g \in G \mid \sigma_g(x) = x\}$. A configuration $x \in \mathcal{A}^G$ is

strongly periodic if $\text{stab}_{\sigma}(x)$ has finite index or, equivalently, if $\text{orb}_{\sigma}(x)$ is finite;

strongly aperiodic if $\text{stab}_{\sigma}(x) = \{1_G\}$.

weakly periodic if it is not strongly aperiodic.

weakly aperiodic if it is not strongly periodic;

More generally, a subshift $X \subset \mathcal{A}^G$ is weakly/strongly aperiodic if every configuration on X is weakly/strongly aperiodic.

Example 2.2. In $G = \mathbb{Z}^2$,

- the configuration x such that $x_g = 0$ for all g is strongly periodic;
- the configuration x such that $x_{g_1^n} = 0$ for all n , and $x_g = 1$ otherwise, is weakly periodic and weakly aperiodic;
- the configuration x such that $x_{(0,0)} = 0$ for all n , and $x_g = 1$ otherwise, is strongly aperiodic.

2.2 Wang tiles, NNSFT and graphs

Definition 2.3 (Wang tiles, Wang subshifts). Let $G = \langle S \mid \mathcal{R} \rangle$ be a finitely generated group and \mathcal{C} a finite set of colors. A *Wang tile* on \mathcal{C} and S is a map $S \cup S^{-1} \rightarrow \mathcal{C}$.

Given a set T of Wang tiles, the corresponding G -Wang subshift is defined as:

$$X_T = \{(x_g) \in T^G \mid \forall g \in G, s \in S \cup S^{-1}, x_g(s) = x_{gs}(s^{-1})\}$$

We call the elements in X_T G -Wang tilings.

Notice that the definition of a Wang tile depends only on the chosen set of generators, so that the same Wang tile can be used for \mathbb{F}_2 and \mathbb{Z}^2 , for example.

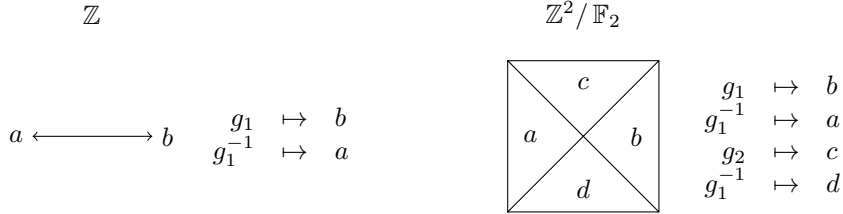


Figure 1: Examples of Wang tiles with colors $\mathcal{C} = \{a, b, c, d\}$ on one and two generators, respectively, with their corresponding maps.

Take any G -NNSFT X on alphabet \mathcal{A} , where $G = \langle \{g_1, \dots, g_d\} \mid \mathcal{R} \rangle$ is an arbitrary finitely generated group. Let \mathcal{F} be a set of forbidden patterns with each support of the form $\{1_G, g_i\}$.

We associate to X a set of d graphs $\Gamma_1, \dots, \Gamma_d$, where the set of vertices is \mathcal{A} for all Γ_i , and

$$\forall a, b \in \mathcal{A}, \quad a \rightarrow b \text{ in } \Gamma_i \iff \begin{cases} 1_G & \rightarrow & a \\ g_i & \rightarrow & b \end{cases} \notin \mathcal{F}.$$

By definition of a G -NNSFT, it follows that a configuration x belongs to X if, and only if, $x_h \rightarrow x_{g_i h}$ is an edge in Γ_i for all $h \in G$ and all $1 \leq i \leq d$.

Definition 2.4 (Cycles). A *cycle* on a graph Γ is a path - with possible edge and vertex repetitions - that starts and ends on the same vertex. A cycle through the vertices $a_1, \dots, a_n a_1$, with $a_i \in \mathcal{A}$, is denoted $\overline{a_1, \dots, a_n}$.

A cycle is *simple* if it does not contain any vertex repetition. Denote $\mathcal{SC}(\Gamma)$ the set of simple cycles on Γ , which is a finite set.

Remark 2.5. In graph theory, cycles are sometimes called *closed walks*, in which case cycle means simple cycle. We decided to follow Piantadosi's conventions [?] for convenience.

Let w be a cycle and $a \in \mathcal{A}$. We define:

$$|w|_a = \#\{i \mid w_i = a, 1 \leq i \leq |w|\}.$$

In any cycle, the path between the closest repetitions is a simple cycle. By removing this simple cycle and iterating the argument, we can see that any cycle w can be decomposed into simple cycles, in the sense that there are integers λ_ω for $\omega \in \mathcal{SC}(\Gamma)$ such that:

$$\forall a \in \mathcal{A}, |w|_a = \sum_{\omega \in \mathcal{SC}(\Gamma)} \lambda_\omega |\omega|_a.$$

We say that two G -subshifts $X, Y \subset \mathcal{A}^G$ are (topologically) *conjugate* if there is an homeomorphism Φ such that $\Phi(X) = Y$. In the context of G -subshifts, an homeomorphism corresponds to a reversible cellular automaton: there is a finite subset $H \subset G$ and a local rule $\varphi : \mathcal{A}^H \rightarrow \mathcal{A}$ such that

$$\forall x \in X, \forall g \in G, \Phi(x)_g = \varphi((x_{gh})_{h \in H}),$$

and Φ^{-1} is itself a cellular automaton.

Proposition 2.6. *For any set of generators, each G -SFT is conjugate to a G -NNSFT and each G -NNSFT is conjugate to a G -Wang subshift.*

This is folklore. A detailed proof for the SFT - NNSFT part can be found in [?] (Propositions 1.6 and 1.7), and a proof of the NNSFT - Wang subshift part in [?].

Since the conjugacy between a G -NNSFT and a G -Wang subshift can be chosen letter-to-letter (i.e. $H = \{1_G\}$ in the definition), it is easy to see that the conjugacy does not depends on G , so we could say that a set of graphs and a set of Wang tiles are conjugate.

Proposition 2.7. *Let X and Y be two conjugate G -subshifts. X admits a valid tiling if and only if Y admits a valid tiling. The same is true for weakly/strongly (a)periodic tilings.*

3 Piantadosi's and Chazottes-Gambaudo-Gautero's conditions

3.1 State of the art on the free group and \mathbb{Z}^2

The first two condition were introduced by Piantadosi in the context of symbolic dynamics on the free group \mathbb{F}_d .

Definition 3.1 ([?]). A family of graphs $\Gamma = \{\Gamma_i\}_{1 \leq i \leq d}$ on alphabet \mathcal{A} satisfies condition (\star) if and only if there is some $\mathcal{A}' \subset \mathcal{A}$ with a *coloring function* $\Psi : \mathcal{A}' \times \mathcal{S} \rightarrow \mathcal{A}'$ such that, for any color $a \in \mathcal{A}'$ and any generator $g_i \in \mathcal{S}$, $a \rightarrow \Psi(a, g_i)$ is an edge in Γ_i .

Theorem 3.2 ([?]). Let X be a \mathbb{F}_d -NNSFT on alphabet \mathcal{A} . X is nonempty if and only if the corresponding set of graphs satisfies condition (\star) .

This theorem provides a decision procedure for Domino problem in free groups of any rank: find a subalphabet such that every letter admits a valid neighbour in the subalphabet for every generator.

Definition 3.3 ([?]). Consider a family of graphs $\Gamma = \{\Gamma_i\}_{1 \leq i \leq d}$ and $\mathcal{SC}(\Gamma_i) = \{\omega_i^j\}_{1 \leq j \leq \#\mathcal{SC}(\Gamma_i)}$ the set of simple cycles for each graph Γ_i .

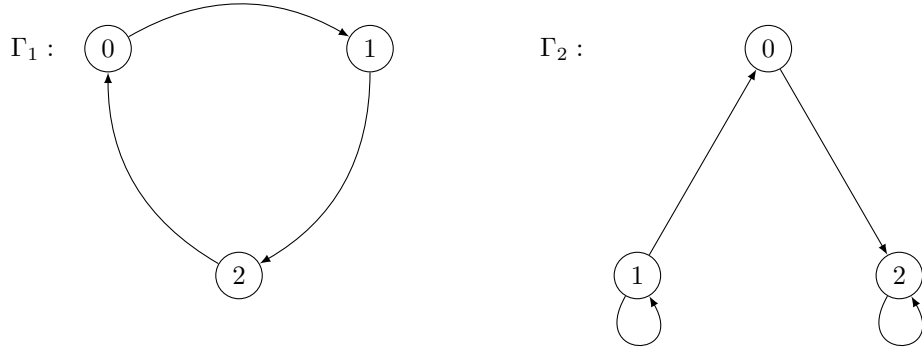
We denote by $(\star\star)$ the following equation on $x_{i,j}$:

$$\forall a \in \mathcal{A}, \quad \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j} |\omega_1^j|_a = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_2)} x_{2,j} |\omega_2^j|_a = \cdots = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_d)} x_{d,j} |\omega_d^j|_a.$$

We say that the graph family satisfies condition $(\star\star)$ if equation $(\star\star)$ admits a nontrivial positive solution.

Theorem 3.4. [?] A \mathbb{F}_d -NNSFT contains a strongly periodic configuration if and only if the associated family of graphs satisfies condition $(\star\star)$.

Example 3.5. We illustrate Piantadosi's conditions on the following example:



The corresponding \mathbb{F}_2 -NNSFT admits a tiling, because it satisfies condition (\star) on alphabet $\mathcal{A}' = \mathcal{A}$. However, it does not admit a periodic tiling: the simple cycles of Γ_1 are (up to shifting) $\{\overline{012}\}$ and the simple cycles of Γ_2 are $\{\overline{1}, \overline{2}\}$, so Equation $(\star\star)$ is:

$$\begin{aligned} x_{1,1} &= 0 & (a = 1) \\ x_{1,1} &= x_{2,1} & (a = 2) \\ x_{1,1} &= x_{2,2} & (a = 3) \end{aligned}$$

which obviously doesn't admit a solution. As we will see later, the corresponding \mathbb{Z}^2 -NNSFT doesn't admit any tiling.

Remark 3.6. For example, if all graphs Γ_i share a common cycle w , then condition $(\star\star)$ admits a solution and therefore the corresponding \mathbb{F}_d -NNSFT contains a periodic configuration.

Definition 3.7 ([?]). Let T be a set of Wang tiles on colors \mathcal{C} and set of generators S . For each $g \in S \cup S^{-1}$ and each color $c \in \mathcal{C}$, define c_g the subset of Wang tiles $\tau_i \in T$ such that $\tau_i(g) = c$. We call $(\star\star)'$ the following equation :

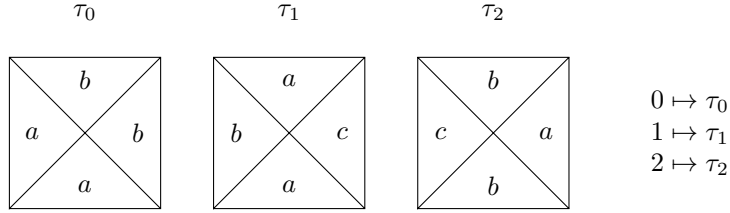
$$\forall g \in S, \forall c \in \mathcal{C}, \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j$$

We say that T satisfies condition $(\star\star)'$ if Equation $(\star\star)'$ admits a positive non-trivial solution.

Theorem 3.8 ([?]). *If a set T of Wang tiles admits a valid tiling of \mathbb{Z}^2 , then it satisfies condition $(\star\star)'$.*

This condition and result was introduced in [?], but a much easier presentation in our context is given in [?].

Example 3.9. Example 3.5 is conjugate to the following set of Wang tiles:



Equation $(\star\star)'$ becomes the following, where next to each equation is the corresponding generator and color:

$(g_1, a) \quad x_2 = x_0$	$(g_2, a) \quad x_1 = x_0 + x_1$
$(g_1, b) \quad x_0 = x_1$	$(g_2, b) \quad x_0 + x_2 = x_2$
$(g_1, c) \quad x_1 = x_2$	$(g_2, c) \quad 0 = 0$

This equation does not admit a positive nontrivial solution, so the corresponding \mathbb{Z}^2 -Wang subshift is empty.

3.2 Conditions $(\star\star)$ and $(\star\star)'$ are equivalent

Although conditions $(\star\star)$ and $(\star\star)'$ were introduced in very different contexts (periodic tilings of the free group and tilings of the Euclidean plane, respectively), it turns out that they are equivalent. The fact that $(\star\star)$ is a condition on graphs (NNSFTs) and $(\star\star)'$ is a condition on sets of Wang tiles (Wang subshifts) is only cosmetic since Proposition 2.6 lets us go from one model to the other.

Theorem 3.10. *Let T be a set of Wang tiles over the set of colors \mathcal{C} and the set of generators S .*

T satisfies condition $(\star\star)'$ if, and only if, the associated graphs satisfy condition $(\star\star)$.

Proof. (\Leftarrow) Let $(x_{i,j})$ be a nonnegative solution to equation $(\star\star)$. For every tile τ_i , put $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j} |\omega_1^j|_{\tau_i}$.

Because each simple cycle of Γ_1 is a cycle, it contains as many tiles in c_{g_1} as in $c_{g_1^{-1}}$; that is, $\sum_{\tau_i \in c_{g_1}} |\omega_1^j|_{\tau_i} = \sum_{\tau_j \in c_{g_1^{-1}}} |\omega_1^j|_{\tau_j}$. Summing over all simple cycles ω_1^j , we get $\sum_{\tau_i \in c_{g_1}} x_i = \sum_{\tau_j \in c_{g_1^{-1}}} x_j$.

Since $(x_{i,j})$ is a solution to Equation $(\star\star)$, we also have $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_n)} x_{n,j} |\omega_n^j|_{\tau_i}$ for every n , so the same argument shows that (x_i) is a nonnegative solution of equation $(\star\star)'$.

(\Rightarrow) Because equation $(\star\star)'$ admits a solution, it admits an rational solution, and therefore an integer solution. Let (x_i) be an integer, nonnegative solution of equation $(\star\star)'$.

For the generator g_1 , define an auxiliary graph G_1 on vertices $\{\tau_i^k\}_{1 \leq i \leq n, 1 \leq k \leq x_i}$ as follows.

Because

$$\forall c \in \mathcal{C}, \sum_{\tau_i \in c_{g_1}} x_i = \sum_{\tau_j \in c_{g_1^{-1}}} x_j,$$

we can fix an arbitrary bijection :

$$\Psi_1 : \{\tau_i^k : \tau_i \in c_{g_1}, k \leq x_i\} \rightarrow \{\tau_{i'}^{k'} : \tau_{i'} \in c_{g_1^{-1}}, k' \leq x_{i'}\}$$

and put an edge $\tau_i^k \rightarrow \tau_{i'}^{k'}$ if and only if $\Psi_1(\tau_i^k) = \tau_{i'}^{k'}$. Because each vertex has indegree and outdegree 1, it is a (not necessarily connected) Eulerian graph and admits a finite set of cycles covering every vertex exactly once. Decompose this finite set of cycles into simple cycles and denote $x_{1,j}$ the total number of each simple cycle ω_1^j obtained in this way.

Because each tile τ_i was present in the graph as a vertex in x_i copies, we have for every i $\sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j} |\omega_1^j|_{\tau_i} = x_i$.

Now apply the same argument for each generator $g_2 \dots g_n$ and the variables $(x_{i,j})$ thus obtained are a solution to equation $(\star\star)$. \square

4 Necessary conditions for tiling arbitrary groups

Since the above conditions apply on sets of Wang tiles or set of graphs, they actually are conditions on a family of G -SFT where G range over all groups with a fixed number of generators. The following proposition relates the properties of these SFT. It can be found (under a different form) in [?] (Proposition 10 and remark below)

Proposition 4.1. *Fix a set of d graphs $\Gamma_1, \dots, \Gamma_d$ on alphabet \mathcal{A} .*

Let $G_1 = \langle \{g_1 \dots g_d\} | r_1 \dots r_n \rangle$, $G_2 = \langle \{g_1 \dots g_d\} | r_1 \dots r_{n-1} \rangle$ be finitely generated groups. Consider the canonical surjective morphism $\pi : G_2 \rightarrow G_1$ defined by $\pi(g_i) = g_i$, $\forall 1 \leq i \leq d$. Let $\Phi : \mathcal{A}^{G_1} \rightarrow \mathcal{A}^{G_2}$ be defined by $\Phi(x)_g = x_{\pi(g)}$. Let X_1 and X_2 be the corresponding G_1 -NNSFT and G_2 -NNSFT respectively, such that X_2 has the same local rules that X_1 .

We have:

1. *If x is a valid tiling for X_1 then $\Phi(x)$ is a valid tiling for X_2 .*

2. If x is weakly periodic then $\Phi(x)$ is weakly periodic. In particular, if X_1 admits a weakly periodic tiling, then X_2 admits a weakly periodic tiling.
3. If x is weakly aperiodic then $\Phi(x)$ is weakly aperiodic. In particular, if X_1 admits a weakly aperiodic tiling, then X_2 admits a weakly aperiodic tiling.

The strong properties are not preserved by Φ , but of course the image of a strongly (a)periodic tiling remains weakly (a)periodic. Stronger versions with different hypotheses can be found in [?, ?].

Proof. 1. Since X_2 is an NNSFT, it is enough to check that, for all $h \in G_2$ and all $1 \leq i \leq d$, $\Phi(x)_h \rightarrow \Phi(x)_{hg_i}$ is an edge in Γ_i , that is to say, that it is not a forbidden pattern for X_2 . By definition of Φ , $\Phi(x)_h = x_{\pi(h)}$ and $\Phi(x)_{hg_i} = x_{\pi(h)\pi(g_i)} = x_{\pi(h)g_i}$. Because x is a valid tiling for X_1 , we have that $x_{\pi(h)} \rightarrow x_{\pi(h)g_i}$ is an edge in Γ_i , which proves the result.

2. If x is a weakly periodic tiling in X_1 , then $\text{stab}_\sigma(x)$ is nontrivial by definition. We have:

$$\begin{aligned} \text{stab}_\sigma(\Phi(x)) &= \{g \in G_2 : \forall h \in G_2, \Phi(x)_{hg} = \Phi(x)_h\} \\ &= \{g \in G_2 : \forall h \in G_2, x_{\pi(h)\pi(g)} = x_{\pi(h)}\} \end{aligned}$$

Since π is surjective, this means that $\pi(\text{stab}_\sigma(\Phi(x))) = \text{stab}_\sigma(x)$. $\text{stab}_\sigma(x)$ is nontrivial so $\text{stab}_\sigma(\Phi(x)) = \pi^{-1}(\text{stab}_\sigma(x))$ is nontrivial as well.

3. If x be a weakly aperiodic tiling in X_1 , then $\text{stab}_\sigma(x)$ does not have finite index. The canonical morphism $\pi : G_2 \rightarrow G_1$ yields a morphism on the quotient:

$$\tilde{\pi} : G_2/\pi^{-1}(\text{stab}_\sigma(x)) \rightarrow G_1/\text{stab}_\sigma(x)$$

and $\tilde{\pi}$ is surjective since π is surjective. Remember that $\text{stab}_\sigma(\Phi(x)) = \pi^{-1}(\text{stab}_\sigma(x))$ by the previous point. Since $\text{stab}_\sigma(x)$ does not have finite index, $G_1/\text{stab}_\sigma(x)$ is infinite, so $G_2/\pi^{-1}(\text{stab}_\sigma(x))$ is infinite as well, and $\text{stab}_\sigma(\Phi(x)) = \pi^{-1}(\text{stab}_\sigma(x))$ does not have finite index. □

Remark 4.2. In the last proposition, the converse of the point (1) does not hold. For instance, if consider $G = \mathbb{Z}^2 = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$. Example 2.2 provided an example of a set of graphs that satisfied condition (\star) (so the corresponding \mathbb{F}_2 -NNSFT admits a valid tiling) but does not satisfies the conditions $(\star\star)$ (so the corresponding \mathbb{Z}^2 -NNSFT does not admit any valid tiling).

To understand why, notice that $\ker(\pi)$ contains $g_1 g_2 g_1^{-1} g_2^{-1}$, so if a tiling $x \in \mathcal{A}^{\mathbb{F}_2}$ is such that $x_{1_{\mathbb{F}_2}} \neq x_{g_1 g_2 g_1^{-1} g_2^{-1}}$, then $\Phi^{-1}(x) = \emptyset$. If this happens for all $x \in X_2$ then X_1 is empty.

Corollary 4.3. *Let $\Gamma_1, \dots, \Gamma_d$ be a set of graphs that does not satisfy the condition (\star) . Then the corresponding G -NNSFT is empty for an arbitrary group G with d generators.*

Proof. If there was a valid tiling in $G = \langle g_1, \dots, g_d \mid \mathcal{R} \rangle$ then, applying Proposition 4.1 iteratively, we would obtain a tiling on \mathbb{F}_d , which is in contradiction with Theorem 3.2. □

5 Necessary conditions for tiling amenable groups

Definition 5.1 (Følner sequence). Let G be a finitely generated group. A *Følner sequence* for G is a sequence of finite subsets $S_n \subset G$ such that:

$$G = \bigcup_n S_n \quad \text{and} \quad \forall g \in G, \frac{\#(S_n g \Delta S_n)}{\#S_n} \xrightarrow{n \rightarrow \infty} 0,$$

where $S_n g = \{hg : h \in S_n\}$ and $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.

In the previous definition, it is easy to see that the second condition only has to be checked for g in a finite generating set. $S_n g \Delta S_n$ can be understood as the border of the group, so an element of a Følner sequence must have a small border relative to its interior.

Definition 5.2 (Amenable group). A finitely generated group G is *amenable* if it admits a Følner sequence.

A few examples :

- \mathbb{Z}^d is amenable and a Følner sequence is given by $S_n = [-n, n]^d$. Indeed, if $(g_i)_{1 \leq i \leq d}$ is the canonical set of generators, then $\#S_n = (2n+1)^d$ and $\#(S_n + g_i) \Delta S_n = 2 \cdot (2n+1)^{d-1}$.
- \mathbb{F}_d for $d \geq 2$ is not amenable. In particular, the balls S_n of radius n - that is, reduced¹ words of length $\leq n$ on the set of generators $(g_i)_{1 \leq i \leq d}$ - are not a Følner sequence. Indeed, one can easily check that $\#S_n = O(d^n)$ and $\#(S_n g_i \Delta S_n) = O(d^n)$.

The following theorem was conjectured in [?], Section 3.1.

Theorem 5.3 (Heuristic for tiling an amenable group). *Let G be a finitely generated amenable group, S a finite set of generators, and T a set of Wang tiles.*

If there is a tiling of G with the tiles T , then condition $(\star\star)$ (or equivalently $(\star\star)'$) is satisfied.

Proof. Let $x \in T^G$ be a tiling of G and S_n be a Følner sequence for G . Using notations from Definition 3.7, for a color $c \in \mathcal{C}$ and a generator $g \in S$, c_g is the set of tiles τ such that $\tau(g) = c$.

For any $h \in S_n \cap S_n g^{-1}$, we have $x_h \in c_g \Leftrightarrow x_{hg} \in c_{g^{-1}}$ (and in this case, $x_{hg} \in S_n \cap S_n \cdot g$). This means that, for all $c \in \mathcal{C}$, $g \in S$ and $n \in \mathbb{N}$:

$$\#\{h \in S_n \cap S_n g^{-1} : x_h \in c_g\} = \#\{h \in S_n \cap S_n g : x_h \in c_{g^{-1}}\}$$

so in particular $|\#\{h \in S_n : x_h \in c_g\} - \#\{h \in S_n : x_h \in c_{g^{-1}}\}| \leq \#S_n g \Delta S_n + \#S_n g^{-1} \Delta S_n$.

For each tile τ_i , let $x_i^n = \frac{\#\{h \in S_n : x_h = \tau_i\}}{\#S_n}$. The previous computation implies that:

$$\forall g \in S, \forall c \in \mathcal{C}, \left| \sum_{\tau_i \in c_g} x_i^n - \sum_{\tau_j \in c_{g^{-1}}} x_j^n \right| \leq \frac{\#S_n g \Delta S_n}{\#S_n} + \frac{\#S_n g^{-1} \Delta S_n}{\#S_n}.$$

¹with no $g_i^{-1} g_i$ or $g_i g_i^{-1}$ factors

Notice that the right-hand side tends to 0 as n tends to infinity by definition of a Følner sequence. Consider the sequence of vectors $((x_i^n)_i)_{n \in \mathbb{N}}$ and, by compactity, let (x_i) be any limit point of this sequence. We have

$$\forall g \in S, \forall c \in \mathcal{C}, \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j,$$

which is exactly condition $(\star\star)$. Condition $(\star\star)'$ follows by Theorem 3.10. \square

6 Counterexamples

It is clear that none of the (\star) , $(\star\star)$ or $(\star\star)'$ conditions can be a sufficient condition to admit a \mathbb{Z}^d -tiling, since it would be a decision procedure for the Domino problem; this argument applies to any group where the Domino problem is undecidable. For completeness, we provide explicit counterexamples for any non-free finitely generated group.

Theorem 6.1. *Let G be an arbitrary finitely generated group. If G is not free, then there exists an SFT that satisfies the three conditions (\star) , $(\star\star)$ and $(\star\star)'$ and which is empty.*

Proof. Write $G = \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle$, with $r_1 : w_1 \dots w_\ell = 1$.

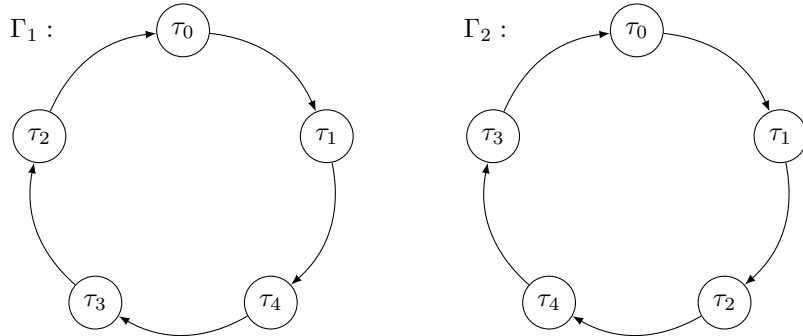
We build a family of graphs Γ_n on vertices $\{0, \dots, n\}$ that satisfy the following properties :

$$\forall i \leq \ell, \begin{cases} \text{if } w_i = g_j, \text{ then } \Gamma_j \text{ has an edge } i-1 \rightarrow i; \\ \text{if } w_i = g_j^{-1}, \text{ then } \Gamma_j \text{ has an edge } i \rightarrow i-1. \end{cases}$$

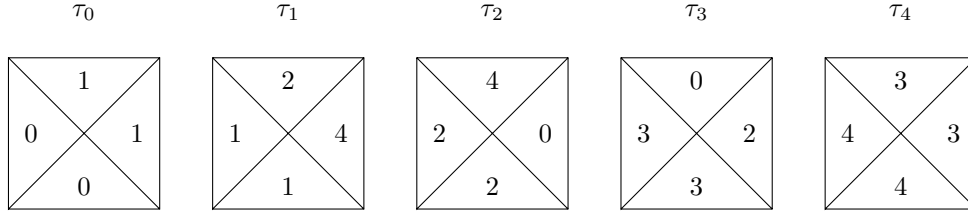
Because the previous process did not create any cycle, we can complete every Γ_j to be isomorphic to a n -cycle graph C_n .

Now we define a set of $n+1$ Wang tiles on $n+1$ colors $\{0 \dots n\}$ as follows. Tile τ_i has the following colors: $g_j^{-1} \rightarrow i$ for all j and $g_j \rightarrow k$ if there is an edge $\tau_i \rightarrow \tau_k$ in Γ_j .

Example 6.2. For \mathbb{Z}^2 , we have $r_1 : g_1 g_2 g_1^{-1} g_2^{-1} = 1$. Therefore Γ_1 contains $0 \rightarrow 1$ and $3 \rightarrow 2$, and Γ_2 contains $1 \rightarrow 2$ and $4 \rightarrow 3$. One possible completion for Γ_1 and Γ_2 is the following:



and the corresponding set of Wang tiles:



This tiling satisfies condition (\star) since we can assign the same weight $\frac{1}{n}$ to each tile.

It is clear that a tiling x of G using tiles τ_0, \dots, τ_n must contain every tile. Assume w.l.o.g that $x_1 = \tau_0$. By construction we must have $x_{w_1} = \tau_1$, $x_{w_1 w_2} = \tau_2$, and by an easy induction $x_w = \tau_n$. But since $w = 1$ in G , we have $\tau_0 = x_1 = x_w = \tau_n$, a contradiction. Therefore there is no tiling of G using tiles τ_0, \dots, τ_n . \square

7 Conclusion

We would like to mention the two following conjectures that relate the fact of admitting a valid (periodic) tiling and the underlying group structure:

Conjecture 7.1 ([?]). *A finitely generated group has a decidable domino problem if and only if it is virtually free.*

Conjecture 7.2 ([?]). *A finitely generated group has an SFT with no strongly periodic point if and only if it is not virtually cyclic.*

In both cases, the “if” direction is proven and the “only if” direction is open.

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