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Giovanni S. Alberti, Yves Capdeboscq

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COMBINING THE RUNGE APPROXIMATION AND THE WHITNEY EMBEDDING THEOREM IN HYBRID IMAGING

GIOVANNI S. ALBERTI AND YVES CAPDEBOSCQ

ABSTRACT. This paper addresses enforcing non-vanishing constraints for solutions to a second order elliptic partial differential equation by appropriate choices of boundary conditions. We show that, in dimension $d \geq 2$, under suitable regularity assumptions, the family of $2d$ solutions such that their Jacobian has maximal rank in the domain is both open and dense. The case of less regular coefficients is also addressed, together with other constraints, which are relevant for applications to recent hybrid imaging modalities. Our approach is based on the combination of the Runge approximation property and the Whitney projection argument [Greene and Wu, Ann. Inst. Fourier (Grenoble), 25(1, vii):215–235, 1975]. The method is very general, and can be used in other settings.

1. INTRODUCTION

We consider a general second-order elliptic equation

$$(1) \quad Lu := -\operatorname{div}(a\nabla u + bu) + c \cdot \nabla u + qu = 0 \quad \text{in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded and smooth domain. We assume that L is uniformly elliptic, namely,

$$(2) \quad a(x) \xi \cdot \xi \geq \lambda |\xi|^2, \quad \text{a.e. } x \in \Omega, \xi \in \mathbb{R}^d,$$

for some $\lambda > 0$. The parameters of equation (1) are assumed to satisfy mild regularity assumptions, namely either

$$(3) \quad a \in C^{\ell-1, \alpha}(\overline{\Omega}; \mathbb{R}^{d \times d}), b \in C^{\ell-1, \alpha}(\overline{\Omega}; \mathbb{R}^d), c \in W^{\ell-1, \infty}(\Omega; \mathbb{R}^d), q \in W^{\ell-1, \infty}(\Omega; \mathbb{R}),$$

with $\ell \geq 1$ and $\alpha \in (0, 1)$, or

$$(4) \quad a \in L^\infty(\Omega; \mathbb{R}^{d \times d}), b, c \in L^\infty(\Omega; \mathbb{R}^d), q \in L^\infty(\Omega; \mathbb{R}^d),$$

which will be referred to as $\ell = 0$. By classical elliptic regularity theory [43, 42, 60], the solutions to (1) belong to $C_{\text{loc}}^{\ell, \alpha}(\Omega; \mathbb{R})$ and, provided that the boundary conditions are chosen in the appropriate trace space, such a regularity extends up to the boundary, namely $u \in C^{\ell, \alpha}(\overline{\Omega}; \mathbb{R})$. In the case $\ell = 0$, the Hölder exponent α is for example the one given by the De Giorgi–Nash–Moser theorem.

This paper focuses on how to enforce pointwise constraints on the solutions of (1). Our motivation for studying such a question comes from hybrid imaging. Hybrid, or multi-physics, imaging problems are a type of parameter identification problems

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that in many cases involve the reconstruction of the coefficients of a PDE from the knowledge of some internal functional of its solutions [16, 65, 23, 49, 20, 9].

Amongst all these constraints, the most ubiquitous one is the non-vanishing Jacobian problem. It can be reworded as follows: given L as in (1) and a compact set $K \subseteq \Omega$, how can one choose boundary conditions g_1, \dots, g_N such that

$$(5) \quad \text{rank}(\nabla u_1, \dots, \nabla u_N) = d \text{ everywhere in } K,$$

where

$$\begin{cases} Lu_i = 0 & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases} \quad i = 1, \dots, N?$$

The difficulty here is that, apart from the fact that a, b, c and q are relatively smooth and coercive, nothing is known about these coefficients, which are the unknowns of the inverse problem.

When $d = 2, b = 0, c = 0$ and $q = 0$, that is, L is simply

$$Lu = -\text{div}(a\nabla u),$$

it turns out that the Radó–Kneser–Choquet Theorem can be extended to this setting (without regularity assumptions) [11, 12, 13, 14]. Only two boundary conditions, independent of the matrix valued function a , are required for the constraint to be satisfied globally. This result cannot be extended to higher dimensions [66, 53, 39, 7, 9], even locally: it is not possible to find suitable boundary conditions independently of the (unknown) coefficient. For more general models, when b, c or q are not null, such as the Helmholtz equation, no solace can be found in any dimension, since the Radó–Kneser–Choquet Theorem, whose proof uses the maximum principle, does not apply.

One is therefore drawn to ask whether using a large number of boundary conditions would help. Again, some counter-examples can be derived to the most optimistic claims [9, Corollary 6.18]; nevertheless, it is possible to construct open sets of boundary conditions valid for open sets of parameters for the relevant elliptic operator L . Two main strategies have been used to achieve this goal: complex geometrical optics (CGO) solutions [63, 33, 30, 29, 18, 48, 58, 23, 57, 25, 35, 19, 31, 27] and the Runge approximation property [27, 59, 35, 31]. Other approaches based on frequency variations [2, 4, 3, 5, 8, 6, 21] or dynamical systems [26] were also developed: these are not discussed here.

CGO solutions are only available for isotropic coefficients a , that is, $a = \gamma I_d$ where γ is a real-valued function. The CGO solution method provides a non-vanishing Jacobian globally inside the domain for a suitable choice of (d complex-valued) boundary conditions. This approach requires high regularity assumptions on the coefficients. On the other hand, the Runge approximation property holds provided that the unique continuation property holds [54], such a property being enjoyed by a much larger class of problems [15]. A drawback is that the argument is local, applied on a covering of the domain by small balls, and so many boundary conditions are needed. Further, while CGO solutions are constructed (depending on the coefficients), the Runge approximation provides an existence result of suitable solutions, but not a constructive method to derive them.

In this work, we combine a Whitney projection argument [64], as described in [45, 44], with the Runge approximation. Not only do we provide an explicit bound on the number of boundary conditions to be considered, but we also obtain that these constitute an open and dense set. For instance, the set of $2d$ boundary

conditions such that (5) is satisfied is open and dense in $H^{1/2}(\partial\Omega; \mathbb{R})^{2d}$. Our result applies to more general constraints than (5), so that it is in particular applicable to a variety of imaging problems (see section 2 for details). Our result confirms what has been observed in numerical simulations in the setting of scalar (isotropic) diffusion coefficients, where good reconstructions are obtained for a relatively small set of boundary conditions [58, 59, 26, 35]. After the first version of this manuscript was published, we were made aware of the recent preprint [41], where similar techniques are used for the fractional Calderón problem.

This paper is structured as follows. In section 2, we state our main results and discuss some open problems. Section 3 is devoted to the Runge approximation property. Finally, in section 4 we provide the proof of the main result.

2. MAIN RESULTS

Let $K \subseteq \bar{\Omega}$ be a smooth compact set and

$$\zeta: C^{\ell, \alpha}(K) \rightarrow C^{0, \alpha}(K)^n$$

be a continuous linear map, with $n \geq 1$. Let $\mathcal{H}(K)$ denote the set of solutions to (1) that are smooth in K , namely

$$(6) \quad \mathcal{H}(K) = \{u \in C^{\ell, \alpha}(K) \cap H^1(\Omega) : Lu = 0 \text{ in } \Omega\},$$

equipped with the norm $\|u\|_{\mathcal{H}(K)} = \|u\|_{C^{\ell, \alpha}(K)} + \|u\|_{H^1(\Omega)}$. We are interested in solutions $u_i \in \mathcal{H}(K)$ satisfying the constraint

$$(7) \quad \det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x) \neq 0,$$

in K , locally or globally.

Example 1. Constraints of the form (7) appear in various problems.

- When $n = 1$, $\ell = 0$ and $\zeta(u) = u$, the constraint corresponds to avoiding nodal points, namely $u_1(x) \neq 0$. This is useful whenever a division by u_1 is required.
- When $n = d$, $\ell = 1$, and $\zeta(u) = \nabla u$ (taken as row vector), the constraint imposes a non-vanishing Jacobian. In that case, (u_1, \dots, u_d) defines a local C^2 diffeomorphism. This is the case discussed in the introduction.
- When $n = d + 1$, $\ell = 1$, and $\zeta(u) = [u \quad \nabla u]$ (taken as a row vector) the constraint imposes a non-vanishing “augmented” Jacobian. The additional potential u may represent a scaled time derivative, in a time harmonic model. This constraint may also correspond to the non-vanishing Jacobian for $v_i = \frac{u_{i+1}}{u_1}$, $i = 1, \dots, d$.

Such constraints appear in quantitative photoacoustic tomography [22, 35, 32, 29], in quantitative thermoacoustic tomography [30, 19, 3], in acousto-electric tomography (also known as electrical impedance tomography by elastic deformation or ultrasound modulated electrical impedance tomography) [17, 40, 57, 25, 50, 59, 51], in microwave imaging by elastic deformation [18, 2, 10, 63], in current density imaging [47, 55, 61, 27], in dynamic elastography [22, 34, 35] and in other hybrid imaging modalities. We refer to [9] for additional methods and further explanations on some of the models we have mentioned.

We introduce the following notation.

Definition 2. The *candidate set* $\mathcal{C}(K)$ is the set of all $x \in K$ for which there exist $u_1, \dots, u_n \in \mathcal{H}(K)$ so that

$$\det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x) \neq 0.$$

The *admissible set* $\mathcal{E}(K)$ is the set of all $u = (u_1, \dots, u_{\lfloor \frac{d+n}{\alpha} \rfloor}) \in \mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$ such that

$$\sum_{i_1, \dots, i_n=1}^{\lfloor \frac{d+n}{\alpha} \rfloor} \left| \det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x) \right| > 0, \text{ for all } x \in K.$$

Here, $\lfloor \frac{d+n}{\alpha} \rfloor = \max \{N \in \mathbb{N} : N \leq \frac{d+n}{\alpha}\}$ denotes the integer part of $\frac{d+n}{\alpha}$.

Remark 3. In other words, $u \in \mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$ belongs to $\mathcal{E}(K)$ if and only if for every $x \in K$ there exist $i_1, \dots, i_n \in \{1, \dots, \lfloor \frac{d+n}{\alpha} \rfloor\}$ such that

$$\det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x) \neq 0,$$

namely, if and only if the desired constraint is satisfied everywhere in K for a suitable subset of the solutions $u_1, \dots, u_{\lfloor \frac{d+n}{\alpha} \rfloor}$. The candidate set is the subset of K where satisfying the constraint pointwise is possible at all. If $\mathcal{C}(K) \neq K$, then $\mathcal{E}(K)$ is empty.

Remark 4. If the coefficients of the PDE are smooth enough, so that $\alpha > \frac{d+n}{d+n+1}$, then the number of solutions is $d+n = \lfloor \frac{d+n}{\alpha} \rfloor$.

In this general setting, we have the following result.

Theorem 5. *Take a smooth compact set $K \subseteq \overline{\Omega}$. Assume that the candidate set satisfies*

$$(8) \quad \mathcal{C}(K) = K.$$

Then the admissible set $\mathcal{E}(K)$ is open and dense in $\mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$.

Remark 6. Note that, since a finite intersection of open and dense sets is open and dense, the result immediately extends to the case when finitely many constraints are imposed simultaneously.

In section 3 we observe, using the Runge Approximation Property, that assumption (8) is satisfied for a large class of examples, since $\mathcal{C}(\overline{\Omega}) = \overline{\Omega}$.

Our initial focus was on boundary value problems, since such problems are relevant for non-invasive imaging methods, as explained in the introduction. The following corollary is a rewording of our result for boundary value problems.

Corollary 7. *Take a compact set $K \subseteq \Omega$. Suppose that for every $g \in H^{1/2}(\partial\Omega)$ the problem*

$$(9) \quad \begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $u^g \in H^1(\Omega)$. If (8) holds true, then the set

$$\left\{ \left(g_1, \dots, g_{\lfloor \frac{d+n}{\alpha} \rfloor} \right) \in H^{1/2}(\partial\Omega)^{\lfloor \frac{d+n}{\alpha} \rfloor} : \left(u^{g_1}, \dots, u^{g_{\lfloor \frac{d+n}{\alpha} \rfloor}} \right) \in \mathcal{E}(K) \right\}$$

is open and dense in $H^{1/2}(\partial\Omega)^{\lfloor \frac{d+n}{\alpha} \rfloor}$.

Proof. Consider the map

$$\psi: H^{1/2}(\partial\Omega)^{\lfloor \frac{d+n}{\alpha} \rfloor} \rightarrow H^1(\Omega)^{\lfloor \frac{d+n}{\alpha} \rfloor}, \quad \left(g_1, \dots, g_{\lfloor \frac{d+n}{\alpha} \rfloor} \right) \mapsto \left(u^{g_1}, \dots, u^{g_{\lfloor \frac{d+n}{\alpha} \rfloor}} \right),$$

where u^{g_i} is defined by (9). Because problem (9) is well-posed, we have $\|u^g\|_{H^1(\Omega)} \leq C(L)\|g\|_{H^{1/2}(\partial\Omega)}$ for every $g \in H^{1/2}(\partial\Omega)$. Further, because of our outstanding regularity assumptions on the coefficients (3) we also have $\|u^g\|_{C^{\ell,\alpha}(K)} \leq C(L)\|g\|_{H^{1/2}(\partial\Omega)}$. This shows that the map $\psi: H^{1/2}(\partial\Omega)^{\lfloor \frac{d+n}{\alpha} \rfloor} \rightarrow \mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$ is continuous. Its inverse is given by the trace operator acting component-wise, and is also continuous. In other words, ψ is an isomorphism, and the result immediately follows from Theorem 5, since the set under consideration is $\psi^{-1}(\mathcal{E}(K))$. \square

Remark 8. Corollary 7 was stated for simplicity only if K is a proper subset of Ω . When K touches $\partial\Omega$, for instance if $K = \Omega$, the same result holds, provided that $H^{1/2}(\partial\Omega)$ is replaced with a suitable trace space consisting of smoother functions.

Future perspectives. The regularity assumptions we made are important in all generality, because we use a Unique Continuation Principle argument. For a specific problem, with a given geometry and/or coefficient structure, appropriate extension can often be envisioned (see e.g. [37] for a strategy on how to handle a large class of piecewise regular coefficients). We have limited ourselves to elliptic PDE with real coefficients. Considering the case of complex valued coefficients (which appear in thermo-acoustic tomography [30, 19]) is a natural extension of this work. Maxwell's equation [62, 46, 36, 4, 3] and linear elasticity [38, 56, 52, 24, 28] are not considered here and are natural frameworks where this method could be applied. Finally, note that while a rough description of our result could be that a ‘‘random’’ choice of $\lfloor \frac{d+n}{\alpha} \rfloor$ boundary condition suffices, we have not established such a claim. It would be interesting to move from an open and dense set of admissible boundary conditions to a random choice of boundary conditions with high probability (or indeed probability 1).

3. THE RUNGE APPROXIMATION PROPERTY AND ASSUMPTION (8)

For simplicity of exposition, in this section we restrict ourselves to considering only the constraints associated to the maps ζ given in Example 1, namely:

- $n = 1, \ell = 0, \zeta(u) = u;$
- $n = d, \ell = 1, \zeta(u) = \nabla u;$
- or $n = d + 1, \ell = 1, \zeta(u) = [u \quad \nabla u].$

However, with minor modifications to the argument, many other constraints can be considered, since this approach is very general. The main tool to satisfy (8), namely to show that there always exist global solutions satisfying the desired constraints locally, is the following result: it is sufficient to build suitable solutions of the PDE with constant coefficients, and without lower order terms.

Proposition 9. *Let L be the elliptic operator defined in (1). In addition to (2), (3) if $\ell = 1$ and (4) if $\ell = 0$, assume that $a(x)$ is a symmetric matrix for every $x \in \overline{\Omega}$ and, if $d \geq 3$, $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{d \times d})$. Take $x_0 \in \overline{\Omega}$. Let $r > 0$ and $u_1, \dots, u_n \in C^{\ell, \alpha}(\overline{\Omega}; \mathbb{R})$ be solutions to the constant coefficient problem*

$$-\operatorname{div}(a(x_0) \nabla u_i) = 0 \quad \text{in } B(x_0, r), \quad i = 1, \dots, n.$$

If

$$(10) \quad \det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x_0) \neq 0,$$

then $x_0 \in \mathcal{C}(\overline{\Omega})$.

Remark 10. This result can in some cases be extended to operators L with piecewise Lipschitz coefficients with possibly countably many pieces, following the strategy given in [37].

Proof. This result, even though not in this exact form, was first derived in [35], and later discussed in [9, Section 7.3] (only in the case $x_0 \in \Omega$). Here we provide only a sketch of the proof in order to highlight the main features; the reader is referred to the references mentioned for the details of the argument.

The proof is split into three steps.

Step 1: approximation of u_i with local solutions v_i to $Lv_i = 0$. Using standard elliptic regularity estimates, it is possible to find $\tilde{r} \in (0, r]$ and $v_i \in H^1(B(x_0, \tilde{r}) \cap \Omega)$ such that $Lv_i = 0$ in $B(x_0, \tilde{r}) \cap \Omega$ and $\|u_i - v_i\|_{C^1(\overline{B(x_0, \tilde{r})} \cap \overline{\Omega})}$ is arbitrarily small (provided that \tilde{r} is chosen small enough). It is worth observing that, even if in [35] the lower order terms are kept in the PDE with constant coefficients, that is not needed [9, Proposition 7.10].

Step 2: approximation of v_i with global solutions w_i to $Lw_i = 0$. Thanks to the regularity assumptions on the coefficients, the elliptic operator L enjoys the unique continuation property [15]. This is equivalent to the Runge approximation property [54], by which it is possible to approximate local solutions with global solutions. Thus, in our setting, there exist $w_i \in H^1(\Omega)$ solutions to $Lw_i = 0$ in Ω such that $\|w_i - v_i\|_{H^1(B(x_0, \tilde{r}) \cap \Omega)}$ is arbitrarily small. By elliptic regularity, we can ensure that $\|w_i - v_i\|_{C^1(\overline{B(x_0, \tilde{r}/2)} \cap \overline{\Omega})}$ is arbitrarily small too.

Step 3: (w_1, \dots, w_n) satisfy the constraint in x_0 . Combining the previous steps, we have that $\|u_i - w_i\|_{C^1(\overline{B(x_0, \tilde{r}/2)} \cap \overline{\Omega})}$ is arbitrarily small. For the maps ζ considered above, this immediately implies that $|\zeta(u_i) - \zeta(w_i)|(x_0)$ is arbitrarily small. Thus, by (10), \tilde{r} and w_i may be chosen in such a way that

$$\det \begin{bmatrix} \zeta(w_1) \\ \vdots \\ \zeta(w_n) \end{bmatrix} (x_0) \neq 0,$$

which shows that $x_0 \in \mathcal{C}(\overline{\Omega})$. □

Let us now verify that for the maps ζ mentioned above, we always have $\mathcal{C}(\overline{\Omega}) = \overline{\Omega}$; in other words, the assumptions of Theorem 5 are satisfied with $K = \overline{\Omega}$.

Corollary 11. *Let L be the elliptic operator defined in (1). In addition to (2), (3) if $\ell = 1$ and (4) if $\ell = 0$, assume that $a(x)$ is a symmetric matrix for every $x \in \overline{\Omega}$ and, if $d \geq 3$, $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{d \times d})$. If ζ is one of the maps considered in Example 1, then $\mathcal{C}(K) = K$ for any $K \subseteq \overline{\Omega}$.*

Proof. We consider the three constraints separately:

- $n = 1, \ell = 0, \zeta(u) = u$: set $u_1 = 1$.
- $n = d, \ell = 1, \zeta(u) = \nabla u$: set $u_1 = x_1, \dots, u_n = x_n$.
- $n = d + 1, \ell = 1, \zeta(u) = [u \quad \nabla u]$: set $u_1 = 1, u_2 = x_1, \dots, u_{n+1} = x_n$.

Given $x_0 \in \overline{\Omega}$, for any $a(x_0)$, there holds

$$-\operatorname{div}(a(x_0)\nabla u_i) = 0 \quad \text{in } \mathbb{R}^d, \quad i = 1, \dots, n,$$

and

$$\det \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_n) \end{bmatrix} (x_0) \neq 0.$$

The conclusion follows from Proposition 9. \square

4. PROOF OF THEOREM 5

We need two lemmata. For $k \geq 2$ and $a \in \mathbb{R}^{k-1}$ let $P_a: \mathbb{R}^k \rightarrow \mathbb{R}^{k-1}$ denote the linear map given by

$$P_a(y) = (y_1 - a_1 y_k, \dots, y_{k-1} - a_{k-1} y_k).$$

In the following, we shall identify the matrices in $\mathbb{R}^{k \times n}$ with k rows and n columns with the linear maps from \mathbb{R}^n into \mathbb{R}^k . We shall denote the Lebesgue measure in \mathbb{R}^m by $|\cdot|_m$.

Lemma 12. *Take a smooth and compact set $K \subseteq \mathbb{R}^d$ and a positive integer $k > \frac{d+n}{\alpha}$. Let $F: K \rightarrow \mathbb{R}^{k \times n}$ be of class $C^{0,\alpha}$ and such that $F_x: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is injective for all $x \in K$. Let $G \subseteq \mathbb{R}^{k-1}$ be the set of those $a \in \mathbb{R}^{k-1}$ for which $P_a \circ F_x: \mathbb{R}^n \rightarrow \mathbb{R}^{k-1}$ is injective for all $x \in K$, namely*

$$G = \bigcap_{x \in K} \{a \in \mathbb{R}^{k-1} : P_a \circ F_x: \mathbb{R}^n \rightarrow \mathbb{R}^{k-1} \text{ is injective}\}.$$

Then $|\mathbb{R}^{k-1} \setminus G|_{k-1} = 0$.

Proof. Note that $\ker P_a = \operatorname{span}\{(a_1, \dots, a_{k-1}, 1)\}$. Thus, since F_x is injective, we have for $x \in K$

$$\begin{aligned} P_a \circ F_x \text{ is injective} &\iff \operatorname{ran} F_x \cap \ker P_a = \{0\} \\ &\iff (a_1, \dots, a_{k-1}, 1) \notin \operatorname{ran} F_x. \end{aligned}$$

Then, $a \in \mathbb{R}^{k-1} \setminus G$ if and only if there exists $x \in K$ such that $(a_1, \dots, a_{k-1}, 1) \in \operatorname{ran} F_x$, namely $(a_1, \dots, a_{k-1}, 1) \in \cup_{x \in K} \operatorname{ran} F_x$. Therefore, using the projection $\pi: \mathbb{R}^k \rightarrow \mathbb{R}^{k-1}$, $(b_1, \dots, b_k) \mapsto (b_1, \dots, b_{k-1})$, we can express $\mathbb{R}^{k-1} \setminus G$ as

$$\mathbb{R}^{k-1} \setminus G = \pi(B), \quad B = \left(\bigcup_{x \in K} \operatorname{ran} F_x \right) \cap \{b \in \mathbb{R}^k : b_k = 1\}.$$

Hence, it remains to prove that

$$\mathcal{H}^{k-1}(B) = 0,$$

where \mathcal{H}^{k-1} denotes the Hausdorff measure of dimension $k-1$.

Consider the map

$$f: K \times \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad (x, v) \mapsto F_x v.$$

By construction, $\text{ran } f = \bigcup_{x \in K} \text{ran } F_x$. Note that f is of class $C^{0,\alpha}$ and that $\dim(K \times \mathbb{R}^n) = d+n < \alpha k$, so that $\mathcal{H}^{\alpha k}(K \times \mathbb{R}^n) = 0$. By elementary properties of Hausdorff measures [1, section 4.1], we have $\mathcal{H}^k(\bigcup_{x \in K} \text{ran } F_x) = \mathcal{H}^k(\text{ran } f) = 0$. By linearity of $v \mapsto F_x v$, the set $\bigcup_{x \in K} \text{ran } F_x$ is closed under scalar multiplication, and so $\bigcup_{x \in K} \text{ran } F_x \supseteq \mathbb{R}_+ \cdot B$, which implies

$$\mathcal{H}^k(\mathbb{R}_+ \cdot B) = 0.$$

Using the change of variables formula and Tonelli theorem, we deduce $\mathcal{H}^{k-1}(B) = 0$, as desired. \square

Lemma 13. *Take a smooth and compact set $K \subseteq \bar{\Omega}$ and a positive integer $k > \frac{d+n}{\alpha}$. Let $u_1, \dots, u_k \in \mathcal{H}(K)$ be such that*

$$\text{rank} \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_k) \end{bmatrix} (x) = n, \quad x \in K.$$

Let $G \subseteq \mathbb{R}^{k-1}$ be the set of those $a \in \mathbb{R}^{k-1}$ such that

$$\text{rank} \begin{bmatrix} \zeta(u_1 - a_1 u_k) \\ \zeta(u_2 - a_2 u_k) \\ \vdots \\ \zeta(u_{k-1} - a_{k-1} u_k) \end{bmatrix} (x) = n, \quad x \in K.$$

Then $|\mathbb{R}^{k-1} \setminus G|_{k-1} = 0$.

Proof. Let $F: K \rightarrow \mathbb{R}^{k \times n}$ be the map of class $C^{0,\alpha}$ defined by

$$F_x = \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_k) \end{bmatrix} (x): \mathbb{R}^n \rightarrow \mathbb{R}^k.$$

By assumption, we have that F_x is injective for all $x \in K$. Observe that, by linearity of ζ , we have

$$P_a \circ F_x = \begin{bmatrix} \zeta(u_1)(x) - a_1 \zeta(u_k)(x) \\ \vdots \\ \zeta(u_{k-1})(x) - a_{k-1} \zeta(u_k)(x) \end{bmatrix} = \begin{bmatrix} \zeta(u_1 - a_1 u_k) \\ \vdots \\ \zeta(u_{k-1} - a_{k-1} u_k) \end{bmatrix} (x)$$

and so the conclusion immediately follows by Lemma 12. \square

We are now ready to prove Theorem 5.

Proof of Theorem 5.

Step 1: $\mathcal{E}(K)$ is open in $\mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$.

Take $u \in \mathcal{E}(K)$. By definition, we have

$$\sum_{i_1, \dots, i_n=1}^{\lfloor \frac{d+n}{\alpha} \rfloor} |\det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x)| > 0, \quad x \in K.$$

Since $\zeta(u_{i_j})$ are $C^{0,\alpha}$ maps, and in particular continuous, and K is compact, we have that

$$\sum_{i_1, \dots, i_n=1}^{\lfloor \frac{d+n}{\alpha} \rfloor} \left| \det \begin{bmatrix} \zeta(u_{i_1}) \\ \vdots \\ \zeta(u_{i_n}) \end{bmatrix} (x) \right| \geq C, \quad x \in K,$$

for some constant $C > 0$. Finally, since the map ζ is continuous itself, if $v \in \mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$ is chosen close enough to u we have

$$\sum_{i_1, \dots, i_n=1}^{\lfloor \frac{d+n}{\alpha} \rfloor} \left| \det \begin{bmatrix} \zeta(v_{i_1}) \\ \vdots \\ \zeta(v_{i_n}) \end{bmatrix} (x) \right| \geq \frac{C}{2}, \quad x \in K,$$

which implies $v \in \mathcal{E}(K)$. This concludes the first step.

Step 2: $\mathcal{E}(K)$ is dense in $\mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$.

Take $H = (h_1, \dots, h_{\lfloor \frac{d+n}{\alpha} \rfloor}) \in \mathcal{H}(K)^{\lfloor \frac{d+n}{\alpha} \rfloor}$. By assumption, for all $x \in K$ there exist $u_{1,x}, \dots, u_{n,x} \in \mathcal{H}(K)$ such that

$$\left| \det \begin{bmatrix} \zeta(u_{1,x}) \\ \vdots \\ \zeta(u_{n,x}) \end{bmatrix} (x) \right| > 0.$$

By continuity of $\zeta(u_{i,x})$, there exist neighbourhoods $U^x \ni x$ such that

$$\left| \det \begin{bmatrix} \zeta(u_{1,x}) \\ \vdots \\ \zeta(u_{n,x}) \end{bmatrix} (y) \right| > 0, \quad y \in U^x \cap K.$$

Since $K \subseteq \cup_{x \in K} U^x$, by compactness there exist $x_1, \dots, x_N \in K$ such that $K \subseteq \cup_{j=1}^N U^{x_j}$. Thus, for all $x \in K$ there exists $j = 1, \dots, N$ such that

$$\left| \det \begin{bmatrix} \zeta(u_{1,x_j}) \\ \vdots \\ \zeta(u_{n,x_j}) \end{bmatrix} (x) \right| > 0.$$

Consider all the $M = nN$ corresponding solutions

$$(u_1, \dots, u_M) = (u_{1,x_1}, \dots, u_{n,x_1}, u_{1,x_2}, \dots, u_{n,x_2}, \dots, u_{1,x_N}, \dots, u_{n,x_N}),$$

so that

$$\text{rank} \begin{bmatrix} \zeta(u_1) \\ \vdots \\ \zeta(u_M) \end{bmatrix} (x) = n, \quad x \in K.$$

In particular, we have

$$\text{rank} \begin{bmatrix} \zeta(h_1) \\ \vdots \\ \zeta(h_{\lfloor \frac{d+n}{\alpha} \rfloor}) \\ \zeta(u_1) \\ \vdots \\ \zeta(u_M) \end{bmatrix} (x) = n, \quad x \in K.$$

By Lemma 13, since $k = \lceil \frac{d+n}{\alpha} \rceil + M > \lceil \frac{d+n}{\alpha} \rceil$, we have $k > \frac{d+n}{\alpha}$, and so for almost every $a \in \mathbb{R}^{k-1}$ we have

$$\text{rank} \begin{bmatrix} \zeta(h_1 - a_1 u_M) \\ \vdots \\ \zeta(h_{\lceil \frac{d+n}{\alpha} \rceil} - a_{\lceil \frac{d+n}{\alpha} \rceil} u_M) \\ \zeta(u_1 - a_{\lceil \frac{d+n}{\alpha} \rceil + 1} u_M) \\ \vdots \\ \zeta(u_{M-1} - a_{k-1} u_M) \end{bmatrix} (x) = n, \quad x \in K.$$

Repeating this argument M times (as long as $k > \lceil \frac{d+n}{\alpha} \rceil$) with very small weights a , we obtain that there exist $\xi_{i,j} \in \mathbb{R}$ ($i = 1, \dots, \lceil \frac{d+n}{\alpha} \rceil$, $j = 1, \dots, M$) which can be chosen arbitrarily small such that

$$\text{rank} \begin{bmatrix} \zeta(h_1 - \xi_{1,j} u_j) \\ \vdots \\ \zeta(h_{\lceil \frac{d+n}{\alpha} \rceil} - \xi_{\lceil \frac{d+n}{\alpha} \rceil, j} u_j) \end{bmatrix} (x) = n, \quad x \in K,$$

where we used Einstein summation convention of repeated indices. This implies that

$$(h_1 - \xi_{1,j} u_j, \dots, h_{\lceil \frac{d+n}{\alpha} \rceil} - \xi_{\lceil \frac{d+n}{\alpha} \rceil, j} u_j) \in \mathcal{E}(K),$$

which, since the weights $\xi_{i,j}$ are chosen arbitrarily small, concludes the proof. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GENOA, VIA DODECANESO 35, 16146 GENOVA, ITALY

Email address: `giovanni.alberti@unige.it`

UNIVERSITÉ DE PARIS, CNRS, SORBONNE UNIVERSITÉ, LABORATOIRE JACQUES-LOUIS LIONS UMR7598, PARIS, FRANCE

Email address: `capdeboscq@ljl1.univ-paris-diderot.fr`