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BOUNDARY NULL-CONTROLLABILITY OF COUPLED PARABOLIC SYSTEMS WITH ROBIN CONDITIONS

KUNTAL BHANDARI* AND FRANCK BOYER*†

Abstract. The main goal of this paper is to investigate the boundary controllability of a coupled parabolic system in the cascade form in the case where the boundary conditions are of Robin type. In particular, we prove that the associated controls satisfy suitable uniform bounds with respect to the Robin parameters, that let us show that they converge towards a Dirichlet control when the Robin parameters go to infinity. This is a justification of the popular penalisation method for dealing with Dirichlet boundary data in the framework of the controllability of coupled parabolic systems.

Key words. Control theory, parabolic systems, moments method, spectral estimates.

AMS subject classifications. 35B30 - 35K20 - 93B05

1. Introduction.

1.1. The problem under study. This paper is concerned with the boundary null-controllability problem for linear coupled parabolic systems with less controls than equations. It is by now well-known that it can be a difficult problem in various situations and that there is still no complete theory in the literature. We will concentrate here on a particular case which is in the so-called cascade form and that can be written as follows

$$(1.1) \quad \begin{cases} \partial_t y_1 - \operatorname{div}(\gamma(x)\nabla y_1) = 0 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma(x)\nabla y_2) + y_1 = 0 & \text{in } (0, T) \times \Omega, \\ y_1(0, \cdot) = y_{0,1} & \text{in } \Omega, \\ y_2(0, \cdot) = y_{0,2} & \text{in } \Omega, \end{cases}$$

where the control v will be acting only on the component y_1 on some part Γ_0 of the boundary Γ of the domain Ω . Since we want to control both components of the system and v has no direct influence in the equation for y_2 , the role of the coupling term y_1 in the second equation is fundamental: it acts as an indirect control term. We refer for instance to the review paper [5] for a general presentation of different results on that topic.

The usually studied case is the one of a Dirichlet control, which means that the above system is supplemented with the following boundary conditions

$$(1.2) \quad \begin{cases} y_1 = \mathbf{1}_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\ y_2 = 0 & \text{on } (0, T) \times \Gamma. \end{cases}$$

In the present work we would like to analyse the controllability properties of the same system when one considers, instead of the Dirichlet boundary conditions, a set of Robin boundary conditions with two non negative parameters β_1, β_2

$$(1.3) \quad \begin{cases} \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = \mathbf{1}_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 & \text{on } (0, T) \times \Gamma, \end{cases}$$

where the conormal derivative operator associated to the diffusion tensor γ is defined by

$$\frac{\partial}{\partial \nu_\gamma} = \nu \cdot (\gamma \nabla \cdot).$$

1.2. Motivations and overview of the paper. Our motivation for studying the above problem is two-fold.

The first one comes from the fact that it is an instance of the very popular penalisation approach to deal with boundary condition that have never been studied, as far as we know, in the framework of the controllability of coupled parabolic systems. From a numerical point of view, for instance when considering a Galerkin approximation of an elliptic or parabolic equation, this approach consists in replacing a Dirichlet boundary condition $y = g$ by

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33 a Robin boundary condition $\frac{1}{\beta}\partial_{\nu_\gamma}y + y = g$, with a large penalisation parameter β . It generally induces more
 34 flexibility and robustness in the computational code. This approach is indeed proposed in many finite element
 35 libraries and softwares. Moreover, it is also a suitable way to deal with data g that have a regularity lower than the
 36 one expected to solve the problem in the usual energy spaces (typically if $g \notin H^{1/2}(\Gamma)$ when solving the Laplace
 37 equation). This approach was initially studied in [7] for elliptic problems or in [9] for parabolic problems. In the
 38 framework of control theory, this penalisation approach was for instance analyzed in [8, 13] for solving optimal
 39 control of elliptic equations. In each case, it is proven that the solution of the penalised problem actually converges
 40 to the one of the original problem, with some estimate of the rate of convergence.

41 Our motivation is thus to investigate the same kind of issues for the coupled parabolic system (1.1) with a
 42 single boundary control and in particular to show that, not only the problem (1.1)-(1.3) is null-controllable, but
 43 more importantly, that it is possible to prove estimates on the controls that are (in some sense that will be precised
 44 later) uniform with respect to the Robin (penalisation) parameter. It will follow that the corresponding controlled
 45 solution converges towards a controlled solution of the Dirichlet problem when those parameters go to infinity.

46 Another motivation for this analysis, related to the discussion above, is that Robin boundary conditions have a
 47 regularizing effect on the boundary data. Indeed, as it will be recalled at the beginning of Section 2, the functional
 48 analysis adapted to boundary controls in L^2 for parabolic systems is a little intricate since, with such a low regularity
 49 of the data, we cannot expect solutions to exist in the usual energy space $C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ and to
 50 satisfy a standard weak formulation. Instead the solutions are known to live in a larger space $C^0([0, T], H^{-1}(\Omega)) \cap$
 51 $L^2(0, T, L^2(\Omega))$, the boundary condition being understood in a weak sense. When changing the Dirichlet boundary
 52 condition into Robin (or Neumann) boundary conditions, the functional setting is more comfortable and we recover
 53 the expected regularity for weak solutions even if the boundary data is only in L^2 .

54 Finally, we want to recall here that we lack of general mathematical techniques to deal with the controllability
 55 issue for those systems; applicability of the few available methods is very dependent on the structure of the underlying
 56 operators. Therefore, the analysis of each significantly new system needs to develop more elements (of spectral
 57 nature in our case) that are interesting by themselves and possibly useful in other situations. More precisely,
 58 there are not that many works regarding the controllability of coupled parabolic systems with less controls than
 59 equations, especially for boundary controls. This is mainly due to the fact that the very powerful *Carleman estimates*
 60 *approach* is essentially inefficient in that context. In particular, we recall that the boundary controllability for such
 61 systems is no longer equivalent with distributed controllability as it has been observed for instance in [18], see
 62 also [5]. In particular, most of the available controllability results concern the 1D setting since they are based on
 63 the moments method (that we will discuss below) which is not straightforward to implement in higher dimension.
 64 Among the few results available, we mention [4] where the authors proved the controllability to trajectories of
 65 a 1D system of n parabolic equations when $m < n$ number of controls are exerted on a part of the boundary
 66 through Dirichlet conditions. They actually proved that a general Kalman condition is a necessary and sufficient
 67 controllability condition for this problem. In the multi-dimensional case, we quote [10, 2], where controllability
 68 results are obtained in particular cylindrical geometries by exploiting on the one hand a sharp estimate of the
 69 control for the associated 1D problem and on the other hand spectral Lebeau-Robbiano inequalities, see also the
 70 discussion in Section 2.4.2. We also mention [1] where symmetric parabolic coupled systems are analysed in any
 71 dimension, provided that the control region satisfies the Geometric Control Condition.

72 *Paper organisation.* In Section 2, we first recall the different notions of solutions for (1.1) with boundary
 73 conditions (1.2) or (1.3), that we will need in the paper and we give the associated wellposedness and regularity
 74 results. In Section 2.4, we give the precise statements of our main results. As mentioned above, those results
 75 essentially say that the coupled parabolic system with Robin boundary condition is null-controllable at any time
 76 $T > 0$ and that we can find uniform bounds on the control that allow to justify the convergence towards a control
 77 for the Dirichlet problem when the Robin parameters are large. The proofs are given in Sections 3,4 and 5. They
 78 are based on the moments method [17] and on its recent extension called block moments method [11]; they require
 79 in particular a careful analysis of spectral properties of the underlying operators, with estimates uniform with
 80 respect to the parameters. Some of those spectral estimates are particularly difficult to obtain when the two Robin
 81 parameters are different, that is why in that case we restrict our analysis to a constant diffusion coefficient γ .

82 *Notations.* Throughout this paper C or C' denotes a generic positive constant (that may vary from line to
 83 line) which does not depend on T , y_0 nor the parameters β_1, β_2 but may depend on the diffusion coefficient γ .
 84 Sometimes, we will make emphasis on the dependence of a constant on some quantities $\alpha_1, \alpha_2, \dots, \alpha_n$ ($n \geq 1$) by
 85 $C_{\alpha_1, \alpha_2, \dots, \alpha_n}$.

Moreover, we shall use the following notation

$$((a, b)) := (\min\{a, b\}, \max\{a, b\}), \quad \text{for any } a, b \in \mathbb{R},$$

86 which is an open interval in \mathbb{R} .

87 The euclidean inner product in \mathbb{R}^d , $d \geq 1$, will be denoted by $\xi_1 \cdot \xi_2$ for any $\xi_1, \xi_2 \in \mathbb{R}^d$.

88 **2. General setting and main results.** In this section, we will first discuss about the well-posedness for our
89 parabolic system with Dirichlet and Robin boundary condition with L^2 data. We will be particularly interested in
90 estimates on the solutions that are uniform with respect to the Robin parameters. Then, we will give our main
91 results concerning the associated control problems.

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and $\gamma : \Omega \rightarrow M_d(\mathbb{R})$ be a smooth bounded field of symmetric matrices which are uniformly coercive: there is a $\gamma_{min} > 0$ such that

$$(\gamma(x)\xi) \cdot \xi \geq \gamma_{min}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \forall x \in \Omega.$$

92 We will first study the scalar problem before discussing the coupled cascade system.

93 2.1. The scalar problem.

94 **2.1.1. Dirichlet boundary data.** We first recall the usual setting adapted to the analysis of the Dirichlet
95 problem

$$96 \quad (2.1) \quad \begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ y = g & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases}$$

97 with non smooth data. In the case where $g = 0$, we can easily solve the above problem in a weak sense in
98 $C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ for given $y_0 \in L^2(\Omega)$. This can be done by using the continuous semigroup in
99 $L^2(\Omega)$ associated with the operator $-A_D = \operatorname{div}(\gamma \nabla \cdot)$ with the domain $D(A_D) = H^2(\Omega) \cap H_0^1(\Omega)$. However, if one
100 considers the case where g is any data in $L^2((0, T) \times \Gamma)$ which is the usual framework in control theory, we cannot
101 define as easily a good notion of weak solution because of a lack of regularity of the data. Instead, we have the
102 following well-posedness result in a dual sense, see for instance [14, 25].

103 **PROPOSITION 2.1.** *For any $y_0 \in L^2(\Omega)$, $f \in L^2((0, T) \times \Omega)$, $g \in L^2((0, T) \times \Gamma)$, there exists a unique $y \in$
104 $C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ solution of (2.1) in the following sense: for any $t \in [0, T]$ and $\zeta \in H_0^1(\Omega)$, we
105 have*

$$106 \quad \langle y(t), \zeta \rangle_{H^{-1}, H_0^1} = (y_0, e^{-tA_D^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f \left(e^{-(t-\cdot)A_D^*} \zeta \right) - \int_0^t \int_{\Gamma} g \frac{\partial}{\partial \nu_{\gamma}} \left(e^{-(t-\cdot)A_D^*} \zeta \right).$$

107 **REMARK 2.2.** *The operator A_D being self-adjoint, we could have replaced A_D^* by A_D in the previous statement
108 but we prefer to keep it in order to be consistent with the non-scalar case that we will consider in Section 2.3.*

109 **2.1.2. Homogeneous Robin boundary data.** For any $\beta \in [0, +\infty)$, we consider now the following parabolic
110 problem

$$111 \quad (2.2) \quad \begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial \nu_{\gamma}} + \beta y = 0 & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases}$$

112 where the regularity of y_0 and f will be precised below.

If the data are regular enough, the semigroup theory also gives a solution for this problem. Indeed, if one introduces the (self-adjoint) unbounded operator $A_{\beta} = -\operatorname{div}(\gamma \nabla \cdot)$ in $L^2(\Omega)$ associated with the domain

$$D(A_{\beta}) = \left\{ u \in H^2(\Omega) \mid \frac{\partial y}{\partial \nu_{\gamma}} + \beta y = 0 \text{ on } \Gamma \right\},$$

113 then we can prove that $-A_{\beta}$ generates a continuous semigroup in $L^2(\Omega)$. Hence, the following result holds.

PROPOSITION 2.3. *Let $\beta \in [0, +\infty)$ be given. For any $y_0 \in D(A_\beta)$ and $f \in C^1([0, T], L^2(\Omega))$, there exists a unique strong solution y to (2.2) in $C^1([0, T], L^2(\Omega)) \cap C^0([0, T], D(A_\beta))$, which is given by*

$$y(t) = e^{-tA_\beta} y_0 + \int_0^t e^{-(t-s)A_\beta} f(s) ds.$$

114 *Moreover, this solution satisfies the energy estimates*

$$115 \quad (2.3) \quad \|y\|_{L^\infty(0,T,L^2(\Omega))} + \|y\|_{L^2(0,T,H^1(\Omega))} + \sqrt{\beta}\|y\|_{L^2((0,T)\times\Gamma)} \leq C_T(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}),$$

116 *and*

117

$$118 \quad (2.4) \quad \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T)\times\Omega)} + \|y\|_{L^2(0,T,H^2(\Omega))} + \sqrt{\beta}\|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ 119 \quad \leq C_T(\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta}\|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0,T)\times\Omega)}),$$

121 *where $C_T > 0$ does not depend on β .*

122 *In particular, if $y_0 \in D(A_\beta) \cap H_0^1(\Omega)$, we have an estimate whose right-hand side does not depend on β*

123

$$124 \quad (2.5) \quad \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T)\times\Omega)} + \|y\|_{L^2(0,T,H^2(\Omega))} + \sqrt{\beta}\|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ 125 \quad \leq C_T(\|\nabla y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}).$$

127 *Proof.* The existence of a unique strong solution is a standard result from semigroup theory, see for instance
128 [15, Corollary 7.6]. We only sketch the proof of the estimates. The weak estimate (2.3) simply comes by multiplying
129 the equation by y and using that

$$130 \quad (2.6) \quad (A_\beta \zeta, \zeta)_{L^2} = \int_\Omega (\gamma \nabla \zeta) \cdot \nabla \zeta + \beta \int_\Gamma |\zeta|^2, \quad \forall \zeta \in D(A_\beta).$$

To prove the stronger estimate, we first assume that $y_0 \in D(A_\beta^2)$ and that $f \in C^1([0, T], D(A_\beta))$, the final result being deduced by a density argument. With this regularity of the data we can justify that

$$\frac{1}{2} \frac{d}{dt} (A_\beta y, y)_{L^2} = (A_\beta y, \partial_t y)_{L^2} = -\|\partial_t y\|_{L^2}^2 + (f, \partial_t y)_{L^2}.$$

Using the Cauchy-Schwarz inequality, and integrating in time, we get

$$(A_\beta y(t), y(t))_{L^2} + \int_0^t \|\partial_t y\|_{L^2}^2 \leq (A_\beta y_0, y_0)_{L^2} + \int_0^t \|f\|_{L^2}^2.$$

131 By (2.6), it follows that

132

$$133 \quad (2.7) \quad \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T)\times\Omega)} + \sqrt{\beta}\|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ 134 \quad \leq C_T(\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta}\|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0,T)\times\Omega)}).$$

It remains to prove the $L^2(0, T, H^2(\Omega))$ estimate. To this end, we observe that

$$\|A_\beta y\|_{L^2((0,T)\times\Omega)} \leq \|f\|_{L^2((0,T)\times\Omega)} + \|\partial_t y\|_{L^2((0,T)\times\Omega)},$$

and thus the claim is just a consequence of (2.7) and of the following elliptic regularity property: there exists a $C > 0$, independent of $\beta \in [0, +\infty)$, such that

$$\|\zeta\|_{H^2(\Omega)} \leq C(\|\zeta\|_{L^2(\Omega)} + \|A_\beta \zeta\|_{L^2(\Omega)}), \quad \forall \zeta \in D(A_\beta).$$

136 This can be proved, for instance, as in [12, Theorems III.4.2 and III.4.3] and using the fact that $\beta \geq 0$ to obtain a
137 constant which is independent of β . \square

138 **2.1.3. Non-homogeneous Robin boundary data.** Let us now consider the same problem but with a non-
139 homogeneous boundary data

$$(2.8) \quad \begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial \nu_\gamma} + \beta y = g & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases}$$

141 The theory developed in [23] for this problem gives the following result concerning existence and uniqueness of a
142 solution in the natural energy spaces.

143 **PROPOSITION 2.4.** *Let $\beta \in [0, +\infty)$ be given. For any $y_0 \in L^2(\Omega)$, $f \in L^2((0, T) \times \Omega)$, $g \in L^2((0, T) \times \Gamma)$, there
144 exists a unique weak solution $y \in C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ to (2.8) in the following sense:*

- 145 • $y(0) = y_0$.
- 146 • For any test function $\psi \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$, and any $t_1, t_2 \in [0, T]$ we have

$$(2.9) \quad - \int_{t_1}^{t_2} \int_{\Omega} y \psi_t + \int_{t_1}^{t_2} \int_{\Omega} (\gamma \nabla y) \cdot \nabla \psi + \beta \int_{t_1}^{t_2} \int_{\Gamma} y \psi \\ = \int_{\Omega} y(t_1) \psi(t_1) - \int_{\Omega} y(t_2) \psi(t_2) + \int_{t_1}^{t_2} \int_{\Omega} f \psi + \int_{t_1}^{t_2} \int_{\Gamma} g \psi.$$

151 Moreover, it satisfies the estimate

$$(2.10) \quad \|y\|_{C^0([0, T], L^2(\Omega))} + \|y\|_{L^2(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2(0, T, H^{-1}(\Omega))} \\ \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)} + \|g\|_{L^2((0, T) \times \Gamma)}),$$

156 for some $C_T > 0$ independent of β .

REMARK 2.5 (Strong estimates do not pass to the limit). Note that if the boundary data for the Robin problem
(2.8) is chosen in the form $g = \beta g_D$, with $g_D \in L^2((0, T) \times \Gamma)$ then the boundary condition reads

$$\frac{1}{\beta} \frac{\partial y}{\partial \nu_\gamma} + y = g_D, \quad \text{for } \beta \in (0, +\infty),$$

157 and we can formally expect the solution to converge, when $\beta \rightarrow \infty$, towards the one associated with the Dirichlet
158 boundary condition $y = g_D$, that is to a solution of (2.1).

159 However, the estimate in the proposition above reads

$$\|y\|_{C^0([0, T], L^2(\Omega))} + \|y\|_{L^2(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2(0, T, H^{-1}(\Omega))} \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)} + \beta \|g_D\|_{L^2((0, T) \times \Gamma)}),$$

161 which is not uniform with respect to β and therefore we cannot a priori prove that the associated solution y is
162 bounded when $\beta \rightarrow \infty$. This is due to the fact that, considering only L^2 boundary data, we cannot expect a uniform
163 bound in $L^2(0, T, H^1(\Omega))$ of the solution that would necessitate at least g_D to be in $L^2(0, T, H^{1/2}(\Gamma))$.

164 For the reasons above, we need to introduce a weaker formulation of the Robin problem that will allow to
165 analyse the limit towards the Dirichlet problem with L^2 data in a convenient way.

166 **PROPOSITION 2.6.** *We consider the same assumption as in Proposition 2.4.*

1. The weak solution y to the problem (2.8) is the unique function belonging to $C^0([0, T], L^2(\Omega))$ and satisfying, for any $\zeta \in L^2(\Omega)$ and any $t \in [0, T]$,

$$(y(t), \zeta)_{L^2} - (y_0, e^{-tA_\beta^*} \zeta)_{L^2} = \int_0^t \int_{\Omega} f \left(e^{-(t-\cdot)A_\beta^*} \zeta \right) + \int_0^t \int_{\Gamma} g e^{-(t-\cdot)A_\beta^*} \zeta$$

167 in addition with the estimate (2.10).

2. The weak solution y to the problem (2.8) with $\beta \in (0, +\infty)$, is also the unique function belonging to
168 $C^0([0, T], H^{-1}(\Omega))$ and satisfying, for any $\zeta \in D(A_\beta^*) \cap H_0^1(\Omega)$ and any $t \in [0, T]$,

$$(y(t), \zeta)_{H^{-1}, H_0^1} - (y_0, e^{-tA_\beta^*} \zeta)_{L^2} = \int_0^t \int_{\Omega} f \left(e^{-(t-\cdot)A_\beta^*} \zeta \right) - \int_0^t \int_{\Gamma} \frac{g}{\beta} \frac{\partial}{\partial \nu_\gamma} \left(e^{-(t-\cdot)A_\beta^*} \zeta \right).$$

170

Moreover, this weak solution satisfies the estimate

$$(2.11) \quad \|y\|_{C^0([0,T],H^{-1}(\Omega))} + \|y\|_{L^2((0,T)\times\Omega)} + \|\partial_t y\|_{L^2(0,T,H^{-2}(\Omega))} \\ \leq C_T \left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \right),$$

where $C_T > 0$ does not depend on β .

REMARK 2.7. As explained in Remark 2.2, we decided to keep the adjoint notation A_β^* instead of A_β in the previous statement, even though it is unnecessary.

Proof. 1. Let us first consider any $\zeta \in D(A_\beta^*) = D(A_\beta)$ and let us choose as a test function in (2.9) the strong solution of the homogeneous backward problem $t \mapsto \psi(t) = e^{-(T-t)A_\beta^*} \zeta$. The integration by parts are well justified and naturally lead to the expected formula. By density of $D(A_\beta)$ in $L^2(\Omega)$ and by the estimate (2.3), we can extend the equality to any $\zeta \in L^2(\Omega)$.
2. In the case where $\zeta \in D(A_\beta) \cap H_0^1(\Omega)$, we know that $\psi(t) \in D(A_\beta)$ for any t , and in particular we have the equality $\frac{\partial \psi}{\partial \nu_\gamma} + \beta \psi = 0$ on $(0, T) \times \Gamma$, which gives the claimed equality. Now, applying the estimates (2.3) and (2.5) to ψ , we obtain for any $t \in [0, T]$ and $\beta \in (0, +\infty)$,

$$|\langle y(t), \zeta \rangle_{H^{-1}, H_0^1}| \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}) \|\zeta\|_{L^2(\Omega)} + C_T \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \|\nabla \zeta\|_{L^2(\Omega)}.$$

Since $D(A_\beta) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, we get the expected bound on $\|y\|_{C^0([0,T],H^{-1}(\Omega))}$. Let us show now the bound in $L^2((0, T) \times \Omega)$. Consider any $h \in C_c^\infty((0, T) \times \Omega)$ and let ψ be the unique strong solution (as given by Proposition 2.3) to the backward problem

$$(2.12) \quad \begin{cases} -\partial_t \psi - \operatorname{div}(\gamma \nabla \psi) = h & \text{in } (0, T) \times \Omega, \\ \frac{\partial \psi}{\partial \nu_\gamma} + \beta \psi = 0 & \text{on } (0, T) \times \Gamma, \\ \psi(T, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By (2.5), we have the following estimate, uniformly with respect to the parameter β

$$(2.13) \quad \|\psi\|_{L^\infty(0,T,H^1(\Omega))} + \|\psi\|_{L^2(0,T,H^2(\Omega))} + \|\partial_t \psi\|_{L^2(0,T,L^2(\Omega))} \leq C_T \|h\|_{L^2((0,T)\times\Omega)}.$$

Putting this test function in (2.9) and integrating by parts lead to

$$\int_0^T \int_\Omega y h = \int_\Omega y_0 \psi(0) + \int_0^T \int_\Omega f \psi - \int_0^T \int_\Gamma \frac{g}{\beta} \frac{\partial \psi}{\partial \nu_\gamma}, \quad \text{for } \beta \in (0, +\infty),$$

where we have used the boundary condition satisfied by ψ at each time t in the boundary term. Using the Cauchy-Schwarz inequality and (2.13), we finally get

$$\left| \int_0^T \int_\Omega y h \right| \leq C (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}) \|\psi\|_{L^\infty(0,T,L^2(\Omega))} + C \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \|\psi\|_{L^2(0,T,H^2(\Omega))} \\ \leq C_T \left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \right) \|h\|_{L^2((0,T)\times\Omega)}.$$

Since $C_c^\infty((0, T) \times \Omega)$ is dense in $L^2((0, T) \times \Omega)$, we obtain the expected estimate by duality.

Finally, we can easily see that the weak solution y satisfies, in the distribution sense, the equation $\partial_t y - \operatorname{div}(\gamma \nabla y) = f$, and the bound of $\partial_t y$ in $L^2(0, T, H^{-2}(\Omega))$ immediately follows. \square

REMARK 2.8 (Weak estimates pass to the limit). Going back to the situation described in Remark 2.5, that is if $g = \beta g_D$, $g_D \in L^2((0, T) \times \Gamma)$, we deduce now a bound of the associated solution which is uniform when $\beta \rightarrow +\infty$, yet in weaker norms than above. We shall see in the next section that those estimates allow us to pass to the limit towards the Dirichlet problem.

205 **2.2. Passing to the limit to the Dirichlet problem.** With the above existence results and estimates, we
 206 can now state and prove a convergence result of the solutions of a suitable Robin problem to the one of a Dirichlet
 207 problem.

208 **THEOREM 2.9** (Convergence towards the Dirichlet problem). *Let $y_0 \in L^2(\Omega)$ be a given initial data. For any*
 209 *$\beta > 0$ we consider a source term $f_\beta \in L^2((0, T) \times \Omega)$ and a boundary data $g_\beta \in L^2((0, T) \times \Gamma)$, and we denote by*
 210 *y_β the associated weak solution to (2.8).*

211 *We assume that, for some $f_D \in L^2((0, T) \times \Omega)$, $g_D \in L^2((0, T) \times \Gamma)$ we have the L^2 -weak convergences*

$$212 \quad (2.14) \quad f_\beta \xrightarrow{\beta \rightarrow +\infty} f_D, \quad \frac{g_\beta}{\beta} \xrightarrow{\beta \rightarrow +\infty} g_D,$$

213 *Then y_β converges, when $\beta \rightarrow +\infty$, weakly in $L^2((0, T) \times \Omega)$ and strongly in $L^2(0, T, H^{-1}(\Omega))$ towards the*
 214 *unique solution $y_D \in C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ to the Dirichlet problem (2.1) associated to the data f_D*
 215 *and g_D .*

216 *Moreover, for any $t \in [0, T]$, $y_\beta(t) \xrightarrow{\beta \rightarrow +\infty} y_D(t)$ weakly in $H^{-1}(\Omega)$.*

217 *Proof.* From the hypothesis, we have a bound on the quantities $\|g_\beta/\beta\|_{L^2((0, T) \times \Gamma)}$ and $\|f_\beta\|_{L^2((0, T) \times \Omega)}$ uniform
 218 with respect to $\beta \geq 1$. Hence, from (2.11), we deduce that, for some $C_{T, y_0} > 0$, uniform in β , we have

$$219 \quad (2.15) \quad \|y_\beta\|_{C^0([0, T], H^{-1}(\Omega))} + \|y_\beta\|_{L^2((0, T) \times \Omega)} + \|\partial_t y_\beta\|_{L^2(0, T, H^{-2}(\Omega))} \leq C_{T, y_0}.$$

221 We can then find some $y_D \in C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ and a subsequence, still denoted by $(y_\beta)_\beta$ such
 222 that

$$223 \quad (2.16) \quad \begin{cases} y_\beta \xrightarrow{\beta \rightarrow +\infty} y_D & \text{weakly in } L^2((0, T) \times \Omega), \\ \partial_t y_\beta \xrightarrow{\beta \rightarrow +\infty} \partial_t y_D & \text{weakly in } L^2(0, T, H^{-2}(\Omega)), \\ y_\beta \xrightarrow[\beta \rightarrow +\infty]{*} y_D & \text{weakly-* in } C^0([0, T], H^{-1}(\Omega)), \\ y_\beta \xrightarrow[\beta \rightarrow +\infty} y_D & \text{strongly in } L^2(0, T, H^{-1}(\Omega)). \end{cases}$$

225 The last strong convergence comes from the compactness of the embeddings $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow$
 226 $H^{-2}(\Omega)$ and the Aubin-Lions lemma.

227 All we need to show is that this limit y_D is indeed the solution to the corresponding Dirichlet problem. By
 228 uniqueness of the solution of Dirichlet problem (2.1) with the data f_D, g_D , the convergence of the whole family
 229 $(y_\beta)_\beta$ will be established.

230 Let us consider a final data $\zeta \in C_c^\infty(\Omega) \subset D(A_\beta^*) \cap H_0^1(\Omega)$ for the adjoint homogeneous problem. The corre-
 231 sponding strong solution is given by $\psi_\beta(t) = e^{-(T-t)A_\beta^*} \zeta$ and, thanks to (2.5), we have

$$232 \quad (2.17) \quad \|\psi_\beta\|_{C^0([0, T], H^1(\Omega))} + \|\psi_\beta\|_{L^2(0, T, H^2(\Omega))} + \|\partial_t \psi_\beta\|_{L^2((0, T) \times \Omega)} \leq C_T \|\zeta\|_{H_0^1(\Omega)},$$

234 where C_T is uniform in β .

235 We can then extract a subsequence, still denoted by $(\psi_\beta)_\beta$, such that

$$236 \quad (2.18) \quad \begin{cases} \psi_\beta \xrightarrow{\beta \rightarrow +\infty} \psi_D & \text{weakly in } L^2(0, T, H^2(\Omega)), \\ \partial_t \psi_\beta \xrightarrow{\beta \rightarrow +\infty} \partial_t \psi_D & \text{weakly in } L^2((0, T) \times \Omega), \\ \psi_\beta \xrightarrow[\beta \rightarrow +\infty]{*} \psi_D & \text{weakly-* in } C^0([0, T], H^1(\Omega)), \\ \psi_\beta \xrightarrow[\beta \rightarrow +\infty} \psi_D & \text{strongly in } L^2(0, T, H^1(\Omega)), \end{cases}$$

238 for some $\psi_D \in L^2(0, T, H^2(\Omega)) \cap C^0([0, T], H^1(\Omega))$. Here also we have used the Aubin-Lions lemma to obtain the
 239 last strong convergence.

Moreover, from the boundary condition satisfied by ψ_β , we have $\psi_\beta = -\frac{1}{\beta} \frac{\partial \psi_\beta}{\partial \nu_\gamma}$ on the boundary where the
 quantity $\left\| \frac{\partial \psi_\beta}{\partial \nu_\gamma} \right\|_{L^2((0, T) \times \Gamma)}$ is bounded for any large β (using (2.17)). Hence, it follows that

$$\psi_\beta \xrightarrow{\beta \rightarrow \infty} 0 \quad \text{in } L^2((0, T) \times \Gamma),$$

240 which actually implies that $\psi_D = 0$ on the boundary $(0, T) \times \Gamma$. Moreover, by passing to the limit in the equation
 241 in the distribution sense, we finally find that ψ_D is the unique solution to the backward homogeneous Dirichlet
 242 problem, that is $\psi_D(t) = e^{-(T-t)A_D^*} \zeta$.

243 Now, using the trace theorem, we observe that

$$244 \quad \int_0^T \int_{\Gamma} \left| \frac{\partial \psi_{\beta}}{\partial \nu_{\gamma}} - \frac{\partial \psi_D}{\partial \nu_{\gamma}} \right|^2 \leq C \int_0^T \|\psi_{\beta} - \psi_D\|_{H^1(\Omega)} \|\psi_{\beta} - \psi_D\|_{H^2(\Omega)}$$

$$245 \quad \leq C \|\psi_{\beta} - \psi_D\|_{L^2(0, T, H^1(\Omega))} \|\psi_{\beta} - \psi_D\|_{L^2(0, T, H^2(\Omega))}.$$

247 By (2.18) we see that the first factor of the very right hand side of the above inequality converges to 0 as $\beta \rightarrow +\infty$,
 248 whereas the second factor is bounded. Thus, we have

$$249 \quad (2.19) \quad \frac{\partial \psi_{\beta}}{\partial \nu_{\gamma}} \xrightarrow{\beta \rightarrow +\infty} \frac{\partial \psi_D}{\partial \nu_{\gamma}} \quad \text{in } L^2((0, T) \times \Gamma).$$

251 In particular, from the third point of (2.18) we have

$$252 \quad (2.20) \quad \psi_{\beta}(0, \cdot) \xrightarrow{\beta \rightarrow +\infty} \psi_D(0, \cdot) \quad \text{weakly in } H^1(\Omega).$$

254 With the convergence results (2.19) and (2.20) above together with (2.14) we get

$$255 \quad (y_0, \psi_{\beta}(0))_{L^2} + \int_0^t \int_{\Omega} f_{\beta} \psi_{\beta} - \int_0^t \int_{\Gamma} \frac{g_{\beta}}{\beta} \frac{\partial \psi_{\beta}}{\partial \nu_{\gamma}} \xrightarrow{\beta \rightarrow +\infty} (y_0, \psi_D(0))_{L^2} + \int_0^t \int_{\Omega} f_D \psi_D - \int_0^t \int_{\Gamma} g_D \frac{\partial \psi_D}{\partial \nu_{\gamma}}.$$

256 Using the weak formulation (2.9) satisfied by y_{β} with ψ_{β} as a test function we see that we have actually proved that

$$257 \quad (y_{\beta}(t), \zeta)_{L^2} \xrightarrow{\beta \rightarrow +\infty} (y_0, \psi_D(0))_{L^2} + \int_0^t \int_{\Omega} f_D \psi_D - \int_0^t \int_{\Gamma} g_D \frac{\partial \psi_D}{\partial \nu_{\gamma}},$$

and in the same time, by (2.16), we have

$$(y_{\beta}(t), \zeta)_{L^2} = \langle y_{\beta}(t), \zeta \rangle_{H^{-1}, H_0^1} \xrightarrow{\beta \rightarrow +\infty} \langle y_D(t), \zeta \rangle_{H^{-1}, H_0^1}.$$

258 As a conclusion, we have proved that y_D satisfies

$$259 \quad \langle y_D(t), \zeta \rangle_{H^{-1}, H_0^1} = (y_0, e^{-tA_D^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f_D \left(e^{-(t-\cdot)A_D^*} \zeta \right) - \int_0^t \int_{\Gamma} g_D \frac{\partial}{\partial \nu_{\gamma}} \left(e^{-(t-\cdot)A_D^*} \zeta \right),$$

260 which is exactly the definition of the solution of (2.1) with the data f_D, g_D , see Proposition 2.1. \square

261 **REMARK 2.10** (Convergence towards the Neumann problem). *By similar, and in fact simpler, arguments one*
 262 *can prove that if $(f_{\beta})_{\beta}$ and $(g_{\beta})_{\beta}$ both weakly converge, when $\beta \rightarrow 0$, towards some f_N and g_N in $L^2((0, T) \times \Omega)$*
 263 *and $L^2((0, T) \times \Gamma)$ respectively, then the corresponding solution y_{β} converges, when $\beta \rightarrow 0$, to the solution y_N of*
 264 *the corresponding non-homogeneous Neumann problem.*

265 **2.3. The coupled system.** We can now move to the cascade coupled parabolic systems we are interested in,
 266 namely the one with Dirichlet boundary condition

$$267 \quad (2.21) \quad \begin{cases} \partial_t y_1 - \operatorname{div}(\gamma \nabla y_1) & = f_1 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma \nabla y_2) + y_1 & = f_2 & \text{in } (0, T) \times \Omega, \\ & y_1 = g_1 & \text{on } (0, T) \times \Gamma, \\ & y_2 = g_2 & \text{on } (0, T) \times \Gamma, \\ & y_1(0, \cdot) = y_{0,1} & \text{in } \Omega, \\ & y_2(0, \cdot) = y_{0,2} & \text{in } \Omega, \end{cases}$$

268 and the one with Robin boundary conditions

$$\begin{cases}
 \partial_t y_1 - \operatorname{div}(\gamma \nabla y_1) &= f_1 & \text{in } (0, T) \times \Omega, \\
 \partial_t y_2 - \operatorname{div}(\gamma \nabla y_2) + y_1 &= f_2 & \text{in } (0, T) \times \Omega, \\
 \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 &= g_1 & \text{on } (0, T) \times \Gamma, \\
 \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 &= g_2 & \text{on } (0, T) \times \Gamma, \\
 y_1(0, \cdot) &= y_{0,1} & \text{in } \Omega, \\
 y_2(0, \cdot) &= y_{0,2} & \text{in } \Omega.
 \end{cases}
 \tag{2.22}$$

270 We can obviously solve those two systems by simply using the results on the scalar case: we first solve the equation
 271 for y_1 then we solve the scalar equation for y_2 by considering the coupling term y_1 as an additional L^2 source term.

272 **THEOREM 2.11.** *We suppose given $y_0 := (y_{0,1}, y_{0,2}) \in (L^2(\Omega))^2$, $f := (f_1, f_2) \in (L^2((0, T) \times \Omega))^2$ and $g :=$
 273 $(g_1, g_2) \in (L^2((0, T) \times \Gamma))^2$.*

- 274 1. *There exists a unique solution $y = (y_1, y_2) \in (C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega))^2$ of (2.21), that is, for
 275 any $i = 1, 2$, y_i satisfies the corresponding scalar problem in the sense of [Proposition 2.1](#).*
- 276 2. *For any $\beta_1, \beta_2 \in [0, +\infty)$, there exists a unique solution $y = (y_1, y_2) \in (C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)))^2$
 277 of (2.22), that is, for any $i = 1, 2$, y_i satisfies the corresponding scalar problem in the sense of [Proposi-
 278 tion 2.4](#).*
- 279 3. *For any $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$, we suppose given $f_{i,\beta} \in L^2((0, T) \times \Omega)$ and $g_{i,\beta} \in L^2((0, T) \times \Gamma)$, for
 280 $i = 1, 2$ such that*

$$\frac{g_{i,\beta}}{\beta_i} \xrightarrow{\beta \rightarrow +\infty} g_i, \quad f_{i,\beta} \xrightarrow{\beta \rightarrow +\infty} f_i.$$

282 *Then, the solution y_β of (2.22) corresponding to the data f_β, g_β converges weakly in $(L^2((0, T) \times \Omega))^2$ and
 283 strongly in $(L^2(0, T, H^{-1}(\Omega)))^2$ towards the unique solution of the corresponding Dirichlet problem.*

284 For the analysis of the control problem, it is not convenient to make appear the component y_1 of the solution
 285 as a source term in the equation for y_2 since it breaks down the cascade structure of the system which is essential
 286 to prove its controllability with only one control. That is the reason why it is necessary to introduce the following
 287 unbounded operators in $(L^2(\Omega))^2$: let $\mathcal{A}_{\beta_1, \beta_2}$ and \mathcal{A}_D be defined by the same formal expression

$$\begin{pmatrix} -\operatorname{div}(\gamma \nabla \cdot) & 0 \\ 1 & -\operatorname{div}(\gamma \nabla \cdot) \end{pmatrix},$$

but with the different domains

$$D(\mathcal{A}_{\beta_1, \beta_2}) := \left\{ y \in (H^2(\Omega))^2 \left| \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = 0, \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 \text{ on } \Gamma \right. \right\},$$

and

$$D(\mathcal{A}_D) := (H^2(\Omega) \cap H_0^1(\Omega))^2,$$

290 respectively. Those operators are no more self-adjoint and we define their adjoints by $D(\mathcal{A}_{\beta_1, \beta_2}^*) = D(\mathcal{A}_{\beta_1, \beta_2})$ and
 291 $D(\mathcal{A}_D^*) = D(\mathcal{A}_D)$ and the same formal expression

$$\begin{pmatrix} -\operatorname{div}(\gamma \nabla \cdot) & 1 \\ 0 & -\operatorname{div}(\gamma \nabla \cdot) \end{pmatrix}.$$

294 Standard elliptic theory shows that $-\mathcal{A}_{\beta_1, \beta_2}$, and $-\mathcal{A}_{\beta_1, \beta_2}^*$ as well as $-\mathcal{A}_D$ and $-\mathcal{A}_D^*$ generate continuous
 295 semigroups in $(L^2(\Omega))^2$. A similar analysis as in Section [2.1.3](#) for the scalar case, leads to the following result.

296 **PROPOSITION 2.12.** *We suppose given any $y_0 \in (L^2(\Omega))^2$, $f \in (L^2((0, T) \times \Omega))^2$ and $g \in (L^2((0, T) \times \Gamma))^2$.*

- 297 1. *The solution to (2.21) is the unique element $y \in (C^0([0, T], H^{-1}(\Omega)))^2$ satisfying, for any $\zeta \in (H_0^1(\Omega))^2$
 298 and any $t \in [0, T]$*

$$\langle y(t), \zeta \rangle_{H^{-1}, H_0^1} = (y_0, e^{-t\mathcal{A}_D^*} \zeta)_{L^2} + \int_0^t \int_\Omega f \cdot (e^{-(t-\cdot)\mathcal{A}_D^*} \zeta) - \int_0^t \int_\Gamma g \cdot \frac{\partial}{\partial \nu_\gamma} (e^{-(t-\cdot)\mathcal{A}_D^*} \zeta).$$

- 300 2. For any $\beta_1, \beta_2 \in [0, +\infty)$, the solution to (2.22) is the unique element $y \in (C^0([0, T], L^2(\Omega)))^2$ satisfying,
 301 for any $\zeta \in (L^2(\Omega))^2$ and any $t \in [0, T]$

$$302 \quad (y(t), \zeta)_{L^2} = (y_0, e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f \cdot \left(e^{-(t-\cdot)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right) - \int_0^t \int_{\Gamma} g \cdot \left(e^{-(t-\cdot)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right).$$

303 **2.4. Main results.** Let now Γ_0 be a part of Γ . Using the analysis presented in the previous sections we can
 304 now formulate the null-control problems we are interested in as follows.

305 PROPOSITION 2.13. Let $y_0 \in (L^2(\Omega))^2$ be given.

- 306 1. A function $v \in L^2((0, T) \times \Gamma)$ is a null-control at time T for the Dirichlet problem (1.1)-(1.2), if and only
 307 if it satisfies: for any $\zeta \in (H_0^1(\Omega))^2$

$$308 \quad (2.25) \quad (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \int_0^T \int_{\Gamma_0} 1_{\Gamma_0} v \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial \nu_{\gamma}} \left(e^{-(T-\cdot)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right).$$

- 309 2. A function $v \in L^2((0, T) \times \Gamma)$ is a null-control at time T for the Robin problem (1.1)-(1.3), if and only if
 310 it satisfies: for any $\zeta \in (L^2(\Omega))^2$

$$311 \quad (2.26) \quad - (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \int_0^T \int_{\Gamma} 1_{\Gamma_0} v \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \left(e^{-(T-\cdot)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right).$$

312 **2.4.1. The 1D case.** We start with a discussion of the 1D setting since, as we will see in the next section, we
 313 can deduce some multi-D results from the 1D analysis.

314 Hence, we particularize the above control problem to the 1D situation where $\Omega = (0, 1)$, $\Gamma_0 = \{0\}$ and the
 315 diffusion coefficient is simply a scalar function $\gamma \in C^1([0, 1])$ with $\gamma_{min} = \inf_{[0, 1]} \gamma > 0$ and $\gamma_{max} = \sup_{[0, 1]} \gamma < +\infty$.
 316 In that case the control we are looking for is just a scalar function $v \in L^2(0, T)$ and the formulations (2.25), (2.26)
 317 just reads

$$318 \quad (2.27) \quad - (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \gamma(0) \int_0^T v(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial x} \Big|_{x=0} \left(e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right) dt,$$

319 for the Dirichlet problem and

$$320 \quad (2.28) \quad - (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \int_0^T v(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \left(e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right) \Big|_{x=0} dt,$$

321 for the Robin problem with the same notations for the adjoint of the diffusion operators as in multi-D. It is convenient
 322 to introduce the observation operator \mathcal{B}^* (that does not depend on the Robin parameters β_1, β_2) defined as follows

$$323 \quad (2.29) \quad \mathcal{B}^* : \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in (H^1(0, 1))^2 \mapsto \zeta_1(0),$$

324 in such a way that (2.28) becomes

$$325 \quad (2.30) \quad - (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \int_0^T v(t) \mathcal{B}^* \left(e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta \right) dt.$$

326 Most of the work in Sections 4 and 5 will consist in solving this problem with suitable estimates of the control with
 327 respect to the parameters β_1 and β_2 . Our main result in that direction is the following.

328 THEOREM 2.14. Let $y_0 \in (L^2(0, 1))^2$ and $T > 0$ be given.

1. Let $\beta \in (0, +\infty)$ and set $\beta_1 = \beta_2 = \beta$. Then, there exists a null-control $v_{\beta} \in L^2(0, T)$ for the 1D problem
 (2.30) that satisfies in addition the estimate

$$\|v_{\beta}\|_{L^2(0, T)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{L^2(0, 1)},$$

329 where $C > 0$ does not depend on β and T .

2. Assume that γ is a positive constant and let $\beta^* > 0$ be given. Let $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$ be any couple of
 Robin parameters. Then, there exists a null-control $v_{\beta} \in L^2(0, T)$ for the 1D problem (2.30) that satisfies
 in addition the estimate

$$\|v_{\beta}\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0, 1)},$$

330 as soon as either $\beta_1, \beta_2 \in (0, \beta^*]$, or $\beta_1, \beta_2 \in [\beta^*, +\infty)$, where $C_{T, \beta^*} > 0$ does not depend on β .

331 COROLLARY 2.15 (Convergence towards Dirichlet control). *Let $\beta_n = (\beta_{1,n}, \beta_{2,n}) \in (0, +\infty)^2$ be any sequence*
 332 *of Robin parameters such that $\beta_{i,n} \rightarrow +\infty$ when $n \rightarrow \infty$, for $i = 1, 2$. If the diffusion coefficient γ is not a constant,*
 333 *we assume in addition that $\beta_{1,n} = \beta_{2,n}$ for any n . For each n , let v_n (resp. y_n) be the unique null-control of minimal*
 334 *$L^2(0, T)$ norm (resp. the associated trajectory) for the problem (2.30) with Robin parameters $\beta_{1,n}$ and $\beta_{2,n}$.*

Then, there exists a subsequence $(n_k)_k$ such that

$$\frac{v_{n_k}}{\beta_{1,n_k}} \xrightarrow[k \rightarrow \infty]{} v_D, \quad \text{in } L^2(0, T),$$

$$y_{n_k} \xrightarrow[k \rightarrow \infty]{} y_D, \quad \text{strongly in } (L^2(0, T, H^{-1}(0, 1)))^2 \text{ and weakly in } (L^2((0, T) \times (0, 1)))^2,$$

338 where v_D (resp. y_D) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (2.27).

339 REMARK 2.16 (Convergence towards Neumann control). *With the same notation as in the previous corollary,*
 340 *if we assume that $\beta_{i,n} \rightarrow 0$ when $n \rightarrow \infty$, for $i = 1, 2$, then we obtain the convergence, up to a subsequence, of the*
 341 *null-control v_n (resp. of the trajectory y_n) towards a null-control v_N (resp. the trajectory y_N) corresponding to the*
 342 *Neumann boundary conditions on both components.*

343 REMARK 2.17 (The Dirichlet/Neumann case). *In point 2 of Theorem 2.14, we needed to assume that either*
 344 *the two Robin parameters are both smaller than some β^* or that they are both higher than some β^* . It is worth*
 345 *noticing that we cannot expect to prove a similar result without those assumptions.*

346 *Indeed, if we were able to prove the estimate $\|v_\beta\| \leq C_T(1 + \beta_1)\|y_0\|_{L^2(0,1)}$, for any couple of parameters*
 347 *β_1 and β_2 , then by following the same lines as in Corollary 2.15, we would be able to prove the convergence,*
 348 *up to a subsequence, of $v_n/\beta_{1,n}$ when $\beta_{1,n} \rightarrow +\infty$ and $\beta_{2,n} \rightarrow 0$ to some v_{DN} that would be a null-control for the*
 349 *Dirichlet/Neumann problem (that is system (1.1) in 1D, with a Dirichlet boundary condition for the first component*
 350 *y_1 and a Neumann boundary condition for the second component y_2). However, we know that this last problem is*
 351 *not even approximately controllable since the underlying operator $\mathcal{A}_{\infty,0}^*$ has eigenspaces of dimension higher than*
 352 *1, which prevents the Fattorini-Hautus criterion (see [16, 24]) from being satisfied.*

353 *The same remark holds for the Neumann/Dirichlet problem, that is when $\beta_{1,n} \rightarrow 0$ and $\beta_{2,n} \rightarrow +\infty$.*

354 **2.4.2. A multi-D result.** By using the methodology described in [2, 10] it is possible, starting from a suitable
 355 null-controllability result for the 1D problem, at least when both Robin parameters are the same, to deduce the
 356 corresponding result in any cylinder of \mathbb{R}^d for $d \geq 2$, see Figure 1.

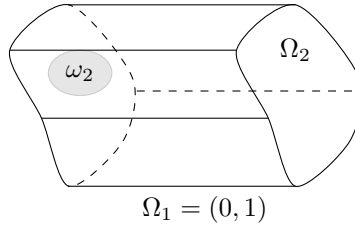


FIGURE 1. The cylindrical geometry

More precisely, we consider a domain $\Omega = (0, 1) \times \Omega_2$ in \mathbb{R}^d where Ω_2 is a bounded smooth connected domain in \mathbb{R}^{d-1} . The variable in Ω will be denoted by (x, \bar{x}) , with $x \in \Omega_1$ and $\bar{x} \in \Omega_2$ and we assume that the diffusion tensor has the following form

$$\gamma(x, \bar{x}) = \begin{pmatrix} \gamma(x) & 0 & \cdots & 0 \\ 0 & \left(\bar{\gamma}(\bar{x}) \right) \\ \vdots & & & \\ 0 & & & \end{pmatrix},$$

357 with $\gamma : \Omega_1 \rightarrow \mathbb{R}$ and $\bar{\gamma} : \Omega_2 \rightarrow M_{d-1}(\mathbb{R})$. Let $\omega_2 \subset \Omega_2$ be a non empty open subset of Ω_2 . The control region we
 358 will consider is $\Gamma_0 = \{0\} \times \omega_2$ so that the control problem is the following

$$(2.31) \quad \begin{cases} \partial_t y_1 - \partial_x(\gamma(x)\partial_x y_1) - \operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} y_1) = 0 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \partial_x(\gamma(x)\partial_x y_2) - \operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} y_2) + y_1 = 0 & \text{in } (0, T) \times \Omega, \\ y_1(0, \cdot) = y_{0,1} & \text{in } \Omega, \\ y_2(0, \cdot) = y_{0,2} & \text{in } \Omega, \end{cases}$$

360 associated with either Dirichlet boundary conditions

$$361 \quad (2.32) \quad \begin{cases} y_1 = \mathbb{1}_{\{0\} \times \omega_2} v & \text{on } (0, T) \times \Gamma, \\ y_2 = 0 & \text{on } (0, T) \times \Gamma, \end{cases}$$

362 or Robin boundary conditions with the same parameter

$$363 \quad (2.33) \quad \begin{cases} \frac{\partial y_1}{\partial \nu_\gamma} + \beta y_1 = \mathbb{1}_{\{0\} \times \omega_2} v & \text{on } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta y_2 = 0 & \text{on } (0, T) \times \Gamma. \end{cases}$$

364 Note that the result below is restricted to the case $\beta_1 = \beta_2$ for two reasons. The main one is that when the two
365 parameters are different, the problem has no more a suitable tensor product structure that is crucial in the analysis
366 (see [2]). The second one is that the constant C_{T, β^*} in point 2 of [Theorem 2.14](#) is not explicit enough with respect
367 to T ; we would need an exponential dependence of the constant for the analysis to apply directly.

THEOREM 2.18. *Let $y_0 \in (L^2(\Omega))^2$ be given. For any $T > 0$ and any $\beta \in (0, +\infty)$, there exists a null-control $v_\beta \in L^2((0, T) \times \Gamma)$ for the multi-D problem (2.31)-(2.33) that satisfies in addition the estimate*

$$\|v_\beta\|_{L^2((0, T) \times \Gamma)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{L^2(\Omega)},$$

368 where $C > 0$ neither depends on β nor on T .

369 *Proof.* The proof is mainly based on the strategy developed in [2, 10] which needs the sharp estimate with respect
370 to T of the 1D control cost given by point 1 of [Theorem 2.14](#) and a Lebeau-Robbiano spectral inequality, uniform
371 in $\beta \in (0, +\infty)$, relative to our control region ω_2 , for the eigenfunctions of the diffusion operator $-\operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} \cdot)$
372 with homogeneous Robin boundary condition in Ω_2 .

373 The required Lebeau-Robbiano inequality has been proved in [22, Theorem 1.2] for the eigenfunctions of Laplace-
374 Beltrami operator in a multi-dimensional connected compact C^1 -smooth Riemannian manifold M with the boundary
375 condition $\tilde{l} \frac{\partial u}{\partial \bar{x}} + lu = 0$ ($u \in H^2(M)$) for $\tilde{l} \equiv 1$ and any $l \in L^\infty(\Gamma)$ with $l \geq 0$. Although they did not mention it in
376 the paper, a careful look at their computations ensures us that the Lebeau-Robbiano inequality in this reference is
377 in fact uniform with respect to $l \geq 0$. Thus, the required inequality holds true for our operator $-\operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} \cdot)$ in
378 Ω_2 with homogeneous Robin boundary condition with any parameter $\beta \in (0, +\infty)$. \square

379 **COROLLARY 2.19** (Convergence towards Dirichlet control). *Let $(\beta_n)_n$ be any sequence of positive Robin pa-
380 rameters such that $\beta_n \rightarrow +\infty$ when $n \rightarrow +\infty$. For each n we define v_n (resp. y_n) to be the null-control of minimal
381 L^2 norm (resp. the associated trajectory) for the problem (2.31)-(2.33) with Robin parameter β_n .*

There exists a subsequence $(n_k)_k$ such that

$$\frac{v_{n_k}}{\beta_{n_k}} \xrightarrow[k \rightarrow \infty]{} v_D, \quad \text{in } L^2((0, T) \times \Gamma),$$

382

$$383 \quad y_{n_k} \xrightarrow[k \rightarrow \infty]{} y_D, \quad \text{strongly in } (L^2(0, T, H^{-1}(\Omega)))^2$$

384

$$\text{and weakly in } (L^2((0, T) \times \Omega))^2,$$

386 where v_D (resp. y_D) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (2.31)-(2.32)
387 for the same initial data.

388 **REMARK 2.20** (Convergence towards Neumann control). *When $(\beta_n)_n$ goes to 0, we obtain the convergence
389 towards a null-control for the Neumann problem as in [Remark 2.16](#).*

390 **2.5. Outline.** The rest of the paper is dedicated to the proof of our main theorem for the 1D case, namely
391 [Theorem 2.14](#). First of all, we establish useful spectral properties for the 1D Robin eigenvalue problem in Section
392 3. Then, we prove in Section 4 the controllability result in the case of an arbitrary diffusion coefficient but for the
393 same Robin parameter for both components (point 1 of [Theorem 2.14](#)). Finally, in Section 5, we investigate the
394 case of a constant diffusion coefficient with two different Robin parameters (point 2 of [Theorem 2.14](#)).

395 **3. Some spectral properties of the 1D Robin eigenvalue problem.** In this section, we develop some
396 properties of the eigenvalue-eigenfunctions of the 1D scalar operator A_β as introduced in Section 2.1.2. Note that
397 we use same notation as for the general higher dimension case. Those results will be used to draw some spectral
398 properties of our main operator $\mathcal{A}_{\beta_1, \beta_2}^*$.

399 **3.1. The case of a non-constant diffusion coefficient.** We begin with the following scalar eigenvalue
 400 problem

$$(3.1) \quad \begin{cases} -\partial_x(\gamma(x)\partial_x\varphi) = \lambda\varphi & \text{in } (0, 1), \\ -\gamma(0)\partial_x\varphi(0) + \beta\varphi(0) = 0, \\ \gamma(1)\partial_x\varphi(1) + \beta\varphi(1) = 0, \end{cases}$$

402 where β is any non-negative parameter and γ is chosen as in Section 2.4.1. Let us denote the eigenvalue-eigenfunction
 403 pairs of the Sturm-Liouville problem (3.1) as $(\lambda_{k,\gamma}^\beta, \varphi_{k,\gamma}^\beta)_{k \geq 0}$. We recall that the eigenvalues are simple and real
 404 and can be numbered in such a way that

$$(3.2) \quad 0 \leq \lambda_{0,\gamma}^\beta < \lambda_{1,\gamma}^\beta < \dots < \lambda_{k,\gamma}^\beta < \lambda_{k+1,\gamma}^\beta \nearrow +\infty,$$

407 see for instance [6, Theorem 8.4.5]. Also it is well-known that the family $(\varphi_{k,\gamma}^\beta)_{k \geq 0}$ is a Hilbert basis of $L^2(0, 1)$,
 408 as soon as they are normalized, and indeed each of $\varphi_{k,\gamma}^\beta$ belongs to the domain of the corresponding differential
 409 operator in (3.1).

410 **REMARK 3.1.** Observe that for $\beta = 0$, the problem (3.1) reduces to a Neumann eigenvalue problem where we
 411 denote the Neumann eigenvalues by $\lambda_{k,\gamma}^N := \lambda_{k,\gamma}^0$ for $k \geq 0$.

412 On the other hand, for $\beta = +\infty$, (3.1) degenerates into a Dirichlet eigenvalue problem and we denote by
 413 $\lambda_{k,\gamma}^D := \lambda_{k,\gamma}^\infty$ for $k \geq 0$, the Dirichlet eigenvalues.

For any $\beta \in [0, +\infty)$ and any $\varphi \in H^1(0, 1)$, $\varphi \neq 0$, we define the Rayleigh quotient associated with (3.1),

$$R_\beta(\varphi) := \frac{\int_0^1 \gamma |\partial_x \varphi|^2 + \beta(|\varphi(0)|^2 + |\varphi(1)|^2)}{\int_0^1 |\varphi|^2}.$$

For $\beta = +\infty$, we set

$$R_\infty(\varphi) := \begin{cases} \frac{\int_0^1 \gamma |\partial_x \varphi|^2}{\int_0^1 |\varphi|^2}, & \text{if } \varphi \in H_0^1(0, 1), \varphi \neq 0, \\ +\infty, & \text{if } \varphi \notin H_0^1(0, 1). \end{cases}$$

414 Conventionally we set $R_\beta(0) = 0$ for any $\beta \in [0, +\infty]$.

415 We recall that, for any $\beta \in [0, +\infty]$ the eigenvalues of our problem can be characterised by the min-max formula

$$(3.3) \quad \lambda_{k,\gamma}^\beta = \inf_{\substack{X_{k+1} \subset H^1(0,1) \\ \dim X_{k+1} = k+1}} \sup_{\varphi \in X_{k+1}} R_\beta(\varphi).$$

417 **REMARK 3.2.** A first consequence of the above formula is that, for any $0 < \beta < \beta^*$, we can bound from below
 418 the smallest eigenvalue $\lambda_{0,\gamma}^\beta$ as follows

$$419 \quad \lambda_{0,\gamma}^\beta = \inf_{\substack{\varphi \in H^1(0,1) \\ \varphi \neq 0}} R_\beta(\varphi) \geq \frac{\beta}{\beta^*} \inf_{\substack{\varphi \in H^1(0,1) \\ \varphi \neq 0}} R_{\beta^*}(\varphi) = \frac{\beta}{\beta^*} \lambda_{0,\gamma}^{\beta^*}.$$

420 **LEMMA 3.3.** For any two parameters $0 \leq \alpha < \beta \leq +\infty$, we have the following strict inequality

$$421 \quad \lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^\beta, \quad \forall k \geq 0$$

423 In particular, for any parameter $0 < \beta < +\infty$, we have

$$424 \quad \lambda_{k,\gamma}^N < \lambda_{k,\gamma}^\beta < \lambda_{k,\gamma}^D, \quad \forall k \geq 0.$$

426 *Proof.* From (3.3), we write

$$(3.4) \quad \begin{aligned} \lambda_{k,\gamma}^\alpha &= \inf_{\substack{X_{k+1} \subset H^1(0,1) \\ \dim X_{k+1} = k+1}} \sup_{\varphi \in X_{k+1}} R_\alpha(\varphi) \\ &\leq \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi) \\ &\leq \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\beta(\varphi) \\ &= R_\beta(\varphi_{k,\gamma}^\beta) = \lambda_{k,\gamma}^\beta. \end{aligned}$$

428 Let us show that the inequality is in fact strict.

Assume first that $\beta < +\infty$ and that there exists some $k \geq 0$ such that $\lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta$. This implies that all the inequalities above are, in fact, equalities. Thus, there is some $\tilde{\varphi} = \sum_{j=0}^k a_j \varphi_{j,\gamma}^\beta$ with $\sum_{j=0}^k |a_j|^2 = 1$, such that

$$\lambda_{k,\gamma}^\alpha = \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi) = R_\alpha(\tilde{\varphi}),$$

429 which yields that

$$430 \quad (3.5) \quad \int_0^1 \gamma |\partial_x \tilde{\varphi}|^2 + \alpha (|\tilde{\varphi}(0)|^2 + |\tilde{\varphi}(1)|^2) = \lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta.$$

432 On the other hand, since each $\varphi_{j,\gamma}^\beta$ is a L^2 -normalized eigenfunction of the operator $-\partial_x(\gamma(x)\partial_x)$ with the Robin
433 boundary condition with parameter β , corresponding to eigenvalue $\lambda_{j,\gamma}^\beta$ for $0 \leq j \leq k$, we see that $\tilde{\varphi}$ enjoys the
434 following

$$435 \quad (3.6) \quad \int_0^1 \gamma |\partial_x \tilde{\varphi}|^2 + \beta (|\tilde{\varphi}(0)|^2 + |\tilde{\varphi}(1)|^2) = \sum_{j=0}^k \lambda_{j,\gamma}^\beta |a_j|^2 \leq \lambda_{k,\gamma}^\beta \sum_{j=0}^k |a_j|^2 = \lambda_{k,\gamma}^\beta.$$

Since $\beta < +\infty$, and $\alpha < \beta$, we can compare (3.5) and (3.6) to deduce that $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$ and moreover

$$\sum_{j=0}^k \lambda_{j,\gamma}^\beta |a_j|^2 = \lambda_{k,\gamma}^\beta \sum_{j=0}^k |a_j|^2.$$

436 By (3.2), this equality implies that $a_j = 0$ for any $0 \leq j \leq k-1$ and then that $\tilde{\varphi}$ is proportional to $\varphi_{k,\gamma}^\beta$. However,
437 such an eigenfunction cannot vanish at $x = 0$ (see for instance Theorem 3.5 below) which is a contradiction.

In the case where $\beta = +\infty$, we use the previous results to simply write

$$\lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^{\alpha+1} \leq \lambda_{k,\gamma}^\infty,$$

438 and the proof of the lemma is complete. □

439 **REMARK 3.4.** Let $0 \leq \beta \leq +\infty$. We denote by $\lambda_{k,\gamma_{\min}}^\beta$ and $\lambda_{k,\gamma_{\max}}^\beta$ for $k \geq 0$, the eigenvalues to the operator
440 $-\gamma_{\min} \partial_x^2$ and $-\gamma_{\max} \partial_x^2$ respectively, with Robin boundary conditions with parameter β . Then from (3.3), one has
441 the following inequality

$$442 \quad \lambda_{k,\gamma_{\min}}^\beta \leq \lambda_{k,\gamma}^\beta \leq \lambda_{k,\gamma_{\max}}^\beta, \quad \forall k \geq 0 \quad \text{and} \quad \forall \beta \in [0, +\infty].$$

444 Let us observe now that, for any non-trivial eigenfunction $\varphi_{k,\gamma}^\beta$ of our problem (3.1), the quantity $\varphi_{k,\gamma}^\beta(0)$ (and
445 hence $(\varphi_{k,\gamma}^\beta)'(0)$) is non-zero for any $k \geq 0$ and $\beta \in (0, +\infty)$. In fact, we prove the following theorem that give
446 bounds from below for those quantities.

447 **THEOREM 3.5.** There exists a constant $C > 0$ depending only on the diffusion coefficient γ such that we have

$$448 \quad (3.7) \quad |\varphi_{k,\gamma}^\beta(0)|^2 \left(1 + \frac{\beta^2}{\gamma(0)\lambda_{k,\gamma}^\beta} \right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty),$$

449
450

$$451 \quad (3.8) \quad |(\varphi_{k,\gamma}^\beta)'(0)|^2 \left(\frac{(\gamma(0))^2}{\beta^2} + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} \right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty),$$

452
453

$$454 \quad (3.9) \quad \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta}, \quad \forall k \geq 0, \quad \beta \in [0, +\infty].$$

456 We first state the following lemma which is a straightforward consequence of [3, Lemma 2.2 and Lemma 2.3].

LEMMA 3.6. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and $\lambda > 0$. Suppose that $u : [0, 1] \rightarrow \mathbb{R}$ is smooth and satisfies the following second-order differential equation (without any assumptions on the boundary conditions):

$$(3.10) \quad -\partial_x(\gamma(x)\partial_x u)(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1),$$

then there exists $C' > 0$, depending only on γ , such that we have

$$(3.11) \quad |u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 \leq C' \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 + \frac{1}{\lambda} \int_0^1 \frac{|f(x)|^2}{\gamma(x)} \right), \quad \forall x, y \in (0, 1).$$

REMARK 3.7. We recall that in [3, Lemma 2.2], the authors have assumed that $\lambda \geq 1$ but this was related to the fact that they considered the slightly more general second-order differential equation

$$-\partial_x(\gamma(x)\partial_x u)(x) + q(x)u(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1).$$

In our case we have $q(x) \equiv 0$ in (3.10), and so having a careful look at the proof of [3, Lemma 2.3], one can observe that we simply need $\lambda > 0$ for the constant C' in Lemma 3.6 to be uniform with respect to λ .

Proof of Theorem 3.5.

- We recall the eigenvalue problem (3.1) and apply Lemma 3.6 to $u = \varphi_{k,\gamma}^\beta$, $\lambda = \lambda_{k,\gamma}^\beta$ and $f = 0$ to obtain for each $k \geq 0$ and $\beta \in (0, +\infty)$ that

$$|\varphi_{k,\gamma}^\beta(x)|^2 + \frac{\gamma(x)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(x)|^2 \geq \frac{1}{C'} \left(|\varphi_{k,\gamma}^\beta(y)|^2 + \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 \right), \quad \forall x, y \in (0, 1).$$

Putting $x = 0$ above and integrating over $y \in (0, 1)$, we obtain

$$|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'} \left(\|\varphi_{k,\gamma}^\beta\|_{L^2(\Omega)}^2 + \int_0^1 \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 dy \right).$$

Thanks to the normalizing condition $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$ and due to the positivity of the second integral in the right hand side of the last inequality, we have

$$|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'}.$$

- In one hand, we use the boundary condition of $\varphi_{k,\gamma}^\beta$ at $x = 0$ to express $(\varphi_{k,\gamma}^\beta)'(0)$ as a function of $\varphi_{k,\gamma}^\beta(0)$ and obtain (3.7).
- On the other hand, we use the same boundary condition to express $\varphi_{k,\gamma}^\beta(0)$ as a function of $(\varphi_{k,\gamma}^\beta)'(0)$ and obtain (3.8).
- Secondly, for any $k \geq 0$ and $\beta \in [0, +\infty]$, we define

$$u(x) := \varphi_{k+1,\gamma}^\beta(x)\varphi_{k,\gamma}^\beta(0) - \varphi_{k,\gamma}^\beta(x)\varphi_{k+1,\gamma}^\beta(0), \quad \forall x \in (0, 1),$$

which satisfies

$$-\partial_x(\gamma(x)\partial_x u)(x) = \lambda_{k+1,\gamma}^\beta u(x) + f(x), \quad \forall x \in (0, 1)$$

with

$$f(x) = \left(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \right) \varphi_{k,\gamma}^\beta(x) \varphi_{k+1,\gamma}^\beta(0).$$

Moreover, we observe that $u(0) = 0$ and $u'(0) = 0$, from the construction of u . So, by taking $x = 0$ in the inequality (3.11), we see

$$|u(y)|^2 + \frac{\gamma(y)}{\lambda_{k+1,\gamma}^\beta} |u'(y)|^2 \leq C' \frac{\left(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \right)^2}{\lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2 \int_0^1 \frac{1}{\gamma(s)} |\varphi_{k,\gamma}^\beta(s)|^2 ds,$$

492 for all $y \in (0, 1)$. Thanks to the normalizing condition $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$ and the definition of u together
 493 implies

$$494 \quad |\varphi_{k+1,\gamma}^\beta(y)\varphi_{k,\gamma}^\beta(0) - \varphi_{k,\gamma}^\beta(y)\varphi_{k+1,\gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta)^2}{\gamma_{\min} \lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2,$$

496 for all $y \in (0, 1)$. Now integrating the left hand side over $y \in (0, 1)$ and using the L^2 -orthonormality
 497 condition of $(\varphi_{k,\gamma}^\beta)_{k \geq 0}$ we have

$$498 \quad |\varphi_{k,\gamma}^\beta(0)|^2 + |\varphi_{k+1,\gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta)^2}{\gamma_{\min} \lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2,$$

500 which yields that

$$501 \quad \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta}, \quad \forall k \geq 0 \text{ and } \beta \in [0, +\infty].$$

503 where the constant C depends only on γ .

504 The proof is complete. \square

505 **3.2. The case of a constant diffusion coefficient.** In this case, without loss of generality we can assume
 506 that $\gamma \equiv 1$ on $[0, 1]$ and so we can find a more explicit form of the eigenfunctions to the following problem

$$507 \quad (3.12) \quad \begin{cases} -\partial_x^2 \varphi = \lambda \varphi & \text{in } (0, 1), \\ -\partial_x \varphi(0) + \beta \varphi(0) = 0, \\ \partial_x \varphi(1) + \beta \varphi(1) = 0. \end{cases}$$

508 Let us first assume that $\beta \in (0, +\infty)$. Using the boundary condition at $x = 0$, and solving explicitly the differential
 509 equation, we shall look for φ_k^β in the following form

$$510 \quad (3.13) \quad \varphi_k^\beta(x) = \frac{\sqrt{\lambda_k^\beta}}{\beta} \cos \sqrt{\lambda_k^\beta} x + \sin \sqrt{\lambda_k^\beta} x, \quad \forall x \in (0, 1), \quad \forall \beta > 0 \text{ and } \forall k \geq 0,$$

512 where the eigenvalue λ_k^β will be required to satisfy the following transcendental equation

$$513 \quad (3.14) \quad 2\beta \sqrt{\lambda_k^\beta} \cos \sqrt{\lambda_k^\beta} + (\beta^2 - \lambda_k^\beta) \sin \sqrt{\lambda_k^\beta} = 0, \quad \forall \beta > 0 \text{ and } \forall k \geq 0.$$

515 This equation is obtained from the boundary condition that φ_k^β should satisfy at $x = 1$.

516 Notice that, in order to simplify the formulas, we do not assume here that φ_k^β is normalised in L^2 . This will
 517 not be a problem in the sequel since we will only use the fact that this family is complete in L^2 .

REMARK 3.8. 1. We know that the family of eigenvalue-eigenfunctions of the operator $-\partial_x^2$ with Dirichlet and Neumann boundary conditions are

$$\varphi_k^D(x) = \sin((k+1)\pi x), \quad x \in [0, 1] \text{ with } \lambda_k^D = (k+1)^2\pi^2, \quad \forall k \geq 0, \text{ and}$$

$$\varphi_k^N(x) = \cos(k\pi x), \quad x \in [0, 1] \text{ with } \lambda_k^N = k^2\pi^2, \quad \forall k \geq 0.$$

518 From above, our first obvious observation is $\lambda_k^D = \lambda_{k+1}^N$, $\forall k \geq 0$.

519 2. Secondly, one has $\lambda_k^\beta \in (k^2\pi^2, (k+1)^2\pi^2)$, $\forall k \geq 0$ and $\beta \in (0, +\infty)$, thanks to [Lemma 3.3](#). To be more
 520 precise, λ_k^β is the unique solution of (3.14) in the interval $(k^2\pi^2, (k+1)^2\pi^2)$ for each $k \geq 0$.

REMARK 3.9. For any fixed $\lambda \in (k^2\pi^2, (k+1)^2\pi^2)$, $k \geq 0$, the following quadratic equation

$$2\beta\sqrt{\lambda} \cos \sqrt{\lambda} + (\beta^2 - \lambda) \sin \sqrt{\lambda} = 0,$$

for the unknown β , has one and only one positive solution. Indeed, we see that the solutions of the above equation are given by

$$\beta' = \sqrt{\lambda} \tan \frac{\sqrt{\lambda}}{2} \text{ and } \beta'' = -\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2},$$

521 that clearly have different signs. More precisely, since $\sqrt{\lambda} \in (k\pi, (k+1)\pi)$ we can see that

- 522 • for k even we have $\beta' > 0$ and $\beta'' < 0$,
- 523 • for k odd, we have $\beta' < 0$ and $\beta'' > 0$.

524 REMARK 3.10. One can obtain from the transcendental equation (3.14) that

$$525 \quad \sqrt{\lambda_k^\beta} = k\pi + \frac{2\beta}{k\pi} + O_\beta\left(\frac{1}{k^3}\right), \quad \forall k \geq 1.$$

526 See, for instance [19, Problem Ib].

527 We have seen in Remark 3.8 that the sequence of eigenvalues for the Dirichlet boundary condition and the one
 528 for the Neumann boundary condition almost coincide. The following lemma shows that, for other pairs of Robin
 529 parameters the corresponding sequences of eigenvalues are in fact disjoint.

530 LEMMA 3.11. Consider two parameters $\beta_1, \beta_2 \in [0, +\infty]$, such that $\beta_1 < \beta_2$. If for some k, l we have

$$\lambda_k^{\beta_2} = \lambda_l^{\beta_1},$$

then we necessarily have

$$\beta_1 = 0, \quad \beta_2 = +\infty, \quad \text{and} \quad l = k + 1.$$

531 *Proof.* If $\beta_1 > 0$, then $\lambda_l^{\beta_1} \in (l^2\pi^2, (l+1)^2\pi^2)$ and thus $\sqrt{\lambda_k^{\beta_2}} \notin \pi\mathbb{N}^*$ and thus $\beta_2 < +\infty$. Similarly, if we
 532 assume $\beta_2 < +\infty$ then we necessarily have $\beta_1 > 0$.

533 Therefore, there are now two cases:

- 534 • First case $(\beta_1, \beta_2) = (0, +\infty)$: the result follows from Remark 3.8.
- 535 • Second case $0 < \beta_1 < \beta_2 < +\infty$: the common value λ of $\lambda_l^{\beta_1}$ and $\lambda_k^{\beta_2}$ simultaneously belongs to $(l^2\pi^2, (l+1)^2\pi^2)$ and $(k^2\pi^2, (k+1)^2\pi^2)$, which implies that $k = l$ and thus we have a contradiction with Lemma 3.3. \square

537 **4. Boundary controllability of the 1D problem with single Robin parameter.** This section is devoted
 538 to establish the one-dimensional boundary null-controllability of our cascade system with same non negative Robin
 539 parameter on both components and for any diffusion coefficient γ as defined in section 2.4.1. In that case, the
 540 system (1.1)-(1.3) simply reads as

$$541 \quad (4.1) \quad \begin{cases} \partial_t y_1 - \partial_x(\gamma(x)\partial_x y_1) = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma(x)\partial_x y_2) + y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \gamma(x)\frac{\partial y_1}{\partial \nu}(t, x) + \beta y_1(t, x) = \mathbb{1}_{\{x=0\}}v(t) & \text{on } (0, T) \times \{0, 1\}, \\ \gamma(x)\frac{\partial y_2}{\partial \nu}(t, x) + \beta y_2(t, x) = 0 & \text{on } (0, T) \times \{0, 1\}, \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases}$$

542 which is associated with the operator $\mathcal{A}_{\beta, \beta}$ as introduced in Section 2.3, but specialized here to the one-dimensional
 543 setting, that is for $\Omega = (0, 1)$. To simplify the notation, we will simply denote this operator by \mathcal{A}_β , since the two
 544 Robin parameters are equal.

4.1. Spectrum of \mathcal{A}_β^* . We consider the eigenvalue problem

$$\mathcal{A}_\beta^* u = \lambda u, \quad \lambda \in \mathbb{C},$$

545 for a complex-valued function u , that is

$$546 \quad (4.2) \quad \begin{cases} -\partial_x(\gamma(x)\partial_x u_1) + u_2 = \lambda u_1 & \text{in } (0, 1), \\ -\partial_x(\gamma(x)\partial_x u_2) = \lambda u_2 & \text{in } (0, 1), \\ \gamma(x)\frac{\partial u_1}{\partial \nu}(x) + \beta u_1(x) = 0 & \text{for } x \in \{0, 1\}, \\ \gamma(x)\frac{\partial u_2}{\partial \nu}(x) + \beta u_2(x) = 0 & \text{for } x \in \{0, 1\}. \end{cases}$$

- 547 • Assume first that $u_2 \neq 0$. Multiplying the second equation by $\overline{u_2}$ and integrating by parts, we obtain that
548 $\lambda = \lambda_{k,\gamma}^\beta$ for some k and that we can assume that $u_2 = \varphi_{k,\gamma}^\beta$. Moreover, taking the real or imaginary part,
549 we can assume that u_1 is real-valued, then multiplying the first equation by u_2 and integrating by parts,
550 we obtain that $\int_0^1 |u_2|^2 = 0$ which is a contradiction.
 - 551 • We have proved that, necessarily, $u_2 = 0$. From the first equation we deduce that $\lambda = \lambda_{k,\gamma}^\beta$ for some k and
552 that, up to a multiplicative constant, we have $u_1 = \varphi_{k,\gamma}^\beta$.
- 553 Hence, the eigenfunctions of \mathcal{A}_β^* are

$$554 \quad (4.3) \quad \Phi_{k,\gamma}^\beta := \begin{pmatrix} \varphi_{k,\gamma}^\beta \\ 0 \end{pmatrix} \quad \text{corresponding to the eigenvalues } \lambda_{k,\gamma}^\beta, \quad \forall k \geq 0.$$

We observe that the set $\{\Phi_{k,\gamma}^\beta\}_{k \geq 0}$ is not sufficient generate the whole space $(L^2(0,1))^2$ because the second component of $\Phi_{k,\gamma}^\beta$ is 0 for each $k \geq 0$. Hence we need to look for the generalized eigenfunctions by solving the following problem

$$\mathcal{A}_\beta^* u = \lambda_{k,\gamma}^\beta u + \Phi_{k,\gamma}^\beta, \quad \forall k \geq 0,$$

556 that is

$$557 \quad (4.4) \quad \begin{cases} -\partial_x(\gamma(x)\partial_x u_1) + u_2 = \lambda_{k,\gamma}^\beta u_1 + \varphi_{k,\gamma}^\beta & \text{in } (0,1), \\ -\partial_x(\gamma(x)\partial_x u_2) = \lambda_{k,\gamma}^\beta u_2 & \text{in } (0,1), \\ \gamma(x)\frac{\partial u_1}{\partial \nu}(x) + \beta u_1(x) = 0 & \text{for } x \in \{0,1\}, \\ \gamma(x)\frac{\partial u_2}{\partial \nu}(x) + \beta u_2(x) = 0 & \text{for } x \in \{0,1\}. \end{cases}$$

558 The second equation shows that $u_2 = a\varphi_{k,\gamma}^\beta$ for some $a \in \mathbb{R}$. But multiplying the first equation by $(u_2 - \varphi_{k,\gamma}^\beta)$, i.e.,
559 $(a-1)\varphi_{k,\gamma}^\beta$ and performing an integration by parts yields us that the only admissible value for a is 1.

560 Now its enough to take $u_1 = 0$, which is by default an admissible solution of the system (4.4) and hence the
561 generalized eigenfunctions can be interpreted as

$$562 \quad (4.5) \quad \Psi_{k,\gamma}^\beta = \begin{pmatrix} 0 \\ \varphi_{k,\gamma}^\beta \end{pmatrix}, \quad \forall k \geq 0.$$

564 We observe now that, the family $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta\}_{k \geq 0}$ is a Riesz basis of $(L^2(0,1))^2$, made of eigenfunctions and
565 generalized eigenfunctions of the operator \mathcal{A}_β^* . By construction we simply have

$$566 \quad (4.6) \quad \begin{cases} e^{-t\mathcal{A}_\beta^*} \Phi_{k,\gamma}^\beta = e^{-t\lambda_{k,\gamma}^\beta} \Phi_{k,\gamma}^\beta, & \forall t \in [0, T], \\ e^{-t\mathcal{A}_\beta^*} \Psi_{k,\gamma}^\beta = e^{-t\lambda_{k,\gamma}^\beta} (\Psi_{k,\gamma}^\beta - t\Phi_{k,\gamma}^\beta), & \forall t \in [0, T]. \end{cases}$$

567

REMARK 4.1 (Approximate controllability). *We observe that the eigenfunctions of \mathcal{A}_β^* are observable, in the sense that*

$$\mathcal{B}^* \Phi_{k,\gamma}^\beta = \varphi_{k,\gamma}^\beta(0) \neq 0, \quad \forall k \geq 0,$$

568 where \mathcal{B}^* is given by (2.29).

569 *By using the Fattorini-Hautus test (the hypothesis of which are fulfilled in our case, see for instance [16, 24]),*
570 *we deduce that the control system (4.1), with any $\beta > 0$, is approximately controllable at any time $T > 0$.*

571 **4.2. Null-controllability.** We are now in position to prove the null-controllability of our system, with a
572 precise bound of the control with respect to β , that is the first point of [Theorem 2.14](#).

573 **4.2.1. The moments problem.** We recall that $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta\}_{k \geq 0}$ (defined by (4.3)-(4.5)) forms a complete
574 family in $(L^2(0,1))^2$, so it is enough to check the controllability equation (2.30) (with the operator \mathcal{A}_β^* here) for
575 $\Phi_{k,\gamma}^\beta$ and $\Psi_{k,\gamma}^\beta$ for each $k \geq 0$. This indeed tells us, for any $y_0 \in (L^2(0,1))^2$, that the input $v \in L^2(0, T)$ is a null
576 control for (4.1) if and only if we have

$$577 \quad \begin{cases} -e^{-T\lambda_{k,\gamma}^\beta} (y_0, \Phi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-(T-t)\lambda_{k,\gamma}^\beta} \mathcal{B}^* \Phi_{k,\gamma}^\beta dt, & \forall k \geq 0, \\ -e^{-T\lambda_{k,\gamma}^\beta} (y_0, \Psi_{k,\gamma}^\beta - T\Phi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-(T-t)\lambda_{k,\gamma}^\beta} \mathcal{B}^* (\Psi_{k,\gamma}^\beta - (T-t)\Phi_{k,\gamma}^\beta) dt, & \forall k \geq 0, \end{cases}$$

578

579 using the formulas given by (4.6).

580 Now since $\mathcal{B}^* \Phi_{k,\gamma}^\beta = \varphi_{k,\gamma}^\beta(0) \neq 0$ and $\mathcal{B}^* \Psi_{k,\gamma}^\beta = 0$ for each $k \geq 0$, we can simplify the above set of equations as

$$581 \quad (4.7) \quad \begin{cases} -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-\lambda_{k,\gamma}^\beta(T-t)} dt, & \forall k \geq 0, \\ -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} \left(T(y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} - (y_{0,2}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} \right) = \int_0^T v(t) (T-t) e^{-\lambda_{k,\gamma}^\beta(T-t)} dt, & \forall k \geq 0. \end{cases}$$

582 The above set of equations is the moments problem that we shall solve in our case.

583 **4.2.2. Existence of a bi-orthogonal family to real exponentials.** To construct our control v by solving
584 the moments problem above, the existence of a suitable bi-orthogonal family to time-dependent exponential functions
585 is one the most important ingredient. In this context, it is worth mentioning [10, Theorem 1.5] where the authors
586 proved the existence of bi-orthogonal families to $(t^j e^{-\Lambda_k t})_{k \geq 0, 0 \leq j \leq \eta}$ ($\eta \in \mathbb{N}$) for a complex sequence $(\Lambda_k)_{k \geq 0}$ with
587 non-decreasing modulus. This proof is based on a proper gap condition of $|\Lambda_k - \Lambda_n|$ for all $k \neq n$ and some property
588 of the counting function associated with $(\Lambda_k)_{k \geq 0}$ which has been introduced by point 5 and 6 of their proof. In
589 fact, concerning this hypothesis on the counting function, a slightly more general version has been introduced in [2,
590 Remark 4.3] and we indeed make use of this fact in the present study.

591 We deal with the real sequence $(\lambda_{k,\gamma}^\beta)_{k \geq 0}$ and we show that this sequence satisfies all the assumptions of [10,
592 Theorem 1.5] uniformly with respect to the parameter β .

593 1. *The gap condition:*

594 Without loss of generality we assume that $k > n$ and therefore, $k = n + m$ for some $m \in \mathbb{N}$. We recall (3.9)
595 and Lemma 3.3 to observe that

$$596 \quad (4.8) \quad \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta} \geq C \sqrt{\lambda_{k+1,\gamma}^N}, \quad \forall k \geq 0, \quad \forall \beta \geq 0.$$

597 Also, by Remark 3.4, we have $\lambda_{k+1,\gamma}^N \geq \lambda_{k+1,\gamma_{\min}}^N$, and since it is easy to observe from Remark 3.8 that
598 $\lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$ for each $k \geq 0$, so the inequality (4.8) is simplified as

$$600 \quad (4.9) \quad \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \gamma_{\min} \pi (k+1), \quad \forall k \geq 0, \quad \forall \beta \geq 0,$$

601 which gives us

$$602 \quad \lambda_{n+m,\gamma}^\beta - \lambda_{n,\gamma}^\beta \geq C \gamma_{\min} \pi \sum_{j=n+1}^{n+m} j = C \gamma_{\min} \pi \left[mn + \frac{m(m+1)}{2} \right] = \frac{C}{2} \gamma_{\min} \pi [(m+n)^2 - n^2 + m].$$

603 Thus for any k, n with $k \geq n+1$, and for any $\beta \geq 0$, we have

$$604 \quad (4.10) \quad \lambda_{k,\gamma}^\beta - \lambda_{n,\gamma}^\beta \geq \rho(k^2 - n^2), \quad \forall k, n : k - n \geq 1.$$

606 with $\rho := \frac{C}{2} \gamma_{\min} \pi$.

607 2. *The counting function:* Let \mathcal{N} be the counting function associated with the sequence $(\lambda_{k,\gamma}^\beta)_{k \geq 0}$, defined by

$$\mathcal{N}(r) = \#\{k : \lambda_{k,\gamma}^\beta \leq r\}, \quad \forall r > 0.$$

608 We observe that, the function \mathcal{N} is piecewise constant and non-decreasing in the interval $[0, +\infty)$. Also for
609 every $r \in [0, +\infty)$ we have $\mathcal{N}(r) < +\infty$ and $\lim_{r \rightarrow +\infty} \mathcal{N}(r) = +\infty$. Moreover,

$$610 \quad \mathcal{N}(r) = k \iff \lambda_{k,\gamma}^\beta \leq r \text{ and } \lambda_{k+1,\gamma}^\beta > r,$$

611 so that, in particular, if $\mathcal{N}(r) = k$, we have

$$612 \quad \sqrt{\lambda_{k,\gamma}^\beta} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^\beta},$$

613 which yields, by Lemma 3.3 and Remark 3.4, that

$$614 \quad \sqrt{\lambda_{k,\gamma_{\min}}^N} \leq \sqrt{\lambda_{k,\gamma}^N} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^D} \leq \sqrt{\lambda_{k+1,\gamma_{\max}}^D}$$

615
616

617 But we have $\lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$ and $\lambda_{k+1,\gamma_{\max}}^D = \gamma_{\max} (k+2)^2 \pi^2$, hence

$$618 \quad \sqrt{\gamma_{\min}} k \pi \leq \sqrt{r} < \sqrt{\gamma_{\max}} (k+2) \pi.$$

620 Replacing k by $N(r)$, we determine that

$$621 \quad \frac{1}{\sqrt{\gamma_{\max}} \pi} \sqrt{r} - 2 < N(r) \leq \frac{1}{\sqrt{\gamma_{\min}} \pi} \sqrt{r} < \frac{1}{\sqrt{\gamma_{\min}} \pi} \sqrt{r} + 2,$$

623 which is the point 6 given just before [2, Remark 4.3] with $\alpha = 2$, $p_{\min} = \frac{1}{\sqrt{\gamma_{\max}} \pi}$ and $p_{\max} = \frac{1}{\sqrt{\gamma_{\min}} \pi}$.

624 From the discussion above, and any given $T > 0$, we can ensure that the existence of a bi-orthogonal family in
625 $L^2(0, T)$, denoted by $(q_{k,j}^\beta)_{k \geq 0, 0 \leq j \leq 1}$, to the family of exponential functions $((T-\cdot)^i e^{-\lambda_{k,\gamma}^\beta (T-\cdot)})_{k \geq 0, 0 \leq i \leq 1}$. Moreover,
626 this family satisfies the following estimates

$$627 \quad (4.11) \quad \|q_{k,j}^\beta\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_{k,\gamma}^\beta} + \frac{C}{T}}, \quad \forall k \geq 0, j = 0, 1,$$

629 where the constant $C > 0$ is independent on $T > 0$ and uniform with respect to $k \geq 0$ and to the parameter $\beta \geq 0$
630 since all the quantities ρ , α , p_{\min} and p_{\max} introduced above do not depend on the Robin parameter β .

631 **4.2.3. The controllability result.** We can now proceed to the proof of the null-controllability result in that
632 case.

633 *Proof of Point 1 of Theorem 2.14.* Consider

$$634 \quad (4.12) \quad v_\beta(t) = \sum_{k \geq 0} v_k^\beta(t), \quad \forall t \in (0, T), \quad \text{where}$$

$$637 \quad v_k^\beta(t) = -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} q_{k,0}^\beta(t) - \frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} \left(T(y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} - (y_{0,2}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} \right) q_{k,1}^\beta(t).$$

638 With this choice of $v = v_\beta$, one can observe that the set of moment equations in (4.7) are formally satisfied.
639 Now, all we have to check is the convergence of the series (4.12) in $L^2(0, T)$, with explicit bounds with respect to
640 $\beta \in (0, +\infty)$. To this end, for each $k \geq 0$, we compute

$$641 \quad \|v_k^\beta\|_{L^2(0,T)} \leq \frac{\|y_{0,1}\|_{L^2(0,1)}}{|\varphi_{k,\gamma}^\beta(0)|} e^{-T\lambda_{k,\gamma}^\beta} \|q_{k,0}^\beta\|_{L^2(0,T)} + \frac{T\|y_{0,1}\|_{L^2(0,1)} + \|y_{0,2}\|_{L^2(0,1)}}{|\varphi_{k,\gamma}^\beta(0)|} e^{-T\lambda_{k,\gamma}^\beta} \|q_{k,1}^\beta\|_{L^2(0,T)},$$

642 thanks to the normalizing condition $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$. Moreover, the result (3.7) gives us

$$643 \quad (4.13) \quad \frac{1}{|\varphi_{k,\gamma}^\beta(0)|} \leq C \left(1 + \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} \right), \quad \forall k \geq 0, \beta \in (0, +\infty),$$

645 where C depends only on γ .

646 Now, using (4.13) and the bounds on bi-orthogonal functions in (4.11), we deduce for each $k \geq 0$ and for any
647 finite $T > 0$ that

$$648 \quad (4.14) \quad \|v_k^\beta\|_{L^2(0,T)} \leq C(T+2) \left(1 + \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} \right) e^{-T\lambda_{k,\gamma}^\beta} e^{C\sqrt{\lambda_{k,\gamma}^\beta} + \frac{C}{T}} \|y_0\|_{L^2(0,1)},$$

650 since $\gamma(0) > \gamma_{\min} > 0$. Now, Young's inequality gives us

$$651 \quad (4.15) \quad C\sqrt{\lambda_{k,\gamma}^\beta} \leq \frac{T}{2}\lambda_{k,\gamma}^\beta + \frac{C^2}{2T}, \quad \forall k \geq 0.$$

653 Thus, we see

$$654 \quad (4.16) \quad e^{-T\lambda_{k,\gamma}^\beta + C\sqrt{\lambda_{k,\gamma}^\beta} + C/T} \leq e^{-\frac{T}{2}\lambda_{k,\gamma}^\beta + \frac{C}{T}}, \quad \forall k \geq 0.$$

656 But we have, $\lambda_{k,\gamma}^\beta \geq \lambda_{k,\gamma}^N \geq \lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$ for $k \geq 0$, and so, in one hand, we have

$$657 \quad (4.17) \quad \sum_{k \geq 0} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq \sum_{k \geq 0} e^{-Ck^2 \frac{T}{2}} \leq \frac{1}{1 - e^{-CT/2}} \leq C \left(1 + \frac{1}{T}\right).$$

658
659 On the other hand, we see

$$660 \quad (4.18) \quad \sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{0,\gamma}^\beta} + \frac{\beta}{\pi \sqrt{\gamma_{\min}}} \sum_{k \geq 1} \frac{1}{k} e^{-\pi^2 \gamma_{\min} k^2 \frac{T}{2}}$$

$$661 \quad \leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} + C\beta \sum_{k \geq 1} e^{-Ck^2 \frac{T}{2}}.$$

662
663 The second quantity in the right hand side of (4.18) has the bound $C\beta \left(1 + \frac{1}{T}\right)$ where the bound of first quantity
664 is not so obvious because we see that $\lambda_{0,\gamma}^\beta$ is getting smaller with respect to smaller $\beta > 0$. But, we have

$$665 \quad \left\{ \begin{array}{l} \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq \frac{\sqrt{\beta}}{\sqrt{\lambda_{0,\gamma}^1}} \leq C(1 + \beta), \text{ for } 0 < \beta < 1, \text{ by Remark 3.2, and} \\ \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq C\beta, \text{ for } \beta \geq 1, \text{ which is easy to observe.} \end{array} \right.$$

666
667 Now, using this in (4.18), one can deduce that

$$668 \quad (4.19) \quad \sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq C(1 + \beta) \left(1 + \frac{1}{T}\right).$$

669
670 Now, we take summation over $k \geq 0$ in (4.14), and using the estimates (4.16), (4.17) and (4.19), we get for any
671 $\beta \in (0, +\infty)$ and finite $T > 0$ that

$$672 \quad \sum_{k \geq 0} \|v_k^\beta\|_{L^2(0,T)} \leq C(1 + \beta) e^{C/T} \|y_0\|_{L^2(0,1)}.$$

673 This completes the proof. □

674 **5. Boundary controllability result of the 1D problem with different Robin parameters.** In this
675 section, we discuss about the boundary controllability of the system (1.1)-(1.3) in 1D with two different parameters
676 $\beta_1 \neq \beta_2$ with $\beta_1, \beta_2 \in (0, +\infty)$ for the two components of the system and as mentioned in the introduction of this
677 paper, we assume now that γ is a positive constant that we arbitrarily choose to be equal to 1. We rewrite the
678 control system (1.1)-(1.3) in that setting below

$$679 \quad (5.1) \quad \left\{ \begin{array}{ll} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \frac{\partial y_1}{\partial \nu}(t, x) + \beta_1 y_1(t, x) = \mathbf{1}_{\{x=0\}} v(t) & \text{on } (0, T) \times \{0, 1\}, \\ \frac{\partial y_2}{\partial \nu}(t, x) + \beta_2 y_2(t, x) = 0 & \text{on } (0, T) \times \{0, 1\}, \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1). \end{array} \right.$$

680 In this case, we recall that the associated operator is $\mathcal{A}_{\beta_1, \beta_2}$, as defined in Section 2.3, specified here for $\Omega = (0, 1)$
681 and for $\gamma \equiv 1$.

682 The main difference between the present section and Section 4 concerns the spectral properties of the adjoint
683 operators. Unlike the previous case, we will have here a possible condensation of eigenvalues with two different sets
684 of eigenfunctions that form a complete family of the state space, instead of having well-separated eigenvalues and
685 associated generalized eigenfunctions.

686 **5.1. Spectrum of $\mathcal{A}_{\beta_1, \beta_2}^*$.** In the present situation, the eigenvalue problem associated with $\mathcal{A}_{\beta_1, \beta_2}^*$ is explicitly
 687 given by

$$688 \quad (5.2) \quad \begin{cases} -\partial_x^2 u_1 + u_2 = \lambda u_1 & \text{in } (0, 1), \\ -\partial_x^2 u_2 = \lambda u_2 & \text{in } (0, 1), \\ \frac{\partial u_1}{\partial \nu} + \beta_1 u_1 = 0 & \text{on } \{0, 1\}, \\ \frac{\partial u_2}{\partial \nu} + \beta_2 u_2 = 0 & \text{on } \{0, 1\}. \end{cases}$$

689 *First case:* Assume that $u_2 = 0$, then our system (5.2) reduces to the Robin eigenvalue problem (3.12) with positive
 690 parameter β_1 and this gives us the solution $u_1 = \varphi_k^{\beta_1}$ corresponding $\lambda = \lambda_k^{\beta_1}$ which is real for any $k \geq 0$ (recall that,
 691 φ_k^β has already been given by (3.13) for all $k \geq 0$ and $\beta \in (0, +\infty)$). This gives us the following set of eigenfunctions
 692 (without normalizing) of $\mathcal{A}_{\beta_1, \beta_2}^*$

$$693 \quad (5.3) \quad \Phi_{k,1} := \begin{pmatrix} \varphi_k^{\beta_1} \\ 0 \end{pmatrix} \text{ corresponding to the eigenvalues } \lambda_k^{\beta_1}, \quad \forall k \geq 0.$$

695 *Second case:* Assume now that $u_2 \neq 0$, then we first solve the second set of equations of (5.2), i.e.,

$$696 \quad \begin{cases} -\partial_x^2 u_2 = \lambda u_2 & \text{in } (0, 1), \\ \frac{\partial u_2}{\partial \nu} + \beta_2 u_2 = 0 & \text{on } \{0, 1\}, \end{cases}$$

697 which gives $u_2 = \varphi_k^{\beta_2}$, up to a multiplicative constant (which we can take 1), corresponding to $\lambda = \lambda_k^{\beta_2}$ for all $k \geq 0$.
 698 Now by implementing $u_2 = \varphi_k^{\beta_2}$ for each $k \geq 0$ to the first equation of (5.2) address us the following problem

$$699 \quad (5.4) \quad \begin{cases} -\partial_x^2 u_1 + \varphi_k^{\beta_2} = \lambda_k^{\beta_2} u_1 & \text{in } (0, 1), \\ -\partial_x u_1(0) + \beta_1 u_1(0) = 0, \\ \partial_x u_1(1) + \beta_1 u_1(1) = 0. \end{cases}$$

700 The existence and uniqueness of the solution to (5.4) follows from the Fredholm alternative theorem and to the fact
 701 that $\lambda_k^{\beta_2} \notin (\lambda_l^{\beta_1})_{l \geq 0}$ for any $k \geq 0$ and $\beta_1 \neq \beta_2$ (by Lemma 3.11). Let us denote the unique solution u_1 of (5.4) by
 702 $\psi_k^{\beta_1, \beta_2}$, for $k \geq 0$ and hence the second set of eigenfunctions (without normalizing) of $\mathcal{A}_{\beta_1, \beta_2}^*$ is given by

$$703 \quad (5.5) \quad \Phi_{k,2} := \begin{pmatrix} \psi_k^{\beta_1, \beta_2} \\ \varphi_k^{\beta_2} \end{pmatrix} \text{ corresponding to the eigenvalues } \lambda_k^{\beta_2}, \quad \forall k \geq 0.$$

705 The family $\{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0}$ is complete in $(L^2(0, 1))^2$, and we observe that

$$706 \quad (5.6) \quad \begin{cases} e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \Phi_{k,1} = e^{-t\lambda_k^{\beta_1}} \Phi_{k,1}, & \forall k \geq 0, \\ e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \Phi_{k,2} = e^{-t\lambda_k^{\beta_2}} \Phi_{k,2}, & \forall k \geq 0. \end{cases}$$

707 **5.1.1. More on spectral properties and approximate controllability.** This section is devoted to show
 708 some properties of the first component $\psi_k^{\beta_1, \beta_2}$ of the eigenfunction $\Phi_{k,2}$ and how the spectral gap $|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|$ depends
 709 on the parameters β_1, β_2 (for any $k \geq 0$). We need all these to find a proper bound of our null-control.

710 Proving the estimates of this section for any non constant diffusion coefficient γ is still an open problem, that
 711 is why we restrict here our attention to the constant coefficient case.

712 **LEMMA 5.1.** *Let $\beta_1 \neq \beta_2$ be any two real parameters with $\beta_1, \beta_2 \in (0, +\infty)$ and*
 713 *$(\psi_k^{\beta_1, \beta_2})_{k \geq 0}$ be the set of solutions to (5.4) as introduced before. Then, we have*

$$714 \quad |\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{\lambda_k^{\beta_2} + \beta_2^2}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} |\beta_1 - \beta_2|}, \quad \forall k \geq 0.$$

716 *Proof.* 1. Observe that $\psi_k^{\beta_1, \beta_2}$ satisfies the second order ordinary differential equation (5.4), i.e.,

$$717 \quad (5.7) \quad \frac{d^2 \psi_k^{\beta_1, \beta_2}}{dx^2} + \lambda_k^{\beta_2} \psi_k^{\beta_1, \beta_2} = \varphi_k^{\beta_2}, \quad \text{for each } k \geq 0.$$

719 To solve this, recall the explicit form of $\varphi_k^{\beta_2}$ from (3.13) (with $\beta = \beta_2$), and hence for each $k \geq 0$, we are
720 looking for $\psi_k^{\beta_1, \beta_2}$ in the following form

$$721 \quad (5.8) \quad \psi_k^{\beta_1, \beta_2}(x) = (Ax + B) \cos(\sqrt{\lambda_k^{\beta_2}} x) + (Cx + D) \sin(\sqrt{\lambda_k^{\beta_2}} x), \quad \forall x \in [0, 1].$$

723 Substituting (5.8) into the equation (5.7), we get

$$724 \quad (5.9) \quad A = -\frac{1}{2\sqrt{\lambda_k^{\beta_2}}} \quad \text{and} \quad C = \frac{1}{2\beta_2}.$$

726 Then, from the boundary conditions at $x = 0$ and 1 satisfying by $\psi_k^{\beta_1, \beta_2}$, one can obtain

$$727 \quad D\sqrt{\lambda_k^{\beta_2}} = \beta_1 B - A, \quad \text{and}$$

$$731 \quad (5.10) \quad B \left(2\beta_1 \sqrt{\lambda_k^{\beta_2}} \cos \sqrt{\lambda_k^{\beta_2}} + (\beta_1^2 - \lambda_k^{\beta_2}) \sin \sqrt{\lambda_k^{\beta_2}} \right) \\ 732 \quad = \left(\frac{\beta_1 \beta_2 - \lambda_k^{\beta_2}}{2\beta_2} \right) \cos \sqrt{\lambda_k^{\beta_2}} - \left(\frac{(1 + \beta_1 + \beta_2) \lambda_k^{\beta_2} + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda_k^{\beta_2}}} \right) \sin \sqrt{\lambda_k^{\beta_2}}$$

734 respectively. We must mention here that the coefficient of B in the left hand side of (5.10) never vanishes
735 due to Remark 3.9.

736 2. By (3.14), it is known that the eigenvalue $\lambda_k^{\beta_2}$ is the unique solution to

$$737 \quad (5.11) \quad 2\beta_2 \sqrt{\lambda_k^{\beta_2}} \cos \sqrt{\lambda_k^{\beta_2}} + (\beta_2^2 - \lambda_k^{\beta_2}) \sin \sqrt{\lambda_k^{\beta_2}} = 0, \quad \text{for each } k \geq 0.$$

739 Now by substituting the expression of $\cos \sqrt{\lambda_k^{\beta_2}}$ from (5.11) into (5.10) and replacing B by $\psi_k^{\beta_1, \beta_2}$, we get

$$740 \quad \frac{1}{\beta_2} \psi_k^{\beta_1, \beta_2}(0) \left[\beta_1 (\lambda_k^{\beta_2} - \beta_2^2) + \beta_2 (\beta_1^2 - \lambda_k^{\beta_2}) \right] = \frac{(\beta_1 \beta_2 - \lambda_k^{\beta_2})(\lambda_k^{\beta_2} - \beta_2^2)}{4\beta_2^2 \sqrt{\lambda_k^{\beta_2}}} - \frac{(1 + \beta_1 + \beta_2) \lambda_k^{\beta_2} + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda_k^{\beta_2}}},$$

741 where we omitted $\sin \sqrt{\lambda_k^{\beta_2}}$ from both sides since $\sin \sqrt{\lambda_k^{\beta_2}} \neq 0$ for all $k \geq 0$ and $\beta_2 \in (0, +\infty)$. Now by
742 simplifying the above equality provides us

$$743 \quad (5.12) \quad \psi_k^{\beta_1, \beta_2}(0) = \frac{-(\lambda_k^{\beta_2})^2 - \beta_1 \beta_2 \lambda_k^{\beta_2} - \beta_2^2 \lambda_k^{\beta_2} - 2\beta_2 \lambda_k^{\beta_2} - 2\beta_1 \beta_2^2 - \beta_1 \beta_2^3}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} \left[\beta_1 (\lambda_k^{\beta_2} - \beta_2^2) + \beta_2 (\beta_1^2 - \lambda_k^{\beta_2}) \right]} =: \frac{I_k}{J_k}.$$

745 Here one can rewrite the quantity J_k as

$$746 \quad (5.13) \quad J_k = 4\beta_2 \sqrt{\lambda_k^{\beta_2}} (\beta_1 - \beta_2) (\lambda_k^{\beta_2} + \beta_1 \beta_2), \quad \forall k \geq 0,$$

747 whereas, I_k enjoys the following

$$748 \quad (5.14) \quad |I_k| \geq (\lambda_k^{\beta_2})^2 + \beta_1 \beta_2 \lambda_k^{\beta_2} + \beta_2^2 \lambda_k^{\beta_2} + \beta_1 \beta_2^3 = (\lambda_k^{\beta_2} + \beta_2^2) (\lambda_k^{\beta_2} + \beta_1 \beta_2), \quad \forall k \geq 0.$$

750 To this end, we use (5.13) and (5.14) in the expression (5.12) to deduce that

$$751 \quad |\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{(\lambda_k^{\beta_2} + \beta_2^2)}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} |\beta_1 - \beta_2|}, \quad \forall k \geq 0,$$

753 and this concludes the lemma. \square

REMARK 5.2 (Approximate controllability). *The control problem (5.1) is approximately controllable for any finite time $T > 0$.*

To prove this, we will again use Fattorini-Hautus test as mentioned in Remark 4.1, that is to show $\mathcal{B}^ \Phi_{k,1} \neq 0$ and $\mathcal{B}^* \Phi_{k,2} \neq 0$ for each $k \geq 0$ (where \mathcal{B}^* has been defined in (2.29)). But (3.13) and Lemma 5.1 respectively ensure us*

$$\mathcal{B}^* \Phi_{k,1} = \varphi_k^{\beta_1}(0) \neq 0 \text{ and } \mathcal{B}^* \Phi_{k,2} = \psi_k^{\beta_1, \beta_2}(0) \neq 0, \quad \forall k \geq 0,$$

which gives the claim.

LEMMA 5.3. *Let $\beta_1 \neq \beta_2$ be any two parameters such that $\beta_1, \beta_2 \in (0, +\infty)$ and $\lambda_k^{\beta_1}, \lambda_k^{\beta_2}, k \geq 0$, be defined as before. Let $\beta^* > 0$ be a fixed finite number. Then*

1. *for $0 < \beta_1, \beta_2 < \beta^*$, we have*

$$|\beta_1 - \beta_2| \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0,$$

2. *for $\beta_1, \beta_2 \geq \beta^*$, we have*

$$\left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0.$$

Proof. We begin with the fact that any $\beta \in (0, +\infty)$ can be represented by

$$(5.15) \quad \begin{cases} \beta = \sqrt{\lambda_k^\beta} \tan \frac{\sqrt{\lambda_k^\beta}}{2}, & \text{for } k \text{ even and} \\ \beta = -\sqrt{\lambda_k^\beta} \cot \frac{\sqrt{\lambda_k^\beta}}{2}, & \text{for } k \text{ odd (by Remark 3.9).} \end{cases}$$

Also, since we have $\sqrt{\lambda_k^\beta} \in (k\pi, (k+1)\pi)$ by point 2 of Remark 3.8, so one can write

$$(5.16) \quad \sqrt{\lambda_k^{\beta_j}} = k\pi + \delta_k^{\beta_j} \text{ for } j = 1, 2, \text{ and } \sqrt{\lambda_k^{\beta^*}} = k\pi + \delta_k^{\beta^*}, \quad \forall k \geq 0,$$

where $\delta_k^{\beta_j}, \delta_k^{\beta^*} \in (0, \pi)$.

1. Now, we assume that $0 < \beta_1, \beta_2 < \beta^*$ which implies $0 < \delta_k^{\beta_1}, \delta_k^{\beta_2} < \delta_k^{\beta^*} < \pi, \forall k \geq 0$. We denote

$$(5.17) \quad 0 < \delta_{\beta^*} := \sup_{k \geq 0} \delta_k^{\beta^*} < \pi,$$

where $\delta_{\beta^*} < \pi$ since the quantity $\delta_k^{\beta^*}$ is getting smaller as $k \geq 1$ getting larger due to the asymptotic behavior of $\sqrt{\lambda_k^{\beta^*}}$ given by Remark 3.10.

We discuss the proof for k even, for odd k the steps will be similar. We have

$$(5.18) \quad \begin{aligned} \beta_1 - \beta_2 &= (k\pi + \delta_k^{\beta_1}) \tan \left(\frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - (k\pi + \delta_k^{\beta_2}) \tan \left(\frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right) \\ &= k\pi \left(\tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right) + \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2}. \end{aligned}$$

Applying Mean value theorem to the functions $\tan \frac{\mu}{2}$ and $\mu \tan \frac{\mu}{2}$ on $\mu \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2}))$, we have for some δ'_k and δ''_k in $((\delta_k^{\beta_1}, \delta_k^{\beta_2}))$, that

$$\left| \tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \sec^2 \frac{\delta'_k}{2} \leq C_{\beta^*} |\delta_k^{\beta_1} - \delta_k^{\beta_2}|,$$

and

$$\begin{aligned} \left| \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2} \right| &\leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \left| \frac{\delta''_k}{2} \sec^2 \frac{\delta''_k}{2} + \tan \frac{\delta''_k}{2} \right| \\ &\leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \frac{\delta''_k}{2} \left| \sec^2 \frac{\delta''_k}{2} + \frac{\sin(\delta''_k/2)}{\delta''_k/2} \sec \frac{\delta''_k}{2} \right| \\ &\leq C_{\beta^*} (\delta_k^{\beta_1} + \delta_k^{\beta_2}) |\delta_k^{\beta_1} - \delta_k^{\beta_2}|, \end{aligned}$$

789 where we make use of the fact that the quantities $\sec \frac{\delta'_k}{2}$ and $\sec \frac{\delta''_k}{2}$ can be bounded by $\sec \frac{\delta_{\beta^*}}{2}$ which is
 790 some constant C_{β^*} . Now, we turn back to (5.18) to deduce that

$$\begin{aligned} 791 \quad |\beta_1 - \beta_2| &\leq C_{\beta^*} (k\pi + \delta_k^{\beta_1} + \delta_k^{\beta_2}) |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \\ 792 \quad &= C_{\beta^*} \left(\sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \left| \sqrt{\lambda_k^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right| \\ 793 \quad &\leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|. \end{aligned}$$

795 2. Here also, we demonstrate the result for k even. We recall (5.15) to observe that

$$\begin{aligned} 796 \quad (5.19) \quad \frac{1}{\beta_1} - \frac{1}{\beta_2} &= \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \left(\frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \left(\frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right) \\ 797 \quad &= \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \frac{\delta_k^{\beta_1}}{2} - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \frac{\delta_k^{\beta_2}}{2}, \end{aligned}$$

Let us define the function

$$g(\mu) = \frac{1}{k\pi + \mu} \cot \frac{\mu}{2}, \quad \text{for } \mu \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})) \subset (0, \pi).$$

Consequently,

$$g'(\mu) = -\frac{1}{2(k\pi + \mu)} \csc^2 \frac{\mu}{2} - \frac{1}{(k\pi + \mu)^2} \cot \frac{\mu}{2},$$

799 and $|g'|$ is monotonically decreasing function in $(0, \pi)$. Now, applying Mean value theorem on $g(\mu)$, we
 800 have from (5.19)

$$801 \quad (5.20) \quad \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| |g'(\tilde{\delta}_k)|, \quad \text{for some } \tilde{\delta}_k \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})).$$

803 But we have $\delta_k^{\beta_j}, \tilde{\delta}_k \geq \delta_k^{\beta^*}$ ($j = 1, 2$), since $\beta_1, \beta_2 \geq \beta^*$, and hence

$$\begin{aligned} 804 \quad (5.21) \quad |g'(\tilde{\delta}_k)| &\leq \frac{1}{k\pi + \tilde{\delta}_k} \left[\frac{1}{2 \sin^2 \left(\frac{\tilde{\delta}_k}{2} \right)} + \frac{1}{k\pi + \tilde{\delta}_k} \cot \frac{\tilde{\delta}_k}{2} \right] \\ 805 \quad &\leq \frac{1}{k\pi + \delta_k^{\beta^*}} \left[\frac{1}{2 \sin^2 \left(\frac{\delta_k^{\beta^*}}{2} \right)} + \frac{1}{k\pi + \delta_k^{\beta^*}} \frac{1}{\sin \left(\frac{\delta_k^{\beta^*}}{2} \right)} \right]. \end{aligned}$$

Let us recall the asymptotic formula of $\sqrt{\lambda_k^{\beta^*}}$ from Remark 3.10 to observe

$$\delta_k^{\beta^*} = \frac{2\beta^*}{k\pi} + O_{\beta^*} \left(\frac{1}{k^3} \right) \geq \frac{\bar{\beta}}{k\pi}, \quad \forall k \geq 1,$$

for some $\bar{\beta} > 0$ depending only on β^* . As a consequence, $\sin \frac{\delta_k^{\beta^*}}{2} \geq \sin \frac{\bar{\beta}}{2k\pi}$, since $\sin x$ is monotonically
 increasing on $(0, \pi/2)$. Also, since $\sin x \geq \frac{2}{\pi}x$ for all $x \in (0, \pi/2)$, eventually we have

$$\sin \frac{\bar{\beta}}{2k\pi} \geq \frac{\bar{\beta}}{k\pi^2}, \quad \forall k \geq 1.$$

807 Now we come back to (5.21), we obtain for any $k \neq 0$ even, that

$$\begin{aligned} 808 \quad |g'(\tilde{\delta}_k)| &\leq \frac{1}{\sqrt{\lambda_k^{\beta^*}}} \left[\frac{k^2 \pi^4}{2\bar{\beta}^2} + \frac{k\pi^2}{\bar{\beta} \sqrt{\lambda_k^{\beta^*}}} \right] \\ 809 \quad &\leq C_{\beta^*} \sqrt{\lambda_k^{\beta^*}} \leq C_{\beta^*} \left(\sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right). \end{aligned}$$

810

Implementing this estimate into (5.20), we obtain that

$$(5.22) \quad \begin{aligned} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| &\leq C_{\beta^*} \left| \sqrt{\lambda_k^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right| \left(\sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \\ &= C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \neq 0 \text{ even.} \end{aligned}$$

For $k = 0$, one can observe from (5.21) that

$$(5.23) \quad \begin{aligned} |g'(\tilde{\delta}_0)| &\leq \frac{1}{\tilde{\delta}_0} \left[\frac{1}{2 \sin^2(\tilde{\delta}_0/2)} + \frac{1}{\tilde{\delta}_0} \cot \frac{\tilde{\delta}_0}{2} \right] \\ &\leq \frac{1}{\delta_0^{\beta^*}} \left[\frac{1}{2 \sin^2(\delta_0^{\beta^*}/2)} + \frac{1}{\delta_0^{\beta^*}} \frac{1}{\sin(\delta_0^{\beta^*}/2)} \right] \leq C_{\beta^*}, \end{aligned}$$

and so (5.20) now provides us the same inequality (5.22) for $k = 0$. The proof is complete. \square

Lemma 5.3 now helps us to prove the following proposition which is the key point to obtain a uniform $L^2(0, T)$ -bound of a control that we construct in next section, with respect to the parameters β_1, β_2 .

PROPOSITION 5.4. *Let $\beta_1 \neq \beta_2$ be any two parameters with $\beta_1, \beta_2 \in (0, +\infty)$ and $(\lambda_k^{\beta_1}, \Phi_{k,1})$, $(\lambda_k^{\beta_2}, \Phi_{k,2})$ be the eigenvalue-eigenfunction pairs of the operator $\mathcal{A}_{\beta_1, \beta_2}^*$ for each $k \geq 0$. Also, assume that β^* be any positive real number. Then, for any $k \geq 0$, we have*

$$(5.23) \quad \frac{1}{|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|} \left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} - \frac{\Phi_{k,2}}{\mathcal{B}^* \Phi_{k,2}} \right\|_{L^2(0,1)} \leq \begin{cases} C_{\beta^*}, & \text{if } 0 < \beta_1, \beta_2 < \beta^*, \\ C_{\beta^*} \beta_1 \sqrt{\lambda_k^{\beta_2}}, & \text{if } \beta_1, \beta_2 \geq \beta^*, \end{cases}$$

where \mathcal{B}^* is defined in (2.29).

Proof. Since $\mathcal{B}^* = \mathbf{1}_{\{x=0\}} (1 \ 0)$ (introduced in (2.29)), and using the definitions (5.3) and (5.5), the quantity we want to estimate can be denoted by

$$\Theta_k := \begin{pmatrix} \Theta_k^1 \\ \Theta_k^2 \end{pmatrix},$$

where the two components are

$$\begin{aligned} \Theta_k^1(x) &:= \frac{1}{(\lambda_k^{\beta_1} - \lambda_k^{\beta_2})} \left(\frac{\varphi_k^{\beta_1}(x)}{\varphi_k^{\beta_1}(0)} - \frac{\psi_k^{\beta_1, \beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right), \quad x \in (0, 1), \\ \Theta_k^2(x) &:= \frac{1}{(\lambda_k^{\beta_1} - \lambda_k^{\beta_2})} \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)}, \quad x \in (0, 1). \end{aligned}$$

In order to ease the computations, we will denote by $\mu_i = \sqrt{\lambda_k^{\beta_i}}$ for $i = 1, 2$, the dependence in k being now implicit.

1. We first assume that $0 < \beta_1, \beta_2 < \beta^*$. Using this assumption, and the fact that $\lambda_0^{\beta_i} \geq \frac{\beta_i}{\beta^*} \lambda_0^{\beta^*}$ (by Remark 3.2) and simply $\lambda_k^{\beta_i} \geq k^2 \pi^2$, $\forall k \geq 1$ (by point 2 of Remark 3.8), we obtain uniformly in $k \geq 0$, that

$$(5.24) \quad \mu_i \geq C_{\beta^*} \sqrt{\beta_i}, \quad i = 1, 2.$$

- Estimate of the first component Θ_k^1 :

Recall the expression $\varphi_k^{\beta_1}$ (with $\beta = \beta_1$) and $\psi_k^{\beta_1, \beta_2}$ from (3.13) and (5.8) respectively and following some steps of computations we obtain

$$(5.25) \quad \begin{aligned} \left(\frac{\varphi_k^{\beta_1}(x)}{\varphi_k^{\beta_1}(0)} - \frac{\psi_k^{\beta_1, \beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right) &= \cos(\mu_1 x) - \cos(\mu_2 x) + \beta_1 \left[\frac{\sin \mu_1 x}{\mu_1} - \frac{\sin \mu_2 x}{\mu_2} \right] \\ &\quad - \frac{Ax}{B} \cos(\mu_2 x) - \frac{Cx}{B} \sin(\mu_2 x) + \frac{A}{B\mu_2} \sin(\mu_2 x). \end{aligned}$$

Let us bound the contribution of each term in the L^2 norm of Θ_k^1 , recalling that $(\lambda_k^{\beta_1} - \lambda_k^{\beta_2}) = \mu_1^2 - \mu_2^2$.

840 – First, it is easy to deduce that for any $\beta_1 \neq \beta_2$ and any $x \in (0, 1]$, we have

$$841 \quad (5.26) \quad \left| \frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} \right| = \frac{x^2}{2} \left| \frac{\sin\left(\frac{\mu_1 - \mu_2}{2} x\right) \sin\left(\frac{\mu_1 + \mu_2}{2} x\right)}{\left(\frac{\mu_1 - \mu_2}{2} x\right) \left(\frac{\mu_1 + \mu_2}{2} x\right)} \right| \leq \frac{1}{2}.$$

842 – For $x \in (0, 1]$ fixed, let us define the function $\mu \mapsto f(\mu) := \frac{\sin(\mu x)}{\mu}$, whose derivative is $f'(\mu) =$
843 $\frac{1}{\mu} x \cos(\mu x) - \frac{1}{\mu^2} \sin(\mu x)$. Applying Mean value theorem, we have

$$844 \quad |f(\mu_1) - f(\mu_2)| \leq |\mu_1 - \mu_2| |f'(\tilde{\mu})|, \quad \text{for some } \tilde{\mu} \in ((\mu_1, \mu_2)).$$

846 Now, if $0 < \tilde{\mu} < 1$ (consequently $0 < \tilde{\mu} x < 1$ for $x \in (0, 1]$), then we have

$$847 \quad |f'(\tilde{\mu})| = \left| \frac{1}{\tilde{\mu}} x \cos(\tilde{\mu} x) - \frac{1}{\tilde{\mu}^2} \sin(\tilde{\mu} x) \right| \\ 848 \quad = \left| \tilde{\mu} x^3 \left(\frac{\cos(\tilde{\mu} x) - 1}{(\tilde{\mu} x)^2} - \frac{\sin(\tilde{\mu} x) - \tilde{\mu} x}{(\tilde{\mu} x)^3} \right) \right| \\ 849 \quad \leq \tilde{\mu} x^3 \leq 1, \quad \forall x \in (0, 1],$$

851 since $\left| \frac{\cos(\tilde{\mu} x) - 1}{(\tilde{\mu} x)^2} \right| \leq C$ and $\left| \frac{\sin(\tilde{\mu} x) - \tilde{\mu} x}{(\tilde{\mu} x)^3} \right| \leq C$, for $0 < \tilde{\mu} x < 1$.

852 On the other hand, for $\tilde{\mu} \geq 1$, it is quite obvious to see that $|f'(\tilde{\mu})| \leq C$, and so finally we have
853 uniformly in x ,

$$854 \quad |f(\mu_1) - f(\mu_2)| \leq C |\mu_1 - \mu_2|, \quad \forall \beta_1 \neq \beta_2 \text{ positive.}$$

856 This implies that for $0 < \beta_1 < \beta^*$ one has

$$857 \quad (5.27) \quad \frac{\beta_1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\sin(\mu_1 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right| \leq \frac{C \beta_1}{\mu_1 + \mu_2} \leq C_{\beta^*},$$

859 where the last inequality follows from (5.24).

– For estimating the remaining three terms, we use the values of A and C from (5.9) and Lemma 5.1, that gives the bound from below

$$|B| \geq \frac{\mu_2}{4\beta_2 |\beta_1 - \beta_2|}.$$

860 With the inequality proved in the first point of Lemma 5.3 and (5.24) we see that

$$861 \quad (5.28) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B} \cos(\mu_2 x) \right| \leq \frac{2\beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \leq \frac{C_{\beta^*} \beta_2}{\mu_2^2} \leq C_{\beta^*}.$$

863 Similarly, we have

$$864 \quad (5.29) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Cx}{B} \sin(\mu_2 x) \right| \leq \frac{|\beta_1 - \beta_2|}{|\mu_1^2 - \mu_2^2|} \left| \frac{x^2 \sin(\mu_2 x)}{\mu_2 x} \right| \leq C_{\beta^*},$$

866 and

$$867 \quad (5.30) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B\mu_2} \sin(\mu_2 x) \right| \leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A}{B} \right| \leq \frac{2\beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \leq \frac{C_{\beta^*} \beta_2}{\mu_2^2} \leq C_{\beta^*}.$$

868 Hence, gathering all the estimates from (5.26)-(5.30), one can deduce that $\|\Theta_k^1\|_{L^2(0,1)} \leq C_{\beta^*}$ for
869 any $k \geq 0$.

870 • Estimate of the second component Θ_k^2 :

871 By using the expression of $\varphi_k^{\beta_2}$ from (3.13), we have for each $k \geq 0$ that

$$872 \quad \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| \leq \frac{\mu_2 |\cos(\mu_2 x)|}{\beta_2 |\psi_k^{\beta_1, \beta_2}(0)|} + \frac{|\sin(\mu_2 x)|}{|\psi_k^{\beta_1, \beta_2}(0)|} \\ 873 \quad \leq 4|\beta_1 - \beta_2| + 4\beta_2 |\beta_1 - \beta_2| \left| \frac{x \sin(\mu_2 x)}{\mu_2 x} \right| \\ 874 \quad \leq 4(1 + \beta_2) |\beta_1 - \beta_2| \leq C_{\beta^*} (1 + \beta_2) |\mu_1^2 - \mu_2^2|,$$

876 where we make use the facts that $|\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{\mu_2}{4\beta_2 |\beta_1 - \beta_2|}$ and the estimate in the first point of
877 Lemma 5.3. Consequently, we deduce that $\|\Theta_k^2\|_{L^2(0,1)} \leq C_{\beta^*}$ for any $k \geq 0$.

- This completes the proof of the uniform estimate of Θ_k in L^2 for $0 < \beta_1, \beta_2 < \beta^*$, $\beta_1 \neq \beta_2$.
2. Let us assume now that $\beta_1, \beta_2 \geq \beta^* > 0$. Using this assumption, and the fact that $\lambda_0^{\beta_i} \geq \lambda_0^{\beta^*}$ and $\lambda_k^{\beta_i} \geq k^2 \pi^2$, $\forall k \geq 1$, we obtain uniformly in $k \geq 0$, that

$$(5.31) \quad \mu_i \geq C_{\beta^*}, \quad i = 1, 2.$$

We need to prove that $\frac{1}{\beta_1} \Theta_k$ is bounded uniformly in k , β_1 and β_2 .

- Estimate of the first component Θ_k^1 :

We still start from (5.25) and we estimate each term as follows.

- Analogous to (5.26) and (5.27), we respectively have

$$(5.32) \quad \left| \frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} \right| \leq \frac{1}{2},$$

and

$$(5.33) \quad \frac{\beta_1}{|\mu_1^2 - \mu_2^2|} \left[\frac{\sin(\mu_1 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right] \leq \frac{\beta_1}{\mu_1} \leq C_{\beta^*} \beta_1,$$

by (5.31).

- Now using the fact $|B| \geq \frac{\mu_2^2 + \beta_2^2}{4\beta_2 \mu_2 |\beta_1 - \beta_2|}$ from Lemma 5.1, and the second point of Lemma 5.3, we can bound the remaining three terms as follows

$$(5.34) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B} \cos(\mu_2 x) \right| \leq \frac{2|\beta_1 - \beta_2|}{\beta_2} \frac{\beta_2^2}{|\mu_1^2 - \mu_2^2| (\mu_2^2 + \beta_2^2)} \leq \frac{2\beta_1}{|\mu_1^2 - \mu_2^2|} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq C_{\beta^*} \beta_1.$$

Similarly, we have

$$(5.35) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Cx}{B} \sin(\mu_2 x) \right| \leq \frac{|\beta_1 - \beta_2|}{\beta_2} \frac{2\beta_2 \mu_2}{|\mu_1^2 - \mu_2^2| (\mu_2^2 + \beta_2^2)} \leq C_{\beta^*} \beta_1,$$

and finally

$$(5.36) \quad \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B\mu_2} \sin(\mu_2 x) \right| \leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A}{B} \right| \leq C_{\beta^*} \beta_1,$$

which is obtained by a similar type of computations as in (5.34).

Gathering all the estimates from (5.32)-(5.36), we get that $\|\Theta_k^1\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1$ for any $k \geq 0$.

- Estimate of the second component Θ_k^2 :

Using the same ingredients as before, we compute

$$\begin{aligned} \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| &\leq \frac{\mu_2}{\beta_2} \frac{|\cos(\mu_2 x)|}{|B|} + \frac{|\sin(\mu_2 x)|}{|B|} \\ &\leq 2\mu_2 \frac{2\beta_1 \beta_2 \mu_2}{(\mu_2^2 + \beta_2^2)} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| + 4\mu_2 \frac{\beta_1 \beta_2^2}{(\mu_2^2 + \beta_2^2)} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \\ &\leq C_{\beta^*} \mu_2 \beta_1 |\mu_1^2 - \mu_2^2|, \end{aligned}$$

by the second point of Lemma 5.3. This implies

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| \leq C_{\beta^*} \beta_1 \mu_2,$$

and thus the expected bound $\|\Theta_k^2\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1 \mu_2$ for any $k \geq 0$. \square

5.2. Null-controllability. We can now prove the null-controllability of our system, with a precise bound of the control with respect to β_1 and β_2 , that is the second point of Theorem 2.14.

912 **5.2.1. The moments problem.** In the present context, we recall that the family $\{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0}$ (defined
 913 by (5.3)-(5.5)) is complete in $(L^2(0,1))^2$ and so, by checking the equation (2.28) for $\Phi_{k,1}$ and $\Phi_{k,2}$ for each $k \geq 0$,
 914 indeed tells us that for any $y_0 \in (L^2(0,1))^2$ the input $v \in L^2(0, T)$ is a null control for (5.1) if and only if one has

$$915 \quad (5.37) \quad \begin{cases} -e^{-T\lambda_k^{\beta_1}} \frac{(y_0, \Phi_{k,1})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,1}} = \int_0^T v(t) e^{-\lambda_k^{\beta_1}(T-t)} dt, & \forall k \geq 0, \\ -e^{-T\lambda_k^{\beta_2}} \frac{(y_0, \Phi_{k,2})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,2}} = \int_0^T v(t) e^{-\lambda_k^{\beta_2}(T-t)} dt, & \forall k \geq 0, \end{cases}$$

916 where we used the formulas given in (5.6).

917 **5.2.2. The block moment method.** It is known that for any $k \geq 0$ the eigenvalue λ_k^β is continuous with
 918 respect to the parameter $\beta \in [0, +\infty]$, see for instance [21, Theorem 3.1] and as a consequence, it may occur that
 919 the two eigenvalues $\lambda_k^{\beta_1}$ and $\lambda_k^{\beta_2}$ are arbitrarily close if β_1 and β_2 are close. This phenomenon is called *spectral*
 920 *condensation* and may, in general, prevent us from obtaining uniform bounds on the controls when β_1 and β_2 are
 921 getting closer (see for instance a discussion on the influence of the condensation index on controllability properties
 922 of parabolic systems in [20]).

923 Indeed, the classic way to solve the moments problem, as we did in Section 4 is inadequate. More precisely,
 924 it is not true anymore that any bi-orthogonal family to $(e^{-\lambda_k^{\beta_j}(T-t)})_{k \geq 0, j=0,1}$ will satisfy uniform $L^2(0, T)$ -bound
 925 with respect to the parameters β_1, β_2 since the gap $\inf_k |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|$ may be arbitrary small when $|\beta_1 - \beta_2|$ is small
 926 (see Lemma 5.3).

927 To overcome this situation, and still prove uniform controllability result, we will use the block moment approach
 928 developed in [11] to solve problems like (5.37) when a weak gap condition holds, instead of a usual uniform gap
 929 condition. This method let us take benefit of the condensation of eigenfunctions that actually compensate the
 930 condensation of the eigenvalues. Let us go into the details.

931 We first define $\Lambda^{\beta_i} := \{\lambda_k^{\beta_i}, k \geq 0\}$ for $i = 1, 2$, the two families of eigenvalues we are concerned with and we
 932 set $\Lambda^{\beta_1, \beta_2} = \Lambda^{\beta_1} \cup \Lambda^{\beta_2}$.

933 As we have seen in (4.9), each of the two families satisfies a uniform spectral gap property

$$934 \quad (5.38) \quad \inf_k |\lambda_{k+1}^{\beta_i} - \lambda_k^{\beta_i}| \geq C\pi, \quad i = 1, 2,$$

935 and their reciprocal values are uniformly summable in the sense that, there exists a function $\mathcal{N} : (0, +\infty) \rightarrow (0, +\infty)$
 936 that does not depend on β_1 and β_2 , such that

$$937 \quad (5.39) \quad \sum_{\substack{\lambda \in \Lambda^{\beta_i} \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} \leq \varepsilon,$$

938 for any $\varepsilon > 0$ and any $i = 1, 2$.

939 Therefore, by [11, Lemma 2.1], we know that the union family $\Lambda^{\beta_1, \beta_2}$ satisfies a weak-gap property : for any
 940 $\rho > 0$ (independent of β_1 and β_2) such that

$$941 \quad (5.40) \quad \rho < C\pi,$$

942 we have that $\Lambda^{\beta_1, \beta_2} \cap [\mu, \mu + \rho)$, contains at most 2 elements for any $\mu > 0$. Moreover, the reciprocal values of
 943 $\Lambda^{\beta_1, \beta_2}$ are also uniformly summable as in (5.39) but with a possibly different function \mathcal{N} .

By [11, Proposition 7.1] we know that, for each value of β_1 and β_2 , we can find a family of disjoint non empty
 groups $(G_n)_n$ each of them having a cardinal less or equal than 2 and such that

$$\Lambda^{\beta_1, \beta_2} = \bigcup_n G_n,$$

$$(\min G_{n+1}) - (\max G_n) \geq \rho/2,$$

$$\text{diam}(G_n) < \rho.$$

944 Let us prove now that, for ρ small enough, the structure of those groups is actually simple.

945 LEMMA 5.5. Let $\beta^* > 0$ be fixed and for $\beta_1 \neq \beta_2$ assume that either $\beta_1, \beta_2 < \beta^*$ or $\beta_1, \beta_2 \geq \beta^*$.

There exists ρ^* depending only on β^* such that, if we assume that $\rho < \rho^*$ in the above construction in addition to (5.40), then for any group G_n of cardinal 2, there exists an integer k such that

$$G_n = \{\lambda_k^{\beta_1}, \lambda_k^{\beta_2}\}.$$

Proof. Without loss of generality, we assume that $\beta_1 < \beta_2$. Since the diameter of G_n is less than ρ and using (5.38) and (5.40) we know that G_n contains exactly one element from Λ^{β_1} and one element from Λ^{β_2} , that is

$$G_n = \{\lambda_j^{\beta_1}, \lambda_k^{\beta_2}\},$$

946 for some integers k and j . We want to show that $j = k$.

947 By Lemma 3.3 and Lemma 3.11 we know that

$$948 \quad (5.41) \quad \lambda_k^{\beta_1} < \lambda_k^{\beta_2} < \lambda_{k+1}^{\beta_1},$$

949 thus the only possibilities are $j = k$ or $j = k + 1$.

950 • First, we treat the case when $0 < \beta_1 < \beta_2 < \beta^*$. We have

$$951 \quad \sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \geq \sqrt{\lambda_{k+1}^N} - \sqrt{\lambda_k^{\beta^*}} = \pi - \delta_{\beta^*} > 0,$$

953 since $\lambda_{k+1}^N = (k+1)^2\pi^2$ by point 1 of Remark 3.8; $\lambda_k^{\beta^*}$ and δ_{β^*} have been introduced in (5.16) and (5.17)
954 respectively. Hence, for all $k \geq 0$, we have

$$955 \quad \lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} = \left(\sqrt{\lambda_{k+1}^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \left(\sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right) \\ 956 \quad \geq \pi(\pi - \delta_{\beta^*}) > 0.$$

958 We choose $\rho^* = \pi(\pi - \delta_{\beta^*})$. From the computation above we see that if $\rho < \rho^*$, then $\lambda_{k+1}^{\beta_1}$ and $\lambda_k^{\beta_2}$ cannot
959 belong to the same group G_n and thus we necessarily have $j = k$ and the claim is proved.

960 • Assume now that $\beta_2 > \beta_1 \geq \beta^* > 0$. Then, from the asymptotic formula given by Remark 3.10, we see

$$961 \quad \sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \geq \sqrt{\lambda_{k+1}^{\beta^*}} - \sqrt{\lambda_k^D} = \frac{2\beta^*}{(k+1)\pi} + O_{\beta^*} \left(\frac{1}{k^2} \right), \quad \forall k \geq 1,$$

963 since $\lambda_k^D = (k+1)^2\pi^2$ by point 1 of Remark 3.8. Now, it is obvious that $\sqrt{\lambda_{k+1}^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \geq (k+1)\pi$, for all
964 $k \geq 0$ and so, there exists a $k_{\beta^*} \geq 1$, depending only on β^* such that

$$965 \quad \lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} \geq \beta^* > 0, \quad \forall k \geq k_{\beta^*}.$$

It remains to deal with the other values of $k = 0, 1, \dots, k_{\beta^*} - 1$. We simply use the fact that

$$\lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} \geq \lambda_{k+1}^{\beta^*} - \lambda_k^D,$$

to define

$$\rho^* := \min \left(\beta^*, \min_{0 \leq k < k_{\beta^*}} (\lambda_{k+1}^{\beta^*} - \lambda_k^D) \right) > 0.$$

967 Here also we conclude that if $\rho < \rho^*$, $\lambda_{k+1}^{\beta_1}$ and $\lambda_k^{\beta_2}$ cannot be in the same group G_n and thus $j = k$, and
968 the proof is complete. \square

969 We can now proceed to the proof of our main result concerning the uniform null-controllability of the system
970 with two different Robin parameters.

971 *Proof of Point 2 of Theorem 2.14.* We proved above that the sequence of eigenvalues $\Lambda^{\beta_1, \beta_2}$ satisfy the good
972 weak-gap and summability conditions required by the block moment method. More precisely, we can apply [11,
973 Theorem 2.1] to find a solution to the set of equations (5.37) as an infinite sum of terms, each of them corresponding
974 to the resolution of the contribution of the group G_n . In our case, we can observe that, by Lemma 5.5, the set
975 $\{\lambda_k^{\beta_1}, \lambda_k^{\beta_2}\}$ for any $k \geq 0$, is either exactly one of the groups G_n or the union $G_n \cup G_{n+1}$ of two distinct groups of
976 cardinal 1.

977 It follows that, the result of [11, Theorem 2.1] can be reformulated as follows: there exist functions $v_k^{\beta_1, \beta_2} \in$
 978 $L^2(0, T)$ for each $k \geq 0$, which satisfy the following

$$979 \quad (5.42) \quad \begin{cases} \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^{\beta_1}(T-t)} dt = -e^{-T\lambda_k^{\beta_1}} \frac{(y_0, \Phi_{k,1})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,1}}, \\ \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^{\beta_2}(T-t)} dt = -e^{-T\lambda_k^{\beta_2}} \frac{(y_0, \Phi_{k,2})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,2}}, \\ \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_i^{\beta_i}(T-t)} dt = 0, \quad \forall l \neq k, \forall i = 1, 2. \end{cases}$$

980 and satisfying the following bound, for any $\varepsilon > 0$,

$$981 \quad (5.43) \quad \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \varepsilon, \mathcal{N}, \rho^*} e^{(\varepsilon - T)\lambda_k^{\beta_1}} \max \left\{ \left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} \right\|_{L^2(0,1)}, \frac{\left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} - \frac{\Phi_{k,2}}{\mathcal{B}^* \Phi_{k,2}} \right\|_{L^2(0,1)}}{|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|} \right\} \|y_0\|_{L^2(0,1)}.$$

982 Note that in [11] it is assumed that all the eigenvalues in the system are greater than 1, whereas in our case we only
 983 know that they are non-negative (we recall that λ_0^β goes to 0 when $\beta \rightarrow 0$). However, one can check that this does
 984 not change significantly the result since it simply amounts to add a factor e^T in front of the constant $C_{T, \varepsilon, \mathcal{N}, \rho^*}$ in
 985 the estimate.

986 We now define v_β as

$$987 \quad (5.44) \quad v_\beta(t) := \sum_{k \geq 0} v_k^{\beta_1, \beta_2}(t), \quad \forall t \in [0, T],$$

989 so that v_β formally satisfies the set of moments problem (5.37), it remains to show that the series converges and to
 990 obtain the expected bound on v_β .

- 991 • In the case when $\beta_1 = \beta_2$, the result is just a particular case of point 1 of Theorem 2.14.
- 992 • Assume that $\beta_1 \neq \beta_2$. We observe that $\left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} \right\|_{L^2(0,1)} = \left\| \frac{\varphi_k^{\beta_1}}{\varphi_k^{\beta_1}(0)} \right\|_{L^2(0,1)}$ which can be bounded by $C(1 + \beta_1)$

993 for any $\beta_1 \in (0, +\infty)$ (recall the expression of $\varphi_k^{\beta_1}$ from (3.13)).

994 We can then choose $\varepsilon = T/2$ and apply Proposition 5.4 to obtain that, for $0 < \beta_1, \beta_2 < \beta^*$,

$$995 \quad (5.45) \quad \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) e^{-\frac{T}{2}\lambda_k^{\beta_1}} \|y_0\|_{L^2(0,1)},$$

996 and for $\beta_1, \beta_2 \geq \beta^*$,

$$997 \quad (5.46) \quad \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) \sqrt{\lambda_k^{\beta_2}} e^{-\frac{T}{2}\lambda_k^{\beta_1}} \|y_0\|_{L^2(0,1)}.$$

998 From (5.44), it follows that

$$999 \quad \|v_\beta\|_{L^2(0, T)} \leq \sum_{k \geq 0} \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)},$$

1000 in which we can plug (5.45) or (5.46) to finally obtain

$$1001 \quad \|v_\beta\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0,1)},$$

1002 where $C_{T, \beta^*} > 0$ does not depend explicitly on the parameters β_1, β_2 because the two series $\sum_{k \geq 0} e^{-\frac{T}{2}\lambda_k^{\beta_1}}$
 1003 and $\sum_{k \geq 0} \sqrt{\lambda_k^{\beta_2}} e^{-\frac{T}{2}\lambda_k^{\beta_1}}$ converges uniformly with respect to the parameters due to the fact $\lambda_k^N < \lambda_k^{\beta_i} < \lambda_k^D$,
 1004 $\forall k \geq 0, i = 1, 2$, by Lemma 3.3.

1005 The proof of the theorem is complete. \square

- 1007 [1] F. ALABAU-BOUSSOIRA AND M. LÉAUTAUD, *Indirect controllability of locally coupled wave-type systems and applications*, Journal
1008 de Mathématiques Pures et Appliquées, 99 (2013), pp. 544–576.
- 1009 [2] D. ALLONSIUS AND F. BOYER, *Boundary null-controllability of semi-discrete coupled parabolic systems in some multi-dimensional*
1010 *geometries*, Mathematical Control and Related Fields, (2018), <https://hal.archives-ouvertes.fr/hal-01827044>. to appear.
- 1011 [3] D. ALLONSIUS, F. BOYER, AND M. MORANCEY, *Spectral analysis of discrete elliptic operators and applications in control theory*,
1012 Numerische Mathematik, 140 (2018), pp. 857–911.
- 1013 [4] F. AMMAR-KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *The Kalman condition for the boundary*
1014 *controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures
1015 Appl. (9), 96 (2011), pp. 555–590.
- 1016 [5] F. AMMAR-KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Recent results on the controllability of linear*
1017 *coupled parabolic problems: a survey*, Math. Control Relat. Fields, 1 (2011), pp. 267–306.
- 1018 [6] F. V. ATKINSON, *Discrete and continuous boundary problems*, Mathematics in Science and Engineering, Vol. 8, Academic Press,
1019 New York-London, 1964.
- 1020 [7] I. BABUSKA, *The finite element method with penalty*, Mathematics of Computation, 27 (1973), p. 221.
- 1021 [8] F. B. BELGACEM, H. E. FEKIH, AND H. METOUI, *Singular perturbation for the dirichlet boundary control of elliptic problems*,
1022 ESAIM: Mathematical Modelling and Numerical Analysis, 37 (2003), pp. 833–850.
- 1023 [9] F. BEN BELGACEM, H. EL FEKIH, AND J. P. RAYMOND, *A penalized Robin approach for solving a parabolic equation with nonsmooth*
1024 *Dirichlet boundary conditions*, Asymptot. Anal., 34 (2003), pp. 121–136.
- 1025 [10] A. BENABDALLAH, F. BOYER, M. GONZÁLEZ-BURGOS, AND G. OLIVE, *Sharp estimates of the one-dimensional boundary control*
1026 *cost for parabolic systems and application to the n -dimensional boundary null controllability in cylindrical domains*, SIAM
1027 Journal on Control and Optimization, 52 (2014), pp. 2970–3001.
- 1028 [11] A. BENABDALLAH, F. BOYER, AND M. MORANCEY, *A block moment method to handle spectral condensation phenomenon in*
1029 *parabolic control problems*. working paper or preprint, Dec. 2018, <https://hal.archives-ouvertes.fr/hal-01949391>.
- 1030 [12] F. BOYER AND P. FABRIE, *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*,
1031 vol. 183 of Applied Mathematical Sciences, Springer, New York, 2013.
- 1032 [13] E. CASAS, M. MATEOS, AND J.-P. RAYMOND, *Penalization of dirichlet optimal control problems*, ESAIM: Control, Optimisation
1033 and Calculus of Variations, 15 (2008), pp. 782–809.
- 1034 [14] J.-M. CORON, *Control and nonlinearity*, vol. 136 of Mathematical Surveys and Monographs, American Mathematical Society,
1035 Providence, RI, 2007.
- 1036 [15] K.-J. ENGEL AND R. NAGEL, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics,
1037 Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafunne, G. Nickel, D. Pallara,
1038 C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- 1039 [16] H. O. FATTORINI, *Some remarks on complete controllability*, SIAM J. Control, 4 (1966), pp. 686–694.
- 1040 [17] H. O. FATTORINI AND D. L. RUSSELL, *Uniform bounds on biorthogonal functions for real exponentials with an application to the*
1041 *control theory of parabolic equations*, Quart. Appl. Math., 32 (1974/75), pp. 45–69.
- 1042 [18] E. FERNÁNDEZ-CARA, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Boundary controllability of parabolic coupled equations*, J.
1043 Funct. Anal., 259 (2010), pp. 1720–1758.
- 1044 [19] H. HOCHSTADT, *Asymptotic estimates for the Sturm-Liouville spectrum*, Comm. Pure Appl. Math., 14 (1961), pp. 749–764.
- 1045 [20] F. A. KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Minimal time for the null controllability of parabolic*
1046 *systems: The effect of the condensation index of complex sequences*, Journal of Functional Analysis, 267 (2014), pp. 2077–2151.
- 1047 [21] Q. KONG AND A. ZETTL, *Eigenvalues of regular Sturm-Liouville problems*, J. Differential Equations, 131 (1996), pp. 1–19.
- 1048 [22] Q. LÜ, *A lower bound on local energy of partial sum of eigenfunctions for Laplace-Beltrami operators*, ESAIM Control Optim.
1049 Calc. Var., 19 (2013), pp. 255–273.
- 1050 [23] R. NITTKA, *Inhomogeneous parabolic neumann problems*, Czechoslovak Mathematical Journal, 64 (2014), pp. 703–742.
- 1051 [24] G. OLIVE, *Boundary approximate controllability of some linear parabolic systems*, Evol. Equ. Control Theory, 3 (2014), pp. 167–
1052 189.
- 1053 [25] M. TUCSNAK AND G. WEISS, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher.
1054 [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.