On a Ring-Shaped Service of Collective Taxis, Part 1: Traffic Physics and Service Quality

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Abstract

In the digital era of mobility, the generalization of connectivity for customers as well as for vehicles has enabled for the development of different kinds of platform-based shared mobility services. Here we consider a service of collective taxis that combines on-demand access and egress of customers and "transit cycles" of vehicles along a ring-shaped circuit to serve mobility demand.

The paper brings about a physical theory of customer and vehicle traffic in an abstract ring-shaped service of collective taxis. The traffic system is parameterized by (i) on the infrastructure side, the ring circumference and the average run speed, (ii) on the vehicle side, the cab passenger capacity and the stop time required to pick up and drop off a passenger, (iii) on the service side, the fleet size and the time span of daily operations, (iv) on the demand side, the trip volume and the average ride length. Spatial and temporal homogeneity are postulated to devise a mathematical formulation of (i) cab occupancy and the availability of vacant places to potential customers, (ii) Circulating, Alighting and Boarding phases (CAB), (iii) average service speed, in the operator perspective, (iv) average commercial speed, in the user perspective, yielding typical ride time, (v) access length and typical access time to an incoming user.

The mathematical treatment is based on a stochastic Markov chain model of one collective cab with state vector that combines the C/A/B status with the number of riders. We establish the stationary regime of this dynamical stochastic system and characterize its stochastic equilibrium as a Fundamental Equation of Service Traffic in one scalar variable of load factor. The solution is unique and it exists if a capacity requirement is satisfied. Closed-form analytical formulas are provided to characterize every model outcome, together with sensitivity properties with respect to all system parameters.

Keywords

Traffic model; Load index; Fundamental traffic diagram; Stochastic equilibrium; Vehicle availability; Service access time; Service speed; Commercial speed; Sensitivity analysis

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1. Introduction

1.1 Background and literature review

In the big cities of emerging countries, for example Nairobi and Addis Ababa in Africa, collective taxis often account for 20 or 30% of passenger transport (Ferro, 2015). The reasons for this high modal share are twofold: on the one hand, a relative paucity of public transport lines and private cars; and on the other hand a large local supply of collective taxis, which attract large numbers of customers, making it possible to spread production costs and offer attractive prices.

In the developed countries, collective taxi services are only present in certain niche markets: for example point-to-point links between an airport and the city center, or on-demand transport services targeting a particular customer population (particularly people without private cars) and with relatively impractical operating conditions (booking a day in advance, setting a time window for pickup, risk of significant waiting times, detours…) (Orski, 1977; Jansson et al., 2016).

However, with the digital revolution, mobility has become connected and shared mobility services are developing fast, particularly in big cities in developed countries. The meteoric rise of the Uber platform is one emblematic example in the taxi industry, alongside other competing firms (Lyft in the US, Didi in China, Ola in India, etc.). Collective taxi services seem to be the next source of growth for these companies: for example, Uber Pool, Lyft Line, Padam, etc. allow travelers to share the use of vehicles in real time, at much lower fares than the individual taxi (around $5 or €5 per ride in city central areas). Moreover, the anticipated arrival of self-driving vehicles will sharply reduce driving costs, which will in turn lower production costs by half or two thirds: this can be expected to have similar effects on use as the impact on taxis of the dramatic fall in transaction costs brought about by online platforms.

Collective taxi services (CTS) have given rise to an academic literature in transportation analysis that splits in two streams according to their orientation towards either operations or planning. Service operations have been modeled in a microscopic way, detailing each vehicle and each customer request, so as to develop specific algorithms for vehicle dispatching and vehicle-customer matching (e.g. Malucelli et al., 1999, Lioris, 2010): such models involve two levels of analysis, the lower one for dynamic microsimulation with much detail in both space and time, the higher one to manage service operations and optimize the assignment of resources. Yet service demand is exogenous as it is generated randomly from a stochastic process with given parameters. Micro-simulation models of Personal Rapid Transit systems belong to that category (e.g. Andreasson, 1994, 2003, 2005), as do traffic modeling studies for cities, for instance the simulation of self-driving shared cabs for Lisbon city (ITF, 2015), the simulation study of some shared mobility services for the Zurich area (Hörl et al., 2019) and the Barcelona MaaS study by PTV (Lenz, 2019).

The second stream of literature is focused on system planning: it involves the demand side and especially Travel Demand Modeling in order to derive the potential usage, frequentation, customership and fare revenues of a given CTS. CTS usage is modeled per individual trip-maker as the outcome of travel decisions, from trip generation to mode choice in which CTS stands as a specific option that competes with rival modes of private car, mass transit, maybe also bike and walk. Recent contributions include Fagnant and Kockelman (2014, 2016) and Fagnant et al. (2015).

The next step is to combine a supply-oriented model of service operations and a demand-oriented model of service planning in order first to ensure physical consistency (e.g. Berrada et al., 2019) and then to look for profitable business models (e.g. Horl, 2019).

On comparing the CTS academic literature to the taxi one, a prominent difference is the shortage of microeconomic theory for CTS. The microeconomic theory of taxi services has been pioneered by Douglas (1972) who identified four main components in a taxi service as a system of supply and demand: namely combining (i) a demand function with respect to price and wait time, (ii) a pricing rule linking the tariff fare to trip time, (iii) a production cost function proportional to taxi time occupied and vacant, (iv) a "delay distribution" i.e. a function relating the taxi unoccupied time from customer drop-off to next customer pick-up, to the density of vacant taxis and also the car speed.

The Douglas model has given rise to subsequent contributions, notably Arnott (1996) who considered homogenous 2D space to demonstrate the collective interest to subsidy taxi supply in a less aggregative perspective, by Yang et al. (2002) who explored the effects of different regulation policies on the market equilibrium on the basis of a network model of taxi supply and demand in the Hong-Kong urban area, and by Yang et al. (2010) who put forward a matching function to derive the respective wait times of taxis and customers to fleet size and demand volume.

Indeed, microeconomic analysis is required to better understand taxi sharing, especially the interaction between several customers using the same vehicle at the same time, thereby imposing delays to each other.

1.2 Objective and contribution

The paper brings about a stylized analytical model of a CTS as a system of supply and demand. A basic set of modeling assumptions are provided both for the supply side and the demand side, so as to model their dynamic interaction and derive the "usage conditions" of vehicle occupancy, vehicle availability times for customers and ride time per user trip. Thus the model deals primarily with traffic physics: yet the physical representation is stylized in order to constitute a microeconomic model of a CTS as a "technology function" that is suitable for further economic analysis (such as supply-demand equilibrium, optimal service management, regulation policy).

More precisely, the demand is represented by an overall volume of demand trips over a time period, its spread in space and the average trip length. Service supply is represented by fleet size, vehicle capacity, run speed, average time to pick up and drop off an individual passenger. From these parameters, postulating a specific spatial configuration as a ring shape, specific service operational processes, on-demand service and a stationary traffic regime, we obtain mathematical formulas for the probability of vehicle availability, the statistical distribution of vehicle occupancy by passengers, the average access time, the average ride time.

The formulas are relatively simple, especially for vehicle capacity of 1 and 2. It is then easy to assess the respective performances of different service specifications under a given pattern of demand.

Key to the analysis is a postulate of ring shape for both demand and supply: the spatial distribution of customer origins is postulated homogenous along a ring (such as a circular road) and the service vehicles are assumed to run continuously, each in one direction along the ring. While it may seem highly restrictive, the ring shape in fact matches the basic

principle of vehicle cycles in the operation of transit lines. The specific postulate induces a spatial homogeneity that is essential to our mathematical analysis.

1.3 Methodology

Our modeling methodology combines elements from traffic microsimulation, queuing theory and stochastic modeling. Space is represented as a specific configuration of places: this enables for the identification of different positions in space as well as of distances between points along the ring.

As for traffic microsimulation, we consider two kinds of entities: collective taxis as vehicles and individual customers that use the vehicles as passengers. Each entity is an individual element of the set of its kind: individual situations are modeled in an abstract way owing to stochastic modeling that enables for generic treatment and formal calculus.

The traffic model of CTS, with its vehicles as servers and its customers, is related to queuing theory in which a number of servers are available under certain conditions to a flow of customers. The classical interpretation in queuing theory is that servers have fixed positions and that customers get to them as “arrival streams” of service requests (e.g. Kleinrock, 1975). In our model, the servers are mobile; every customer waits for one of them at their own place prior to boarding, riding and alighting. Using properties of spatial homogeneity together with some additional simplifications, we model every vehicle as a stochastic system with states and transitions: the state vector combines the number of passengers on board and the “logistical status” either Circulating, Boarding and Alighting. The transition rates from one state to another depend on the physical situation, notably the number of on-board passengers, the average trip length, the flow of customer arrivals and the base circulation speed.

Under the Markov postulate that the next system state depends only on the current one, we study the stochastic equilibrium of the system by solving the balance equations of probability flows between the elementary states. The resulting stationary distribution of probability enables us to characterize the conditions of traffic and usage in a fairly simple way.

Coming to the time dimension, stochastic equilibrium corresponds to a stationary regime of system dynamics. The equilibrium distributions of access times, of vehicle occupancy and availability are endowed with fairly simple analytical formulas.

In all, we obtain an analytical model of the service conditions (intensity of usage, quality of service) according to supply and demand characteristics.

1.4 Paper structure

The rest of the paper is organized in eight parts. We begin by specifying the territorial framework and the service process (section 2). Then we model the access time of a given customer to the service as a random variable depending on the number of available vehicles (section 3). Next, we build up the stochastic model of one vehicle by identifying the elementary states and the transitions between them, and by specifying the transition rates as functions of the state variable and exogenous parameters (section 4). We then study the stationary regime and characterize its determination as a “Fundamental Equation of Service Traffic” (FEST) in one scalar variable only, called the “load factor” and denoted by x (section 5). Based on the solution of the FEST, we derive the availability probability and also the access length (section 6). We also derive the service speed and the commercial speed that correspond to vehicles and customers, respectively, and we emphasize the difference between them as a traffic relativity effect typical of collective transit (section 7). The last consequences pertain to the ride time and the access time (section 8). To conclude, we synthesize the model

framework and also the directions of influence of the diverse factors onto the different model outcomes (section 9).

Table 1: Notation

C ring circumference, R ring radius, Ω ring geometric centre Ω

ℓ bandwidth of catchment area along the ring

M point along the ring, referred to by angle θ between $\overrightarrow{\Omega M_0}$ and $\overrightarrow{\Omega M}$

v_0 reference speed of vehicle running, excluding stops

N fleet size (number of vehicles)

H period of service operations in a typical day

t_S^+ (resp. t_S^-) delay caused to the vehicle to allow a customer to board (resp. alight)

K total number of places in a collective taxi (its passenger "capacity")

Q demand volume: number of trips per day

L_R average length of cab ride requested by a customer; F_R CDF of ride lengths

$t_0 \equiv L_R / v_0$ base ride time (under free flow speed)

$t_R \equiv L_R / v^o$ effective ride time, under service speed v^o

L_A average access length from cab position to waiting customer; F_A CDF of access lengths

$t_A \equiv L_A / v^o$ effective ride time, under service speed v^o

n the number of customers on board a collective taxi at a given moment h

σ logistical status either C for Circulating, A for Alighting or B for Boarding

$p_{\sigma.n}$ the probability that the collective taxi will be in state $\sigma.n$ at a given moment h

P_A Availability rate is the probability of cab having one place available at a given moment h

P'_A pseudo-availability rate

P_C probability of Circulating phase

y ratio between demand and supply

ρ composite load index

x load factor

Ψ_K characteristic function involved in Fundamental Equation of Service Traffic

2. Territory, mobility and collective taxi system

Let us define the service of collective taxis by specifying first the territory and the ring shape (§ 2.1), then the mobility demand to be served along the ring arterial (§ 2.2), next the roadway infrastructure and its traffic conditions (§ 2.3) and lastly the service quality and its operational processes (§ 2.4).

2.1 Territory and ring shape

Here the "territory" refers to the geographical space that is a set of places which accommodates a human population. Each place is identified by a point M in 2D space at which some activities can take place.

The ideal territorial configuration for an urban transit system is that of a ring-shaped city, as claimed by Maupu (2006) and studied by Leurent (2019) for a taxi service. Indeed, an arterial ring road enables collective taxis to circulate and also to stop for passenger alighting and boarding.

Let us postulate that such an arterial road exists in the city and exhibits some ring shape: circle, ellipse or a somewhat less regular circuit. Consider a reference point Ω as "ring center" and a reference "major axis" passing through it. Then, each point M along the ring is characterized by polar coordinates: angle $\theta \in [0,2\pi]$ between the vectors $\overrightarrow{\Omega \mathrm{M}_0}$ on the major axis and $\overrightarrow{\Omega \mathrm{M}}$, and radius r_θ from the center.

The main physical parameter of the circuit is its circumference, denoted C. If the ring is a circle of radius R, then $r_\theta = \mathrm{R}$ for every angle θ and we have that $C = 2\pi\mathrm{R}$. More generally, we consider as "ring radius" the average radius $\mathrm{R} = C/(2\pi)$. A second parameter is the width of the ring, say 2ℓ on average assuming that each position on the ring can "catch up" trips for places up to distance ℓ from it.

Thus the ring extends in two dimensions of space not only as a circuit but more broadly as a band of land around it, called its catchment area (cf. Figure 1).

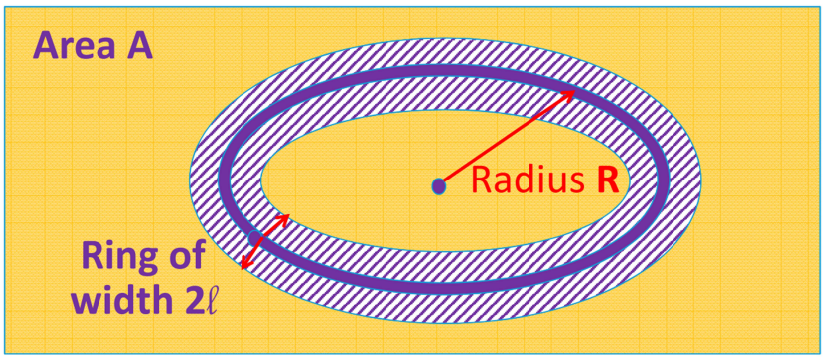

Fig. 1. Ring circuit and its catchment area.

2.2 Mobility demand

We consider the mobility demand of individual users as the set of trips that they make individually. On a given day, during service period of time length H, there are Q such trips. Each user trip is a 3-leg sequence of Walk, Ride and Walk legs. We focus on the Ride legs and consider the ring arterial as the locus of Ride origins and destinations.

Spatial homogeneity is postulated: that each point M along the arterial generates ride trips with uniform rate of generation and uniform statistical distribution of ride length from M (in each traffic direction). We shall denote by F_R the CDF of ride lengths and L_R its average value. Uniform origins generating i.i.d. trips yield uniform distribution of destination points along the circuit.

We moreover postulate temporal homogeneity: by time unit within the service interval, Q/H rides are generated.

2.3 Transport infrastructure and speed of travel

We postulate that a two-way arterial road runs through all the points on the ring and serves all the places along it. The following modelling assumptions are made:

[I1] in each direction of traffic, vehicles run at an average speed denoted by v_0;

[I2] a collective taxi may stop at any point to park, or pick up or drop off a customer, without disturbing the flow;

[I3] each collective taxi travels in only one direction: it never changes direction, whether or not there are customers on board;

[I4] servicing customers entails manoeuvres in order to leave the traffic flow, to stop, to open a door, to let the customer board or alight, to close the door, to move off again and re-enter the traffic flow. Each manoeuvre generates a delay for the vehicle, and therefore for the customers on board and for customers who have booked and are waiting. We will take into account the cumulative effect of these manoeuvres in the service speed.

2.4 The collective taxi service: quality of service and service process

We further assume that rides are provided by a single collective taxi operator, under the following conditions:

[S1] a given vehicle type, with uniform level of comfort and a number of K places for simultaneous transport of passengers;

[S2] a ride time on board, denoted by t_R, proportional to the distance covered L_R and inversely proportional to the commercial speed;

[S3] the "customer process" in terms of the sequence Plan-Book-Ticket is managed by a web application: for each ride, the transaction time for the customer is $\tilde{t}_\mathrm{T}$. For the collective taxi, a customer's boarding (resp. alighting) generates a delay of t_S^+ (resp. t_S^-).

[S4] an access time denoted by t_A between the moment the customer makes a request and the moment the collective taxi arrives to pick them up. This time is proportional to the length L_A that the collective taxi needs to cover between the booking and actual boarding.

In summary, the quality of service level for the customer is characterised in terms of comfort level and times t_R, $\tilde{t}_\mathrm{T}$ and t_A.

Concerning the operational management of the service, we assume the following service process. We denote the size of the operator's collective taxi fleet, and therefore the number of vehicles, by N. We ignore the proportion of vehicles off the road for maintenance or repair.

At a given moment h:

[P1] the spaces occupied in the vehicles are those assigned to a customer;

[P2] available vehicles are those where there is at least one empty place, e.g. a seat with no occupant;

[P3] when a new customer arrives, they inform the system that they need a ride, and the system assigns the request to the direction of travel that minimises the distance covered on board. Among the collective taxis travelling in that direction, the customer is assigned to the nearest vehicle with a seat available at that moment;

[P4] the customer waits for this designated vehicle to arrive in order to board and be transported to their destination point;

[P5] in each direction, the number of available vehicles is $\frac{1}{2}NP_{\mathrm{A}}$, where the probability of the vehicle being available, i.e. having at least one unoccupied space, is denoted by P_{A}. It holds that $P_{\mathrm{A}} = 1 - p_K$, where p_K is the probability of a collective taxi being full with its K places occupied;

From [P3], each new customer request is treated using the shortest route in terms of ride distance. Assuming that the ride lengths L are distributed according to Cumulative Distribution Function F_{R} in each traffic direction, then

$$L_{\mathrm{R}} \equiv \mathrm{E}[L] = \int_0^{C/2} L \,\mathrm{dF}_{\mathrm{R}}(L). \tag{2.1}$$

3. Access length

In a given direction of traffic, at a given moment, the k available vehicles have random positions that are distributed uniformly and independently from each other on the road. At every point where a request is generated, the distance between the customer position and that of the i-th available vehicle is a random variable L_i with uniform distribution in interval $[0, C]$. The access length L_{A} is the minimum of the distances L_i.

Random variable L_i has Cumulative Distribution Function as follows:

$$\mathrm{F}_i(x) \equiv \Pr\{L_i \le x\} = \frac{\min\{x, C\}}{C} 1_{\{x \ge 0\}}. \tag{3.1}$$

The available vehicle closest to the customer to serve the ride in the assigned traffic direction is located at distance $L_{\mathrm{A}} \equiv \min_{i \in \{1,..k\}} L_i$. This distance is distributed with CDF as follows:

$$\begin{aligned}
\mathrm{F}_{\mathrm{A}}(x) &\equiv \Pr\{L_{\mathrm{A}} \le x\} = \Pr\{\min_i L_i \le x\} \\
&= 1 - \Pr\{L_i > x : \forall i\} \\
&= 1 - \textstyle\prod_i \Pr\{L_i > x\} \quad \text{as the positions are independently distributed}
\end{aligned}$$

Thus
$$1 - \mathrm{F}_{\mathrm{A}}(x) = \textstyle\prod_i (1 - \mathrm{F}_i(x)) = ((1 - \frac{x}{C})^+)^k \quad \text{for} \quad x \le C. \tag{3.2}$$

From this stems the average access length conditionally to k:

$$\mathrm{E}[L_\mathrm{A} \mid k] = \int_0^C x \,\mathrm{d}\mathrm{F}_\mathrm{A}(x) = [x\,\mathrm{F}_\mathrm{A}(x)]_0^C - \int_0^C \mathrm{F}_\mathrm{A}(x)\,\mathrm{d}x$$
$$= C - C\int_0^1 [1-(1-u)^k]\,\mathrm{d}u = C\int_0^1 (1-u)^k\,\mathrm{d}u$$

Thus
$$\mathrm{E}[L_\mathrm{A} \mid k] = \frac{C}{k+1}. \tag{3.3}$$

This average value is conditional to the number k of available cabs. In turn, this number comes from the $N_2 \equiv N/2$ cabs assigned to the traffic direction. Taking the availability status of cabs as independent Bernoulli-distributed variables with success probability P_A, then k is distributed binomial with parameters N_2 and P_A. Its probability distribution function is:

$$\forall k \in \{0,1..N_2\},\ P_k = \binom{N_2}{k}.P_\mathrm{A}^k.(1-P_\mathrm{A})^{N_2-k}. \tag{3.4}$$

The unconditional mean is then

$$\mathrm{E}[L_\mathrm{A}] = \sum_{k=0}^{N_2} P_k\,\mathrm{E}[L_\mathrm{A} \mid k] = \sum_{k=0}^{N_2} P_\mathrm{A}^k.(1-P_\mathrm{A})^{N_2-k}\,\frac{N_2!}{k!\,(N_2-k)!}\frac{C}{k+1}$$
$$= C.\sum_{k=0}^{N_2}\frac{P_\mathrm{A}^k.(1-P_\mathrm{A})^{N_2-k}N_2!}{(k+1)!\,(N_2-k)!} = \frac{C}{(N_2+1)P_\mathrm{A}}.\sum_{k=0}^{N_2}\frac{P_\mathrm{A}^{k+1}.(1-P_\mathrm{A})^{N_2-k}(N_2+1)!}{(k+1)!\,(N_2-k)!}$$
$$= \frac{C}{(N_2+1)P_\mathrm{A}}.\sum_{k=1}^{N_2+1}\frac{P_\mathrm{A}^k.(1-P_\mathrm{A})^{N_2+1-k}(N_2+1)!}{k!\,(N_2+1-k)!}$$
$$= \frac{C}{(N_2+1)P_\mathrm{A}}.(1-(1-P_\mathrm{A})^{N_2+1})$$

Neglecting $(1-P_\mathrm{A})^{N_2+1}$ and approaching $N_2+1 \approx N_2$, we obtain the following approximation for $\mathrm{E}[L_\mathrm{A}]$:

$$\mathrm{E}[L_\mathrm{A}] = \frac{C}{N_2 P_\mathrm{A}}. \tag{3.5}$$

4. Stochastic model of one cab

Let us now model the dynamical state of one cab as a stochastic Markov chain. To keep the model simple enough, we consider a state vector that is reduced to the logistical status and the number of passengers on board. We shall first define the elementary states and the transitions between them (§ 4.1). Then, we model the transition rates so as to capture the physics of service traffic: some simplifications are required to make the model Markovian (§ 4.2). Based on the transition rates, we derive the stationary distribution of probability that characterizes a CAB system in stochastic equilibrium (§ 4.3).

4.1 States and transitions

In real time, a collective taxi is in one or other of the following states, depending on the number $n \le K$ of passengers on board and the logistical phase σ:

- on the move with $n > 0$, in a fixed direction and at speed v_0: a state denoted by C.n (C standing for Circulating);

- on the move with no passengers on board or boarding or alighting: a state denoted by $\mathrm{C}.0$ i.e. $\mathrm{C}.n$ with $n=0$;
- stationary for an additional passenger to board: a state denoted by $\mathrm{B}.n$ (B for Boarding);
- stationary for a passenger to alight: a state denoted by $\mathrm{A}.n$ (A for Alighting).

As customers arrive individually and independently, the transitions from one state to another occur between "neighbouring states" as in a Birth-Death process i.e. from n to $n+1$ or from $n+1$ to n, and only between B and C or between A and C:

- from $\mathrm{C}.n$ where $n<K$, to $\mathrm{B}.n+1$, each time the vehicle has to stop to pick up a customer.
- from $\mathrm{B}.n+1$ to $\mathrm{C}.n+1$ when the boarding stop takes end.
- from $\mathrm{C}.n$ where $n\geq 1$ to $\mathrm{A}.n$: each time one of the passenger's rides ends up and the vehicle has to stop so that the customer can alight.
- from $\mathrm{A}.n$ where $n\geq 1$ to $\mathrm{C}.n-1$ when the alighting stop ends up.

Figure 2 describes the typical states and the transitions.

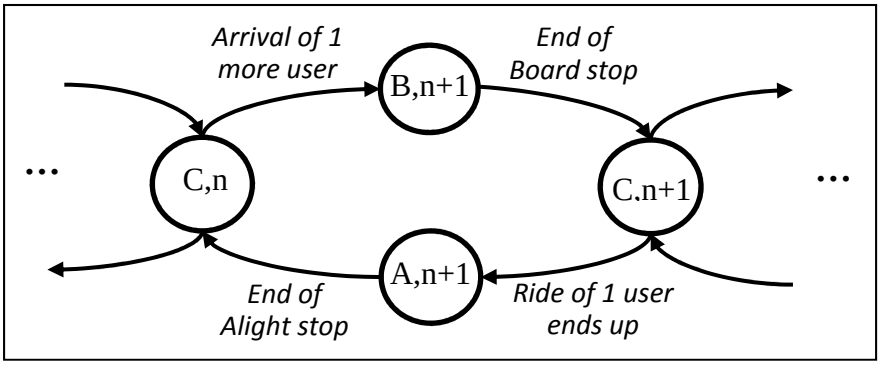

Fig. 2. Topology of states and transitions in CAB Markov chain.

4.2 Transition rates

We reduce the system description to the pairs $\sigma.n$ of logistical status $\sigma\in\{\mathrm{C},\mathrm{A},\mathrm{B}\}$ and the number $n\in\{0,1,..\mathrm{K}\}$ of passengers on board. By taking $\sigma.n$ as state vector, we omit such things as the cab current position along the ring, the locations of pending customer requests and the destinations of the riders. These are reduced to residual ride lengths from the cab current position. Furthermore, we take the residual lengths as independent random variables identically distributed according to an exponential law. Similarly, we take the boarding and alighting times as independent random variables with exponential laws.

These approximations enable us to consider time-independent transition rates that suffice to model the system dynamics and derive its stationary state (e.g. Kleinrock, 1975).

Between states $\sigma.n$, transitions occur with positive rates only in the following cases.

From $\mathrm{C}.n$ where $n<K$, to $\mathrm{B}.n+1$, each time the vehicle stops to pick up a new customer: per traffic direction, the total flow of customers has rate $Q/(2\mathrm{H})$ and it is split into $N_2 P_\mathrm{A}$ cabs. Then, per cab the customers arrive according to the following rate:

$$\lambda \equiv \frac{y}{P_\mathrm{A}}, \quad \text{wherein} \quad y \equiv \frac{Q}{\mathrm{H}\,N}. \tag{4.1}$$

This rate applies on average at any instant. Yet in the model only the system states $\mathrm{C}.n$ with $n<K$ can support customer arrivals. So, defining $P'_{\mathrm{A}} \equiv \sum_{n=0}^{K-1} \Pr\{C.n\}$ the probability of such supportive states, the effective arrival rate λ' satisfies that $\lambda'.P'_{\mathrm{A}} = y$ instead of $\lambda.P_{\mathrm{A}} = y$, hence

$$\lambda' \equiv \frac{y}{P'_{\mathrm{A}}} = \frac{Q}{\mathrm{H}\, N\, P'_{\mathrm{A}}}. \tag{4.2}$$

We call P'_{A} the pseudo-rate of availability.

Rate λ' applies for any transition from $\mathrm{C}.n$ with $n<K$, to $\mathrm{B}.n+1$.

Let us now consider a transition from $\mathrm{B}.n$ to $\mathrm{C}.n$. As a boarding stop has average duration of t_{S}^{+}, we model its duration by an exponential random variable of identical mean, hence with parameter $1/t_{\mathrm{S}}^{+}$. Thus, the transition from $\mathrm{B}.n$ to $\mathrm{C}.n$ occurs at time rate $1/t_{\mathrm{S}}^{+}$, which is the time rate at which that kind of random variable comes to its end.

Similarly, let us consider a transition from $\mathrm{A}.n$ where $n\geq 1$, to $\mathrm{C}.n-1$. As an alighting stop has average duration of t_{S}^{-}, we model its duration by an exponential random variable of identical mean, hence with parameter $1/t_{\mathrm{S}}^{-}$. Thus, the transition from $\mathrm{A}.n$ to $\mathrm{C}.n-1$ occurs at time rate $1/t_{\mathrm{S}}^{-}$.

Lastly, a transition from $\mathrm{C}.n$ with $n\geq 1$, to $\mathrm{A}.n$ corresponds to the ending of a ride for one on-board passenger. As a ride requests an average run time of $t_0 \equiv L_{\mathrm{R}}/v_0$, we model the duration as an exponential random variable of identical mean, hence with parameter $\mu \equiv 1/t_0$. The transitions from $\mathrm{C}.1$ to $\mathrm{A}.1$ will occur at time rate μ of ride ending. From $\mathrm{C}.n$ to $\mathrm{A}.n$ with $n\geq 1$, the transition occurs with rate equal to n times μ since any of the n riders may end his or her ride. Rate $n.\mu$ is the rate of ending of the ride of minimal residual length among the n ones. Indeed, among n independent exponential variables, the minimum is still an exponential variable, its parameter being the sum of those of its arguments, hence $n.\mu$ if they are identically distributed of parameter μ.

To sum up, denoting by $z_{\sigma.n}^{\tau.m}$ the transition rate from $\sigma.n$ to $\tau.m$, we have:

$$z_{\mathrm{C}.n}^{\mathrm{B}.n+1} = \lambda' \text{ for } n<K, \tag{4.3a}$$

$$z_{\mathrm{B}.n}^{\mathrm{C}.n} = 1/t_{\mathrm{S}}^{+} \text{ for } n>0, \tag{4.3b}$$

$$z_{\mathrm{A}.n+1}^{\mathrm{C}.n} = 1/t_{\mathrm{S}}^{-} \text{ for } n<K, \tag{4.3c}$$

$$z_{\mathrm{C}.n}^{\mathrm{A}.n} = n.\mu = n/t_0 \text{ for } n>0. \tag{4.3d}$$

4.3 Stationary regime and distribution

Along the time dimension, which we index by instant h, state $\sigma.n$ has a probability $p_{\sigma.n}(h)$ that varies owing to transition occurrences. The Chapman-Kolmogorov equation is as follows for a short time laps δh from h:

$$p_{\sigma.n}(h+\delta h) = p_{\sigma.n}(h) - \delta h.p_{\sigma.n}(h).\Big(\sum_{\tau.m\neq\sigma.n} z_{\sigma.n}^{\tau.m}\Big) + \delta h.\Big(\sum_{\tau.m\neq\sigma.n} z_{\tau.m}^{\sigma.n}.p_{\tau.m}(h)\Big). \tag{4.4}$$

The state probabilities are stable over time if for each state the probability flows in and out of it are balanced. This condition is called the local balance equation (dropping the time index):

$$p_{\sigma.n}.(\sum_{\tau.m\neq\sigma.n} z_{\sigma.n}^{\tau.m}) = \sum_{\tau.m\neq\sigma.n} z_{\tau.m}^{\sigma.n}.p_{\tau.m} \,. \tag{4.5}$$

At $\mathrm{B}.n$ the local balance equation is $p_{\mathrm{B}.n}/t_{\mathrm{S}}^{+} = p_{\mathrm{C}.n-1}.\lambda'$: then

$$p_{\mathrm{B}.n} = p_{\mathrm{C}.n-1}.\lambda' t_{\mathrm{S}}^{+} \quad \text{for } n<K \,. \tag{4.6a}$$

At $\mathrm{A}.n$ the local balance equation is $p_{\mathrm{A}.n}/t_{\mathrm{S}}^{-} = p_{\mathrm{C}.n}.n\mu$: then

$$p_{\mathrm{A}.n} = p_{\mathrm{C}.n}.n\mu t_{\mathrm{S}}^{-} \quad \text{for } n>0 \,. \tag{4.6b}$$

Let us now split the set of states in two subsets, by artificially cutting the transitions from $\mathrm{C}.n-1$ to $\mathrm{B}.n$ and from $\mathrm{C}.n$ to $\mathrm{A}.n$. Between the two subsets, the flow balance condition is

$$p_{\mathrm{C}.n-1}.\lambda' = p_{\mathrm{C}.n}.n.\mu \text{ , so that}$$

$$p_{\mathrm{C}.n} = \frac{\lambda'}{n\mu}.p_{\mathrm{C}.n-1} \quad \text{for } n>0 \,. \tag{4.6c}$$

Combining (4.6c) and (4.6b), we recover that

$$p_{\mathrm{A}.n} = p_{\mathrm{C}.n-1}.\lambda' t_{\mathrm{S}}^{-} \quad \text{for } n>0 \,. \tag{4.6d}$$

Let us define a "load factor" $x \equiv \lambda'/\mu$. By induction, it comes out that

$$p_{\mathrm{C}.n} = \frac{x}{n}.p_{\mathrm{C}.n-1} = p_{\mathrm{C}.0}\prod_{i=1}^{n}\frac{x}{i} = \frac{x^n}{n!}.p_{\mathrm{C}.0} \quad \text{for } n>0 \,. \tag{4.7}$$

Denote also $x_n \equiv x^n/n!$, $p_0 \equiv p_{\mathrm{C}.0}$ and $t_{\mathrm{S}} \equiv t_{\mathrm{S}}^{+} + t_{\mathrm{S}}^{-}$.

For every $n \in \{0,1,..\mathrm{K}-1\}$ we can put together the states $\mathrm{C}.n$, $\mathrm{B}.n+1$ and $\mathrm{A}.n+1$: their joint probability amounts to $p_n^{+} \equiv (1+\lambda' t_S).p_{\mathrm{C}.n}$.

Then, the total probability of all states is

$$\begin{aligned}\mathrm{Pr}_{\Sigma} &= p_{\mathrm{C}.K} + \textstyle\sum_{n=0}^{K-1} p_n^{+} = p_0.x_K + \sum_{n=0}^{K-1} p_{\mathrm{C}.n}(1+\lambda' t_{\mathrm{S}}) \\ &= p_0.x_K + \textstyle\sum_{n=0}^{K-1} p_0 x_n (1+x.\mu.t_{\mathrm{S}}) \\ &= p_0(\mathrm{X}+a.x.\dot{\mathrm{X}})\end{aligned}$$

Wherein: $a \equiv t_{\mathrm{S}}\mu = \frac{t_{\mathrm{S}}}{t_0}$, $\mathrm{X} \equiv \sum_{n=0}^{K} x_n$ and $\dot{\mathrm{X}} \equiv \frac{\partial}{\partial x}\mathrm{X} = \sum_{n=0}^{K-1} x_n$.

As the total probability is equal to 1, it follows that

$$p_0 = \frac{1}{\mathrm{X}+a.x.\dot{\mathrm{X}}} \,. \tag{4.8}$$

From this and (4.6) we get that:

$$p_{\mathrm{A}.n} = p_0 \lambda' t_{\mathrm{S}}^{-} \frac{x^{n-1}}{(n-1)!} \quad \text{for } n \in \{1,..\mathrm{K}\} \,. \tag{4.9a}$$

$$p_{\mathrm{B}.n} = p_0 \lambda' t_{\mathrm{S}}^{+} \frac{x^{n-1}}{(n-1)!} \quad \text{for } n \in \{1,..\mathrm{K}\} \,. \tag{4.9b}$$

$$p_{\mathrm{C}.n} = p_0 \frac{x^n}{n!} \quad \text{for } n \in \{0,1,..\mathrm{K}\} \,. \tag{4.9c}$$

Figure 3 depicts the stationary distribution of probability for $K = 4$, $t_0 = 18$ min, $t_S^+ = 45$ s and $t_S^- = 75$ s and $x = 2.26$.

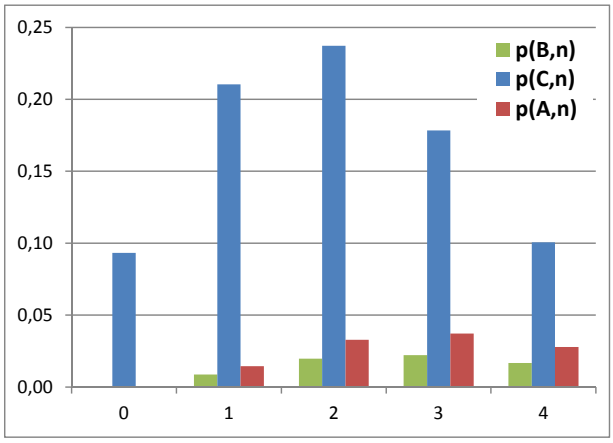

Fig. 3. Stationary distribution.

5. Stochastic equilibrium and FEST

The stationary probability distribution of the CAB states involves the flow rate of customer arrivals per available cab. By combining the associated conditions, we obtain a Fundamental Equation of Service Traffic (FEST) that characterizes the CAB system state in stochastic equilibrium (§ 5.1). The FEST is a mono-dimensional, nonlinear relationship linking a service load factor x as unknown variable, to exogenous conditions of demand volume, service time span and fleet size that are encapsulated in a composite parameter denoted ρ. We shall study this parameter and analyze its sensitivity to its exogenous factors (§ 5.2). Turning to the Ψ_K function that relates x to ρ, we will provide a physical interpretation, establish mathematical properties and its sensitivity with respect to cab capacity K (§ 5.3). Next, we demonstrate the existence and uniqueness of a solution to the FEST and we establish its sensitivities with respect to the different model parameters (§ 5.4). Lastly, we provide a general computation scheme, together with analytical solutions to the FEST with $K = 1$ and $K = 2$ (§ 5.5).

5.1 Fundamental equation of service traffic

Based on demand volume Q, service time span H and fleet size N, all taken as exogenous factors, let us define a composite parameter that is the average cab productivity in rides per day:

$$y \equiv \frac{Q}{\mathrm{H}\,N}. \tag{5.1}$$

The flow rate of customer arrivals per available cab, λ' defined in (4.2), can then be expressed as

$$\lambda' = y / P'_{\mathrm{A}}. \tag{5.2}$$

There are two relations between λ' and the load factor x. The first one consists in the definition of the pseudo-rate of availability, which can be restated as:

$$P'_{\mathrm{A}} \equiv \sum_{n=0}^{K-1} p_{C.n} = p_0\,\dot{\mathrm{X}}. \tag{5.3a}$$

The other relation consists in the definition of x,

$$x \equiv \lambda' t_0 = y\, t_0 / P'_{\mathrm{A}}\,. \tag{5.3b}$$

Combining the two relations, we obtain a unique condition that characterizes x:

$$\begin{aligned} x &= \frac{y\, t_0}{p_0\, \dot{\mathrm{X}}} \\ &= \frac{y\, t_0}{\dot{\mathrm{X}}}(\mathrm{X} + a\, x\, \dot{\mathrm{X}}) \quad \text{after replacing } p_0 \text{ by its expression with respect to } x \end{aligned}\,.$$

As $a = t_{\mathrm{S}} / t_0$, we get that

$$x = \frac{y\, t_0}{\dot{\mathrm{X}}}(\mathrm{X} + \frac{t_{\mathrm{S}}}{t_0}\, x\, \dot{\mathrm{X}}) = y(t_0 \frac{\mathrm{X}}{\dot{\mathrm{X}}} + t_{\mathrm{S}}\, x)\,.$$

Recombining, it comes out that

$$\frac{x\, \dot{\mathrm{X}}_{(K)}}{\mathrm{X}_{(K)}} = \frac{y\, t_0}{1 - t_{\mathrm{S}} y}\,. \tag{5.4}$$

The subscripts (K) remind that X and $\dot{\mathrm{X}}$ involve the cab capacity K.

Let us define:

$$\rho \equiv \frac{y\, t_0}{1 - t_{\mathrm{S}} y}\,. \tag{5.5a}$$

$$\Psi_K(x) \equiv \frac{x\, \dot{\mathrm{X}}_{(K)}}{\mathrm{X}_{(K)}}\,. \tag{5.5b}$$

Proposition 1 (FEST) *(i) Apart from cab capacity K, the exogenous factors in the CAB model are summarized by the composite parameter ρ. (ii) The stationary state depends entirely on a factor x that must satisfy the following equation, called the FEST:*

$$\Psi_K(x) = \rho\,. \tag{5.6}$$

Proof. The modelling assumptions made in Section 4.2 imply (5.4) as a necessary condition. It determines x with respect to exogenous factors K, Q, H, N, t_{S}^{+}, t_{S}^{-}, L_{R} and v_0, via $y \equiv Q/(\mathrm{H}\, N)$, $t_{\mathrm{S}} \equiv t_{\mathrm{S}}^{+} + t_{\mathrm{S}}^{-}$ and $t_0 \equiv L_{\mathrm{R}} / v_0$. Apart from K, all of the exogenous factors influence x via ρ only, making (i) hold true. In turn, under the definitions (5.5), then (5.4) implies (5.6), yielding point (ii).

Figure 4 depicts several Ψ_K graphs for different values of K.

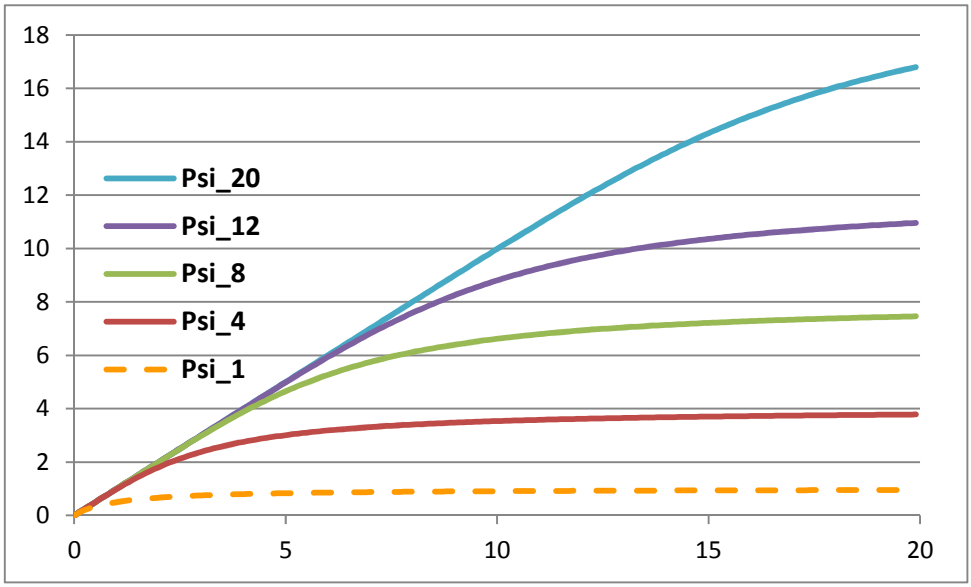

Fig. 4. Graphs of Ψ_K functions.

5.2 Properties of composite parameter ρ

Given fleet size N and time span H, the total vehicle operating time HN must meet two requirements from the Q customers:

- The first requirement pertains to dwell time, at level $Q(t_S^+ + t_S^-) = Q.t_S$: the constraint is $Q.t_S \leq N\,\mathrm{H}$.
- The other one involves base ride times: as there are K places per cab, $Q.t_0 \leq K.N.\mathrm{H}$, yielding a vehicle time requirement of $Q.t_0 / K \leq N.\mathrm{H}$.

As the dwell times and the base ride times are disjoint, it must hold that

$$Q.(t_S + \tfrac{1}{K} t_0) \leq N.\mathrm{H}, \text{ or equivalently}$$

$$y.(t_S + \tfrac{1}{K} t_0) \leq 1.$$

Recombining, we get the following condition that

$$\frac{y\,t_0}{1 - y\,t_S} \leq K, \tag{5.7}$$

In which the composite parameter ρ can be recognized.

This establishes ρ as a cab occupancy index that must be lower than cab capacity K.

Proposition 2: Sensitivity of cab load index ρ. *Provided that* $y \geq 0$, $t_0 \geq 0$ *and* $y.t_S < 1$, *index* ρ *is an increasing function of* Q, L_R, t_S^+ *and* t_S^-, *and a decreasing function of* H, N *and* v_0.

Proof. The sensitivities of ρ to its immediate factors are as follows:

$$\frac{\partial \rho}{\partial t_0} = \frac{\rho}{t_0}$$

$$\frac{\partial \rho}{\partial t_S} = \frac{\rho^2}{t_0}$$

$$\frac{\partial \rho}{\partial y} = \frac{\rho}{y}(1 + \frac{t_S}{t_0} y)$$

All of them are nonnegative under the blanket assumption. Thus ρ is an increasing function of t_0 and in turn it increases with L_R and decreases with v_0 since $t_0 \equiv L_R / v_0$. Furthermore, ρ is an increasing function of t_S, and in turn of t_S^+ and t_S^- since $t_S \equiv t_S^+ + t_S^-$. Lastly, ρ is an increasing function of y: in turn, it increases with Q and decreases with H and N.

5.3 The Ψ_K function

Conditionally to cab running, the passenger load has mean value as follows:

$$\mathrm{E}[n \mid \mathrm{C}] = \frac{\sum_{n=0}^{K} n.p_{C.n}}{\sum_{n=0}^{K} p_{C.n}}.$$

On the one hand, $\sum_{n=0}^{K} n.p_{C.n} = p_0 \sum_{n=0}^{K} \frac{n.x^n}{n!} = p_0 x \sum_{n=1}^{K} \frac{x^{n-1}}{(n-1)!} = p_0\, x\, \dot{\mathrm{X}}$.

On the other hand, $\sum_{n=0}^{K} p_{C.n} = p_0\, \mathrm{X}$. Thus

Thus
$$\mathrm{E}[n \mid \mathrm{C}] = \frac{x\dot{\mathrm{X}}}{\mathrm{X}} = \Psi_K(x). \tag{5.8}$$

We can then interpret cab load index ρ as the mean passenger load of a running cab.

The following proposition is demonstrated in the Appendix, § A.2.

Proposition 3: Properties of the Ψ_K function.

(i) Function Ψ_K is continuous on $[0,+\infty[$ and strictly increasing.

(ii) On that range, the function takes on all the values from 0 to K excluded.

(iii) It is a concave function.

(iv) Given x, the sequence $K \mapsto \Psi_K(x)$ is increasing.

5.4 FEST solution

Proposition 4: Existence and uniqueness of FEST solution*. Provided that $\rho < K$, the FEST admits one solution x_ρ which is unique.*

Proof. From Proposition 3 (i) and (ii), function Ψ_K can be inverted and the inverse function $\Psi_K^{(-1)}$ is increasing from $[0,K[$ to $[0,+\infty[$. As $\rho < K$ then $x_\rho \equiv \Psi_K^{(-1)}(\rho)$ exists and is the unique value such that $\Psi_K(x) = \rho$. Put in other words, x_ρ solves the FEST equation associated to ρ and it is the unique solution to it.

***Proposition 5: Sensitivities of FEST solution to exogenous factors**.*

(i) The solution $x \equiv \Psi_K^{(-1)}(\rho)$ *is an increasing function of* Q, L_R, t_S^+ *and* t_S^-, *and a decreasing function of* H, N *and* v_0.

(ii) A combined variation of these parameters influences x *in the same direction as it influences* ρ.

(iii) Given ρ, *the sequence* $K \mapsto \Psi_K^{(-1)}(\rho)$ *is decreasing.*

Proof. (i) x depends on any said parameter only through ρ: as it increases with ρ, it inherits its sensitivities so it varies in the same direction as ρ with respect to each factor. So point (i) stems from Proposition 2.

(ii) Any combined variation influences x only via ρ, with positive coefficient from ρ to x.

(iii) Let x be the solution of $\Psi_K(x) = \rho$. As $K \mapsto \Psi_K(x)$ is an increasing sequence, necessarily $\Psi_{K+1}(x) \geq \Psi_K(x) = \rho$. As $\Psi_{K+1}^{(-1)}$ is increasing, then $x \geq \Psi_{K+1}^{(-1)}(\rho)$ and in turn $\Psi_K^{(-1)}(\rho) \geq \Psi_{K+1}^{(-1)}(\rho)$.

5.5 General computation scheme and special instances

General computation scheme. In the general case, the FEST can be solved iteratively by setting it as the following fixed-point problem:

$$x = \frac{\rho.\mathrm{X}}{\dot{\mathrm{X}}}. \tag{5.9}$$

The iterative algorithm has initial value $x^{(0)}$ and induction rule from step k to step $k+1$ as follows:

$$x^{(0)} \equiv \rho. \tag{5.10a}$$

$$x^{(k+1)} \equiv \frac{\rho.X^{(k)}}{\dot{X}^{(k)}} \quad \text{with } X^{(k)} \equiv \mathrm{X}(x^{(k)}) \text{ and } \dot{X}^{(k)} \equiv \dot{\mathrm{X}}(x^{(k)}) \ . \tag{5.10b}$$

Special instance with unit passenger capacity. If $K = 1$ then the cab service is individual. This requires $\rho < 1$. It is straightforward to invert $\Psi_1(x) = x/(x+1)$ into

$$\Psi_1^{(-1)}(\rho) = \frac{\rho}{1-\rho}. \tag{5.11}$$

In turn, replacing ρ by its expression depending on y yields that

$$x = \frac{y\,t_0}{1 - y(t_0 + t_S)}. \tag{5.12}$$

More generally, recovering x from ρ at order K amounts to solving a K-th degree equation in x. There exist analytical formulas up to $K = 4$ only (by Abel's theorem).

Special instance with cab capacity of 2. At order $K = 2$, it is required that $\rho < 2$ and we have

$$\Psi_2(x) = \frac{x(x+1)}{1 + x + \frac{1}{2}x^2}.$$

Condition $\Psi_2(x) = \rho$ is equivalent to $\rho(1 + x + \frac{1}{2}x^2) = x + x^2$, hence to

$$x^2 + 2\frac{1-\rho}{2-\rho}x - \frac{2\rho}{2-\rho} = 0.$$

This second degree equation has one positive root only:

$$\Psi_2^{(-1)}(\rho) = \sqrt{(\frac{1-\rho}{2-\rho})^2 + \frac{2\rho}{2-\rho}} - \frac{1-\rho}{2-\rho}. \tag{5.13}$$

6. On Cab Availability and Service Access

In stochastic equilibrium, the primary outcomes of the CAB model consist in the state probabilities which are intimately related to the load factor x. From the elementary state probabilities stem the probability of any subset of states: of particular interest are the availability rates that play a crucial role both on the supply side (the effective rate) and on the demand side (since the true rate is essential to the mean access length).

This section provides characteristic formulas and sensitivity properties first for the availability rates (§ 6.1), second for the mean access length (§ 6.2).

6.1 Availability rates

The effective availability rate P'_A is defined as the probability of a given cab to be running and having at least one place available:

$$P'_{\mathrm{A}} \equiv \sum_{n=0}^{K-1} p_{\mathrm{C}.n}\,. \tag{6.1a}$$

This definition gives rise to characteristic formulas as follows:

$$P'_{\mathrm{A}} = p_0\,\dot{\mathrm{X}} = \frac{\dot{\mathrm{X}}}{\mathrm{X} + a\,x\,\dot{\mathrm{X}}}. \tag{6.2a}$$

$$P'_{\mathrm{A}} = \frac{\Psi_K(x)}{x(1 + a\Psi_K(x))}. \tag{6.2b}$$

$$P'_{\mathrm{A}} = \frac{\rho}{x(1 + a\rho)}. \tag{6.2c}$$

$$P'_{\mathrm{A}} = \frac{t_0\,y}{x}. \tag{6.2d}$$

Each of these equivalent forms may be used to recover the effective rate from x together with either y or ρ.

The following Proposition is demonstrated in the Appendix, §A.3.

Proposition 6: Sensitivities of the pseudo-rate of availability.

(i) Given K, the rate P'_{A} is a decreasing function of x.

(ii) Rate P'_{A} is a decreasing function of t_{S} and L_{R} but an increasing function of v_0.

(iii) Rate P'_{A} is a decreasing function of Q and an increasing function of N and H.

The "true" availability rate P_A is defined as the probability of having at least one place available in a given cab. It has more significance for the quality of service than the effective availability rate. It stems from the distribution of stationary probability as follows:

$$P_A \equiv 1-(p_{C.K} + p_{A.K} + p_{B.K}). \tag{6.2}$$

Thus, $P_A = 1-(1+aK)p_{C.K} = 1-\frac{(1+aK)x_K}{X+ax\dot{X}} = \frac{X-x_K+a(x\dot{X}-Kx_K)}{X+ax\dot{X}} = \frac{\dot{X}+a\,x\ddot{X}}{X+ax\dot{X}}$.

This gives rise to characteristic formulas as follows:

$$P_A = p_0(\dot{X}+a\,x\ddot{X}). \tag{6.3a}$$

$$P_A = p_0\,\dot{X}.(1+a\,\Psi_{K-1}(x)). \tag{6.3b}$$

$$P_A = P'_A.(1+a\,\Psi_{K-1}(x)). \tag{6.3c}$$

From the last expression, it comes out that $P_A \geq P'_A$, all the more so as K is larger. The two rates are identical only if $K=1$ i.e. no real-time sharing of cab capacity K.

The following sensitivity properties are demonstrated in the Appendix, § A.3.

Proposition 7: Sensitivities of the true availability rate P_A.

(i) Given K, the rate P_A is a decreasing function of x.

(ii) Rate P_A is a decreasing function of t_S and L_R but an increasing function of v_0.

(iii) Rate P_A is a decreasing function of Q and an increasing function of N, K and H.

These theoretical properties are in accordance with physical intuition.

Proposition 8: Sensitivities of the rate ratio P_A / P'_A. *(i) The ratio is an increasing function of Q, t_S^+, t_S^- and a decreasing function of N and H. (ii) The twofold influence of t_0 onto the ratio, via x (increasing) and via a (decreasing), is decreasing as a whole: thus the ratio is an increasing function of v_0 and a decreasing function of L_R.*

Illustration. Assume that $t_S = 2$ min and $t_0 = 18$ min, yielding $a = 1/9$. Figure 5 exhibits the variations of the effective rate (on the left side) and those of the true rate (on the right side) with respect to load factor x, for selected values of cab capacity K. The difference between the two rates becomes more and more apparent as x or K increases.

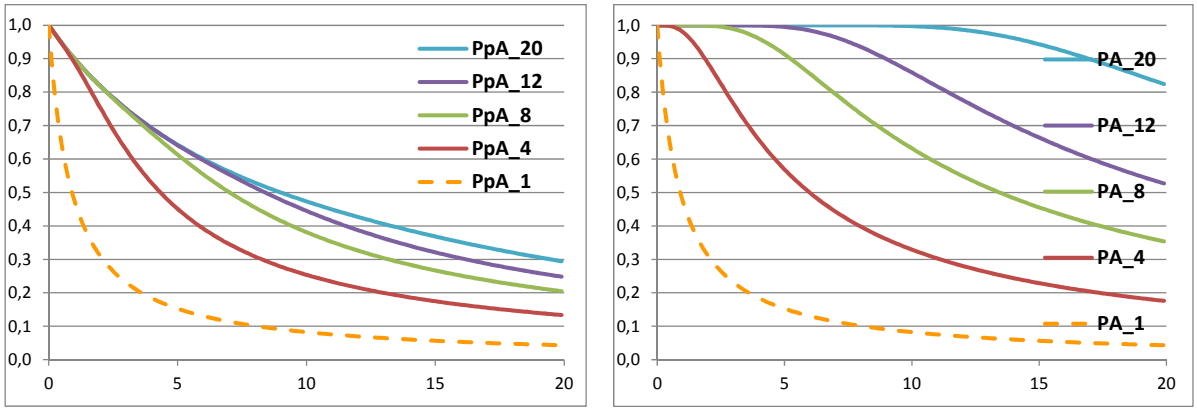

Fig. 5. Effective rate (left part) and True rate (right part) as functions of load factor.

6.2 Access length

The mean access length has been modelled in Section 3 as the circuit circumference divided by the number of available cabs in the traffic direction of interest, i.e. $N_2 P_A$.

$$L_A \equiv \frac{C}{N_2 P_A}. \tag{6.4}$$

It thus inherits its characteristic formulas and sensitivity properties from the true availability rate, but with variations in the inverse direction. Let us state them as follows.

$$L_A = \frac{C}{N_2} \frac{X + a x \dot{X}}{\dot{X} + a x \ddot{X}}. \tag{6.5a}$$

$$L_A = \frac{C}{N_2} \frac{1}{p_0 \dot{X}(1 + a \Psi_{K-1}(x))}. \tag{6.5b}$$

$$L_A = \frac{C}{N_2 P'_A (1 + a \Psi_{K-1}(x))}. \tag{6.5c}$$

Proposition 9: Sensitivities of the access length.

(i) L_A *is a decreasing function of* N *and* H.

(ii) L_A *increases with* Q, L_R *and* t_S *but diminishes with* v_0.

(iii) L_A *is an increasing function of the ring circumference* C.

The sensitivities of L_A to N and C are intuitive. The influences of L_R, t_S and Q are easy to envisage. The influence of v_0 is less obvious, but well-established (see Appendix § A.4).

Illustration. Figure 6 depicts the influence of local factor x on the mean access length for different values of K. In this instance, the parameter values are set up to $R = 4$ km hence $C = 25.1$ km, $N = 100$ cabs, t_0 and t_S as previously so that $a = 1/9$.

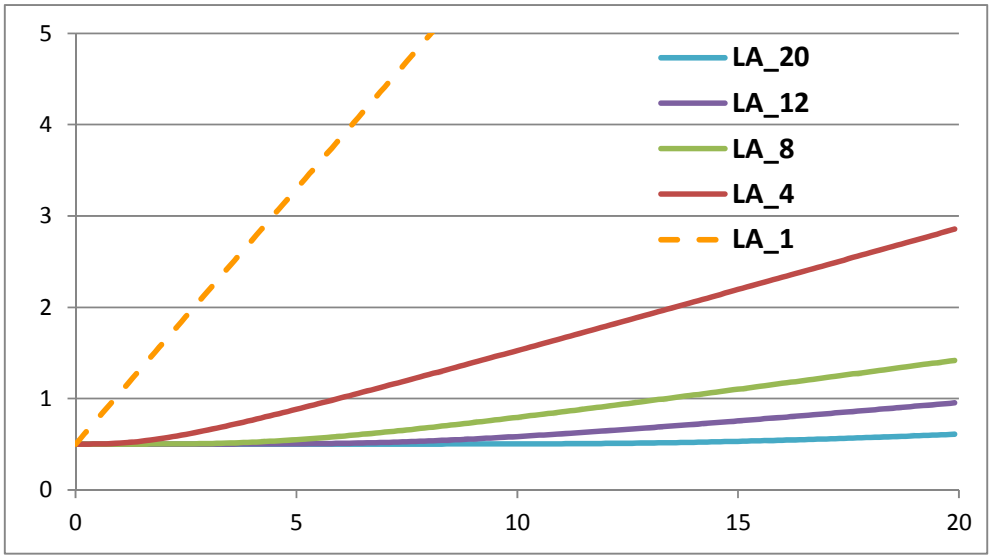

Fig. 6: Mean access length with respect to load factor.

7. On service and commercial speeds

The reference speed v_0 applies to vehicle running on the ring arterial road. In service operations, the cab speed stems from all phases, boarding and alighting as well as circulating. This makes the operator-centric speed lower than the reference speed. Yet on the demand side the associated speed is still a different notion: the speed that is experienced by the users is the commercial speed that stems from the passenger presences on board during all phases, with emphasis on the number of riders as much as on the logistical status. The commercial speed can also be called user-centric.

This section is devoted to the operator- and user-centric speeds and their comparison. After studying firstly the service speed (§ 7.1) and secondly the commercial speed (§ 7.2), we compare the two indicators and demonstrate a relativity effect (§ 7.3).

7.1 Service, operator-centric speed

The reference speed v_0 is the speed of the vehicle during its run phases, for states $\mathrm{C}.n$ for every $n \geq 0$. We also consider that the vehicle runs in state $\mathrm{C}.0$, be it for travelling to meet the next customer, or to avoid delays in the next phase of availability (delays associated to vehicle parking or to driver's engagement in a secondary activity).

The service speed, denoted by v^{o} (superscript o for Operator) arises both from the reference speed v_0 and from the proportion of time spent on the move, denoted by $P_{\mathrm{C}} \equiv \sum_{n=0}^{K} p_{\mathrm{C}.n}$: per unit of time,

$$v^{\mathrm{o}} \equiv v_0 . P_{\mathrm{C}} . \tag{7.1}$$

At the FEST solution x, the circulating probability is endowed with simple characteristic formulas as follows:

$$P_{\mathrm{C}} = p_0 \mathrm{X} = \frac{\mathrm{X}}{\mathrm{X} + a x \dot{\mathrm{X}}} = \frac{1}{1 + a \Psi_K(x)} . \tag{7.2a}$$

$$P_{\mathrm{C}} = \frac{1}{1 + a\rho} = \frac{y t_0}{\rho} . \tag{7.2b}$$

$$P_{\mathrm{C}} = 1 - y t_{\mathrm{S}} . \tag{7.2c}$$

So in fact it is not required to solve the FEST to obtain that probability, as it depends solely on t_{S} and y. The last formula also implies that P_{C} does not depend on t_0 (hence neither on L_{R} nor on v_0) nor on ring circumference C.

It then holds that

$$v^{\mathrm{o}} = v_0 (1 - y t_{\mathrm{S}}) . \tag{7.3}$$

The sensitivities with respect to the exogenous factors are stated in the following Proposition, which is demonstrated in the Appendix, § A.5. In particular, L_{R} does not influences v^{o}.

Proposition 10: Sensitivities of move probability and service speed*. (i) The probability of a vehicle being on the move, P_{C}, is a decreasing function of y and of t_{S}. It does not depend on K, L_{R} and v_0.*

(ii) P_{C} and v^{o} vary in the same direction: they both decrease with Q and t_{S}, and both increase with N, H and v_0. Neither K nor L_{R} influences P_{C} or v^{o}.

7.2 Commercial, user-centric speed

Now, the commercial speed is the average speed experienced by users riding in a collective taxi. Per unit of system time, there are n such users present in states $\sigma.n$, therefore the time they spend on board is:

$$\begin{aligned} T_1^{\mathrm{u}} &= \sum_{n=0}^{K} n.(p_{\mathrm{C}.n} + p_{\mathrm{B}.n} + p_{\mathrm{A}.n}) \\ &= x\frac{\dot{\mathrm{X}} + a\,x\,\ddot{\mathrm{X}}}{\mathrm{X} + a\,x\,\dot{\mathrm{X}}} = x.P_{\mathrm{A}} \end{aligned}$$

We note in passing that this relation allows us to interpret $x.P_{\mathrm{A}}$ as the time average number of users on board a vehicle from the perspective of an outside observer.

Per unit of system time, users cover the following total travel distance:

$$D_1^{\mathrm{u}} = v_0 \sum_{n=0}^{K} n.p_{\mathrm{C}.n} = v_0 \frac{\dot{\mathrm{X}}}{\mathrm{X} + a\,x\,\dot{\mathrm{X}}}.$$

The commercial speed, denoted by v^{u} with a superscript u to indicate the users' perspective, is defined by

$$v^{\mathrm{u}} = \frac{D_1^{\mathrm{u}}}{T_1^{\mathrm{u}}} = v_0 \frac{\dot{\mathrm{X}}}{\dot{\mathrm{X}} + a\,x\,\ddot{\mathrm{X}}} = v_0 \frac{1}{1 + a\Psi_{K-1}} = v_0 \frac{P'_{\mathrm{A}}}{P_{\mathrm{A}}}. \tag{7.4}$$

It possesses the following properties (see Appendix § A.5):

Proposition 11: Sensitivities of the commercial speed.

(i) The ratio v^{u}/v_0 is a proportion that decreases with x and also with a.

(ii) Commercial speed decreases with Q, L_R and t_{S}, and increases with N, H and v_0.

7.3 The user vs. operator relativity of speed

Let us now compare the two speeds. The ratio between the service speed and the commercial speed amounts to

$$\frac{v^{\mathrm{o}}}{v^{\mathrm{u}}} = \frac{1 + a\Psi_{K-1}}{1 + a\Psi_K}. \tag{7.5}$$

Proposition 12: Properties of the ratio between service and commercial speed.

(i) Commercial speed v^{u} is greater than service speed v^{o}.

(ii) Ratio $v^{\mathrm{o}}/v^{\mathrm{u}}$ is a decreasing function of x and also of a.

(iii) With respect to the exogenous factors, ratio $v^{\mathrm{o}}/v^{\mathrm{u}}$ is a decreasing function of Q, L_R and t_{S}, and an increasing function of N, H and v_0.

Proof. Point (i) stems from (7.5) and the increasingness of sequence $K \mapsto \Psi_K(x)$ according to K. Points (ii) to (iv) are demonstrated in the Appendix, § A.5.

The relation that v^{u} is greater than v^{o} may seem paradoxical: we would intuitively tend to imagine that the more users on board, the more dwelling delays, thus the lower experienced speed. But in fact, the more users on board, the more productive are the running phases and the wider is their benefit to the demand on the whole.

The difference between the service, operator-centric speed and the commercial, user-centric speed is a relativity effect: speed depends on the actor's perspective. There are similar effects well-known in transportation: from the relativity of wait time to an arriving user (or a ridesharing passenger) versus headway time to the operator of a transit line (or a ridesharing driver), to the relativity of passenger load in vehicles of a transit line (Leurent et al., 2012).

Illustration. Figure 7 depicts the v^{o}/v^{u} ratio as it varies according to x. Cab capacity is set to $K=4$. Base ride time and stop times are set up as previously.

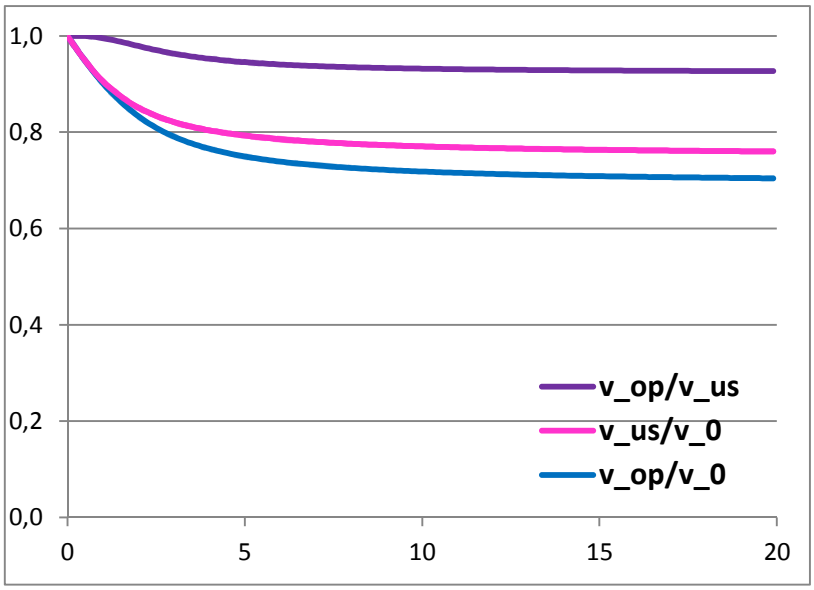

Fig. 7. Speed ratios v^{o}/v^{u}, v^{u}/v_0 and v^{o}/v_0 as functions of load factor.

8. On Ride and Access Times

Vehicle occupancy and service speed characterize the operational performance of the service: they are of interest for supply management. As for the service demand, the characteristics of primary interest are the ride time and the access time per trip. The two kinds of time are key components of service quality. We will consider them for a typical trip with ride length L_R and access length L_A.

This section brings about characteristic formulas and sensitivity properties for both kinds of time. We address first the ride time (§ 8.1) then the access time (§ 8.2).

8.1 Ride time

The average ride time is defined as the ratio between the average ride length L_R and the commercial speed v^{u} since the customer is on board: thus,

$$t_R \equiv \frac{L_R}{v^{u}}. \tag{8.1}$$

Characteristic formulas for t_R are easily derived from those for commercial speed v^{u}:

$$t_R = \frac{L_R}{v_0}(1+a\Psi_{K-1}). \tag{8.2}$$

$$t_R = t_0 + t_S\Psi_{K-1}. \tag{8.3}$$

The last formula reveals the respective influences of base ride time t_0 and stop time t_S in a clear way: at least the direct part of them, since there is also an indirect part via x.

Omitting the relativity effect, an apparent ride time would be defined as the ratio between ride length L_R and service speed v^o, yielding

$$t_R^o \equiv \frac{L_R}{v^o}. \tag{8.4}$$

On comparing the apparent and true, user-centric ride times, the influence of the user speed becomes obvious:

$$t_R^o = t_0(1+a\rho) = t_0(1+a\Psi_K(x)). \tag{8.5a}$$

$$\frac{t_R^u}{t_R^o} = \frac{1+a\Psi_{K-1}(x)}{1+a\Psi_K(x)}. \tag{8.5b}$$

When the user is on board, there is one less place available to other users. So, during his own ride, the particular user is an observer external to the $K-1$ residual places.

Proposition 13: Ride time sensitivities. *(i) Average ride time $t_R \equiv L_R / v^u$ is an increasing function of factors Q, L_R and t_S, and a decreasing function of factors N, H and v_0. (ii) It is an increasing function of L_R.*

The proof is given in the Appendix, § A.6.

8.2 Access time

On waiting for the service, the customer is external to the incoming cab: to him, the length L_A is covered at the service speed v_{K-1}^o since there is at least one place available on board. As $v_{K-1}^o = v_K^u$ (both under load factor x), we must define the access time as the ratio between the access distance L_A and the user speed v^u:

$$t_A \equiv \frac{L_A}{v^u}. \tag{8.6}$$

The access time is endowed with characteristic formulas that result from those for L_A and v^u:

$$t_A = \frac{C}{N_2 P_A v^u} = \frac{C}{N_2 v_0 P_A'} \quad \text{owing to (7.4).} \tag{8.7a}$$

$$t_A = \frac{C}{N_2 v_0}\frac{x}{t_0 y} = \frac{C}{N_2}\frac{x}{L_R y} = x\frac{C\,\mathrm{H}}{Q_2 L_R} \quad \text{with } Q_2 \equiv Q/2. \tag{8.7b}$$

Proposition 14: Access time sensitivities*. The access time $t_A \equiv L_A / v^u$ is an increasing function of factors Q, L_R, t_S and C, and a decreasing function of factors N, H, v_0 and K.*

The proof is given in the Appendix, § A.6.

Illustration. Figure 8 depicts the variations of t_R and t_A as functions of demand volume Q, given $N = 100$ cabs, $\mathrm{H} = 14$ h, $K = 12$, base and stop times as previously. The two functions are increasing: in service dynamic operations, the interaction between customers is not only sharing but also rivalry for available places and hindrance owing to cab dwelling that is a detour at least in time.

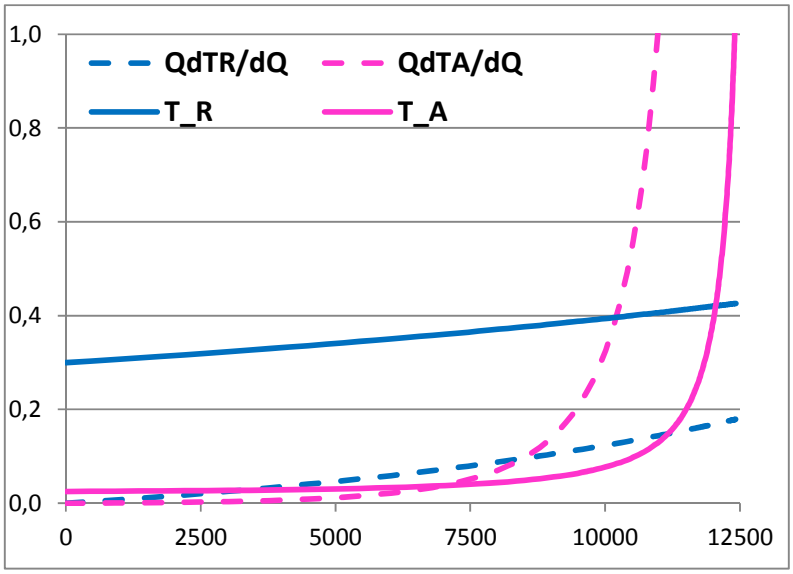

Fig. 8. Average ride time and access time with respect to demand volume.

9. Conclusion

9.1 Summary

We have modelled the dynamic interaction between supply and demand for a collective taxi service shaped as a ring. The exogenous factors in our model are a collective taxi's passenger capacity K, the fleet size N, the passenger boarding and alighting times t_S^+ and t_S^-, the volume of demand Q and the average ride length L_R, together with the speed of travel on the road infrastructure v_0 and the ring circumference C (or equivalently radius R).

The postulates both on demand generation and on service operations, owing to the spatial and temporal heterogeneity, are essential to model the activities of one cab as a Markov chain. The state variable is the number of passengers on board together with the logistical status either Circulating (denoted by C), Alighting (denoted by A) or Boarding (denoted by B). We modelled the transitions between neighbouring states, with rates derived from the average time that is spent in each state.

We showed that the exogenous factors can be combined into a single key parameter ρ which is a ratio between passenger demand and transport supply, since $\rho = y\,t_0/(1-y\,t_S)$ where $y = Q/(\mathrm{H}\,N)$. This ratio must be less than vehicle passenger capacity K.

We also showed that a vehicle's stationary regime satisfies a fundamental equation of service traffic, $\Psi_K(x) = \rho$, which uniquely determines a load factor denoted by x, provided that ρ is less than K.

From the solution x follow all the important system performance indicators: in particular the probability of availability P_A, the pseudo-rate of availability P'_A, the probability of movement P_C, then the average access length to the customer L_A, the service speed v^o and the commercial speed v^u, and finally the ride time t_R and access time t_A. Every outcome is endowed with characteristic formulas and sensitivity properties with respect to every exogenous factor.

In the process, through the distinction between the service speed and the commercial speed, we have shown the relativity of the perspectives between the service operator on one side, and users on the other.

Figure 9 shows the causal chain in the model. Thanks to the properties of the function Ψ_K and the formulas that link the variables, we have established the sensitivities of the indicators to the different exogenous factors. These sensitivities are summarised in Table 2.

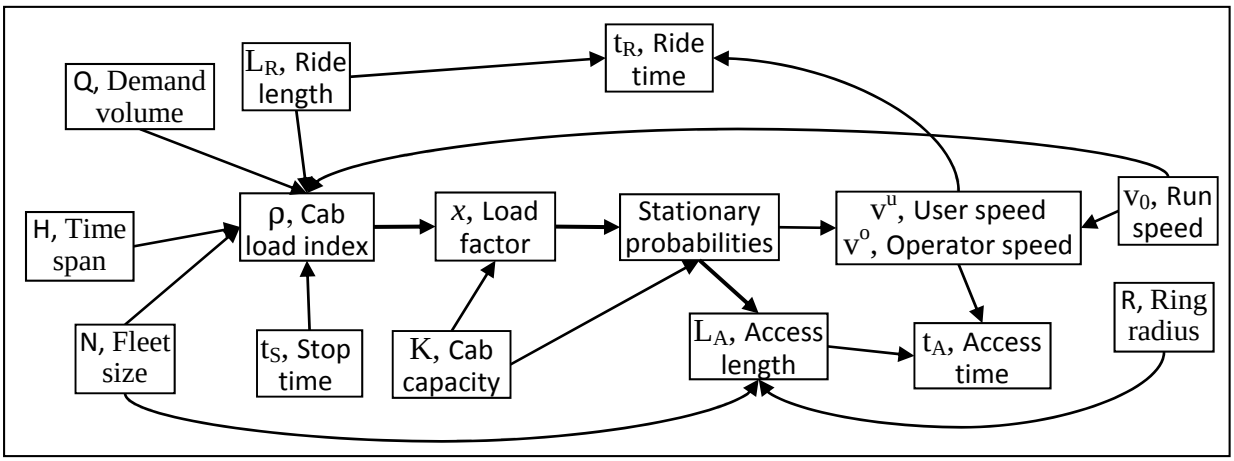

Fig. 9. Causal sequence.

Table 2. Directions of influence from exogenous factors onto model outcomes.

Factor	ρ	x	P'_A	P_A	P_C	L_A	v^o	v^u	t_R	t_A
Q	⊕	⊕	⊖	⊖	⊖	⊕	⊖	⊖	⊕	⊕
L_R	⊕	⊕	⊖	⊖	⊖	⊕	0	⊖	⊕	⊕
t_S	⊕	⊕	⊖	⊖	⊖	⊕	⊖	⊖	⊕	⊕
N	⊖	⊖	⊕	⊕	⊕	⊖	⊕	⊕	⊖	⊖
v_0	⊖	⊖	⊕	⊕	⊕	⊖	⊕	⊕	⊖	⊖
H	⊖	⊖	⊕	⊕	⊕	⊖	⊕	⊕	⊖	⊖
C *or* R	0	0	0	0	0	⊕	0	0	0	⊕
K	0	⊖	⊕	⊕	⊕	⊖	0	⊖	0	⊖
ρ	⊕	⊕	⊖	⊖	⊖	⊕	⊕	⊕	⊕	⊕
x	⊕	⊕	⊖	⊖	⊖	⊕	⊕	⊕	⊕	⊕

CHECK FOR K

9.2 Discussion

We have postulated a set of assumptions in order to represent system things in an explicit model. The idealized and abstract form of the representation makes the model a theoretical one. Owing to mathematical formulation and stochastic modeling, it is also a quantitative model endowed with characteristic formulas suitable to capture and interpret complex influences. Numerical application is easy.

Associated to each postulate is a limitation. Ring shape is a strong assumption: yet the postulate empowers the principle of vehicle cycles inherent to transit operations by augmenting the catchment area.

The postulates of homogeneity in space as well as in time are even stronger. They are essential to derive analytical formulas that constitute theoretical properties.

Modeling the roadway traffic conditions by a certain reference speed v_0 omits local variations, temporal variability, interactions between vehicles and with traffic management on the infrastructure. Would the road have only one lane per traffic direction and no parking lane associated to it, then collective cab operations (at dwelling as well as running) would interact strongly with the roadway traffic, requiring specific model development.

We have also modeled the dwelling time per passenger as a certain parameter t_S. In fact, the individual boarding and alighting times are likely to depend on the number of passengers on board, mostly in an increasing way. The vehicle would better be designed so as to limit that congestion.

Conversely we have considered individual users rather than passenger groups of variable size. We may expect such groups to induce scale economies in dwell times. But their precise effect on system states and especially the transitions between states would be difficult to model.

The model can support a number of theoretical developments, from traffic analysis to economic theory. Another direction for further research is to model an ideal ring using traffic micro-simulation in order first to check the analytical formulas and then to investigate the influence of heterogeneity in space as well as in time.

10. Declarations

10.1 Availability of data and material

The data used in the paper are synthetic ones. All numerical applications illustrated in the figures and tables are available in one Excel spreadsheet.

10.2 Funding

Not applicable.

10.3 Acknowledgements

The author is indebted to Mr. Jean Grébert of the Renault Group for stimulating discussions on car-based shared mobility services, and for putting forward the image of a noria-like service. He is also grateful to Mr. Jaafar Berrada for reading an early draft, useful comments and an introduction to the academic literature.

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A. Appendix

A.1. Short Lemmas

Lemma 1. *Function* $\gamma(x) \equiv x(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2) + \dot{\mathrm{X}}\mathrm{X}$ *satisfies that* $\gamma(x) \geq 0$ *and* $\gamma(x) > 0$ *if* $x > 0$.

Proof. Let us develop

$$\begin{aligned}\gamma(x) &= x(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2) + \dot{\mathrm{X}}\mathrm{X} \\ &= x\,\dot{\mathrm{X}}(x_K - x_{K-1}) - x\,x_K\,x_{K-1} + \dot{\mathrm{X}}.(\dot{\mathrm{X}} + x_K) \\ &= x\,x_K\,(\dot{\mathrm{X}} - x_{K-1}) - x\,x_{K-1}\,\dot{\mathrm{X}} + x_K\,\dot{\mathrm{X}} + \dot{\mathrm{X}}^2 \\ &= x\,x_K\,\ddot{\mathrm{X}} - (K-1)x_K\,\dot{\mathrm{X}} + \dot{\mathrm{X}}^2\end{aligned}$$

Thus $\gamma(r)$ decomposes into the sum of three polynomials:

$$\gamma(x) = \sum_{n=0}^{K-2} \frac{x^{K+1+n}}{n!\,K!} - (K-1)\sum_{n=0}^{K-1} \frac{x^{K+n}}{n!\,K!} + \sum_{\ell,m=0}^{K-1} \frac{x^{\ell+m}}{\ell!\,m!}.$$

To demonstrate that $\gamma(x) \geq 0$ for $x \geq 0$, we only have to show that the coefficients of the power terms in x are all non-negative. As only the median part involves negative coefficients, it is sufficient to show the non-negativity of the coefficients associated to the degrees $K+n$ for $n \in \{0,1..K-1\}$. To do that, let us calculate coefficient c_{K+n} of the term in x^{K+n}:

$$\begin{aligned}c_{K+n} &= \frac{1_{\{n>0\}}}{(n-1)!\,K!} - \frac{K-1}{n!\,K!} + \sum_{\ell=0}^{K-1} \frac{1_{\{0 \leq K+n-\ell \leq K-1\}}}{\ell!\,(K+n-\ell)!} \\ &= \frac{n}{n!\,K!} - \frac{K-1}{n!\,K!} + \sum_{\ell=0}^{K-1} \frac{1_{\{n+1 \leq \ell\}}}{\ell!\,(K+n-\ell)!} \\ &= \frac{n+1-K}{n!\,K!} + \sum_{\ell=n+1}^{K-1} \frac{1}{\ell!\,(K+n-\ell)!}\end{aligned}$$

At $n = K-1$ the first part vanishes and so does the second one, so we get that $c_{2K-1} = 0$.

At $n < K-1$ the second part involves $K-1-(n+1)+1 = K-1+n$ terms $1/[\ell!\,(K+n-\ell)!]$: for each of them we have that

$$\frac{K!}{(K+n-\ell)!} - \frac{\ell!}{n!} = \prod_{i=0}^{\ell-n-1}(K-i) - \prod_{i=0}^{\ell-n-1}(\ell-i) > 0$$

The underlying reason is that each product is composed of $\ell - n$ positive terms and every term in the first product is strictly greater than its counterpart in the second product since $\ell \leq K-1$.

Thus $\frac{K!}{(K+n-\ell)!} > \frac{\ell!}{n!}$, hence $\frac{1}{\ell!\,(K+n-\ell)!} > \frac{1}{n!\,K!}$, and in turn

$\sum_{\ell=n+1}^{K-1} \frac{1}{\ell!\,(K+n-\ell)!} > \frac{K-1+n}{n!\,K!}$, yielding that $c_{K+n} > 0$. QED.

This demonstrates that $\gamma(x) \geq 0$ for $x \geq 0$ and also that $\gamma(x) > 0$ for $x > 0$.

***Lemma 2**. (i) Function $x \mapsto \ddot{\mathrm{X}}\,\mathrm{X} - \dot{\mathrm{X}}^2$ is negative, as is function $x \mapsto \dddot{\mathrm{X}}\,\dot{\mathrm{X}} - \ddot{\mathrm{X}}^2$.*
(ii) Function $x \mapsto \mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}}$ is negative.

***Proof**.* (i) As $\mathrm{X} = \dot{\mathrm{X}} + x_K$ and $\ddot{\mathrm{X}} = \dot{\mathrm{X}} - x_{K-1}$, we have that

$$\begin{aligned}\ddot{X}\,X - \dot{X}^2 &= (\dot{\mathrm{X}} + x_{\mathrm{K}})(\dot{\mathrm{X}} - x_{\mathrm{K}-1}) - \dot{\mathrm{X}}^2 \\ &= x_K\,\dot{\mathrm{X}} - x_{K-1}\,\mathrm{X} = x_{K-1}(\dot{\mathrm{X}}\,x/K - \mathrm{X})\end{aligned}$$

Yet $\dot{\mathrm{X}}\frac{x}{K} = \sum_{n=0}^{K-1} x_n \frac{x}{K} = \sum_{n=0}^{K-1} x_{n+1}\frac{n+1}{K} = \sum_{n=1}^{K} x_n \frac{n}{K}$ so $\dot{\mathrm{X}}\frac{x}{K} - \mathrm{X} = -1 - \sum_{n=1}^{K} x_n(1 - \frac{n}{K}) \le -1 < 0$.

The same property holds for the sign of function $\dddot{\mathrm{X}}\,\dot{\mathrm{X}} - \ddot{\mathrm{X}}^2$ that follows an analogous formula if X is replaced by $\dot{\mathrm{X}}$, which amounts to decrementing K.

(ii) Let us develop

$$\begin{aligned}\mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}} &= (\dot{\mathrm{X}} + x_K)(\ddot{\mathrm{X}} - x_{K-2}) - \dot{\mathrm{X}}\,\ddot{\mathrm{X}} \\ &= x_K\,\ddot{\mathrm{X}} - x_{K-2}\,\mathrm{X} \\ &= x_{K-2}\left(\frac{x^2}{K(K-1)}\ddot{\mathrm{X}} - \mathrm{X}\right)\end{aligned}$$

As $\sum_{n=0}^{K-2} x_n \frac{x^2}{K(K-1)} = \sum_{n=0}^{K-2} x_{n+2}\frac{(n+1)(n+2)}{K(K-1)} = \sum_{n=2}^{K} x_n \frac{n(n-1)}{K(K-1)}$.

Thus $\frac{x^2}{K(K-1)}\ddot{\mathrm{X}} - \mathrm{X} = \sum_{n=2}^{K} x_n\left(\frac{n(n-1)}{K(K-1)} - 1\right) - x - 1 < -1 < 0$.

QED.

A.2. Properties of function Ψ_K

***Proof of Proposition 3**.* (i) Function Ψ_K is well defined on interval $[0,+\infty[$ where it takes on non-negative values, including 0 at $x = 0$. It is continuous and continuously differentiable, as the ratio between two such polynomials with denominator one above 1 on the range of variations.

The first derivative function of Ψ_K, denoted $\dot{\Psi}_K$, satisfies

$$\dot{\Psi}_K(x) \equiv \frac{\partial}{\partial x}\Psi_K(x) = \frac{x\,\ddot{\mathrm{X}} + \dot{\mathrm{X}}}{\mathrm{X}} - \frac{-x\,\dot{\mathrm{X}}^2}{\mathrm{X}^2} = \frac{\dot{\mathrm{X}}\mathrm{X} + x(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2)}{\mathrm{X}^2}.$$

It has the same sign as function $\gamma(x) \equiv x(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2) + \dot{\mathrm{X}}\mathrm{X}$. Thus, from Lemma 1, we have that $\dot{\Psi}_K(x) \ge 0$ and $\dot{\Psi}_K(x) > 0$ if $x > 0$, which implies that Ψ_K is strictly increasing on $[0,+\infty[$.

From this we derive that function Ψ_K is strictly positive over its range.

(ii) The limit value of Ψ_K is $\lim_{x\to+\infty}\Psi_K = \lim_{x\to+\infty}\frac{x.x_{K-1}}{x_K} = K$. Thus the characteristic equation has a solution only if $\rho < K$, which is equivalent to $Q.L_{\mathrm{R}} < Kv_0(HN - t_{\mathrm{S}}Q)$.

(iv) Given x, $\frac{\Psi_{K-1}(x)}{\Psi_K(x)} = \frac{xX_{K-2}}{X_{K-1}} / \frac{xX_{K-1}}{X_K} = \frac{X_{K-2}X_K}{X_{K-1}^2}$: it is less than 1 since $\ddot{\mathrm{X}}\,\mathrm{X} - \dot{\mathrm{X}}^2 \le 0$ from Lemma 2(i). Thus $\Psi_{K-1}(x) \le \Psi_K(x)$ and the sequence $K \mapsto \Psi_K(x)$ is increasing.

(iii) To demonstrate concavity, let us demonstrate that $\ddot{\Psi}_K \le 0$ by formal calculus.

First, we have that

$$\ddot{\Psi}_K(x) \equiv \frac{\partial}{\partial x}\dot{\Psi}_K(x) = \frac{\partial}{\partial x}\left(\frac{\gamma(x)}{\mathrm{X}^2}\right) = \frac{\dot{\gamma}(x).\mathrm{X}^2 - 2\,\mathrm{X}\,\dot{\mathrm{X}}\,\gamma(x)}{\mathrm{X}^4} = \frac{\mathrm{X}\,\dot{\gamma}(x) - 2\,\dot{\mathrm{X}}\,\gamma(x)}{\mathrm{X}^3}.$$

Thus the sign of $\ddot{\Psi}_K$ is that of function $\mathrm{X}\,\dot{\gamma} - 2\,\dot{\mathrm{X}}\,\gamma$, in which:

$$\dot{\gamma}(x) \equiv \frac{\partial}{\partial x}\gamma = \frac{\partial}{\partial x}(\dot{\mathrm{X}}\,\mathrm{X} + x(\mathrm{X}\,\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2)) = 2\,\mathrm{X}\,\ddot{\mathrm{X}} + x(\mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}}).$$

Let us develop expression $\mathrm{X}\,\dot{\gamma} - 2\,\dot{\mathrm{X}}\,\gamma$:

$$\begin{aligned}\mathrm{X}\,\dot{\gamma} - 2\,\dot{\mathrm{X}}\,\gamma &= 2\,\mathrm{X}^2\,\ddot{\mathrm{X}} + x\,\mathrm{X}(\mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}}) - 2\,\dot{\mathrm{X}}^2\,\mathrm{X} - 2x\,\dot{\mathrm{X}}(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2)\\ &= 2(\mathrm{X} - x\,\dot{\mathrm{X}})(\mathrm{X}\ddot{\mathrm{X}} - \dot{\mathrm{X}}^2) + x\,\mathrm{X}(\mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}})\end{aligned}$$

Let us replace $\ddot{\mathrm{X}}\,\mathrm{X} - \dot{\mathrm{X}}^2$ by $x_K\,\dot{\mathrm{X}} - x_{K-1}\,\mathrm{X}$ and $\mathrm{X}\,\dddot{\mathrm{X}} - \dot{\mathrm{X}}\,\ddot{\mathrm{X}}$ by $x_K\,\ddot{\mathrm{X}} - x_{K-2}\,\mathrm{X}$, according to Lemma 2:

$$\begin{aligned}&\mathrm{X}\,\dot{\gamma} - 2\,\dot{\mathrm{X}}\,\gamma\\ &= 2(\mathrm{X} - x\,\dot{\mathrm{X}})(x_K\,\dot{\mathrm{X}} - x_{K-1}\,\mathrm{X}) + x\,\mathrm{X}(x_K\,\ddot{\mathrm{X}} - x_{K-2}\,\mathrm{X})\\ &= 2x_K\,\mathrm{X}\,\dot{\mathrm{X}} - 2x_{K-1}\,\mathrm{X}^2 - 2x.x_K\,\dot{\mathrm{X}}^2 + 2x.x_{K-1}\,\mathrm{X}\dot{\mathrm{X}} + x.x_K\,\mathrm{X}\ddot{\mathrm{X}} - x.x_{K-2}\,\mathrm{X}^2\\ &= 2x_K\,\dot{\mathrm{X}}^2 + 2x_K^2\,\dot{X} - 2x_{K-1}\,\dot{\mathrm{X}}^2 - 4x_{K-1}x_K\,\dot{\mathrm{X}} - 2x_{K-1}x_K^2 - 2x.x_K\,\dot{\mathrm{X}}^2 + 2x.x_{K-1}\,\dot{\mathrm{X}}^2 + 2x.x_{K-1}x_K\,\dot{\mathrm{X}}\\ &\quad + x.x_K\,\dot{\mathrm{X}}^2 + x.x_K^2\,\dot{\mathrm{X}} - x.x_K x_{K-1}\,\dot{\mathrm{X}} - x.x_{K-1}x_K^2 - x.x_{K-2}\,\dot{\mathrm{X}}^2 - 2x.x_{K-2}x_K\,\dot{\mathrm{X}} - x.x_{K-2}x_K^2\\ &= \dot{\mathrm{X}}^2(2x_K - 2x_{K-1} - 2x.x_K + 2x.x_{K-1} + x.x_K - x.x_{K-2})\\ &\quad + \dot{\mathrm{X}}(2x_K^2 - 4x_{K-1}x_K + 2x.x_{K-1}x_K + x.x_K^2 - x.x_K x_{K-1} - 2x.x_{K-2}r_K)\\ &\quad - x_K^2(2x_{K-1} + x.x_{K-1} + x.x_{K-2})\\ &= \frac{x_{K-1}}{K}\dot{\mathrm{X}}^2(-x^2 + 2x(K+1) - (K+1)(3K+2)) + \frac{x_K x_{K-1}}{K}\dot{\mathrm{X}}(x^2 + (K+2)x - 2K(K+1)) - x_K^2 x_{K-1}(x + K + 1)\end{aligned}$$

Up to factor x_{K-1}/K, which is positive, function $\mathrm{X}\,\dot{\gamma} - 2\,\dot{\mathrm{X}}\,\gamma$ is decomposed in three parts depending on whether the power of $\dot{\mathrm{X}}$ is equal to 2, 1 or 0. These terms denoted respectively by T_i for $i \in \{0,1,2\}$ are polynomials in x:

$$T_0 = -Kx_K^2(x + K + 1),$$

$$T_1 = x_K\,\dot{\mathrm{X}}(x^2 + (K+2)x - 2K(K+1)),$$

$$T_2 = \dot{\mathrm{X}}^2(-x^2 + 2x(K+1) - (K+1)(3K+2)).$$

Here are the detailed coefficients, since $\dot{X} = \sum_{n=0}^{K-1} \frac{x^n}{n!}$ hence $\dot{X}^2 = \sum_{\ell,m=0}^{K-1} \frac{x^{\ell+m}}{\ell!\, m!}$ ([2]) :

$$T_0 = -K\frac{x^{2K+1}}{(K!)^2} - K(K+1)\frac{x^{2K}}{(K!)^2},$$

$$T_1 = \sum_{n=0}^{K-1}\frac{x^{n+K+2}}{n!K!} + \sum_{n=0}^{K-1}\frac{(K+2)x^{n+K+1}}{n!K!} - \sum_{n=0}^{K-1}\frac{2K(K+1)x^{n+K}}{n!K!},$$

$$\begin{aligned} T_2 &= \dot{X}^2(-x^2 + 2x(K+1) - (K+1)(3K+2)) \\ &= -\sum_{\ell,m=0}^{K-1}\frac{x^{\ell+m+2}}{\ell!\, m!} + \sum_{\ell,m=0}^{K-1}\frac{2(K+1)x^{\ell+m+1}}{\ell!\, m!} - \sum_{\ell,m=0}^{K-1}\frac{(K+1)(3K+2)x^{\ell+m}}{\ell!\, m!} \end{aligned}$$

Thus the powers of x range from degree $n=0$ to degree $2K+1$.

At order $n=2K+1$ only T_0 and T_1 are contributors and their respective coefficients compensate each other:

$-K\frac{x^{2K+1}}{(K!)^2} + \frac{x^{K-1+K+2}}{(K-1)!K!} = 0$, so that in fact $T_0+T_1+T_2$ is a polynomial of order at most $2K$.

At order $n=2K$ the three terms all contribute to a total coefficient of

$$-\frac{K(K+1)}{(K!)^2} + \frac{1}{(K-2)!K!} + \frac{(K+2)}{(K-1)!K!} - \frac{1}{(K-1)!^2} = \frac{-K(K+1)+K(K-1)+(K+2)(K-1)-K^2}{(K!)^2} = 0,$$

so that the polynomial $T_0+T_1+T_2$ has degree of at most $2K-1$.

At degrees $n \le 2K-1$, T_0 does not contribute any longer, while T_1 contributes by 3 terms:

$$\begin{aligned} T_1^{(n)} &= \frac{1_{\{n\ge K+2\}}}{(n-K-2)^+!K!} + \frac{1_{\{n\ge K+1\}}(K+2)}{(n-K-1)^+!K!} - \frac{1_{\{n\ge K\}}2K(K+1)}{(n-K)^+!K!} \\ &= \frac{1_{\{n\ge K\}}}{(n-K)^+!K!}((n-K-1)^+(n-K)^+ + (n-K)^+(K+2) - 2K(K+1)) \\ &\le \frac{1_{\{n\ge K\}}K}{(n-K)^+!K!}((n-K-1)^+ + (K+2) - 2(K+1)) \quad \text{since} \quad n \le 2K-1 \text{ hence } n-K \le K, \\ &\le \frac{1_{\{n\ge K\}}K}{(n-K)^+!K!}((n-K-1)^+ - K) \\ &\le 0 \end{aligned}$$

As for T_2, its contribution at order $n \le 2K-1$ is

$$T_2^{(n)} = -\sum_{\ell,m=0}^{K-1}\frac{1_{\{\ell+m+2=n\}}}{\ell!\, m!} + 2(K+1)\sum_{\ell,m=0}^{K-1}\frac{1_{\{\ell+m+1=n\}}}{\ell!\, m!} - (K+1)(3K+2)\sum_{\ell,m=0}^{K-1}\frac{1_{\{\ell+m=n\}}}{\ell!\, m!}$$

The amounts are all negative, as only the median term may be positive yet it remains lower than the right-hand side term. Precisely, two cases must be dealt with specifically, depending on whether $n<K$ or $n \ge K$.

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[2] Which can be stated as follows: $\dot{X}^2 = \sum_{n=0}^{2K-2} x^n \sum_{\ell=0}^{\min\{n,K-1\}} \frac{1}{\ell!\,(\min\{n,K-1\}-\ell)!}$

If $n<K$ then the median term involves exactly n couples (ℓ,m) such that $\ell+m+1=n$: one per value of $\ell\in\{0,n-1\}$. To every such couple, we shall associate the pair $(\ell',m')=(\ell+1,m)$ which yields a nonzero element in the right-hand side of $T_2^{(n)}$, because $\ell+1+m=n$ with $\ell+1\le K-1$ since $\ell\le n-1<K-1$, and $m\le K-1$. Then the coefficients associated to the two couples, in their respective terms, are $2(K+1)/(\ell!\,m!)$ and $-(3K+2)(K+1)/(\ell+1)(\ell!\,m!)$. Their sum is negative because:

$$\frac{(K+1)}{(\ell!\,m!)}-\frac{(3K+2)(K+1)}{(\ell+1)(\ell!\,m!)}=\frac{(K+1)}{(\ell!\,m!)}\left(1-\frac{3K+2}{\ell+1}\right)\le -2\frac{(K+1)}{(\ell!\,m!)}.$$

If $n\ge K$ then the midpoint term in T_2 involves exactly $2K-n$ couples (ℓ,m) such that $\ell+m+1=n$: one per value of $\ell\in\{n-K,K-1\}$. To every pair such that $\ell<K-1$ we shall associate pair $(\ell',m')=(\ell+1,m)$: as before, the sum of the coefficients associated to both pairs in their respective terms is negative. As for pair $(\ell,m)=(K-1,n-K)$, the midpoint coefficient $2(K+1)/(\ell!\,m!)$ is compensated by the « right hand side » coefficient that is associated to pair $(\ell',m')=(\ell+1,m)$ which is combined with $(\ell,m)=(n-K,K-1)$ since, in absolute value, the coefficient on the right hand side is more than three times greater than the midpoint coefficient.

In all, polynomial $T_0+T_1+T_2$ has all its coefficients that are negative or zero, hence it is negative for values $x\ge 0$. This implies that $\mathrm{X}\dot{\gamma}-2\dot{\mathrm{X}}\gamma\le 0$ and in turn that $\ddot{\Psi}_K\le 0$.

QED.

A.3. Sensitivities of the availability rates

Proof of Proposition 6 on the pseudo-rate of availability.

Let us study the pseudo-rate of availability by using formula $P'_{\mathrm{A}}=t_0 y/x$ which involves 3 intermediate variables x, t_0 and y. We have that:

$$\frac{\partial P'_{\mathrm{A}}}{\partial x}=-\frac{P'_{\mathrm{A}}}{x}\ ;\ \frac{\partial P'_{\mathrm{A}}}{\partial y}=\frac{P'_{\mathrm{A}}}{y}\ ;\ \frac{\partial P'_{\mathrm{A}}}{\partial t_0}=\frac{P'_{\mathrm{A}}}{t_0}$$

Yet factors t_0 and y influence P'_{A} not just directly but also via x. Their total influence is respectively:

$$\frac{\mathrm{d}P'_{\mathrm{A}}}{\mathrm{d}y}=\frac{\partial P'_{\mathrm{A}}}{\partial y}+\frac{\partial P'_{\mathrm{A}}}{\partial x}\frac{\partial x}{\partial y}=\frac{P'_{\mathrm{A}}}{y}\left(1-\frac{y}{x}\frac{\partial x}{\partial y}\right)$$

$$\frac{\mathrm{d}P'_{\mathrm{A}}}{\mathrm{d}t_0}=\frac{\partial P'_{\mathrm{A}}}{\partial t_0}+\frac{\partial P'_{\mathrm{A}}}{\partial x}\frac{\partial x}{\partial t_0}=\frac{P'_{\mathrm{A}}}{t_0}.\left(1-\frac{t_0}{x}\frac{\partial x}{\partial t_0}\right)$$

We shall restate $\frac{t_0}{x}\frac{\partial x}{\partial t_0}=\left(\frac{t_0}{\rho}\frac{\partial \rho}{\partial t_0}\right).\left(\frac{\rho}{x}\frac{\partial x}{\partial \rho}\right)=\frac{\rho}{x}\frac{\partial x}{\partial \rho}$ as $\frac{t_0}{\rho}\frac{\partial \rho}{\partial t_0}=1$, and

$$\frac{y}{x}\frac{\partial x}{\partial y}=\left(\frac{y}{\rho}\frac{\partial \rho}{\partial y}\right).\left(\frac{\rho}{x}\frac{\partial x}{\partial \rho}\right).$$

It holds that $\frac{\rho}{x}\frac{\partial x}{\partial \rho}=\frac{\Psi_K(x)}{x\dot{\Psi}_K(x)}$ since $\frac{\partial x}{\partial \rho}=\frac{1}{\partial\Psi_K(x)/\partial x}$.

As function Ψ_K is concave, $\Psi_K(x) \geq \Psi_K(0) + (x-0)\dot{\Psi}_K(x)$. Yet $\Psi_K(0) = 0$ so that $\Psi_K(x) \geq x\dot{\Psi}_K(x)$, yielding that $\frac{\rho}{x}\frac{\partial x}{\partial \rho} \geq 1$.

It then follows that $\frac{t_0}{x}\frac{\partial x}{\partial t_0} \geq 1$, hence $\frac{\mathrm{d}P'_A}{\mathrm{d}t_0} \leq 0$.

Furthermore, $\frac{y}{\rho}\frac{\partial \rho}{\partial y} = \frac{y}{\rho}\frac{t_0}{(1-y t_S)^2} = \frac{1}{1-y t_S} \geq 1$ so that $\frac{y}{x}\frac{\partial x}{\partial y} \geq 1$, hence $\frac{\mathrm{d}P'_A}{\mathrm{d}y} \leq 0$.

We are now in position to assert the sensitivities of rate P'_A to the different factors:

+ Q increases y hence its overall influence on P'_A is a decreasing one.

+ N and H each decreases y hence increases P'_A.

+ t_S increases x hence it decreases P'_A.

+ t_0 decreases P'_A, hence L_R decreases P'_A, and v_0 increases P'_A.

+ cab capacity K decreases x hence it increases P'_A.

Proof of Proposition 7 on the true rate of availability.

Concerning the true rate of availability, it depends on x in the following way based on (6.3):

$$P_A = \frac{\dot{X} + a\,x\,\ddot{X}}{X + a\,x\,\dot{X}}.$$

Let us differentiate this formula with respect to x:

$$\frac{\partial P_A}{\partial x} = \frac{(\ddot{X} + a\,\ddot{X} + ax\dddot{X})(X + ax\dot{X}) - (\dot{X} + ax\ddot{X})(\dot{X} + a\,\dot{X} + ax\ddot{X})}{(X + a\,x\,\dot{X})^2}$$

The numerator part satisfies

$$\begin{aligned}\text{Num} &= (\ddot{X} + a\,\ddot{X} + ax\dddot{X})(X + ax\dot{X}) - (\dot{X} + ax\ddot{X})(\dot{X} + a\,\dot{X} + ax\ddot{X}) \\ &= \ddot{X}\,X - \dot{X}^2 + a[X(\ddot{X} + x\dddot{X}) + x\,X\ddot{X} - \dot{X}(\dot{X} + x\ddot{X}) - x\ddot{X}\dot{X}] + a^2[(\ddot{X} + x\dddot{X})x\,X - x\,\ddot{X}(\dot{X}\ddot{X} - \ddot{X}^2)] \\ &= (\ddot{X}\,X - \dot{X}^2)(1+a) + ax(X\dddot{X} - \dot{X}\ddot{X}) + a^2x^2(X\dddot{X} - \ddot{X}^2)\end{aligned}$$

From Lemma 2, every term in the last summation is negative: thus their sum is negative, and so is $\partial P_A / \partial x \leq 0$.

As $\partial P_A / \partial x \leq 0$, true rate P_A is a decreasing function of x, yielding point (i) in Proposition 7.

Furthermore P_A is differentiable with respect to factor a, with partial derivative:

$$\frac{\partial P_A}{\partial a} = \frac{x\ddot{X}(X + ax\dot{X}) - x\dot{X}(\dot{X} + ax\ddot{X})}{(X + a\,x\,\dot{X})^2} = \frac{x(\ddot{X}\,X - \dot{X}^2)}{(X + a\,x\,\dot{X})^2} \text{ which is } \leq 0 \text{ from Lemma 2-(i).}$$

Thus the true rate P_A is also a decreasing function of a.

The sensitivity of P_A to the basic exogenous factors: Q, L_R, H, N, t_S, v_0 depend on the influences of the said factors on ρ hence on x, and also on $a = \mu.t_S = t_S L_R / v_0$:

- Factors Q, H and N influence P_A only via x: Q influences x in an increasing way and in turn P_A in a decreasing way. N (resp. H) influences x in a decreasing way and in turn P_A in an increasing way.
- L_R (resp. t_S) influences P_A on one hand via x and on the other hand via a. As both influences are increasing ones, and as both x and a exert decreasing influences on P_A, the total influence of L_R (resp. t_S) on P_A is decreasing.
- v_0 influences x et a in a decreasing way, so that it will exert a twofold positive influence on P_A.

These properties make up points (ii) and (iii) in Proposition 7.

Proof of Proposition 8 on the rate ratio $\varphi \equiv P_A / P'_A$.

As $\varphi = 1 + a\Psi_{K-1}(x)$, it involves the basic factors via a or x. Let us identify x, t_0 and t_S as intermediary variables for sensitivity analysis. We have that:

$$\frac{\partial \varphi}{\partial x} = a\dot{\Psi}_{K-1}(x)$$

$$\frac{\partial \varphi}{\partial t_S} = \frac{\Psi_{K-1}(x)}{t_0}$$

$$\frac{\partial \varphi}{\partial t_0} = -\frac{a}{t_0}\Psi_{K-1}(x)$$

Yet factors t_0 and t_S influence φ not just directly but also via x. Their total influence is respectively:

$\frac{d\varphi}{dt_S} = \frac{\partial \varphi}{\partial t_S} + \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t_S} = \frac{\Psi_{K-1}(x)}{t_0} + a\dot{\Psi}_{K-1}(x)\frac{\partial x}{\partial t_S}$ which is positive since $\frac{\partial x}{\partial t_S} \geq 0$.

$$\frac{d\varphi}{dt_0} = \frac{\partial \varphi}{\partial t_0} + \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t_0} = -\frac{a}{t_0}\Psi_{K-1}(x) + a\dot{\Psi}_{K-1}(x)\frac{\partial x}{\partial t_0}$$

$$= -\frac{a}{t_0}\Psi_{K-1}(x).(1 - \frac{\dot{\Psi}_{K-1}(x)}{\Psi_{K-1}(x)}t_0\frac{\partial x}{\partial t_0})$$

We have established previously that $t_0\frac{\partial x}{\partial t_0} = \rho\frac{\partial x}{\partial \rho} = \frac{\Psi_K(x)}{\dot{\Psi}_K(x)}$

So we obtain that

$$\frac{d\varphi}{dt_0} = -\frac{a}{t_0}\Psi_{K-1}(x).(1 - \frac{\dot{\Psi}_{K-1}(x)}{\Psi_{K-1}(x)}\frac{\Psi_K(x)}{\dot{\Psi}_K(x)})$$

$$= \frac{t_S}{t_0^2}\Psi_{K-1}(x)\frac{\Psi_K(x)}{\dot{\Psi}_K(x)}.(\frac{\dot{\Psi}_{K-1}(x)}{\Psi_{K-1}(x)} - \frac{\dot{\Psi}_K(x)}{\Psi_K(x)})$$

The bracketed term is the derivative of function $\ln\frac{\Psi_{K-1}(x)}{\Psi_K(x)}$, knowing that $\frac{\Psi_{K-1}(x)}{\Psi_K(x)}=\frac{\ddot{\mathrm{X}}.\mathrm{X}}{\dot{\mathrm{X}}^2}$.

Yet function $x\mapsto \ddot{\mathrm{X}}.\mathrm{X}/\dot{\mathrm{X}}^2$ is decreasing (cf. § A.7): so is its logarithm, implying that the derivative function is negative: thus the bracketed term is negative, and in turn, $\mathrm{d}\varphi/\mathrm{d}t_0 \le 0$.

We are now in position to assert the sensitivities of ratio $\varphi \equiv P_\mathrm{A}/P'_\mathrm{A}$ to the different factors:

+ Q increases x hence increases φ.

+ N and H each decreases x hence decreases φ.

+ t_S increases φ.

+ t_0 decreases φ, hence L_R decreases φ, and v_0 increases φ.

A.4. Sensitivity of Access Length

Proof of Proposition 9 on the sensitivities of the access length.

(i) Factor N exerts a twofold influence: first a direct influence as an increase in N (under fixed P_A) makes L_A decrease, and second an indirect influence via P_A: from Proposition 7, an increase in N increases P_A. Thus, the product $N.P_\mathrm{A}$ increases with N, so that L_A decreases with N.

(ii) Factors Q, v_0, L_R and t_S each influence L_A in a simple way via P_A, hence each in the opposite direction of its own influence onto P_A that is indicated in Proposition 7.

(iii) Being proportional to circumference C (and radius R), L_A varies with it in an increasing, linear way.

A.5. Sensitivities of the service speed and of the commercial speed

Proof of Proposition 10 on the sensitivities of the service speed.

(i) Probability of being on the move, P_C, decreases with respect to y as well as t_S since $\partial P_\mathrm{C}/\partial y = -t_\mathrm{S}$ and $\partial P_\mathrm{C}/\partial t_\mathrm{S} = -y$.

As for factors Q, H and N, each of them influences P_C via y only: Q exerts an increasing influence on y while N (resp. H) exerts a decreasing influence on it, so that Q will have P_C to decrease while N and H will have P_C to increase.

The formula $P_\mathrm{C} = 1 - y.t_S$ also implies that K, L_R and v_0 do not influence P_C.

(ii) K and L_R do not influence P_C nor v_0, so they do not influence v^o. The influence of v_0 onto v^o is linear increasing. The sensitivities of v^o with respect to Q, t_S, H and N are proportional to those of P_C with respect to each factor, with a proportionality coefficient $v_0 \ge 0$: thus both P_C and v^o are increasing functions of H and N, and decreasing functions of Q and t_S.

Proof of Proposition 11 on the sensitivities of the commercial speed.

Let us remark the formal analogy between $\frac{v^{\mathrm{u}}}{v_0} = \frac{\dot{\mathrm{X}}}{\dot{\mathrm{X}} + ax\ddot{\mathrm{X}}}$ and $P_{\mathrm{C}} = \frac{\mathrm{X}}{\mathrm{X} + ax\dot{\mathrm{X}}}$: the ratio v^{u} / v_0 is equal to P_{C} for a collective cab service with one less place per cab, since $\dot{\mathrm{X}}_{(K)} = \mathrm{X}_{(K-1)}$.

(i) The influences of x and a on ratio v^{u} / v_0 are decreasing since an increase in any of them makes the denominator increase in $v^{\mathrm{u}} / v_0 = (1 + a\Psi_{K-1}(\rho))^{-1} = (1 + a\Psi_{K-1} \circ \Psi_K(x))^{-1}$.

(ii) Q (resp. H, N) has an increasing (resp. decreasing) influence on x hence a decreasing (resp. increasing) influence on v^{u} / v_0, hence on v^{u}.

An increase in t_{S} makes both x and a increase, hence it reduces v^{u} / v_0 and in turn v^{u}.

As for t_0, its respective influences on x and a have opposite directions. We calculate:

$$\begin{aligned}
\frac{\partial}{\partial t_0}\frac{v^{\mathrm{u}}}{v_0} &= -\frac{1}{(1 + a\Psi_{K-1}(x))^2}\left(\Psi_{K-1}(x)\frac{\partial a}{\partial t_0} + a\dot{\Psi}_{K-1}(x)\frac{\partial x}{\partial t_0}\right) \\
&= -\frac{1}{(1 + a\Psi_{K-1}(x))^2}\left(\Psi_{K-1}(x)\left(-\frac{a}{t_0}\right) + a\dot{\Psi}_{K-1}(x)\frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial t_0}\right), \\
&= \frac{a\Psi_{K-1}(x)}{t_0(1 + a\Psi_{K-1}(x))^2}\left(1 - \frac{\dot{\Psi}_{K-1}}{\Psi_{K-1}}\frac{\Psi_K}{\dot{\Psi}_K}\right)
\end{aligned}$$

Which is non-negative since the bracketed expression is non-negative, cf. the proof of Proposition 8.

Proof of Proposition 12 on the sensitivities of the $\eta \equiv v^{\mathrm{o}} / v^{\mathrm{u}}$ speed ratio.

Let us denote $\eta \equiv \frac{v^{\mathrm{o}}}{v^{\mathrm{u}}} = \frac{1 + a\Psi_{K-1}(x)}{1 + a\Psi_K(x)}$.

(ii) Let us calculate the partial derivatives with respect to the intermediary variables a and x. We have that:

$$\frac{\partial \eta}{\partial a} = \frac{\Psi_{K-1}(x)}{1 + a\Psi_K(x)} - \frac{\Psi_K(1 + a\Psi_{K-1})}{(1 + a\Psi_K)^2} = \frac{\Psi_{K-1} - \Psi_K}{(1 + a\Psi_K)^2}$$

$$\frac{\partial \eta}{\partial x} = \frac{a\dot{\Psi}_{K-1}(x)}{1 + a\Psi_K(x)} - \frac{\dot{\Psi}_K(x)a(1 + a\Psi_{K-1}(x))}{(1 + a\Psi_K(x))^2} = \frac{a\dot{\Psi}_{K-1}}{(1 + a\Psi_K)^2}\left(1 - \frac{\dot{\Psi}_K}{\dot{\Psi}_{K-1}} - a\Psi_K\right)$$

Both are negative, since $\Psi_{K-1} \le \Psi_K$ for $\partial\eta / \partial a$, while for $\partial\eta / \partial x$ we use the relation that

$\frac{\dot{\Psi}_K}{\dot{\Psi}_{K-1}} \ge \frac{\Psi_K}{\Psi_{K-1}} \ge 1$, yielding a negative value for the bracketed expression.

(iii) Factors Q, H and N exert their respective influence onto η via x only: as Q increases x, it decreases η. As H and N each decreases x, each of them increases η.

As for t_{S}, it increases both a and x: thus its twofold influence on η is decreasing in both ways.

As for t_0, its influences onto η via a and x have opposite directions: overall,

$$\frac{\mathrm{d}\eta}{\mathrm{d}t_0} = \frac{\partial\eta}{\partial a}\frac{\partial a}{\partial t_0} + \frac{\partial\eta}{\partial x}\frac{\partial x}{\partial t_0} = -\frac{a}{t_0}\frac{\partial\eta}{\partial a} + \frac{\rho}{t_0\dot{\Psi}_K}\frac{\partial\eta}{\partial x} \quad \text{since} \quad \frac{\partial x}{\partial t_0} = \frac{\partial x}{\partial\rho}\frac{\partial\rho}{\partial t_0} = \frac{\rho}{t_0}\frac{\partial x}{\partial\rho} = \frac{\rho}{t_0\dot{\Psi}_K}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t_0} = -\frac{a(\Psi_{K-1}-\Psi_K)}{t_0(1+a\Psi_K)^2} + \frac{\rho}{t_0\dot{\Psi}_K}\frac{a\dot{\Psi}_{K-1}}{(1+a\Psi_K)^2}\left(1-\frac{\dot{\Psi}_K}{\dot{\Psi}_{K-1}}-a\Psi_K\right)$$

$$= \frac{a}{t_0(1+a\rho)^2}\left(\rho-\Psi_{K-1}+\rho(1-a\rho)\frac{\dot{\Psi}_{K-1}}{\dot{\Psi}_K}-\rho\right)$$

$$= \frac{a\rho}{t_0(1+a\rho)^2}\left((1-a\rho)\frac{\dot{\Psi}_{K-1}}{\dot{\Psi}_K}-\frac{\Psi_{K-1}}{\Psi_K}\right)$$

Now, as $\frac{\dot{\Psi}_{K-1}}{\dot{\Psi}_K} \le \frac{\Psi_{K-1}}{\Psi_K}$ and $1-a\rho \le 1$, the bracketed expression is negative, so that $\frac{\mathrm{d}\eta}{\mathrm{d}t_0} \le 0$.

Thus, t_0 exerts a decreasing influence on η: so does L_R, whereas v_0 has an increasing influence. QED.

A.6. Sensitivities of user times

Proof of Proposition 13 on the sensitivities of the ride time $t_\mathrm{R} = L_\mathrm{R}/v^\mathrm{u}$**.**

(i) All factors except for L_R influence t_R only through v^u, and they do so in an inversely proportional way. Thus their respective influence on t_R is opposite to that on v^u. From Proposition 11, it follows that t_R in a decreasing function of N, H, K and v_0, while it is an increasing function of Q and t_S.

As for (ii), the influence of L_R on t_R is twofold: the direct one is an increasing one, and so is the indirect one via v^u, so that the overall influence is twofold increasing.

Proof of Proposition 14 on the sensitivities of the access time $t_\mathrm{A} \equiv L_\mathrm{A}/v^\mathrm{u}$**.**

(i) The sensitivity of t_A to C, or equivalently to R is obvious as R does not influence the other variables in the model: it is an increasing and linear influence.

(ii) Formula (8.7a) reveals the influence of factors other than C, N and v_0: each factor among Q, L_R, t_S, H, influences t_A in the opposite way as it influences the effective availability rate P'_A. From Proposition 6, we recover that t_A is an increasing function of Q, L_R, t_S, and a decreasing function of H.

(iii) The influence of N onto t_A is twofold: using definition (8.6), first via L_A, second via v^u. As an increase in N makes L_A decrease and v^u increase, both parts have consistent influences, yielding a decreasing overall sensitivity of t_A to N.

The same applies to v_0: its twofold influence on t_A is a decreasing one.

A.7. Long Lemma

We already know that function $f : x \mapsto 1 - \mathrm{X}.\ddot{\mathrm{X}}/\dot{\mathrm{X}}^2$: is positive. Let us show that it is an increasing function, by demonstrating that its derivative is non-negative,

$\frac{\partial}{\partial x} f = \frac{\partial}{\partial x}(-\frac{\mathrm{X}.\ddot{\mathrm{X}}}{\dot{\mathrm{X}}^2}) = \frac{\partial}{\partial x}(\frac{x_{K-1}}{\dot{\mathrm{X}}^2}(\dot{\mathrm{X}} - \frac{x}{K}\ddot{\mathrm{X}}))$:

It holds that $x_{K-1} = \frac{x^{K-1}}{(K-1)!} = K\frac{x^{K-1}}{K!}$ and $\dot{\mathrm{X}} - \frac{x}{K}\ddot{\mathrm{X}} = \sum_{n=0}^{K-1}(1-\frac{n}{K})x_n$.

Thus $\frac{\partial}{\partial x} f = \frac{\partial}{\partial x}(-\dot{\mathrm{X}}^{-2}\sum_{n=0}^{K-1}(K-n)\frac{x^{n+K-1}}{n!K!})$.

By differentiating each term, we get that

$\frac{\partial}{\partial x} f = \dot{\mathrm{X}}^{-3}(\dot{\mathrm{X}}\sum_{n=0}^{K-1}(K-n)(n+K-1)\frac{x^{n+K-2}}{n!K!} - 2\ddot{\mathrm{X}}\sum_{n=0}^{K-1}(K-n)\frac{x^{n+K-1}}{n!K!})$

$K!\dot{\mathrm{X}}^3\frac{\partial}{\partial x} f = \sum_{\ell,n=0}^{K-1}(K-n)(n+K-1)\frac{x^{n+\ell+K-2}}{n!\ell!} - 2\sum_{n=0}^{K-1}\sum_{\ell=0}^{K-2}(K-n)\frac{x^{n+\ell+K-1}}{n!\ell!}$

$K!\dot{\mathrm{X}}^3\frac{\partial}{\partial x} f = \sum_{\ell,n=0}^{K-1}\frac{(K-n)(n+K-1)}{n!\ell!}x^{n+\ell+K-2} - 2\sum_{n,\ell=0}^{K-1}\frac{(K-n)\ell}{n!\ell!}x^{n+\ell+K-2}$ on passing from ℓ to $\ell' \equiv \ell+1$, so that $1/\ell! = \ell'/\ell'!$.

In all,

$K!\dot{\mathrm{X}}^3\frac{\partial}{\partial x} f = \sum_{\ell,n=0}^{K-1}\frac{(K-n)(n+K-1-2\ell)}{n!\ell!}x^{n+\ell+K-2}$.

In this sum, let us consider the terms with a certain power $\ell+n+K-2$, for every degree $m = \ell+n$. To each pair (ℓ,n) corresponds the symmetric pair (n,ℓ) with respect to $m/2$. By grouping the respective coefficients, we get a numerator of

$(K-n)(n+K-1-2\ell) + (K-\ell)(\ell+K-1-2n)$
$= (K-\frac{m}{2}+(\frac{m}{2}-n))(n+K-1-2\ell) + (K-\frac{m}{2}+(\frac{m}{2}-\ell))(\ell+K-1-2n)$
$= (K-\frac{m}{2})(n+K-1-2\ell+\ell+K-1-2n) - (\frac{m}{2}-\ell)(n+K-1-2\ell-\ell-K+1+2n)$
$= (K-\frac{m}{2})(2K-2-m) - \frac{3}{2}(\ell-n)^2$

By allocating the total between the two indices ℓ and n in a balanced way, we obtain a "symmetrical" term

$C_\ell^{(m)} = C_n^{(m)} = (K-\frac{m}{2})(K-1-\frac{m}{2}) - 3(\ell-\frac{m}{2})^2$.

The selection of the indices that meet condition $\ell \in \{0,1..K-1\}$ constitutes an extraction from a binomial law with number m and proportion $1/2$:

$S_m \equiv \sum_{n,\ell=0}^{K-1}\frac{1_{\{n+\ell=m\}}C_\ell^{(m)}}{n!\ell!} = \sum_{\ell=0}^{m}\frac{\delta_\ell^{(m)}}{(m-\ell)!\ell!}C_\ell^{(m)}$, wherein we denote $\delta_\ell^{(m)} = 1_{\{\ell\le K-1\}}1_{\{m-\ell\le K-1\}}$.

Let us also denote $\binom{m}{\ell} = \frac{m!}{(m-\ell)!\ell!}$ the binomial coefficient and $p_\ell^{(m)} \equiv \binom{m}{\ell}/2^m$: in all,

$S_m = \frac{2^m}{m!}\sum_{\ell=0}^{m}p_\ell^{(m)}\delta_\ell^{(m)}C_\ell^{(m)} = \frac{2^m}{m!}\mathrm{E}[\delta.C]$, hence $m!2^{-m}S_m = \mathrm{E}[\delta.C]$.

If $m \le K-1$ then $\delta_\ell^{(m)} = 1 \ \forall \ell$ hence $\mathrm{E}[\delta] = 1$.

If $m \geq K-1$ then there are $K-1-(m-K-1)+1 = 2K-1-m$ indices ℓ from $m-K-1$ to $K-1$ for which $\delta_\ell^{(m)} = 1$, so that $\mathrm{E}[\delta] = \dfrac{2K-1-m}{m+1}$.

Furthermore, for every index ℓ there is also the symmetrical index $m-\ell$, so that $\mathrm{E}[\delta.\ell] = \frac{1}{2}m$.

This mean is identical to that of the binomial random variable ℓ of parameters m and $\frac{1}{2}$.

Each term $C_\ell^{(m)}$ includes a part $C_{m/2}^{(m)} = (K-\frac{m}{2})(K-1-\frac{m}{2})$ independent of ℓ hence called fixed, and a variable part that is zero at $\frac{1}{2}m$ (hence only if m is even).

On average, the fixed part yields a contribution $\mathrm{E}[\delta]C_{m/2}^{(m)} = \mathrm{E}[\delta](K-\frac{m}{2})(K-1-\frac{m}{2})$.

Also on average, the variable part can be interpreted as -3 times the variance of the indices ℓ such that $\delta = 1$: thus

$$\mathrm{E}[\delta.C] = \mathrm{E}[\delta]C_{m/2}^{(m)} - 3\mathrm{E}[\delta.(\ell-\tfrac{m}{2})^2].$$

Yet $\mathrm{E}[\delta.\ell^2] = (\mathrm{E}[\delta.\ell])^2 + \mathrm{E}[\delta.(\ell-\mathrm{E}[\delta.\ell])^2] = (\dfrac{m}{2})^2 + \mathrm{E}[\delta.(\ell-\dfrac{m}{2})^2]$,

Whereas, by the properties of the binomial law, $\mathrm{E}[\ell^2] = (\mathrm{E}[\ell])^2 + \mathrm{V}[\ell] = (\dfrac{m}{2})^2 + \dfrac{m}{4}$.

By comparison, we obtain the inequality

$$\frac{m!}{2^m} S_m \geq \mathrm{E}[\delta](K-\tfrac{m}{2})(K-1-\tfrac{m}{2}) - \tfrac{3}{4}m.$$

When $m \leq K-1$ then $\mathrm{E}[\delta] = 1$ and $K-1-\frac{m}{2} \geq \frac{m}{2}$, hence

$$\begin{aligned}\frac{m!}{2^m} S_m &\geq (K-\tfrac{m}{2})(K-1-\tfrac{m}{2}) - \tfrac{3}{4}m \\ &\geq (\tfrac{m}{2}+1)\tfrac{m}{2} - \tfrac{3}{4}m = \tfrac{1}{4}m(m-1) \\ &\geq 0 \text{ since } m \geq 1\end{aligned}$$

If $m \geq K-1$, we differentiate two cases depending on whether m is even – thus denoted as $2K-2-2i$ - or odd and thus denoted as $2K-1-2i$.

In the even case, with $m = 2K-2-2i \geq K-1$, then $\frac{m}{2} = K-1-i$ et $K-\frac{m}{2} = 1+i$ hence the fixed term amounts to $C_{m/2}^{(m)} = i(i+1)$.

The variable term of $k \in \{\frac{m}{2},..K-1\}$ is equal to $-3(k-\frac{m}{2})^2 = -3(K-1-i-k)^2$: with respect to the complement to $K-1$ of k, $j \equiv K-1-k$, the variable term amounts to $-3(i-j)^2$.

Thus, $C_k^{(m)} = i(i+1) - 3(i-j)^2$ for $j \in \{0,..i\}$.

The binomial probability $p_k^{(m)}$ is maximal at $k = \frac{m}{2}$. The respective probabilities satisfy that

$$\frac{p_k^{(m)}}{p_{m/2}^{(m)}} = \frac{(m/2)!}{(k)!}\frac{(m/2)!}{(m-k)!} = \frac{1}{\prod_{\ell=1}^{k-m/2}(\frac{m}{2}+\ell)} \prod_{\ell=0}^{k-m/2-1}(\tfrac{m}{2}-\ell),$$

Thus $p_k^{(m)} = p_{m/2}^{(m)} \prod_{\ell=1}^{k-m/2} \frac{\frac{m}{2}-\ell-1}{\frac{m}{2}+\ell}$.

It is a decreasing function of $k \geq \frac{m}{2}$.

Let $k*$ be the largest index $k \geq \frac{m}{2}$ such that $C_k^{(m)} = i(i+1) - 3(i-j)^2 \geq 0$.

We can now gather the non-zero terms in $m!2^{-m}S_m$ by counting with each $k > \frac{m}{2}$ its symmetrical $m-k$:

$$\begin{aligned}
\frac{m!}{2^m} S_m &= i(i+1) + 2\sum_{j=0}^{i-1} (i(i+1) - 3(i-j)^2)\, p_{k(j)}^{(m)} \\
&\geq p_{k*}^{(m)} i(i+1) + 2\, p_{k*}^{(m)} \sum_{j=0}^{i-1} (i(i+1) - 3(i-j)^2) \\
&\geq p_{k*}^{(m)} (i(i+1)(1+2i) - 6\sum_{j=0}^{i-1} (i-j)^2) \\
&\geq p_{k*}^{(m)} (i(i+1)(2i+1) - 6\sum_{\ell=1}^{i} \ell^2)
\end{aligned}$$

Since, for $k \in \{\frac{m}{2}, ..k*\}$, $p_k^{(m)} \geq p_{k*}^{(m)}$ and $i(i+1) \geq 3(i-j)^2$, hence

$p_k^{(m)} C_k^{(m)} \geq p_{k*}^{(m)} (i(i+1) - 3(i-j)^2)$.

For $k > k*$, $p_k^{(m)} \leq p_{k*}^{(m)}$ and $i(i+1) \leq 3(i-j)^2$, so the inequality also holds.

As $\sum_{\ell=1}^{i} \ell^2 = i(i+1)(2i+1)/6$, we derive that $S_m \geq 0$ in the case where m is even and $\geq K-1$.

Let us come to the case of an odd $m = 2K-1-2i$ with $i \leq K/2$ to ensure that $m \geq K-1$. Then $\frac{m}{2} = K - i - \frac{1}{2}$ is not an integer. The relation between the probabilities becomes

$$p_k^{(m)} = p_{K-i}^{(m)} \frac{(K-i)!}{k!} \frac{(K-i-1)!}{(m-k)!} = p_{K-i}^{(m)} \frac{\prod_{\ell=0}^{k-K+i-1} K-i-1-\ell}{\prod_{\ell=1}^{k-K+i} K-i+\ell} = p_{K-i}^{(m)} \prod_{\ell=1}^{k-K+i} \frac{K-i-\ell}{K-i+\ell}.$$

It is a decreasing function of $k \geq \frac{m}{2}$.

In the terms $C_k^{(m)}$, the fixed part is now

$C_{m/2}^{(m)} = (K - \frac{m}{2})(K-1-\frac{m}{2}) = (i+\frac{1}{2})(i-\frac{1}{2}) = \frac{1}{4}(2i+1)(2i-1)$.

Let again $k*$ be the largest index k such that $C_k^{(m)} = C_{m/2}^{(m)} - 3(i-j-\frac{1}{2})^2 \geq 0$.

By associating to each $k > \frac{m}{2}$ its symmetrical $m-k$, we can express $S_m m!/2^m$ as

$\frac{m!}{2^m} S_m = 2\sum_{j=0}^{i-1} (\frac{1}{4}(2i+1)(2i-1) - 3(i-j-\frac{1}{2})^2)\, p_{k(j)}^{(m)}$, hence

$\frac{m!}{2^m} \frac{S_m}{p_{k*}^{(m)}} \geq 2\sum_{j=0}^{i-1} (\frac{1}{4}(2i+1)(2i-1) - 3(i-j-\frac{1}{2})^2) = \frac{1}{2} i(2i-1)(2i+1) - 6\sum_{\ell=0}^{i-1} (\ell - \frac{1}{2})^2$. Or

$$\begin{aligned}\sum_{\ell=0}^{i-1}(\ell-\tfrac{1}{2})^2 &= \tfrac{1}{4}\sum_{\ell=0}^{i-1}(2\ell-1)^2 \\ &= \tfrac{1}{4}(1+\sum_{\ell=0}^{2i-2}\ell^2-\sum_{\ell=0}^{i-1}(2\ell)^2) \\ &= \tfrac{1}{4}(1+\frac{(2i-2)(2i-1)(4i-3)}{6}-4\frac{(i-1)i(2i-1)}{6}) \\ &= \tfrac{1}{4}(1+(i-1)(2i-1)(\frac{4i-3}{3}-2\frac{i}{3}) \\ &= \tfrac{1}{4}(1+\tfrac{1}{3}(i-1)(2i-1)(2i-3))\end{aligned}$$

Thus

$$\begin{aligned}\frac{m!}{2^m}\frac{S_m}{p_{k*}^{(m)}} &\geq \tfrac{1}{4}2i(2i+1)(2i-1)-\tfrac{6}{4}(1+\tfrac{1}{3}(i-1)(2i-1)(2i-3)) \\ &\geq \tfrac{1}{2}(i(2i-1)(2i+1)-3-(i-1)(2i-1)(2i-3)) = \tfrac{1}{2}(2i-1)(i(2i+1)-(i-1)(2i-3)-3) \\ &\geq \tfrac{1}{2}(2i-1)(2i^2+2i-2i^2+5i-3-3) = \tfrac{1}{4}(2i-1)(7i-6) \\ &\geq \tfrac{1}{2}\end{aligned}$$

As soon as $i \geq 1$, which holds since $m \leq 2K-2$.

All in all, we have demonstrated that in all cases the sum S_m is non-negative.

QED.

This implies that $\partial f / \partial x \geq 0$, hence that f is an increasing function.