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Abstract

A (undirected) graph is locally irregular if no two of its adjacent vertices have the same degree. A decomposition of a graph G into k locally irregular subgraphs is a partition $E_1, ..., E_k$ of $E(G)$ into k parts each of which induces a locally irregular subgraph. Not all graphs decompose into locally irregular subgraphs; however, it was conjectured that, whenever a graph does, it should admit such a decomposition into at most three locally irregular subgraphs. This conjecture was verified for a few graph classes in the recent years.

This work is dedicated to the decomposability of degenerate graphs with low degeneracy. Our main result is that decomposable k-degenerate graphs decompose into at most $3k + 1$ locally irregular subgraphs, which improves on previous results whenever $k \leq 9$. We improve this result further for some specific classes of degenerate graphs, such as bipartite cacti, k-trees, and planar graphs.

Keywords: locally irregular decompositions; degenerate graphs; cacti; k-trees; planar graphs.

1. Introduction

A graph G is *locally irregular* if no two of its adjacent vertices have the same degree, i.e., for every edge $uv \in E(G)$ we have $d(u) \neq d(v)$. In general, G might be far from being locally irregular (e.g. when G is regular), and we might then be interested in *decomposing* G into locally irregular subgraphs. The term "decomposition" is, throughout this paper, understood as an edge-partition. A *locally irregular decomposition* of G is then a partition of $E(G)$ into parts $E_1, ..., E_k$ each of which induces a locally irregular subgraph. The least number k such that G can be decomposed into k locally irregular subgraphs is called the *irregular chromatic index* of G, and is denoted $\chi'_{\text{irr}}(G)$. There are contexts where G might admit no locally irregular decomposition at all (consider e.g. any odd-length path), in which case we define $\chi'_{\text{irr}}(G) = \infty$. We call G *decomposable* if $\chi'_{\text{irr}}(G)$ is finite, while we call G *exceptional* otherwise.

Locally irregular decompositions were introduced by Baudon, Bensmail, Przybyło and Woźniak [2] as a tool to deal with some cases of the well-known 1-2-3 Conjecture posed by Karoński, Łuczak and Thomason [6]. The main open question on locally irregular decompositions is about whether every decomposable graph has a decomposition into three locally irregular subgraphs:

Conjecture 1.1 (Baudon, Bensmail, Przybyło, Woźniak [2])**.** *For every decomposable graph G, we have $\chi'_{\text{irr}}(G) \leq 3$.*

For a better understanding of Conjecture 1.1, it is worth mentioning that exceptional graphs comprise exactly three families $\mathcal{P}, \mathcal{C}, \mathcal{T}$ of graphs. The family $\mathcal{P}$ is the one of all odd-length paths, while $\mathcal{C}$ is the family of all odd-length cycles. The definition of $\mathcal{T}$ is recursive:

1. The triangle K_3 belongs to $\mathcal{T}$.
2. Every other graph in $\mathcal{T}$ can be constructed by 1) taking an auxiliary graph H being either an even-length path or an odd-length path with a triangle glued to one of its ends, then 2) choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally 3) identifying v with a vertex of degree 1 of H.

In other words, the graphs in $\mathcal{T}$ consist of disjoint triangles connected in a tree-like fashion, such that, when contracting the triangles, two triangle vertices are joined by an odd-length path, while a triangle vertex and an original degree-1 vertex are joined by an even-length path.

Theorem 1.2 (Baudon, Bensmail, Przybyło, Woźniak [2]). *A connected graph is exceptional if and only if it belongs to $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$.*

Regarding our investigations in this paper, it is worth emphasizing that all exceptional graphs have odd size, maximum degree at most 3, low degeneracy (2), and are planar. Recall that a graph G is said *k-degenerate* if $|V(G)| \leq k+1$ or there exists $v \in V(G)$ with degree at most k such that $G - v$ is k-degenerate.

Conjecture 1.1 was proved for decomposable trees, complete graphs, complete bipartite graphs, some Cartesian products of graphs, regular graphs with degree at least 10^7 [2], graphs with minimum degree at least 10^{10} [9], and decomposable split graphs [7]. For some classes of graphs, there is still some gap between the bound in Conjecture 1.1 and the best known one on the irregular chromatic index. In [4], decomposable bipartite graphs were proved to have irregular chromatic index at most 10, which was improved down to 7 in [8]. In [1], decomposable graphs with maximum degree 3 were proved to have irregular chromatic index at most 5, which was improved down to 4 in [8].

A first constant upper bound of 328 on the irregular chromatic index of decomposable graphs was given in [4] by Bensmail, Merker and Thomassen. The steps of the proof can be roughly sketched as follows:

- In every decomposable graph G, one can find a locally irregular subgraph whose removal leaves a graph whose all connected components are of even size (number of edges), thus decomposable. Thus, towards Conjecture 1.1, one can actually focus on graphs with even size, the price for this being one additional graph in a decomposition.

Theorem 1.3 (Bensmail, Merker, Thomassen [4]). *Let $\mathcal{G}$ be a hereditary family of connected graphs. Then,* $\max\{\chi'_{\rm irr}(G) : G \in \mathcal{G} \text{ is decomposable}\} \leq \max\{\chi'_{\rm irr}(G) : G \in \mathcal{G} \text{ has even size}\} + 1$.

- Once G has even size, it can then be decomposed into two graphs D and H, where D is $(2 \cdot 10^{10} + 2)$-degenerate and has all of its connected components of even size, while H is of minimum degree at least 10^{10}. These two graphs can be further decomposed as follows:
  - H, because of its large minimum degree, can be decomposed into at most three locally irregular subgraphs, according to a result of Przybyło [9].
  - D can be decomposed into at most $\lceil \log_2(2 \cdot 10^{10} + 3) \rceil + 1$ bipartite graphs with even size, which can each be decomposed into at most nine locally irregular subgraphs.

As Lužar, Przybyło and Soták proved that bipartite graphs with even size can even be decomposed into at most six locally irregular subgraphs [8], the exact same proof yields that every decomposable graph has irregular chromatic index at most 220. More precisely, this improved bound is a consequence of the following:

Theorem 1.4 ([4, 8]). *For every connected k-degenerate graph G with even size, we have*

$$\chi'_{\rm irr}(G) \leq 6 \cdot (\lceil \log_2(k+1) \rceil + 1)\,.$$

Consequently, by Theorem 1.3, for every decomposable k-degenerate graph G we have

$$\chi'_{\rm irr}(G) \leq 6 \cdot (\lceil \log_2(k+1) \rceil + 1) + 1.$$

Our results. As a main result in this work, we give, in Section 3, a new upper bound on the irregular chromatic index of graphs with given degeneracy:

Theorem 1.5. *For every connected k-degenerate graph G with even size, we have $\chi'_{\rm irr}(G) \leq 3k$. Consequently, by Theorem 1.3, for every decomposable k-degenerate graph G we have $\chi'_{\rm irr}(G) \leq 3k+1$.*

Theorem 1.5 improves Theorem 1.4 for graphs with low degeneracy. More precisely, $3k < 6 \cdot (\lceil \log_2(k+1) \rceil + 1)$ whenever $k \leq 9$. As notable cases, we get that decomposable 2-degenerate graphs (which include outerplanar graphs, series-parallel graphs, etc.) have irregular chromatic index at most 7 (versus 19 previously), and decomposable planar graphs, which are 5-degenerate, have irregular chromatic index at most 16 (versus 25 previously).

In general, the bound in Theorem 1.5 remains distant from the one conjectured in Conjecture 1.1. As a side aspect in this paper, we also focus on improving Theorem 1.5 for particular classes of degenerate graphs. In Section 2, we prove a key ingredient behind our proof of Theorem 1.5, which is that decomposable bipartite cacti verify Conjecture 1.1. In Section 4, we improve Theorem 1.5 for various classes of degenerate graphs, namely decomposable k-trees (for which we prove $2k$) and decomposable planar graphs (for which we prove 15). Conclusions and perspectives for further works are gathered in Section 5.

2. Decomposing good cacti into locally irregular subgraphs

The proof of our main result, Theorem 1.5, relies of the fact that k-degenerate graphs can, in general, be decomposed into a certain number of particular cacti. Recall that a *cactus* is a graph in which no two cycles intersect in more than one vertex. The cacti we are interested in are those that are bipartite (i.e., which have no odd-length cycle) and have no connected component being an odd-length path (as graphs having such are not decomposable, recall Theorem 1.2). Such cacti we call *good cacti* for convenience.

By Theorem 1.2, note that $\chi'_{\text{irr}}(G)$ is defined for every good cactus G. Furthermore, there exist infinitely many good cacti G verifying $\chi'_{\text{irr}}(G) = 3$. Any cycle with length congruent to 2 modulo 4 is an example. Actually, it is worth recalling that even infinitely many trees with irregular chromatic index 3 exist, as reported in [3]. Still, we prove that Conjecture 1.1 holds for good cacti.

Before proving this claim, let us give a few more details on the case of trees, as some of these will be mentioned in our proofs. In [3], Baudon, Bensmail and Sopena gave a description of trees with irregular chromatic index exactly 3, through the notion of *shrub*. A shrub is a tree being rooted at a root vertex r with degree 1. The authors proved that shrubs all admit "almost" locally irregular decompositions (being locally irregular decompositions where only the root is allowed to belong to a connected component that is not locally irregular) into at most two parts. Now, when considering a general tree T with a vertex v of degree $d \geq 2$, we can see T as d shrubs whose roots were identified into v. The authors proved that, except in very particular cases, almost locally irregular decompositions into at most two parts of these shrubs, when combined, can yield one locally irregular decomposition into at most two parts of T. More precisely, a tree with irregular chromatic index 3 must be constructed using bad shrubs, whose all almost locally irregular decompositions into at most two parts are very particular.

From the investigations in [3], one can in particular deduce the following:

Theorem 2.1 (Baudon, Bensmail, Sopena [3])**.** *Let T be a tree with $\chi'_{\text{irr}}(T) = 3$. Then:*

- *$|E(T)|$ is necessarily odd;*
- *T has no degre-3 or degree-4 vertex adjacent to a leaf.*

We are now ready for proving our main result in this section.

Theorem 2.2. *For every good cactus G, we have $\chi'_{\text{irr}}(G) \leq 3$.*

Proof. Clearly, we may assume that G is connected. The proof is by induction on the number of edges of G. The base cases are the path of length 2 and, more generally, any tree that is an even-length path or any bipartite cycle, in which cases the result holds [2, 3]. Hence, we may assume that G contains at least one cycle and that G is not reduced to this cycle.

Let $v \in V(G)$ be a vertex of G that belongs to a single cycle C. Let us consider the (unique) connected component X containing v after the removal of the two edges of C that are incident to v. If X is a tree (not reduced to the single vertex v), then, in what follows, we denote it by T_v and refer to it as *the tree pending in v*. Similarly, let $v \in V(G)$ be a vertex of G that is not contained in any cycle. If there is an edge incident to v whose removal makes the connected component

containing v to be a tree (with at least one edge), then we denote by T_v this connected component and regard it as the tree pending at v.

First, we prove that G can be assumed to have all its pending trees having a simple form. Here, a *spider* with degree $d \geq 1$ is any graph obtained from a star whose center has degree d by subdividing each edge at most once. That is, a spider is obtained by identifying one end of some paths of length at most 2. Note that every spider is locally irregular unless it is a path P_i with length $i \in \{1, 3, 4\}$.

- Let us first assume that there exists $u \in V(G)$ whose pending tree T_u is a path (u, v, w, x) with three edges such that u is one end of this path. Then[1], $G' = G[V(G) \setminus \{u, v\}]$ is a bipartite cactus (and not an odd-length path) and $|E(G')| < |E(G)|$, and, by the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3. W.l.o.g., $wx \in E(G'_1)$. Then, let G^*_2 be induced by G'_2 and the edges uv and vw (i.e., G^*_2 is the disjoint union of a locally irregular graph G'_2 and of a path (u, v, w) of length 2, i.e., G^*_2 is locally irregular). Then, G'_1, G^*_2, G'_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

- Second, let us assume that there exists $v \in V(G)$ whose pending tree T_v is a path $P_3 = (u, v, w, x)$ of length 3 such that v is not an end of this path. Then, $G' = G[V(G) \setminus \{w, x\}]$ is a bipartite cactus (because G' still contains a cycle and so cannot be an odd-length path) and $|E(G')| < |E(G)|$. Moreover, v has degree at most 3 in G' (by definition of a pending tree); let thus a, b denote its at most two neighbours distinct from u.

  By the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3. W.l.o.g., $uv \in E(G'_1)$. Then, either av or bv (or both) are in $E(G'_1)$ (otherwise, uv would be an isolated edge in G'_1, contradicting that it is locally irregular). W.l.o.g., say $av \in E(G'_1)$. Moreover, if $bv \notin E(G'_1)$, then say, w.l.o.g., that $bv \in E(G'_2)$. Then, let G^*_3 be induced by G'_3 and the edges wv and xw (i.e., G^*_3 is the disjoint union of a locally irregular graph G'_3 and of a path (v, w, x) of length 2, i.e., G^*_3 is locally irregular). Then, G'_1, G'_2, G^*_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

- Third, let us assume that there exists a vertex $v \in V(G)$, not in any cycle, with a pending tree which is a path $P_4 = (a, b, v, c, d)$ of length 4 where v is the center. Then, $G' = G[E(G) \setminus E(T_v)]$ is a bipartite cactus (because G' still contains a cycle and so cannot be an odd-length path) and $|E(G')| < |E(G)|$. Note that v has degree 1 in G'.

  By the induction hypothesis, G' can be partitioned into at most three locally irregular subgraphs G'_1, G'_2, G'_3. W.l.o.g., the edge incident to v in G' belongs to G'_1. Then, let G^*_2 (resp., G^*_3) be induced by $E(G'_2)$ and $\{ab, bv\}$ (resp., induced by $E(G'_3)$ and $\{vc, cd\}$). Then, G'_1, G^*_2, G^*_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

- Finally, let us assume that there exists some vertex $v \in V(G)$ with pending tree T_v that is a spider distinct from P_1, P_3, P_4. As mentioned above, T_v is locally irregular. Then, $G' = G[E(G) \setminus E(T_v)]$ is a bipartite cactus (and not an odd-length path) and $|E(G')| < |E(G)|$. Moreover, v has degree at most 2 in G' (by definition of a pending tree) and let a, b be its at most two neighbours in G'. By the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3. W.l.o.g., $\{va, vb\} \subseteq E(G'_1) \cup E(G'_2)$. Then, let G^*_3 be induced by G_3 and the edges in $E(T_v)$ (so G^*_3 is the disjoint union of two locally irregular graphs). Then, G'_1, G'_2, G^*_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

From the previous reduction rules, we may now assume that, for every pending tree T_v, v belongs to a cycle and T_v is either a single edge or a length-4 path P_4 with v as central vertex. Indeed, otherwise, one of the previous four cases would apply: if v is the single vertex with degree at least 3 in T_v, then the result is obvious; while, otherwise, for w being a vertex of T_v with degree

[1]Throughout this paper, for a given subset $F \subseteq E(G)$ (resp., $X \subseteq V(G)$), we denote by $G[F]$ (resp., $G[X]$) the subgraph of G induced by F (resp., by X).

at least 3 and as far as possible from v, the pending tree T_w is a pending tree to which one of the previous four rules applies.

We are now ready to conclude. If G contains a single cycle, then let C denote this cycle and let $u^* \in V(C)$ be any vertex of C with degree at least 3 (recall that we have assumed that G is not only a cycle). Otherwise, let C, C' be two cycles of G maximizing the distance between them, and let $u^* \in V(C)$ be the (unique) vertex of C that is the closest to C' (hence, all vertices of C except possibly u^* belong to a single cycle). Note that, because G is a bipartite cactus, C has even size at least 4, so some $v^* \in V(C) \setminus N(u^*)$ exists and let a and b the two neighbours of v^* in C. Note also that a, b and v^* belong to a single cycle. There are several cases to be considered.

- **Case 1:** *v^* has degree 2 in G (i.e., v^* has no pending tree).*

  If one, say a, of a, b, has a pending tree T_a, then let $X = \{av^*\} \cup E(T_a)$ and note that X induces a locally irregular graph. Then, $G' = G[E(G) \setminus X]$ is a bipartite cactus (not an odd-length path since u^* has degree at least 3). By the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3. Note that a and v^* have degree 1 in G', so we may assume that they are not incident to any edge in G'_1. Then, let G^*_1 be induced by G_1 and the edges in X (so G^*_1 is the disjoint union of two locally irregular graphs). Then, G^*_1, G'_2, G'_3 are three irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

  So we can assume that both a and b have degree 2, in which case we can define X as the set of (two) edges incident to v^*, and proceed as in the previous case.

- **Case 2:** *v^* has degree 3 in G (i.e., v^* has P_1 as pending tree).*

  Assume not both pending trees T_a and T_b (if they exist) are P_1, say T_a is not P_1 (i.e., T_a is either empty or P_4). Let $X = \{av^*\} \cup E(T_v) \cup E(T_a)$ and note that X induces a locally irregular graph. Then, $G' = G[E(G) \setminus X]$ is a bipartite cactus (not an odd-length path since u^* has degree at least 3). By the induction hypothesis, G' can be decomposed into at most three locally irregular subgraphs G'_1, G'_2, G'_3. Note that a and v^* have degree 1 in G', so we may assume that they are not incident to any edge in G'_1. Then, let G^*_1 be induced by G_1 and the edges in X (so G^*_1 is the disjoint union of two locally irregular graphs). Then, G^*_1, G'_2, G'_3 are three irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

  So we may assume that both T_a and T_b are P_1, in which case we set $X = \{av^*, bv^*\} \cup E(T_{v^*})$ which induces a locally irregular graph. Then, $G' = G[E(G) \setminus X]$ is a bipartite cactus (not an odd-length path since u^* has degree at least 3). By the induction hypothesis, G' can be decomposed into at most three irregular subgraphs G'_1, G'_2, G'_3. Note that a and b have degree 2 in G'. However, both edges incident to a (resp., to b) must belong to the same part of the decomposition (as otherwise T_a, resp., T_b, would induce an isolated edge). Therefore, we may assume that they are not incident to any edge in G'_1. Then, let G^*_1 be induced by G_1 and the edges in X (so G^*_1 is the disjoint union of two locally irregular graphs). Then, G^*_1, G'_2, G'_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

- **Case 3:** *v^* has degree 4 in G (i.e., v^* is the center of a P_4 as pending tree).*

  Assume not both T_a and T_b are P_4, say T_a is not a P_4; then let $X = \{av^*\} \cup E(T_v) \cup E(T_a)$ and note that X induces a locally irregular graph. Then, $G' = G[E(G) \setminus X]$ is a bipartite cactus (not an odd-length path since u^* has degree at least 3). By the induction hypothesis, G' can be decomposed into at most three irregular subgraphs G'_1, G'_2, G'_3. Note that a and v^* have degree 1 in G', so we may assume that they are not incident to any edge in G'_1. Then, let G^*_1 be induced by G_1 and the edges in X (so G^*_1 is the disjoint union of two locally irregular graphs). Then, G^*_1, G'_2, G'_3 are three locally irregular subgraphs decomposing G, i.e., $\chi'_{\text{irr}}(G) \leq 3$.

  The last case is when all of v^*, a, b have degree 4, i.e., each of these vertices has two pending paths of length 2 attached. In that case, let $X = \{av^*, bv^*\} \cup E(T_a) \cup E(T_b) \cup E(T_{v^*})$. Note that this time the subgraph induced by X is not locally irregular.

  Then, $G' = G[E(G) \setminus X]$ is a bipartite cactus (not an odd-length path since u^* has degree at least 3). By the induction hypothesis, G' can be decomposed into at most three irregular subgraphs G'_1, G'_2, G'_3. Note that a and b have degree 1 in G'. Say the edge $a'a$ incident to

a in G' belongs to G'_1 and let d be the degree of a' in G'_1. Also, w.l.o.g., say that the edge incident to b in G' is not in G'_2. It remains two cases to be considered.

- If $d \neq 4$, then let G^*_1 be the subgraph induced by the edges of G'_1 plus $\{av^*\} \cup E(T_a) \cup E(T_{v^*})$ (it is a locally irregular graph) and let G^*_2 be induced by G'_2 and $E(T_b) \cup \{bv^*\}$ (also locally irregular). Then G^*_1, G^*_2, G'_3 is the desired solution.
- If $d = 4$, then let G^*_1 be the subgraph induced by the edges of G'_1 plus $E(T_a)$ (it is a locally irregular graph) and let G^*_2 be induced by G'_2 and $E(T_{v^*}) \cup E(T_b) \cup \{bv^*, av^*\}$ (also locally irregular). Then G^*_1, G^*_2, G'_3 is the desired solution. □

3. Decomposing degenerate graphs into good cacti

In this section, we prove the first part of Theorem 1.5, hence the whole statement. Our proof relies on the following key lemma.

Lemma 3.1. *Let G be a graph, and let (T_1, T_2) be a partial decomposition of G into two good cacti. Consider a vertex v of G belonging to none of T_1, T_2. Then, for every two edges vu, vw incident to v, there exists a partial decomposition (T^*_1, T^*_2) of G into two good cacti, where $E(T^*_1) \cup E(T^*_2) = E(T_1) \cup E(T_2) \cup \{vu, vw\}$.*

Proof. Roughly, T^*_1 and T^*_2 will be obtained starting from T_1 and T_2, possibly switching edges from one of the subgraph to the second graph, and adding vu and vw to some of the subgraphs. We need to consider a few cases. First let us assume that there exists some of T_1 and T_2, say T_1, such that u and w do not belong to the same connected component of the subgraph induced by $E(T_1)$ (possibly, u or w or both are not adjacent to any edge of $E(T_1)$). In that case, let $T^*_2 = T_2$ and let T^*_1 be induced by $E(T_1) \cup \{vu, vw\}$. Clearly, adding these two edges to T_1 cannot create an odd-length cycle (since u and w are not in a same connected component of T_1) and it cannot create an odd-length path (since T_1 is a good cactus). Hence, T^*_1 is a good cactus and we are done.

Second, let us assume that u and w belong to a same connected component, say C_1 (resp., C_2), of T_1 (resp., of T_2).

- Let us first assume that, for $u', w' \in \{u, w\}$, $u' \neq w'$, and for $i, j \in \{1, 2\}$, $i \neq j$, we have:
  - u' has degree 2 in C_x or u' has degree 1 in C_x, and C_x has at least one vertex of degree 3, and
  - w' has degree 2 in C_y or w' has degree 1 in C_y, and C_y has at least one vertex of degree 3.

  In this case, let T^*_x (resp. T^*_y) be obtained by adding $u'v$ to T_x (resp., by adding $w'v$ to T_y). It is easy to see that T^*_1 and T^*_2 are good cacti since no odd-length cycles are created and C_i (for both $i \in \{1, 2\}$) cannot become an odd-length path.
- Second, let us assume that, for some of C_1 or C_2, w.l.o.g., say C_1, the subpath of C_1 from u to w has even length. In that case, let $T^*_2 = T_2$ and let T^*_1 be induced by $E(T_1) \cup \{vu, vw\}$. Clearly, adding these two edges to T_1 cannot create an odd-length cycle and it cannot create an odd-length path (actually, C_1 becomes an even-length cycle plus, possibly, a pending path). Hence, T^*_1 is a good cactus and we are done.
- If none of the previous cases holds, it must be that C_1 and C_2 are even-length paths with a common end in $\{u, w\}$, w.l.o.g., say w, and such that u has degree 2 in both C_1 and C_2. Again, several cases must be considered. For $i \in \{1, 2\}$, let $x_i \in V$ be the end (distinct from w) of C_i and let C^b_i (resp., C^a_i) be the subpath of C_i between x_i and u (resp., between u and w). Moreover, C^a_i and C^b_i ($i \in \{1, 2\}$) have odd length (since C_i has even length and the case when some of C^a_1 or C^a_2 has even length has been considered in the previous item).
  - If C^a_1 is just the edge uw and C^a_2 has odd length (the case when $E(C^a_2) = uw$ and C^a_1 has odd length is symmetric), then let C^*_1 be induced by the edges in $(E(C_1) \setminus \{uw\}) \cup \{uv\}$ (hence C^*_1 is an even-length path) and let C^*_2 be induced by the edges in $E(C_2) \cup \{wv, uw\}$

(hence, C_2^* induces an even-length cycle, since C_2^a is an odd-length path, plus one pending path, C_2^b, and one pending edge wv). Then $T_1^* = (T_1 \setminus C_1) \cup C_1^*$ and $T_2^* = (T_2 \setminus C_2) \cup C_2^*$ are good cacti.

- Now, let us assume that C_1^a and C_2^a have odd length and are not single edges. Since C_1 and C_2 have even length, this implies that C_1^b and C_2^b have odd length too. For $i \in \{1, 2\}$, let q_i be the neighbour of w on C_i.
  - If q_1 has degree 0 in C_2 (the case where q_2 has degree 0 in C_1 is symmetric), then let C_2^* be induced by the edges in $(E(C_2) \setminus \{q_2w\}) \cup \{q_1w\}$ (hence C_2^* is an even-length path) and let C_1^* be induced by the edges in $(E(C_1) \setminus \{q_1w\}) \cup \{vu, vw\}$ (hence C_1^* is acyclic with one vertex, namely u, of degree 3). Then $T_1^* = (T_1 \setminus C_1) \cup C_1^*$ and $T_2^* = (T_2 \setminus C_2) \cup C_2^*$ are good cacti.
  - If q_1 has degree 2 in C_2 (the case where q_2 has degree 2 in C_1 is symmetric), then let C_2^* be induced by the edges in $(E(C_2) \setminus \{q_2w\}) \cup \{q_1w, wv\}$ and let C_1^* be induced by the edges in $(E(C_1) \setminus \{q_1w\}) \cup \{wq_2, vu\}$ (hence C_1^* and C_2^* are acyclic with one vertex, resp. u and q_1, of degree 3). Then $T_1^* = (T_1 \setminus C_1) \cup C_1^*$ and $T_2^* = (T_2 \setminus C_2) \cup C_2^*$ are good cacti.
  - Finally, let us assume that q_1 has degree 1 in C_2 and that q_2 has degree 1 in C_1. This implies that $q_1 = x_2$ and $q_2 = x_1$. Then, let C_2^* be induced by the edges in $(E(C_2) \setminus \{q_2w\}) \cup \{q_1w, vw, vu\}$ (since C_2^a has odd length, this results in C_2^* being an even-length cycle with a pending path) and let C_1^* be induced by the edges in $(E(C_1) \setminus \{q_1w\}) \cup \{q_2w\}$ (hence, C_2^* is an even-length path $(w, q_2 = x_1, ..., u, ..., q_1)$). Finally, $T_1^* = (T_1 \setminus C_1) \cup C_1^*$ and $T_2^* = (T_2 \setminus C_2) \cup C_2^*$ are good cacti. □

We are now ready to prove our main result, the first part of Theorem 1.5.

Proof of Theorem 1.5. We prove the first part in the statement of Theorem 1.5 by proving that every connected k-degenerate graph G with even size decomposes into at most k good cacti, as they can each be further decomposed into at most three locally irregular subgraphs (Theorem 2.2). We prove this by induction on $|E(G)|$. The base case is when G is a path of length 2, in which case the claim obviously holds. Let us now focus on the general case.

Assume first that there exists $v \in V(G)$ such that $E(G)$ can be partitioned into two parts A and B of size at least 2 each, such that, for every $e \in A, f \in B$, if $e \cap f \neq \emptyset$ then $e \cap f = \{v\}$. In other words, v is a cut-vertex of G whose removal results in at least two connected components.

- If A (and so B) has even size, then, by induction on $G[A]$ and $G[B]$, let $A = \bigcup_{j \leq k} T_j^A$ and $B = \bigcup_{j \leq k} T_j^B$ where each T_i^A (resp., T_i^B) induces a (possibly empty) good cactus. Clearly, for every $1 \leq i \leq k$, $T_i = T_i^A \cup T_i^B$ induces a good cactus, and $E(G) = \bigcup_i T_i$.
- Otherwise, $|A|$ and $|B|$ are odd. Let G_A (resp., G_B) be the graph induced by A (resp., by B) with, in addition, a new pending edge p incident to v. Since $|A|, |B| \geq 2$, G_A and G_B have strictly less edges than G and we can proceed by induction. Let $A \cup \{p\} = \bigcup_{j \leq k} T_j^A$ and $B \cup \{p\} = \bigcup_{j \leq k} T_j^B$ where each T_i^A (resp., T_i^B) induces a (possibly empty) good cactus. W.l.o.g., $p \in T_1^A \cap T_1^B$. For every $1 < i \leq k$, let $T_i = T_i^A \cup T_i^B$ and let $T_1 = (T_1^A \cup T_B^1) \setminus \{p\}$. For every $1 \leq i \leq k$, T_i induces a good cactus, and $E = \bigcup_i T_i$.

Second, let us assume that there exists $v \in V(G)$ with degree $2 \leq d(v) \leq k + 1$, incident to a vertex $u \in V(G)$ with degree 1 and such that $G[V(G) \setminus \{u, v\}]$ is connected. Let $w \in N(v) \setminus \{u\}$ and let $E' = E(G) \setminus \{uv, vw\}$. Let $G' = (V', E')$ where $V' = V \setminus \{u\}$ if $d(v) > 2$ and $V' = V(G) \setminus \{uv\}$ otherwise. By the induction hypothesis on G', let $E' = \bigcup_{j \leq k} T_j'$ where each T_i' induces a (possibly empty) good cactus and, w.l.o.g., no edge of T_1' is incident to v in G' (possible since v has degree at most $k - 1$ in G'). Let $T_j = T_j'$ if $j > 1$ and let $T_1 = T_1' \cup \{uv, vw\}$. Then, $(T_j)_{1 \leq j \leq k}$ is a desired solution.

Now, if none of the previous two cases occurs, then, because G is k-degenerate, we may assume that there is a vertex $v \in V(G)$ with $2 \leq d(v) \leq k$ and $G - v$ is connected. Let $u, w \in N(v)$ and let G' be the subgraph induced by $E(G) \setminus \{vu, vw\}$. Since G' is k-degenerate, connected and of even size $|E(G)| - 2$, the induction hypothesis holds and there exists a family $(T_1, ..., T_k)$ of good

cacti decomposing G'. Since v has degree at most $k-2$ in G', there are two good cacti, say T_1 and T_2, such that v is not incident to any edge of $E(T_1) \cup E(T_2)$. All conditions are now met to apply Lemma 3.1, which ensures that the graph induced by the edges $E(T_1) \cup E(T_2) \cup \{uv, uw\}$ can be decomposed into two good cacti T_1^*, T_2^*, hence yielding a desired decomposition $(T_1^*, T_2^*, T_3, ..., T_k)$ of G into k good cacti. □

4. Improved bounds for particular classes of degenerate graphs

Throughout this section, we improve the bound in Theorem 1.5 for two classes of degenerate graphs, namely k-trees and planar graphs.

4.1. k-trees

Let $k \geq 2$ be an integer. A *k-tree* is any graph built recursively as follows. A complete graph with $k+1$ vertices is a k-tree. Given a k-tree and a complete subgraph C of order k in it, a new k-tree is obtained by adding a new vertex adjacent to every vertex of C. Note that k-trees are k-degenerate. Graphs of treewidth at most k are precisely the subgraphs of k-trees.

To any k-tree G may be associated an ordering of its vertices, called *building order*, corresponding to the order in which they have been added to create G (with the initial clique as prefix).

Claim 4.1. *Let G be any k-tree and H be any subgraph of G that is a k-tree. Then there exists a building order of G with the vertices of H as prefix.*

Proof. The proof is by induction on the number n of vertices of G. The result is obvious if $n = k+1$. Let H be a subgraph of G that is a k-tree. If $H \neq G$ (otherwise the result is obvious), then there is a vertex v of degree k in $V(G) \setminus V(H)$. Then, the induction hypothesis can be applied to $G' = G - v$ (which is a k-tree) to obtain a building order $\mathcal{O}$ of G' with $V(H)$ as prefix. The desired building order of G is obtained by adding v at the end of $\mathcal{O}$. ◇

We are now ready to improve Theorem 1.5 for k-trees, which states that their irregular chromatic index is at most $3k+1$. We treat the cases $k = 2$ and $k \geq 3$ separately.

Theorem 4.2. *For every decomposable 2-tree G, we have $\chi'_{\text{irr}}(G) \leq 4$.*

Proof. The fact that G is decomposable implies that G has at least 4 vertices. It is easy to see that the result holds if G has 4 or 5 vertices (there are only three graphs as G in this case, as shown in the left part of Figure 1). Hence, let us assume that G has at least 6 vertices.

First, let us point out two pathological cases where a decomposition of G in at most two locally irregular subgraphs can easily be constructed. On the one hand, if G is a *fan*, i.e., a path (called *external path*) plus a universal vertex, then consider the decomposition, depicted in Figure 2 (D), into two spiders where the center has degree at least 3. If G is a *book*, i.e., a 2-tree with order n and $n-2$ vertices with degree 2, then consider the decomposition, given in Figure 2 (E), into two stars where the center has degree at least 3. From now on, we can thus assume that G is not a fan nor a book.

As shown in Figure 1, there are exactly five non-isomorphic 2-trees with 6 vertices, including one fan and one book. Since G is not a fan nor a book, it must contain some of the three graphs (A), (B), (C) (on Figure 1) as a subgraph. Indeed, let us consider a building order of G. If its prefix (of length 6) induces one of the graphs (A), (B), (C), then we are done. Otherwise, there are two cases to be considered.

- If its prefix (of length 6) induces a fan, then let us consider the largest prefix of it that induces a fan and let v be the next vertex in the order (that vertex must exist since G is not a fan). If v is made adjacent to both ends of an edge of the external path, then this creates a subgraph (A), while it creates a subgraph (C) otherwise.
- If its prefix (of length 6) induces a book, then let us consider the largest prefix of it that induces a book and let v be the next vertex in the order (that vertex must exist since G is not a book). Adding v then creates a subgraph (C).

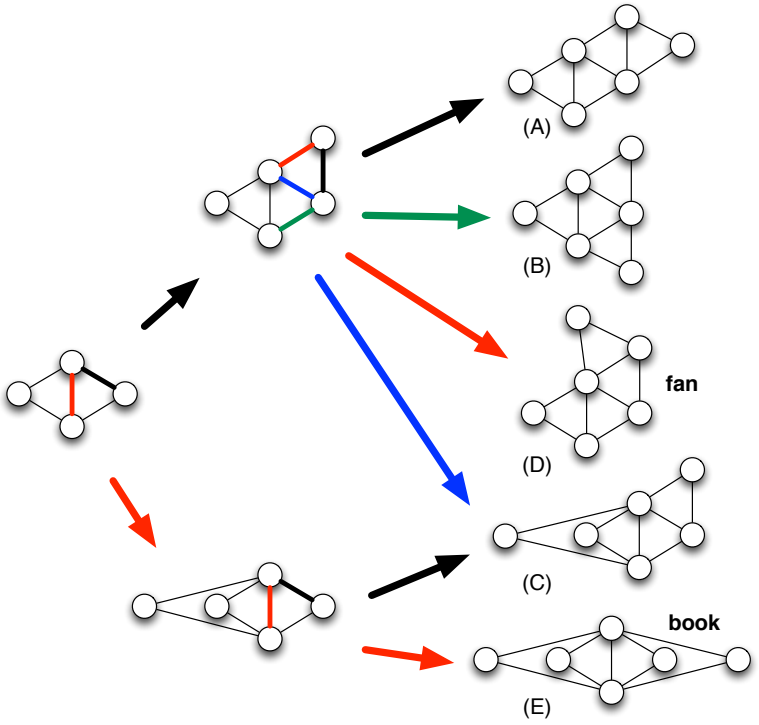

Figure 1: Generation of all non-isomorphic 2-trees on 6 vertices, starting from the single 2-tree on 4 vertices. An arrow coloured with colour x indicates that the new vertex has been made adjacent to the edge colored with colour x.

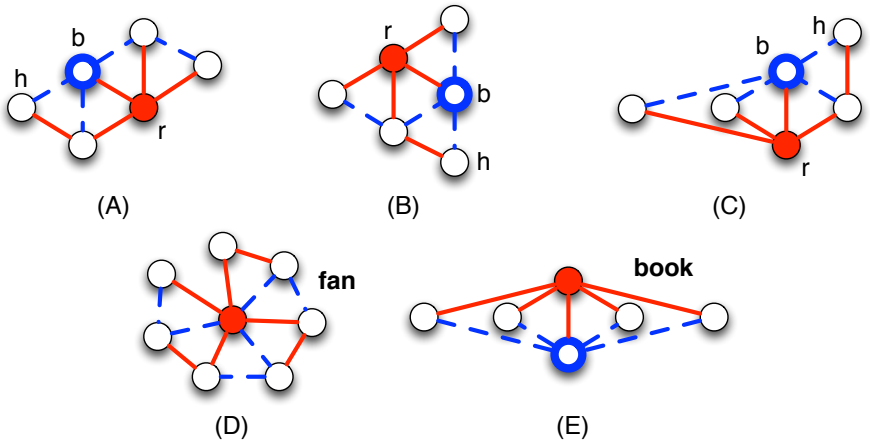

Figure 2: Initial decompositions of the 2-trees (A), (B), (C), and schematic representations of the decompositions of (D) and (E). In all cases, one part is depicted in dotted edges and the second part is in solid line. In both cases (D) (fan) and (E) (book), the two parts induce locally irregular graphs.

From the previous paragraph and Claim 4.1, we may assume that G has a building order whose prefix of size 6 induces either of (A), (B), (C). We consider the three cases simultaneously in what follows. In each of the three cases, let us define the vertices r, b, h as depicted in Figure 2.

By induction on $|V(G)|$ (following the building order), we prove that the edge set of G can be partitioned into two sets X (depicted with solid red edges in Figure 2) and Y (depicted with dotted blue edges in Figure 2) both inducing a tree and such that:

1. vertex r has degree at least 4 in $G[X]$;
2. vertex b has degree 1 in $G[X]$ and degree at least 3 in $G[Y]$;
3. vertex h has degree 1 in $G[Y]$; and
4. every vertex but r has degree at least 1 in both $G[X]$ and $G[Y]$.

The result holds for $|V(G)| = 6$ by defining X and Y as in Figure 2. Now, let us assume that a new vertex v is added as neighbour of $u, w \in V(G)$, where $uw \in E(G)$. If $b \in \{u, w\}$, then set $b = w$. If r, resp., h, belongs to $\{u, w\}$ (note that they cannot both belong to $\{u, w\}$ since $uw \in E(G)$ and $rh \notin E(G)$), then set $u = r$ (resp., $u = h$). Then, add vu to X and vw to Y. It is easy to see that all four properties still hold. Moreover, X and Y induce trees, by the fourth property.

Hence, $E(G)$ can be partitioned into two sets X and Y such that each of $G[X]$ and $G[Y]$ induces a tree containing a vertex with degree at least 3 that is adjacent to some leaf. Any tree T with this property has $\chi'_{\text{irr}}(T) = 2$, recall Theorem 2.1. The result then follows. □

Theorem 4.3. *For every k-tree G with $k \geq 3$, we have $\chi'_{\text{irr}}(G) \leq 2k$.*

Proof. If G is reduced to a clique, then the result follows from [2]. Hence, let us assume that G has $n > k + 1$ vertices. We prove that $E(G)$ can be partitioned into k parts $X_1, ..., X_k$ such that, for every $1 \leq i \leq k$, set X_i induces a tree with a vertex with degree at least 3 that is adjacent to some leaf. Any tree T with this property has $\chi'_{\text{irr}}(T) = 2$ (Theorem 2.1), and so the theorem follows.

The proof is by induction on $|V(G)| > k + 1$. Let us consider a building order of G. Let K be the complete graph induced by the $k + 1$ first vertices, and let us denote by $v_1, ..., v_k, a, b$ the $k + 2$ first vertices of the order. If $|V(G)| = k + 2$, then let $X_1 = \{v_1a, v_1b\} \cup \bigcup_{1<j<k}\{v_1v_j\}$, $X_i = \{v_ia, v_ib\} \cup \bigcup_{i<j\leq k}\{v_iv_j\}$ for $i < k$ and $X_k = \{v_ia, v_ib, v_kv_1\}$. Clearly, $E(G) = \bigcup_{i\leq k} X_i$, each X_i induces a tree, and for every $1 \leq i \leq k$, vertex v_i has degree at least 3 in X_i and vertex v_{i+1} (resp., v_1 for $i = k$) is a neighbour of degree 1 of v_i in X_i. Finally, every vertex but those in $\{v_1, ..., v_k\}$ is adjacent to some edge in X_i for every $1 \leq i \leq k$.

Then, each time a new vertex v is added, exactly one of its k incident edges is added to X_i for every $1 \leq i \leq k$ with the only constraint that, for every $1 \leq i \leq k$, if $vv_i \in E(G)$, then vv_i is added to X_i. It is easy to check that all desired properties are preserved. □

4.2. Planar graphs

Before proceeding with the proof of the main result in this section, let us introduce a few new and existing tools. We first need the notion of *light edges* in planar graphs. An *x-vertex* refers to a vertex with degree precisely x. By an *(x, y)-edge*, we mean an edge whose one end is an x-vertex and other end is a y-vertex.

Recall that a *planar graph* is a graph embedded in the plane in such a way that no two edges cross. It is a well-known fact (from Euler's formula), that planar graphs are 5-degenerate. In planar graphs with minimum degree 5, even lighter structural properties can actually be established.

Theorem 4.4 (see e.g. [5], Theorem 3.1)**.** *Every planar graph with minimum degree 5 has a $(5, 5)$-edge or $(5, 6)$-edge.*

In what follows, whenever referring to a *light edge* we will thus mean a $(5, 5)$-edge or a $(5, 6)$-edge.

We will also be needing the following counterpart of Lemma 3.1 when three edges incident to a same vertex remain to be added to good cacti.

Lemma 4.5. *Let G be a graph, and let T be a subgraph of G that is a good cactus not containing a vertex $v \in V(G)$. Then, for every three edges vu, vw, vx incident to v, there exists a subgraph T^* of G that is a good cactus with $E(T^*) = E(T) \cup \{vu, vw, vx\}$ or $E(T^*) = E(T) \cup \{e\}$ for some $e \in \{vu, vw, vx\}$.*

Proof. If adding all three of vu, vw, vx to T leads to a good cactus, then we are done. Otherwise, this means that at least two of u, w, x, say u, w w.l.o.g., belong to the same connected component of T. If adding vu to T is not correct, then this means that this creates an odd-length path in T, which is possible only when u is an end of an even-length path P of T. By the same argument, w is the second end of P. Now, if P goes through x (meaning x is an internal vertex of P), then adding vx to T yields the correct conclusion, as P would become a tree with a 3-vertex. Thus, we may lastly assume that x does not belong to P, in which case adding all three of vu, vw, vx to T is actually correct: as a result, P becomes an even-length cycle with the edge ux attached, possibly joining the cycle to another connected component of T. □

We are now ready to prove the main result in this section. Since every planar graph is 5-degenerate, from Theorem 1.5 we get that every decomposable planar graph G verifies $\chi'_{\text{irr}}(G) \leq 16$. We slightly improve this bound down to 15.

Theorem 4.6. *For every connected planar graph G with even size, we have $\chi'_{\text{irr}}(G) \leq 14$. Consequently, by Theorem 1.3, for every decomposable planar graph G we have $\chi'_{\text{irr}}(G) \leq 15$.*

Proof. We prove the claim by proving that every connected planar graph G with even size decomposes into four good cacti $T_1, ..., T_4$ and one *even forest* T_5, i.e., a forest whose all trees have even size. This implies the result, as good cacti have irregular chromatic index at most 3 (Theorem 2.2) and even forests have irregular chromatic index at most 2 (Theorem 2.1). This is proved by induction on $|E(G)|$. Since this is trivially true when $|E(G)| = 2$, we focus on the general case. This is done by proving gradually that the claim holds under certain circumstances, until we get to the point where the claim is ultimately proved to hold in general.

We start off by considering bridges. Assume that G has a bridge uv, and let G_u (resp. G_v) denote the connected component of $G - uv$ that contains u (resp. v). Since G has even size, w.l.o.g. G_u has even size while G_v has odd size. Assume $|E(G_u)| \geq 2$. Then both G_u and $G_v + uv$ are planar graphs with even size smaller than that of G, so they admit desired decompositions $(T_1, ..., T_5)$ and $(T'_1, ..., T'_5)$, respectively. Since G_u and $G_v + uv$ intersect only in u, we get that $(T_1 \cup T'_1, ..., T_5 \cup T'_5)$ is a desired decomposition of G, since identifying two vertices from two disjoint good cacti clearly results in a good cactus, and similarly for two even forests. So we may now assume that G contains no such bridge, which we call *good bridges* below. Note that a bridge is not good if and only if one of its end is a 1-vertex.

Claim 4.7. *We may assume that G has no good bridge.*

Since G is planar, it has minimum degree at most 5. In some cases, assuming the existence in G of vertices with small degree, the induction hypothesis can be invoked on a subgraph G' of G to deduce a decomposition that can be extended to G. More precisely:

Claim 4.8. *We may assume that G has not 1-vertex adjacent to a vertex of degree at most 6.*

Proof. Let us assume G has a 1-vertex u that is adjacent to a vertex v of degree at most 6. Let us denote by $w_1, ..., w_d$ the $d \leq 5$ neighbours of v different from u. If one edge vw_i is such that $G' = G - \{uv, vw_i\}$ is connected, then we can deduce a decomposition $(T_1, ..., T_5)$ of G' by induction, and extend it to G by adding uv, vw_i to one of $T_1, ..., T_5$ that does not contain v (one such exists since v has degree at most 4 in G'). This results in a correct decomposition of G, as attaching a pendant path of length 2 to a good cactus or an even forest, respectively, results in a good cactus or an even forest, respectively.

So assume that $G - \{uv, vw_i\}$ is disconnected for every i, i.e., vw_i is a bridge for every i. Since G has even size, note that it cannot be that the connected component that contains w_i in $G - vw_i$ has odd size for every i. Let us thus assume that the connected component that contains w_1 in $G - vw_1$ has even size. Recall that that connected component cannot have size at least 2, as otherwise G would have a good bridge. Thus, by Claim 4.7, w_1 is a 1-vertex. In that case, we consider $G' = G - u - w_1$, which is planar of even size smaller than that of G, and a decomposition $(T_1, ..., T_5)$ of G'. Clearly, adding uv, vw_1 to any of $T_1, ..., T_5$ yields a desired decomposition of G. ◇

Claim 4.9. *We may assume that, for every $x \in \{2, 3, 4\}$, G has no x-vertex.*

Proof. Let us consider the three cases depending on $x \in \{2, 3, 4\}$.

- Assume G has a 2-vertex v with neighbours u_1, u_2.
  - If $G' = G - v$ is connected, then, because it has even size, G' admits a decomposition $(T_1, ..., T_5)$ by induction. Now, since none of the good cacti T_1, T_2 contains v, using Lemma 3.1 we can deduce a desired decomposition $(T_1^*, T_2^*, T_3, ..., T_5)$ of G.
  - Otherwise, if $G - v$ is not connected, then both edges adjacent to v are bridges in G. Since v cannot be adjacent to a 1-vertex and none of vu_1, vu_2 is a good bridge, we deduce that both G_{u_1}, the connected component of $G - v$ that contains u_1, and G_{u_2}, the connected component of $G - v$ that contains u_2, have even size. Now both G_{u_1} and $G_{u_2} + \{u_2v, vu_1\}$ have even size smaller than that of G, so, by the induction hypothesis, they admit good decompositions $(T_1, ..., T_5)$ and $(T_1', ..., T_5')$, respectively. Again, since G_{u_1} and $G_{u_2} + \{u_2v, vu_1\}$ intersect at u_1 only, we get that $(T_1 \cup T_1', ..., T_5 \cup T_5')$ is a desired decomposition of G.

  From now on, we can thus assume that G has no 2-vertex.

- Assume G has a 3-vertex v with neighbours u_1, u_2, u_3.
  - If $G' = G - \{vu_i, vu_j\}$ is connected for any two edges vu_i, vu_j, then, because it has even size, G' admits a decomposition $(T_1, ..., T_5)$ by induction. At least two the four good cacti $T_1, ..., T_4$, say T_1, T_2 without loss of generality, do not contain v. Thus, using Lemma 3.1 we can deduce a desired decomposition $(T_1^*, T_2^*, T_3, ..., T_5)$ of G.
  - Thus, we may suppose that $G - \{vu_1, vu_2\}$ is disconnected, which means that $G - vu_3$ is disconnected as well. In other words, vu_3 is a bridge of G. Since u_3 cannot be a 1-vertex (because v has degree 3 and by Claim 4.8), we get that vu_3 is actually a good bridge, which is impossible.

  Thus, we may now suppose that G has no 3-vertex.

- Assume G has a 4-vertex v with neighbours u_1, u_2, u_3, u_4.
  - If $G - \{vu_i, vu_j\}$ is connected for any two edges vu_i, vu_j, then Lemma 3.1 can again be employed to deduce a desired decomposition of G similarly as in the previous case.
  - Thus, we may suppose that $G - \{vu_1, vu_2\}$ is disconnected. Since v cannot be adjacent to a 1-vertex (by Claim 4.8), none of vu_3, vu_4 can be a bridge, as otherwise it would be a good bridge, a contradiction. However, $G - \{vu_3, vu_4\}$ is disconnected as well. Thus, $G - v$ has two connected components, one of which contains u_1, u_2 while the second one contains u_3, u_4. Then it can be noticed that $G - \{vu_1, vu_3\}$ is connected, a case we have already treated.

  This concludes the proof of the claim. ◇

Thus, we may now suppose that G has no x-vertex for $x = 2, 3, 4$. Furthermore, all 1-vertices of G are adjacent to vertices with degree at least 7, and, by arguments used to deal with 1-vertices above (Claim 4.8), we can also assume that G has no two 1-vertices adjacent to a same vertex. These arguments imply that G^-, the graph obtained from G by removing all 1-vertices, is a planar graph of minimum degree exactly 5. By Theorem 4.4, it has a light edge uv, which is either a $(5, 5)$-edge or a $(5, 6)$-edge. Back in G, by all the previous properties, we deduce that uv is either a $(5, 5)$-edge (**type A**) or a $(5, 6)$-edge where none of u, v is adjacent to a 1-vertex (**type B**), or a $(5, 7)$-edge where the one of u, v with degree 7 is adjacent to a 1-vertex (**type C**). In what follows, we always implicitly assume that $d(u) \leq d(v)$.

Assume first that G has a type-C edge uv. Let us denote by w the 1-vertex adjacent to v. Since G has no good bridge (Claim 4.7), note that $G' = G - \{uv, vw\}$ is connected. Since G' is a planar graph with even size smaller than that of G, it admits a decomposition $(T_1, ..., T_5)$. By that decomposition, there is a T_i of $T_1, ..., T_5$ that does not contain u, since this vertex has degree 4 in G'. Then adding uv, vw to T_i (which either adds a connected component being a path of length 2,

or two pendant vertices attached to a same vertex) results in a good cactus if T_i was one, or in an even forest if T_i was one. We thus get a desired decomposition of G.

To deal with uv being type A, let us focus on 5-vertices that are cut-vertices in G.

Claim 4.10. *We may assume that G has no 5-vertex being a cut-vertex.*

Proof. Assume that G has a 5-vertex x that is a cut-vertex. Since x cannot be adjacent to a 1-vertex, and we have assumed that G has no good bridge (Claim 4.7), we may assume that x is incident to no bridge at all. From this, we deduce that $G - x$ has only two connected components C_1, C_2. Let xy_1 and xy_2 be edges incident to x where y_1 is in C_1 while y_2 is in C_2. Then $G' = G - \{xy_1, xy_2\}$ is a connected planar graph with even size smaller than that of G, which admits a good decomposition $(T_1, ..., T_5)$ by the induction hypothesis. We can now consider one T_i not containing x (which exists since x has degre 3 in G'), and add xy_1, xy_2 to T_i, yielding either a good cactus (if T_i was one), or an even forest (otherwise). This is because joining two good cacti by a path of length 2 yields a good cactus, and similarly for even forests. ◇

Thus we may now assume that G has no 5-vertex being a cut-vertex. Assume now that, in G, uv is a light edge of type A. Let us denote by $w_1, ..., w_4$ the neighbours of u different from v. By the above, we may assume that $G' = G - \{uv, uw_1, uw_2, uw_3\}$ is a connected planar graph with even size smaller than that of G. There thus exists a good decomposition $(T_1, ..., T_5)$ of G'. Since v has degree 4 in G', there is one T_i of $T_1, ..., T_5$ that does not contain v. We consider two cases.

- $i \neq 5$. That is, T_i is a good cactus. If uw_4 does not belong to T_i, then adding w_1u, uv to T_i results in a good cactus. We are then left with treating the two edges w_2u, w_3u, which can be added, via Lemma 3.1 to two of $T_1, ..., T_4$ that are not T_i and do not contain w_4u.

  So now assume that uw_4 belongs to T_i. If adding uv to T_i does not result in T_i becoming an odd-length path, then T_i remains a good cactus. Then w_1u, w_2u, w_3u remain to be treated, and there is so far only one, T_i, of T_1, T_2, T_3, T_4 containing v. Assume $T_i = T_4$ without loss of generality. Now, using Lemma 4.5 either three (in which case we are done) of w_1u, w_2u, w_3u can be added to T_1, T_2, T_3 correctly, or only one of them, say w_1u to T_3. Lemma 3.1 can now be used to add w_2u, w_3u to T_1, T_2, resulting in a desired decomposition of G.

  So lastly assume that adding uv does result in T_i becoming an odd-length path. Then T_i has a connected component being an even-length path P whose one end is u (and P reaches u via w_4). If, say, w_1, does not belong to P, then, by adding both w_1u, uv to T_i, part T_i remains a good cactus (in particular because the P gets included to a subgraph with maximum degree at least 3). Then w_2u, w_3u can again be treated using Lemma 3.1. So we may assume that all of w_1, w_2, w_3 belong to P, which means that at least two of w_1, w_2, w_3 are internal vertices of P, one of which is not adjacent to w_4. Assume w_1 is an internal vertex of P not adjacent to w_4. We here remove w_4u from T_i, and add w_1u, uv to T_i. This way, in T_i, the component P becomes a tree in which w_1 has degree 3. It now remains to deal with w_2u, w_3u, w_4u, while u so far belongs to only one good cactus, T_i. Using Lemmas 3.1 and 4.5, we can eventually do so, and thus construct a desired decomposition of G.

- $i = 5$. That is, T_i is the even forest. Again, if w_4u does not belong to T_i, then we are done as earlier. So we may assume that w_4u belongs to T_i. Then there are only four remaining edges to be treated (w_1u, w_2u, w_3u, uv), all incident to u, while u, this far, does not belong to any of $T_1, ..., T_4$. We can here complete the decomposition by applying Lemma 3.1 twice.

We are now left with the case where uv is a type-B edge. First of all, similarly as for 5-vertices that are cut-vertices (see proof of Claim 4.10), using the same arguments as earlier we can assume that the 6-vertices of G (in particular v) are not cut-vertices. Thus, we can assume that $G - v$ is connected. Since u has degree 4 in $G - v$, the graph $G' = G - u - v$ has at most four connected components $C_1, ..., C_d$. Furthermore, in G, for each C_i, each of u, v has at least one neighbour in C_i (since v is not a cut-vertex). To finish off the proof, we deal with the number d of C_i's, and the connection between these components and u, v in G. We denote by $x_1, ..., x_4$ the four neighbours of u different from v, and by $y_1, ..., y_5$ the five neighbours of v different from u.

If $d = 1$, i.e., G' is connected, then note that G' is a planar graph with even size $|E(G)| - 10$. It thus admits a decomposition $(T_1, ..., T_5)$ be the induction hypothesis. We extend it to G as follows. First, we add uv, vy_5 to T_5, which adds a path of length 2 which is either pendant or isolated in the even forest. To eventually add the eight remaining edges x_1u, x_2u, x_3u, x_4u and y_1v, y_2v, y_3v, y_4, v to the decomposition, we just apply Lemma 3.1 four times: twice on u, and twice on v.

Thus, we may now suppose that G' is not connected, i.e., $d \neq 1$. Note that $G'' = G - v$ is connected and has even size. Thus, by the induction hypothesis, G'' has a good decomposition $(T_1, ..., T_5)$. Since u has degree 4 in G'', one of $T_1, ..., T_5$ does not contain u.

- If T_5 does not contain u, then we add uv and vy_1 to T_5. Note that this adds a path of length 2 that is pendant or isolated in T_5, so it remains an even forest. It remains to add the edges vy_2, vy_3, vy_4, vy_5 to the decomposition, which can be added to some of $T_1, ..., T_4$ by employing Lemma 3.1 twice on v.

- Now assume T_5 contains u. Then, one of $T_1, ..., T_4$, say T_1 w.l.o.g., does not contain u. Since $d \geq 2$, there are y_i, y_j that belong to different connected components of G'. Then we add uv, vy_i, vy_j to T_1. Note that T_1 remains a good cactus, since the three edges we have added are not involved in any cycle of T_1. Furthermore, these edges belong to a connected component with maximum degree at least 3, which is thus not an odd-length path. It remains three edges incident to v to be added to the decomposition, while only T_1 contains v this far. The three edges can then be added to T_2, T_3, T_4 using Lemma 4.5, and then Lemma 3.1 if needed. □

5. Conclusion

In this work, we have improved known upper bounds on the irregular chromatic index of some families of degenerate graphs. Similarly as for other existing proofs, our proofs involve decompositions into auxiliary structures that are to be further decomposed. As a consequence, most known bounds, including ours, are still far from the conjectured one in Conjecture 1.1, even for very particular classes of graphs.

A prime line of research for future work would thus be to aim at lowering those bounds further. As a first appealing case, we believe the case of 2-degenerate graphs is of interest, as many colouring problems tend to become easy for those graphs. Also, it is worth recalling that there is an intricate connection between 2-degenerate graphs and exceptional graphs, as all exceptional graphs are 2-degenerate. Planar graphs also sound interesting to investigate further for the same reasons, and because our bound in Theorem 4.6 leaves more space for improvement. A good compromise could be to consider outerplanar graphs, which are both 2-degenerate and planar. Improving our bound of 7 for these graphs would be an interesting first step.

Our result on 2-trees (Theorem 4.2) is also interesting in that regard, as 2-tree are 2-degenerate. We did not manage to come up with a proof of Conjecture 1.1 for these graphs, which might look surprising due to their very specific structure. Perhaps a way to progress towards all these concerns could be to consider the class of *maximal outerplanar graphs*, which form a subclass of 2-trees.

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