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ON THE QUANTITATIVE ISOPERIMETRIC INEQUALITY IN THE PLANE WITH THE BARYCENTRIC DISTANCE

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ABSTRACT. In this paper we study the following quantitative isoperimetric inequality in the plane: $\lambda_0^2(\Omega) \leq C\delta(\Omega)$ where δ is the isoperimetric deficit and λ_0 is the barycentric asymmetry. Our aim is to generalize some results obtained by B. Fuglede in [10]. For that purpose, we consider the shape optimization problem: minimize the ratio $\delta(\Omega)/\lambda_0^2(\Omega)$ in the class of compact connected sets and in the class of convex sets.

1. INTRODUCTION

In the last thirty years quantitative isoperimetric inequalities have received much attention in the litterature. Several distances between a set and the ball of same measure have been proposed to establish quantitative isoperimetric inequalities, where the isoperimetric deficit

$$\delta(\Omega) = \frac{P(\Omega) - P(B)}{P(B)}\,, \quad |B| = |\Omega|$$

majorizes a power of such a distance. In 1989, Fuglede [9] used the Hausdorff distance of a set Ω from the ball of same volume centered at the barycentre of Ω. He called it the uniform spherical deviation. He proved a series of inequalities for convex sets and nearly spherical sets, that is, star-shaped sets with respect to their barycentre (which may be taken to be 0) written as $\{y \in \mathbb{R}^n : y = tx(1+u(x)), x \in \mathbb{S}^{n-1}, t \in [0,1]\}$, where $u : \mathbb{S}^{n-1} \to \mathbb{R}$ positive Lipschitz, with $\|u\|_{L^\infty} \leq \frac{3}{20n}$ and $\|\nabla u\|_{L^\infty} \leq \frac{1}{2}$. The same inequalities hold for a more general family of sets, as showed in [12], where the minimum of the Hausdorff distance of a set Ω from the ball of same volume as Ω, among all balls of $\mathbb{R}^n$, is used.

L. E. Fraenkel proposed the now called Fraenkel asymmetry to enlarge the family of sets for which a quantitative isoperimetric inequality can hold:

$$\lambda(\Omega) = \inf_{y \in \mathbb{R}^n} \frac{|\Omega \Delta B_y|}{|\Omega|}\,, \quad |B_y| = |\Omega|\,.$$

This distance can be seen as an L^1 distance between Ω and any ball B_y, centered at $y \in \mathbb{R}^n$, with same measure as Ω. On the contrary, the Hausdorff distance is in some sense an L^∞ distance between sets. Many mathematicians studied quantitative isoperimetric inequalities with the Fraenkel asymmetry, establishing sharp inequalities (see for example [14], [15], [1], [7], [13], [4], [11], [6]) and even existence of an optimal set for the optimization problem of the ratio between the isoperimetric deficit and the square of the Fraenkel asymmetry (see [5] and [2]).

In the spirit of the Fraenkel asymmetry, Fuglede proposed in [10] the barycentric asymmetry, which is obviously much easier to compute than the Fraenkel asymmetry:

$$\lambda_0(\Omega) = \frac{|\Omega \Delta B_x|}{|\Omega|}$$

where B_x is a ball centered at the barycentre x of Ω and such that $|\Omega| = |B_x|$. We recall that the barycenter of a set Ω is defined as

$$\frac{1}{|\Omega|}\int_\Omega x\,dx\,.$$

Fuglede proved that there exists a positive constant (depending only on the dimension n) such that

$$\delta(\Omega) \geq C(n)[\lambda_0(\Omega)]^2\,, \quad \forall\, \Omega \subset \mathbb{R}^n \text{ convex}\,. \tag{1}$$

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In this paper we propose two kinds of generalizations of Fuglede's results [10], in dimension $n = 2$.

(1) We will be able to prove that there exists a strictly positive constant C such that inequality (1) holds for compact connected sets (see Section 3). As already observed by Fuglede, the connectedness assumption is necessary (cf. Remark 3.4).
(2) In the class of convex sets, we will prove the existence of a minimizer of the ratio $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ (see Section 4). We will also study the regularity of the optimal set in Section 5 and write different kinds of optimality conditions.

We would like to make some observations about the existence and the shape of an optimal set for the minimization of $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ in the plane.

- For the moment we are not able to prove the existence of an optimal set for the minimization of $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ among compact connected sets, as explained in Remark 3.3. However we formulate a conjecture about its shape.
- Among convex sets, our conjecture is that the optimal set is a stadium, the same found in [1] for the minimization of $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$. In Section 5 we will prove that if the optimal set is a stadium, then it is the minimizer of $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$.
- In [2] our aim was to compute the infimum of $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$. If one can compute the infimum of the ratio $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ then an estimate from below of the infimum of $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$ follows (since $\lambda(\Omega) \leq \lambda_0(\Omega)$). As observed by Fuglede [10], an estimate from below of the infimum of $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$ is given in Lemma 2.1 of [14]: one has $\frac{\delta(\Omega)}{\lambda^2(\Omega)} \geq 0.02$ for every $\Omega \subset \mathbb{R}^2$; see also [7] for an estimate in any dimension. However, we think that one should get a better estimate than the preceding ones (see our conjecture in Remark 3.3).

2. Preliminaries

We denote by E^c the complementary set of E. We denote by E^ε the $\varepsilon-$enlargement of E, that is, $\{x \in \mathbb{R}^2 : d(x, E) \leq \varepsilon\}$ where d is the euclidean distance. We collect here several results which will be useful in the sequel.

For the isoperimetric deficit we will consider the perimeter in the Minkowski sense:

$$P(\Omega) = \lim_{\varepsilon \to 0} (|\Omega^\varepsilon| - |\Omega|)/\varepsilon\,.$$

We will explain later in Remark 3.4 why this notion of perimeter is adapted to our problem and why the classical perimeter in the sense of De Giorgi is not suitable here.

Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Let $\mathcal{K}(\overline{\Omega})$ be the set of all compact connected subsets of $\overline{\Omega}$. We recall that the *Hausdorff distance* between two sets K_1 and K_2 in $\mathcal{K}(\overline{\Omega})$ is defined by

$$d_{\mathcal{H}}(K_1, K_2) := \max\left\{ \sup_{x \in K_1} dist(x, K_2),\ \sup_{x \in K_2} dist(x, K_1) \right\},$$

with the conventions $dist(x, \emptyset) = \mathrm{diam}(\Omega)$ and $\sup \emptyset = 0$. $K = \emptyset$ and $d_{\mathcal{H}}(\emptyset, K) = \mathrm{diam}(\Omega)$ if $K \neq \emptyset$.

We recall the classical Blaschke's Theorem (cfr. Theorem 2.2.3 in [18]):

Theorem 2.1. *Let $\{K_n\}$ be a sequence in $\mathcal{K}(\overline{\Omega})$. Then there exists a subsequence which converges in the Hausdorff metric to a set $K \in \mathcal{K}(\overline{\Omega})$.*

Theorem 2.2. *Let $\{K_n\}$ be a sequence of compact convex sets converging in the Hausdorff metric to a set K. Then K is compact and convex.*

We will also use the following semicontinuity result, analogous to the Golab Theorem for the Minkowski perimeter in the plane, proved by Henrot and Zucco in [19]:

Theorem 2.3. *Let $\{K_n\} \subset \mathbb{R}^2$ be a sequence contained in $\mathcal{K}(\overline{\Omega})$ converging to a set $K \in \mathcal{K}(\overline{\Omega})$ in the Hausdorff metric. Then*

$$P(K) \leq \liminf_{n \to \infty} P(K_n).$$

We will also use the following consequences of the Hausdorff convergence of sets (see Proposition 2.2.21 of [18]). Here χ_K denotes the characteristic function of a set K.

Proposition 2.4. *Let K_n, K in $\mathcal{K}(\overline{\Omega})$. If $K_n \to K$ in the Hausdorff metric, then*

(1) $|K_n \setminus K| \to 0$
(2) $\chi_K \geq \limsup_{n\to\infty} \chi_{K_n}$ *a.e.*
(3) *If $\chi_{K_n} \to \chi$ in $L^1(\Omega)$ (or even weak-star in (L^1, L^∞)), then $\chi \leq \chi_K$.*

We also recall a compactness result about the L^1 convergence of sets, that is, the L^1 convergence of characteristic functions of sets. P^{DG} denotes the De Giorgi perimeter.

Proposition 2.5. *Let K_n be a sequence of sets contained in an open set with finite measure, such that $P^{DG}(K_n) + |K_n|$ is uniformly bounded. Then there exists a set K such that $K_n \to K$ in L^1, up to a subsequence.*

For the proof, see [18].

Remark 2.6. *We also recall that $P^{DG}(K) \leq P(K)$ if $K \subset \mathbb{R}^2$ is a compact connected set, as remarked in* [19].

We recall the following result proved in [2]. There the notion of De Giorgi perimeter was used to define the isoperimetric deficit, but the same results hold with the notion of Minkowski perimeter:

Theorem 2.7. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of planar sets converging to a ball B in the sense that $|B\Delta\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then*

$$\inf\left\{\liminf_{\varepsilon\to 0} \frac{\delta(\Omega_\varepsilon)}{\lambda^2(\Omega_\varepsilon)}\right\} = \frac{\pi}{8(4-\pi)}.$$

We will use the following results in the minimization of $\frac{\delta(\Omega)}{\lambda_0(\Omega)^2}$ among convex sets.

Theorem 2.8. *There exists an optimal set for the minimization problem*

$$\inf_{K\subset\mathbb{R}^2 convex} \frac{\delta(K)}{\lambda^2(K)}\,.$$

The infimum is realized by an explicitely described stadium S and $\min_{K\subset\mathbb{R}^2 convex} \frac{\delta(K)}{\lambda^2(K)} = \frac{\delta(S)}{\lambda^2(S)} \approx 0.406$.

For the proof see [1]. See also Remark 5.5.

Remark 2.9. *In the sequel we will use the set D given by two balls of area $\frac{\pi}{2}$, connected by a segment whose length is equal 2. We will call it dumbbell. We observe that its Minkowski perimeter counts twice the length of the segment and therefore*

$$\frac{\delta(D)}{\lambda_0(D)^2} = \frac{\sqrt{2}-1}{4} + \frac{1}{2\pi} \approx 0.26 < \frac{\delta(S)}{\lambda_0^2(S)} \approx 0.406\,,$$

where S is the stadium of the above theorem.

We will use nearly spherical sets, studied by Fuglede in [9]. Let us consider the star-shaped sets $E = \{y \in \mathbb{R}^2 : y = tx(1+u(x)), x \in \mathbb{S}^1, t \in [0,1]\}$, with $u : \mathbb{S}^1 \to (0,+\infty)$ Lipschitz. Assume that the barycenter of E is 0 and $|E| = \pi$. Let B be the unit ball centered at 0. Then, it is straightforward to check:

$$|E\Delta B| = \frac{1}{2}\int_0^{2\pi} |(1+u)^2 - 1|\,,$$

$$\mathcal{H}^1(\partial E) = \int_0^{2\pi} \sqrt{(1+u)^2 + |u'|^2}\,,$$

$$\int_0^{2\pi} \cos\theta(1+u)^3 = 0 = \int_0^{2\pi} \sin\theta(1+u)^3\,, \quad \int_0^{2\pi} (1+u)^2 = 2\pi\,.$$

We will also use the the following result by Fuglede (Lemma 2.2 in [9]):

Theorem 2.10. *Let K_n be a sequence of convex compact sets of area π, converging in the Hausdorff metric to the unit ball B, written in the form $K_n = \{y \in \mathbb{R}^2 : y = tx(1+u_n(x)), x \in \mathbb{S}^1, t \in [0,1]\}$ where u_n is a Lipschitz function. The following estimate holds:*

$$\|u_n'\|_{L^\infty} \leq 2\frac{1+\|u_n\|_{L^\infty}}{1-\|u_n\|_{L^\infty}} \|u_n\|_{L^\infty}^{\frac{1}{2}}.$$

The following result has been proved in [17]:

Lemma 2.11. *Let R be a real function such that $\displaystyle\int_0^{2\pi} R(t)\sin(t) = \int_0^{2\pi} R(t)\cos(t) = 0$. Then a solution of*

$$\begin{cases} h'' + h = R \\ \\ h \ \ 2\pi\text{-periodic} \\ \\ \displaystyle\int_0^{2\pi} h(t)\sin(t) = \int_0^{2\pi} h(t)\cos(t) = 0 \end{cases}$$

is

$$h(\theta) = \int_{-\pi}^{\pi} G(t)R(\theta+t)dt\,, \qquad G(t) = \frac{1}{2}\left(1 - \frac{|t|}{\pi}\right)\sin|t|\,. \tag{2}$$

In the next result we recall the Riesz inequality, about the symmetric decreasing rearrangement for functions. We will consider functions defined on symmetric intervals $A \subset \mathbb{R}$ with respect to the origin. For a bounded function u we define

$$u^*(0) = ess\,sup(u);$$
$$u^*(s) = \inf\{t : |\{y : u(y) > t\}| < s\}\,, \qquad s > 0\,.$$

The following properties of the symmetric decreasing rearrangement are a direct consequence of the definition.

Proposition 2.12. *Let u, v be two bounded functions on a symmetric interval $A \subset \mathbb{R}$ with respect to the origin. Let $c \in \mathbb{R}$. Then*

$$\int_A u = \int_A u^* \tag{3}$$

$$(u+c)^* = u^* + c \tag{4}$$

$$u \leq v \ \ a.e. \Rightarrow u^* \leq v^* \ \ a.e. \tag{5}$$

The following Riesz inequality is classical, but generally stated for positive functions. For sake of completeness we prove the following version.

Theorem 2.13. *Let $f, g, h : [-T, T] \to \mathbb{R}$ be three bounded functions and let g be $2T$-periodic. Then*

$$\iint_{[-T,T]^2} f(t)g(t-\theta)h(\theta)dtd\theta \leq \iint_{[-T,T]^2} f^*(t)g^*(t-\theta)h^*(\theta)dtd\theta\,.$$

Proof. In [16] the inequality

$$\iint_{\mathbb{R}^2} f_1(t)f_2(t-\theta)f_3(\theta)dtd\theta \leq \iint_{\mathbb{R}^2} f_1^*(t)f_2^*(t-\theta)f_3^*(\theta)dtd\theta$$

was established for positive f_1, f_2, f_3 functions, vanishing at infinity. This inequality can be applied to the integral

$$I = \iint_{\mathbb{R}^2} [f(t)+c_f][g(t-\theta)+c_g][h(\theta)+c_h]dtd\theta\,,$$

where c_f, c_g, c_h are three constants such that $f(t)+c_f, g(t)+c_g, h(t)+c_h$ are positive and extended by 0 for $|t| \geq T$. Therefore $I \leq I^*$, where

$$I^* = \iint_{\mathbb{R}^2} [f(t)+c_f]^*[g(t-\theta)+c_g]^*[h(\theta)+c_h]^*dtd\theta = \iint_{[-T,T]^2} [f(t)^*+c_f][g^*(t-\theta)+c_g][h^*(\theta)+c_h]dtd\theta\,,$$

by (4). We observe that

$$I = \iint_{[-T,T]^2} f(t)g(t-\theta)h(\theta)dtd\theta + \iint_{[-T,T]^2} f(t)g(t-\theta)c_h dtd\theta + \iint_{[-T,T]^2} f(t)h(\theta)c_g dtd\theta + \iint_{[-T,T]^2} f(t)c_g c_h dtd\theta$$
$$+ \iint_{[-T,T]^2} c_f g(t-\theta)h(\theta)dtd\theta + \iint_{[-T,T]^2} c_f g(t-\theta)c_h dtd\theta + \iint_{[-T,T]^2} c_f c_g h(\theta)dtd\theta + \iint_{[-T,T]^2} c_f c_g c_h dtd\theta\,,$$

and

$$I^* = \iint_{[-T,T]^2} f^*(t)g^*(t-\theta)h^*(\theta)dtd\theta + \iint_{[-T,T]^2} f^*(t)g^*(t-\theta)c_h dtd\theta + \iint_{[-T,T]^2} f^*(t)h^*(\theta)c_g dtd\theta$$
$$+ \iint_{[-T,T]^2} f^*(t)c_g c_h dtd\theta + \iint_{[-T,T]^2} c_f g^*(t-\theta)h^*(\theta)dtd\theta + \iint_{[-T,T]^2} c_f g^*(t-\theta)c_h dtd\theta$$
$$+ \iint_{[-T,T]^2} c_f c_g h^*(\theta)dtd\theta + \iint_{[-T,T]^2} c_f c_g c_h dtd\theta\,.$$

By the periodicity of g and property (3), the i-th term of I is equal to the i-th term of I^*, for $i \geq 2$. We deduce that

$$\iint_{[-T,T]^2} f(t)g(t-\theta)h(\theta)dtd\theta \leq \iint_{[-T,T]^2} f^*(t)g^*(t-\theta)h^*(\theta)dtd\theta\,.$$

□

Theorem 2.14. *Let $K \subset \mathbb{R}^2$ be a compact connected set. Then $D(K) \leq \frac{1}{2}P(K)$, where $D(K)$ is its diameter and $P(K)$ is its Minkowski perimeter.*

Proof. We recall that $D(K) = D(coK)$, where coK is the convex hull of K. Since coK is compact and convex, then $D(coK) \leq \frac{1}{2}P^{DG}(coK)$ (see for exemple [23]), where P^{DG} denotes the general notion of perimeter in the sense of De Giorgi. Now, by [8], $P^{DG}(coK) \leq P^{DG}(K)$. Finally, by section 2 of [19], one has $P^{DG}(K) \leq P(K)$. □

3. Minimization of $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ within compact connected sets

In this section, we consider compact connected sets of positive measure (in order the shape functionals δ and λ_0 be well-defined). We are going to prove the following result.

Theorem 3.1. *There exists $C > 0$ such that the inequality $\lambda_0^2(K) \leq C\delta(K)$ holds for any connected compact set $K \subset \mathbb{R}^2$.*

In the proof we will use the following simple lemma:

Lemma 3.2. *Let B_1 and B_2 be two balls such that their area equals π and the distance between their centers equals $a \leq 2$. Then*

$$d_{L^1}(B_1, B_2) = 4a \arcsin\left(\frac{a}{2}\right) + 2a\sqrt{1 - \frac{a^2}{4}} = 4a + o(a).$$

Proof. Up to a rotation we can assume that $B_1 = B_{(0,0)}$ and $B_2 = B_{(a,0)}$, where $B_{(a,0)}$ denote the ball of area π centered at $(a, 0)$, $0 \leq a \leq 2$. Let $\tau = \arcsin(a/2)$. The quantity $d_{L^1}(B_{(0,0)}, B_{(a,0)})$ is equal to 4 times the area of the domain whose boundary is composed by the following three arcs:

(1) $(a + \cos t, \sin t), t \in (0, \alpha), \alpha = \frac{\pi}{2} + \tau$;
(2) $(\cos t, \sin t), t \in (0, \beta), \beta = \frac{\pi}{2} - \tau$;
(3) $(t, 0), t \in (1, 1 + a)$.

By Green's theorem, the area of this domain is given by

$$\frac{1}{2}\int_0^{\frac{\pi}{2}+\tau} (a + \cos\tau)\cos\tau + \sin^2 t - \frac{1}{2}\int_0^{\frac{\pi}{2}-\tau} 1 + 0 = \tau + \frac{a}{2}\cos\tau\,.$$

As $a \to 0$, $d_{L^1}(B_{(0,0)}, B_{(0,a)}) \approx 4a$. □

We are now going to prove Theorem 3.1.

Proof. Let K_n be a minimizing sequence, that is, $\dfrac{\delta(K_n)}{\lambda_0^2(K_n)} \to \inf_E \dfrac{\delta(E)}{\lambda_0^2(E)}$. Without loss of generality, we can assume that all the sets K_n have area π. By Theorem 2.8 one has

$$\frac{\delta(K_n)}{\lambda_0^2(K_n)} \le \frac{\delta(S)}{\lambda_0^2(S)} = \frac{\delta(S)}{\lambda^2(S)} \approx 0.406\,,$$

where S denotes the stadium of Theorem 2.8. Since $\lambda_0(E) \le 2$ for any set E, we get

$$P(K_n) \le 16.6\,. \tag{6}$$

Therefore the sets K_n are all contained in a fixed ball, since they are connected and their perimeter is uniformly bounded. Theorem 2.1 gives us the existence of a connected compact set towards which K_n converges in the Hausdorff metric. Now, there can be two possibilities:

(1) K_n converges to a ball B in the Hausdorff metric;
(2) K_n converges to a set K different from a ball in the Hausdorff metric.

In both cases we are going to prove that $\liminf_{n\to\infty} \dfrac{\delta(K_n)}{\lambda_0^2(K_n)} > 0$. This will imply our result.

(1) In this first case we can assume that $\delta(K_n) \to 0$ and $\lambda(K_n) = 2\varepsilon_n \to 0$, as $n \to \infty$. By Theorem 2.7 one has

$$\delta(K_n) \ge 0.45 \cdot 4\varepsilon_n^2\,.$$

We are now going to prove that

$$|\lambda(K_n) - \lambda_0(K_n)| \le \frac{4A}{\pi}\varepsilon_n \tag{7}$$

for some explicit constant $A > 0$. Therefore

$$\frac{\delta(K_n)}{\lambda_0^2(K_n)} \ge \frac{\delta(K_n)}{\left(\lambda(K_n) + \frac{4A}{\pi}\varepsilon_n\right)^2} \ge \frac{1.8}{\left(2 + \frac{4A}{\pi}\right)^2}\,,$$

which gives the desired estimate in the case of a minimizing sequence K_n converging to a ball in the Hausdorff metric.

To prove (7) it is sufficient to find a positive constant A such that

$$|G_n - F_n| \le A\varepsilon_n \tag{8}$$

where G_n is the barycentre of K_n and F_n is the centre of an optimal ball for $\lambda(K_n)$. Indeed, by the triangle inequality,

$$d_{L^1}(K_n, B_{G_n}) \le d_{L^1}(K_n, B_{F_n}) + d_{L^1}(B_{G_n}, B_{F_n})\,,$$

where B_{F_n} is an optimal ball for the Fraenkel asymmetry. This inequality together with (8) and Lemma 3.2 imply (7).

We are now going to prove (8), which will end the proof of this case. We can always assume that an optimal ball for the Fraenkel asymmetry is centered in 0. We are now going to estimate $x_1^{G_n} = \dfrac{1}{\pi}\displaystyle\int_{K_n} x_1 dx_1 dx_2$. Writing the last integral on $(K_n \setminus B) \cup B \setminus (B \setminus K_n)$ and recalling that $\dfrac{1}{\pi}\displaystyle\int_B x_1 dx_1 dx_2 = 0$, we get

$$|x_1^{G_n}| = \frac{1}{\pi}\left|\int_{K_n\setminus B} x_1 dx_1 dx_2 - \int_{B\setminus K_n} x_1 dx_1 dx_2\right| \le \frac{1}{\pi}\int_{K_n\setminus B} |x_1|\, dx_1 dx_2 + \frac{1}{\pi}\int_{B\setminus K_n} |x_1|\, dx_1 dx_2\,.$$

By using Theorem 2.14 to estimate the first of the last two terms, we get

$$|x_1^{G_n}| \le \frac{8.3\varepsilon_n}{\pi} + \frac{\varepsilon_n}{\pi} = \frac{9.3\varepsilon_n}{\pi}$$

since $P(K_n) \le 16.6$ as observed above. The same estimate can be obtained for $|x_2^{G_n}|$. Therefore $|G_n| \le \sqrt{2}\frac{9.3\varepsilon_n}{\pi}$ and (8) is proved.

(2) We are going to analyse the case where K_n converges to a connected compact set K (in the Hausdorff metric) different from a ball.

Since the sets K_n are connected and their perimeter is uniformly bounded, they are all included in a ball. Therefore there exists a set $\hat{K}$ such that $\chi_{K_n} \to \chi_{\hat{K}}$ in L^1 and $|\hat{K}| = \pi$, by Proposition 2.5 and Remark 2.6. We are going to prove that $\hat{K} = K$ (we note that the only Hausdorff convergence does not allow us to say that $|K| = \pi$).

By Proposition 2.4 (3) applied to K_n^c et K^c, we have

$$\chi_{\hat{K}} \leq \chi_K \,, \tag{9}$$

since $\chi_{K_n} \to \chi_{\hat{K}}$ in L^1. Therefore

$$|K| \geq \pi = |\hat{K}| \,. \tag{10}$$

Since $K_n \to K$ in the Hausdorff metric, $K \subset K_n \subset K_n^\varepsilon$. By the definition of the Minkowski perimeter, we have, for every $\varepsilon > 0$,

$$|K| \leq |K_n^\varepsilon| \leq |K_n| + \varepsilon P(K_n) + o(\varepsilon) = \pi + \varepsilon P(K_n) + o(\varepsilon) \,. \tag{11}$$

Since $P(K_n)$ are uniformly bounded, inequality (11) yields $|K| \leq \pi$. This inequality and (10) imply $|K| = \pi$. We deduce that $K = \hat{K}$ a.e. from (9).

By Theorem 2.3 $P(K) \leq \liminf_{n\to\infty} P(K_n)$. Since $K_n \to K$ in L^1, as $n \to \infty$, we have $\lambda_0(K_n) \to \lambda_0(K)$. Indeed, by the triangle inequality,

$$\pi|\lambda_0(K_n) - \lambda_0(K)| \leq d_{L^1}(K_n, K) + d_{L^1}(B, B_n) \,.$$

The first term in the right hand side tends to 0, as $n \to \infty$ by the L^1 convergence. The second one tends to 0 by Lemma 3.2, since

$$\left|x_1^{G_n} - x_1^G\right| \leq \frac{1}{\pi} \int_{K_n \setminus K} |x_1| dx_1 dx_2 \,,$$

where the last term tends to 0, since the diameter of K_n is uniformly bounded and $|K_n \setminus K| \to 0$, as $n \to \infty$. The same holds for the second coordinate. Therefore

$$\liminf_{n\to\infty} \frac{\delta(K_n)}{\lambda_0^2(K_n)} \geq \frac{\delta(K)}{\lambda_0^2(K)} > 0 \,.$$

□

Remark 3.3. *We conjecture that the infimum of $\frac{\delta}{\lambda_0^2}$ within the connected sets is realized by the dumbbell described in Remark 2.9.*

In the case where the minimizing sequence K_n converges to the ball (in the Hausdorff metric), we get an estimate from below of $\liminf_{n\to\infty} \dfrac{\delta(K_n)}{\lambda_0^2(K_n)}$, but our estimate is lower than the value of $\dfrac{\delta}{\lambda_0^2}$ computed on the dumbbell. This is the reason why we are not able to prove the existence of an optimal set for this problem.

We were not able to find a sort of rearrangement, as in [2], to exclude sequences converging to a ball with the aim to prove that there exists minimizer for $\dfrac{\delta(K)}{\lambda_0^2(K)}$, among connected compacts sets $K \subset \mathbb{R}^2$.

Remark 3.4. *The assumption that Ω is connected is necessary. Indeed one can construct the following sequence of non connected sets Ω_n, given by the union of the disk centered in $(2, 0)$, of radius $R_n = 1 - \frac{1}{n}$, and the disk centered in $\left(-\frac{2(n-1)^2}{2n-1}, 0\right)$, of radius $r_n = \sqrt{\frac{2n-1}{n^2}}$. It is easy to check that $|\Omega_n| = \pi$, the barycentre of Ω_n is the origin, $\delta(\Omega_n) = R_n + r_n - 1 \to 0$ as $n \to \infty$ and $\lambda_0(\Omega_n) = 2$. Thus*

$$\frac{\delta(\Omega_n)}{\lambda_0^2(\Omega_n)} \to 0, \;\; n \to \infty \,.$$

This exemple shows why the classical De Giorgi perimeter is not suitable for the barycentric asymmetry. Indeed, the set $\tilde{\Omega}_n$ obtained by connecting the above two balls by a long segment would have the same De

Giorgi perimeter as the perimeter of Ω_n, since the De Giorgi perimeter of the long segment would be 0. Thus

$$\frac{\delta(\tilde{\Omega}_n)}{\lambda_0^2(\tilde{\Omega}_n)} \to 0, \ \ n \to \infty\,.$$

On the contrary, for the Minkowski perimeter, $\delta(\Omega_n) \to +\infty$, since one has to consider twice the length of the long segment.

Remark 3.5. *The notion of Minkowski perimeter is central in the second part of the above proof, in inequality (11), to prove that $|K| = \pi$.*

4. Minimisation of $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ within compact convex sets

In this section we prove the following theorem :

Theorem 4.1. *There exists an optimal set of* $\inf_{\Omega convex \subset \mathbb{R}^2} \dfrac{\delta(\Omega)}{\lambda_0^2(\Omega)}$.

Proof. Let K_n be a minimizing sequence of convex compact sets. The uniform bound on $\frac{\delta(K_n)}{\lambda_0^2(K_n)}$ and the definition of λ_0 imply that $\delta(K_n)$ is uniformly bounded. Therefore the sets K_n are all contained in a fixed ball, since they are convex and they perimeter is uniformly bounded.

Theorems 2.1 and 2.2 give us the existence of a convex compact set towards which K_n converges in the Hausdorff metric. Now, as in the previous theorem, there can be two possibilities:

(1) K_n converges to a ball B in the Hausdorff metric;
(2) K_n converges to a set K different from a ball in the Hausdorff metric.

In the next theorem we are going to analyse the first case, proving that $\liminf_{n\to\infty} \frac{\delta(K_n)}{\lambda_0^2(K_n)} > 0.406$ which is the value of $\frac{\delta(S)}{\lambda_0^2(S)}$ where S is the stadium of Theorem 2.8. This means that a minimizing sequence cannot converge to a ball. Therefore the only possibility for a minimizing sequence is a second one. In this case we can prove that K is a minimizer with the same arguments as in the proof of case (2) of Theorem 3.1. □

Theorem 4.2. *Let K_n be a sequence of convex compact sets converging to a ball in the Hausdorff metric. Then $\liminf_{n\to\infty} \frac{\delta(K_n)}{\lambda_0^2(K_n)} > 0.41$.*

If a set E has barycenter in 0, it can be written in polar coordinates with respect to 0, as

$$E = \{y \in \mathbb{R}^2 : y = tx(1 + u(x)), x \in \mathbb{S}^1, t \in [0,1]\}\,, \tag{12}$$

with u Lipschitz function. Then we are interested in minimizing the functional $\frac{\delta(E)}{\lambda_0^2(E)}$ which can be written as a function of u defining E (see the computations in Section 2):

$$J(u) = \frac{\pi}{2} \frac{\displaystyle\int_0^{2\pi} \left[\sqrt{(1+u)^2 + u'(\theta)^2} - 1\right] d\theta}{\displaystyle\left[\frac{1}{2}\int_0^{2\pi} |(1+u)^2 - 1| d\theta\right]^2}$$

with the constraints of area and barycentre in 0:

(NL1) $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} (1+u)^2 d\theta = 1$;
(NL2) $\displaystyle\int_0^{2\pi} \cos(\theta)[1+u(\theta)]^3 d\theta = 0 = \int_0^{2\pi} \sin(\theta)[1+u(\theta)]^3 d\theta$;
(NL3) $u(0) = u(2\pi)$.

This leads to a complicated problem in the calculus of variations. Thus, our strategy will consist in replacing this problem by a simpler one which can be seen as a sort of of linearization:

$$m = \inf_{u \in H^1(0,2\pi)} \frac{\displaystyle\int_0^{2\pi} [(u')^2 - u^2] d\theta}{\displaystyle\left[\int_0^{2\pi} |u| d\theta\right]^2} \tag{13}$$

with the constraints:

(L1) $\int_0^{2\pi} u\,d\theta = 0$

(L2) $\int_0^{2\pi} u\cos(\theta)\,d\theta = 0 = \int_0^{2\pi} \sin(\theta)u\,d\theta$

(L3) $u(0) = u(2\pi)$.

Proposition 4.3. *Let $0 < \varepsilon < 1/24$. Let*

$$m_\varepsilon := \inf\{J(u), \|u\|_{L^\infty} = \varepsilon,\ u \text{ satisfies (NL1),(NL2),(NL3)}\}. \tag{14}$$

Then

$$\liminf_{\varepsilon\to 0} m_\varepsilon \geq \frac{\pi}{4} m. \tag{15}$$

Proof. The idea of the proof is the following:

(1) we replace the optimization problem (14) by a new one (problem (19)) which yields a smaller value;
(2) we prove that problem (19) has a minimizer u_ε;
(3) we prove that $v_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ (which is on the unit sphere of L^∞) is bounded in H^1 and converges uniformly to some function v_0;
(4) by passing to the limit as $\varepsilon \to 0$, we prove that v_0 is a test function for the optimization problem (13) whence the desired inequality.

In the sequel of the proof C will denote a constant independent of ε.

Step 1. Since $\int_0^{2\pi}(2u + u^2) = 0$ by (NL1), the minimization of $J(u)$ is equivalent to the minimization (with the same constraints) of

$$J_1(u) = \frac{\pi}{2}\frac{\int_0^{2\pi}\left[\sqrt{(1+u)^2 + u'^2} - 1\right]d\theta - \int_0^{2\pi}(u + u^2/2)d\theta}{\left[\frac{1}{2}\int_0^{2\pi}|(1+u)^2 - 1|d\theta\right]^2}.$$

We are going to estimate the numerator of J_1, that is,

$$\int_0^{2\pi}\left[\sqrt{1 + 2u + u^2 + u'^2} - 1 - (u + u^2/2)\right]d\theta$$

from below. We will assume that $\varepsilon \leq 1/24$ and

$$\|u'\|_{L^\infty} \leq 3\sqrt{\varepsilon} \tag{16}$$

(this is possible by the estimate $\|u\|_{L^\infty} \leq \varepsilon$ and Theorem 2.10). We first observe that

$$\text{for } |\rho| \leq \frac{1}{2},\ \sqrt{1+\rho}\ \geq 1 + \frac{\rho}{2} - \frac{\rho^2}{8} + \frac{\rho^3}{16} - \frac{\rho^4}{8}. \tag{17}$$

By (16), one has the estimate $|2u + u^2 + u'^2| \leq 2\varepsilon + \varepsilon^2 + 9\varepsilon \leq 12\varepsilon \leq \frac{1}{2}$. We can apply (17) to $2u + u^2 + u'^2$ to infer

$$\sqrt{(1+u)^2 + u'^2} - 1 \geq u + \frac{1}{2}u'^2 - \frac{1}{8}u'^4 - \frac{1}{2}uu'^2 + C\varepsilon^3.$$

Therefore, the numerator of J_1 is estimated from below by $\frac{1}{2}\int_0^{2\pi}\left[u'^2 - u^2 - \frac{1}{4}u'^4 - uu'^2\right] + C\varepsilon^3$.

We are going to estimate the denominator of J_1. Since $|2u + u^2| \leq (2+\varepsilon)|u|$ one has that

$$\left[\frac{1}{2}\int_0^{2\pi}|(1+u)^2 - 1|\right]^2 \leq \left(1 + \frac{\varepsilon}{2}\right)^2\left[\int_0^{2\pi}|u|\right]^2.$$

Therefore, under the constraints (NL1), (NL2), (NL3) one has $J(u) = J_1(u) \geq J_2(u)$, where

$$J_2(u) = \frac{\pi}{2}\frac{\frac{1}{2}\int_0^{2\pi}\left[u'^2 - u^2 - \frac{1}{4}u'^4 - uu'^2\right] + C\varepsilon^3}{\left(1 + \frac{\varepsilon}{2}\right)^2\left[\int_0^{2\pi}|u|\right]^2}. \tag{18}$$

Defining

$$m'_\varepsilon := \inf\{J_2(u), \|u\|_{L^\infty} = \varepsilon,\ u \text{ satisfies (NL1),(NL2),(NL3)}\}\,, \tag{19}$$

we have $m_\varepsilon \geq m'_\varepsilon$.

Step 2. We prove that problem (19) has a minimizer u_ε (we notice that here ε is fixed).

Let u_n^ε be a minimizing sequence for J_2. We know that $\|u_n^\varepsilon\|_{L^\infty} = \varepsilon$ and $\|(u_n^\varepsilon)'\|_{L^\infty} \leq 3\sqrt{\varepsilon}$ for every n (by (16)). Therefore $u_n^\varepsilon \to u_\varepsilon$ weakly in $W^{1,\infty}(0,2\pi)$ and uniformly in $(0,2\pi)$, as $n \to \infty$.

To pass to the limit, as $n \to \infty$, in $J_2(u_n^\varepsilon)$, we need to study the integral in the numerator of J_2. We will use a standard argument in the calculus of variations. For small $|s|$ and $|\xi|$ (recall that $|s| \leq \varepsilon \leq \frac{1}{24}$ and $|\xi| \leq 3\sqrt{\varepsilon}$), the function $j(s,\xi) = \xi^2(1-s) - s^2 - \frac{1}{4}\xi^4$, defining the integrand, is convex with respect to ξ. This gives

$$j(s,\xi) \geq j(s,\eta) + \nabla_\xi j(s,\eta)\cdot(\xi - \eta)\,,$$

where $\nabla_\xi j(s,\xi) = 2\xi(1-s) - \xi^3$. Therefore

$$\int_0^{2\pi} j(u_n^\varepsilon, (u_n^\varepsilon)') \geq \int_0^{2\pi} j(u_n^\varepsilon, u'_\varepsilon) + \int_0^{2\pi} \nabla_\xi j(u_n^\varepsilon, u'_\varepsilon)\cdot((u_n^\varepsilon)' - u'_\varepsilon). \tag{20}$$

The uniform convergence of u_n^ε to u_ε implies that $j(u_n^\varepsilon, u'_\varepsilon)$ converges in $L^1(0,2\pi)$ to $j(u^\varepsilon, u'_\varepsilon)$, as $n \to \infty$. Moreover $(u_n^\varepsilon)' - u'_\varepsilon$ converges weakly to 0 in $L^\infty(0,2\pi)$ and $\|\nabla_\xi j(u_n^\varepsilon, u'_\varepsilon)\|_{L^\infty}$ is bounded uniformly in n and ε. Passing to the lim inf in (20), we get

$$\liminf_{n\to+\infty} \int_0^{2\pi} j(u_n^\varepsilon, (u_n^\varepsilon)') \geq \int_0^{2\pi} j(u^\varepsilon, u'_\varepsilon)\,.$$

We deduce the existence of a minimizer u_ε for J_2.

Step 3. We define a new sequence, renormalizing u_ε (the minimizer of problem (19)):

$$v_\varepsilon = \frac{u_\varepsilon}{\varepsilon}\,.$$

We are going to prove some estimates on v_ε which will allow us to compute the limit of v_ε, as $\varepsilon \to 0$. The estimates on v_ε will be established thanks to the test function w_ε, that we are about to define.

Let $a_\varepsilon = \pi/4 - \varepsilon\pi/6$ and $b_\varepsilon = 3\pi/4 - \varepsilon\pi/6$. Let w^ε be the function, piecewise affine, π-periodic, defined by

$$w^\varepsilon(t) = \begin{cases} \varepsilon\frac{t}{a_\varepsilon} & t \in [0, a_\varepsilon]\,, \\ -2\varepsilon\frac{t-a_\varepsilon}{b_\varepsilon - a_\varepsilon} + \varepsilon & t \in [a_\varepsilon, b_\varepsilon]\,, \\ -\varepsilon\frac{\pi - t}{\pi - b_\varepsilon} & t \in [b_\varepsilon, \pi]\,. \end{cases}$$

It is easy to see that w^ε satisfies $\|w^\varepsilon\|_{L^\infty} = \varepsilon$ and (NL3). It also satisfies (NL2), since $(1+w^\varepsilon)^3$ is $\pi-$periodic and therefore orthogonal to sine and cosine. We are going to check (NL1), that is,

$$2\int_0^\pi w^\varepsilon + \int_0^\pi w^{\varepsilon 2} = 0.$$

Elementary calculations provide:

$$\int_0^\pi w^\varepsilon = \frac{1}{2}a_\varepsilon\varepsilon - \frac{1}{2}(\pi - b_\varepsilon)\varepsilon, \quad \int_0^\pi w^{\varepsilon 2} = \pi\frac{\varepsilon^2}{3}\,.$$

Therefore (NL1) is satisfied as soon as $a_\varepsilon + b_\varepsilon - \pi = -\varepsilon/3$ which is true with our choice of a_ε and b_ε. We also remark that

$$\int_0^{2\pi} w^{\varepsilon\prime 2} \leq C\varepsilon^2\,, \quad \int_0^{2\pi} w^{\varepsilon 2} \leq C\varepsilon^2\,, \quad \int_0^{2\pi} w^{\varepsilon\prime 4} \leq C\varepsilon^4\,, \quad \int_0^{2\pi} w^\varepsilon w^{\varepsilon\prime 2} \leq C\varepsilon^3$$

and

$$\int_0^{2\pi} |w^\varepsilon| = \varepsilon\pi \geq \frac{\varepsilon}{2}\,.$$

These estimates imply that $J_2(w^\varepsilon) \le C$ for every ε. Therefore $J_2(u_\varepsilon) \le J_2(w^\varepsilon) \le C$ which yields

$$\int_0^{2\pi} (u_\varepsilon)'^2 \le \int_0^{2\pi} u_\varepsilon^2 + \frac{1}{4}(u_\varepsilon)'^4 + u_\varepsilon (u_\varepsilon)'^2 + C\varepsilon^3 + C\left[\int_0^{2\pi} |u_\varepsilon|\right]^2 .$$

We deduce that $\displaystyle\int_0^{2\pi} (u_\varepsilon)'^2 \le C\varepsilon^2$.

From the definition of v_ε we have $\|v_\varepsilon\|_{L^\infty} = 1$. By the above estimate we get

$$\int_0^{2\pi} v_\varepsilon'^2 \le C. \tag{21}$$

The sequence v_ε is bounded in $H^1(0, 2\pi)$ and, as $\varepsilon \to 0$, up to some sequence, v_ε converges weakly in $H^1(0, 2\pi)$ and uniformly to some v_0. Using (16) we deduce that $\|v_\varepsilon'\|_{L^\infty} \le \frac{3}{\sqrt{\varepsilon}}$, and then using (21), we have

$$\int_0^{2\pi} v_\varepsilon'^4 \le \frac{9}{\varepsilon}\int_0^{2\pi} v_\varepsilon'^2 \le \frac{C}{\varepsilon}.$$

Moreover

$$\left|\int_0^{2\pi} v_\varepsilon v_\varepsilon'^2\right| \le \int_0^{2\pi} v_\varepsilon'^2 \le C$$

by (21) and $\|v_\varepsilon\|_{L^\infty} = 1$.

Step 4. We now prove that the function v_0 found in *Step 3* is a test function for the optimization problem (13). This will allow us to prove the statement of this Proposition.

We observe that, by (19) and the definition of v_ε, we have

$$m_\varepsilon' = \frac{\pi}{2}\frac{\dfrac{1}{2}\displaystyle\int_0^{2\pi}\left[v_\varepsilon'^2 - v_\varepsilon^2 - \frac{\varepsilon^2}{4}v_\varepsilon'^4 - \varepsilon v_\varepsilon v_\varepsilon'^2\right] + C\varepsilon}{\left(1+\dfrac{\varepsilon}{2}\right)^2\left[\displaystyle\int_0^{2\pi}|v_\varepsilon|\right]^2}. \tag{22}$$

Passing to the limit in (22), we get

$$\liminf_{\varepsilon\to 0} m_\varepsilon' \ge \frac{\pi}{2}\frac{\dfrac{1}{2}\displaystyle\int_0^{2\pi}\left[v_0'^2 - v_0^2\right]}{\left[\displaystyle\int_0^{2\pi}|v_0|\right]^2}.$$

On the other hand, passing to the limit in (NL1) and (NL2), we see that v_0 satisfies (L1) and (L2) and therefore is an admissible test function for the optimization problem (13). For exemple, (NL1) is equivalent to $\displaystyle\int_0^{2\pi}(u_\varepsilon^2 + 2u_\varepsilon) = 0$. This implies that $0 \le \displaystyle\int_0^{2\pi} u_\varepsilon^2 = -2\int_0^{2\pi} u_\varepsilon \le C\varepsilon^2$ by the estimate $\|u_\varepsilon\|_{L^\infty} \le \varepsilon$. Therefore, by the definition of v_ε, one has $\displaystyle\int_0^{2\pi} v_\varepsilon \to 0$, as $\varepsilon \to 0$, which gives $\displaystyle\int_0^{2\pi} v_0 = 0$.

The inequality $\liminf_{\varepsilon\to 0} m_\varepsilon \ge \liminf_{\varepsilon\to 0} m_\varepsilon' \ge \frac{\pi}{4} m$ follows.

□

Lemma 4.4. *Let m be defined by (13). If u is a minimizer, then the function $u_0 = \dfrac{u}{m}$ satisfies*

$$\int_{-\pi}^{\pi} |u_0(\theta)|d\theta = \iint_{[-\pi,\pi]^2} sign(u_0)(t)H(\theta - t)sign(u_0)(\theta)dtd\theta\,, \tag{23}$$

where $H(x) = -G(x) + \dfrac{1}{2\pi} + \dfrac{1}{4\pi}\cos(x)$, with G defined by (2). Moreover

$$m = \frac{1}{\displaystyle\int_0^{2\pi}|u_0|}\,. \tag{24}$$

Proof. The existence of a minimizer of the functional in (13) follows easily from the direct methods of the calculus of variations. One can assume that $\int_0^{2\pi} |u| = 1$ by homogeneity. The Euler equation of this problem is

$$-u'' - u = m \cdot sgn(u) + \lambda_0 + \lambda_1 \cos\theta + \lambda_2 \sin\theta\,.$$

By integrating on $(0, 2\pi)$ one finds

$$m \int_0^{2\pi} sgn(u(\theta))d\theta + 2\pi\lambda_0 = 0\,.$$

By multiplying The Euler equation by $\cos\theta$ and $\sin\theta$ and then integrating on $(0, 2\pi)$ one gets, respectively

$$m \int_0^{2\pi} sgn(u(\theta)) \cos\theta d\theta + \pi\lambda_1 = 0\,, \quad m \int_0^{2\pi} sgn(u(\theta)) \sin\theta d\theta + \pi\lambda_2 = 0\,. \tag{25}$$

The function $u_0 = \dfrac{u}{m}$ satisfies

$$u_0'' + u_0 = -sgn(u_0) - \tilde\lambda_0 - \tilde\lambda_1 \cos\theta - \tilde\lambda_2 \sin\theta\,, \tag{26}$$

where

$$\frac{1}{2\pi} \int_0^{2\pi} sgn(u_0(t))dt = -\tilde\lambda_0\,,\ \frac{1}{\pi} \int_0^{2\pi} sgn(u_0(t)) \cos t dt = -\tilde\lambda_1\,,\ \frac{1}{\pi} \int_0^{2\pi} sgn(u_0(t)) \sin t dt = -\tilde\lambda_2$$

and $m = \frac{1}{\int_0^{2\pi} |u_0|}$. It is easy to see that the function at the right hand side of equation (26) is orthogonal to sine and cosine by (25). Therefore Lemma 2.11 applies with

$$R(\theta) = -sgn(u_0(\theta)) + \frac{1}{\pi} \int_{-\pi}^{\pi} sgn(u_0(s)) \left[\frac{1}{2} + \cos(s - \theta)\right] ds$$

and gives

$$u_0(\theta) = \int_{-\pi}^{\pi} G(t)R(\theta + t)dt$$

with G defined by (2), that is, $G(t) = \dfrac{1}{2}\left(1 - \dfrac{|t|}{\pi}\right) \sin|t|$. We observe that

$$u_0(\theta) = -\int_{-\pi}^{\pi} G(t)sgn(u_0(\theta + t))dt + \frac{1}{\pi} \int_{-\pi}^{\pi} sgn(u_0(s)) \int_{-\pi}^{\pi} G(t) \left[\frac{1}{2} + \cos(s - \theta - t)\right] dt\, ds\,.$$

Since

$$\int_{-\pi}^{\pi} G(t) \left[\frac{1}{2} + \cos(s - \theta - t)\right] dt = \frac{1}{2} + \frac{1}{4} \cos(s - \theta)\,,$$

one has

$$u_0(\theta) = \int_{-\pi}^{\pi} sgn(u_0(s)) \left[-G(s - \theta) + \frac{1}{2\pi} + \frac{1}{4\pi} \cos(s - \theta)\right] ds\,.$$

□

We can now prove Theorem 4.2.

Proof. As we already observed, we can write K_n in polar coordinates (see (12)); this implies that

$$\liminf_{n\to\infty} \frac{\delta(K_n)}{\lambda_0^2(K_n)} = \liminf_{\varepsilon\to 0} m_\varepsilon$$

where m_ε is defined by (14). By Proposition 4.3 it is sufficient to prove the estimate

$$\frac{\pi}{4} m > 0.41\,, \tag{27}$$

where m is defined by (13). To do that, we are going to estimate the $L^1(-\pi, \pi)$ norm of u_0, thanks to formula (24). Recall that

$$\int_{-\pi}^{\pi} |u_0(\theta)|d\theta = \iint_{[-\pi,\pi]^2} sign(u_0)(t)H(\theta - t)sign(u_0)(\theta)dtd\theta$$

by (23). By applying Theorem 2.13, one has

$$\int_{-\pi}^{\pi} |u_0(\theta)|d\theta \leq \iint_{[-\pi,\pi]^2} H^*(\theta - t)[\chi_{(-a,a)}(\theta) - \chi_{(-a,a)^c}(\theta)][\chi_{(-a,a)}(t) - \chi_{(-a,a)^c}(t)]dtd\theta \tag{28}$$

where $2a = |I| = 2\pi - |J|$, with $I = \{\theta \in [-\pi, \pi] : u_0(\theta) \geq 0\}$ and $J = \{\theta \in [-\pi, \pi] : u_0(\theta) \leq 0\}$. We are going to analyse separately the four terms of the integral in right hand side of (28). We observe that H^* is even. Let F^* be a primitive of H^*, nul at 0.
The first term gives

$$\iint_{[-\pi,\pi]^2} H^*(\theta - t)\chi_{(-a,a)}(\theta)\chi_{(-a,a)}(t)dtd\theta = \int_{-a}^{a} d\theta \int_{-a}^{a} H^*(\theta - t)dt = 2\int_0^{2a} F^*(s)ds\,.$$

The second term gives

$$\iint_{[-\pi,\pi]^2} H^*(\theta - t)\chi_{(-a,a)^c}(\theta)\chi_{(-a,a)^c}(t)dtd\theta =$$

$$= \int_{-\pi}^{-a} d\theta \int_{-\pi}^{-a} H^*(\theta - t)dt + \int_{-\pi}^{-a} d\theta \int_{a}^{\pi} H^*(\theta - t)dt + \int_{a}^{\pi} d\theta \int_{-\pi}^{-a} H^*(\theta - t)dt + \int_{a}^{\pi} d\theta \int_{a}^{\pi} H^*(\theta - t)dt$$

$$= 4\int_0^{\pi-a} F^*(s)ds + 2\int_{a+\pi}^{2\pi} F^*(s)ds - 2\int_{2a}^{\pi+a} F^*(s)ds\,.$$

The third term gives

$$\int_{-a}^{a} d\theta \int_{(-a,a)^c} H^*(\theta - t)dt = \int_{-a}^{a} d\theta \int_{-\pi}^{-a} H^*(\theta - t)dt + \int_{-a}^{a} d\theta \int_{a}^{\pi} H^*(\theta - t)dt$$

$$= -2\int_0^{2a} F^*(s)ds + 2\int_{\pi-a}^{\pi+a} F^*(s)ds\,.$$

The fourth term gives

$$\int_{(-a,a)^c} d\theta \int_{(-a,a)} H^*(\theta - t)dt = \int_{-\pi}^{-a} d\theta \int_{-a}^{-a} H^*(\theta - t)dt + \int_{a}^{\pi} d\theta \int_{a}^{-a} H^*(\theta - t)dt$$

$$= -2\int_0^{\pi-a} F^*(s)ds + 2\int_{2a}^{\pi+a} F^*(s)ds\,.$$

Summing up, we get

$$\int_{-\pi}^{\pi} |u_0(\theta)|d\theta \leq 4\int_0^{2a} F^*(s)ds - 4\int_{2a}^{\pi+a} F^*(s)ds + 4\int_0^{\pi-a} F^*(s)ds\,.$$

The fonction $a \to 4\displaystyle\int_0^{2a} F^*(s)ds - 4\int_{2a}^{\pi+a} F^*(s)ds + 4\int_0^{\pi-a} F^*(s)ds$ is maximal for $a = \frac{\pi}{2}$. Therefore

$$\int_{-\pi}^{\pi} |u_0(\theta)|d\theta \leq 8\int_0^{\pi} F^*(s)ds = 8\int_0^{\pi} dx \int_0^{x} H^*(t)dt\,. \tag{29}$$

Let $M \geq H$ be defined as follows. Let $x_1 = 0.355$, $x_2 = 0.59$, $x_3 = 1.3$, $x_4 = 1.9$, $x_5 = 2.25$. Let

$$r_5(x) = -\frac{1}{0.99}(x - 2\pi + x_5) \cdot (x - x_5)$$

$$r_1(x) = \frac{r_5(\pi)}{x_1 - x_2}(x - x_2)$$

$$r_2(x) = \frac{1}{4.34}(x - x_2)(x - (x_2 + x_3))$$

$$r_3(x) = -(x - 2.15)(x - (x_3 - 0.85))\frac{r_2(1.3)}{(0.85)^2}$$

$$r_4(x) = \frac{r_3(x_4)}{x_4 - x_5}(x - x_5)$$

$$M(s) = \begin{cases} H(s), & 0 \le s \le x_1 \\ r_1(s), & x_1 \le s \le x_2 \\ r_2(s), & x_2 \le s \le x_3 \\ r_3(s), & x_3 \le s \le x_4 \\ r_4(s), & x_4 \le s \le x_5 \\ r_5(s), & x_5 \le s \le \pi \end{cases}$$

We observe that $M^*(s)$ is the inverse function of

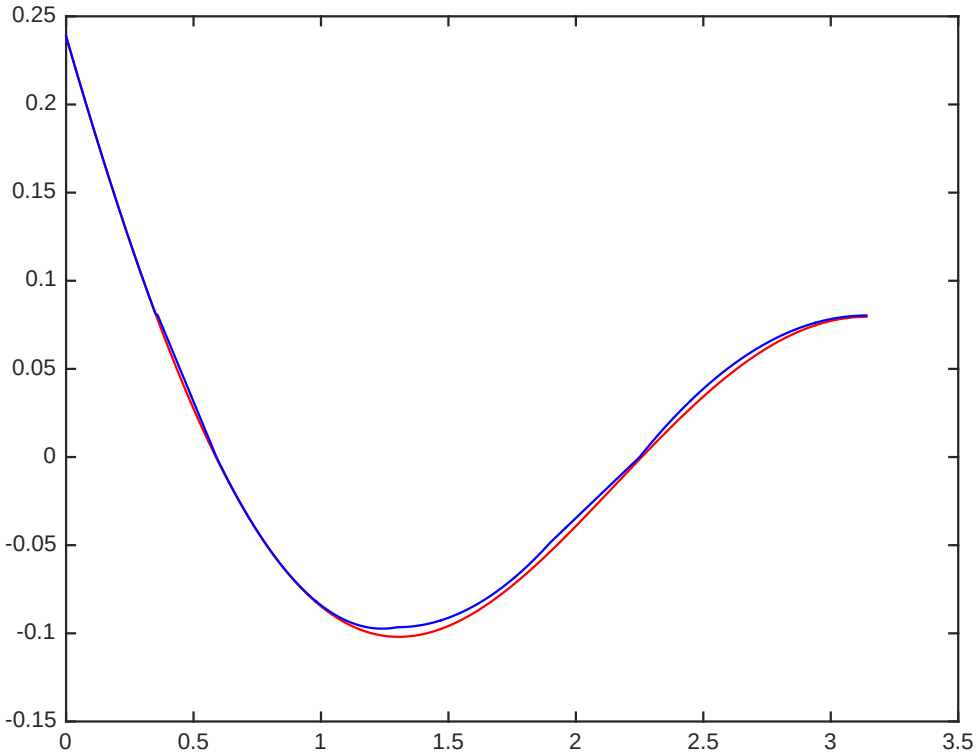

FIGURE 1. The function M ant its symmetrization M^*.

$$t \to |\{x \in [0,\pi] : M(x) > t\}| = \begin{cases} H^{-1}(t), & t \ge H(x_1) \\ r_1^{-1}(t) + \pi - r_5^{-1}(t), & 0 \le t \le H(x_1) \\ r_2^{-1}(t) + \pi - r_4^{-1}(t), & r_4(x_4) \le t \le 0 \\ r_2^{-1}(t) + \pi - r_3^{-1}(t), & r_3(x_3 \le t \le r_4(x_4) \end{cases}$$

and by property (5) $M^* \geq H^*$. By (29) one has

$$\frac{1}{m} = \int_{-\pi}^{\pi} |u_0(\theta)| d\theta \leq 8 \int_0^{\pi} dx \int_0^x H^*(t)dt \leq 8 \int_0^{\pi} dx \int_0^x M^*(t)dt = 8 \int_0^{\pi} (\pi - x) M^*(x) dx \approx 8 \cdot 0.2320$$

which implies (27).

□

Remark 4.5. *Although Fuglede* [9] *was interested in the uniform spherical deviation, that is, the Hausdorff distance of a set E from the ball of same measure centered at the barycenter of E, one can easily deduce from his results the inequality $\delta(E) \geq C(n)[\lambda_0(E)]^2$ for nearly spherical sets (see Theorem 3.1 in* [11]*), where $C(n)$ is a constant depending on the dimension. In particular the following estimate can be proved in the plane:*

$$\delta(E) \geq \frac{1}{16} \lambda_0(E)^2 .$$

However this estimate is not sufficient to exclude sequences converging to the ball.

Our first attempt to prove Theorem 4.2 was the following. For the denominator of J one has

$$\left[\frac{1}{2} \int_0^{2\pi} |(1 + u(\theta))^2 - 1| d\theta \right]^2 \leq \left[\frac{1}{2} \int_0^{2\pi} |u(\theta)|(2 + \varepsilon) d\theta \right]^2 = \left(1 + \frac{\varepsilon}{2} \right)^2 \|u\|_{L^1}^2 .$$

For the numerator

$$\int_0^{2\pi} \left(\sqrt{(1+u)^2 + u'(\theta)^2} d\theta - 1 \right) d\theta$$

$$\geq \int_0^{2\pi} \left[u + \frac{u^2 + (u')^2}{2} - \frac{[4u^2 + 4u^3 + 4uu' + u^4 + u'^4 + 2u^2u'^2]^2}{8} + \frac{8u^3}{16} \right] d\theta$$

$$\geq \int_0^{2\pi} \left[\frac{(u')^2}{2} - \frac{u^2}{2} \right] d\theta + o(2) \geq \frac{c-1}{2} \int_0^{2\pi} u^2 d\theta + o(2)$$

where $c = 4$ is such that

$$\int_0^{2\pi} (u')^2 d\theta \geq c \int_0^{2\pi} u^2 d\theta .$$

Therefore

$$\liminf_{\varepsilon \to 0} J(u) \geq \frac{3\pi}{4} \frac{\displaystyle\int_0^{2\pi} u^2 d\theta}{\left[\displaystyle\int_0^{2\pi} |u| d\theta \right]^2} \geq \frac{3}{8} = 0.375$$

by Hölder inequality. Again, this estimate is not sufficient to exclude sequences converging to the ball. Note also that the minimization problem (13) is difficult to solve exactly due to the three constraints. This explains why we performed this complicated and computational method via symmetrization and Riesz inequality.

5. On the regularity and on the shape of the optimal convex set

In this section we will prove that an optimal set for the minimization of $\frac{\delta}{\lambda_0^2}$ among convex sets in the plane is $C^{1,1}$. About its shape, we conjecture that the stadium S which minimizes $\frac{\delta}{\lambda^2}$ (see Theorem 2.8) also minimizes our functional. Indeed, we will show that among stadia, S is the only one satisfying the optimality conditions that we will write in Theorem 5.4. Unfortunately we are not able to prove that sets different from S do not satisfy the optimality condition.

In the proof of the regularity of the optimal set we will essentially use the first order optimality condition in the spirit of [21]. Let us first recall how to write these optimality conditions, in the case of convexity constraint, when representing the boundary of the convex set with the gauge function about the problem

$$\min \left\{ j(u), u \in H^1(0, 2\pi), u'' + u \geq 0, m(u) := \int_0^{2\pi} \frac{d\theta}{u^2} = m_0 \right\} \tag{30}$$

(see Proposition 2.3.3 of [20]).

Proposition 5.1. *Assume that u_0 solves (30) where $j : H^1(0,2\pi) \to \mathbb{R}$ is C^2. Then there exist ξ_0 nonnegative, $\mu \in \mathbb{R}$ such that $\xi_0 = 0$ on the support of $(u_0'' + u_0)$ and*

$$j'(u_0)v = < \xi + \xi'', v > -\mu m'(u_0)v$$

for every $v \in H^1(0,2\pi)$.

In this section, we prove the following regularity result:

Theorem 5.2. *A minimizer of $\frac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ within convex compact sets of the plane is $C^{1,1}$ and C^∞ on strictly convex parts of the boundary (except at the intersection with the circle of the barycentric disc).*

For the proof, we first express the optimality condition of Proposition 5.1 in our context:

Proposition 5.3. *Let r, θ be the polar coordinates. Let $u(\theta) = \frac{1}{r(\theta)}$ be the gauge function used to describe the boundary of a set. The optimal set satisfies the following condition: there exists $\xi \in H^1$, positive, nul when the boundary is strictly convex, and there exists $\hat\mu_0, \hat\mu_1, \hat\mu_2 \in \mathbb{R}$ such that*

$$-\frac{1}{2\pi\lambda_0^2}\frac{u+u''}{(u^2+u'^2)^{\frac{3}{2}}} - \frac{2\delta}{\pi\lambda_0^3}\frac{sign(u^2-1)}{u^3} = \xi'' + \xi - \frac{\hat\mu_0}{u^3} - 3\frac{\hat\mu_1\cos\theta + \hat\mu_2\sin\theta}{2u^4}\,. \tag{31}$$

Proof. We are going to apply Proposition 5.1, adapted to our constraints. To do that, we need to compute the derivative of $\frac{\delta}{\lambda_0^2}$ and of the constraints.

(1) The derivative of $\frac{\delta}{\lambda_0^2}$ is $\frac{1}{2\pi\lambda_0^2}\cdot P' - 2\frac{\delta}{\lambda_0^3}\cdot\lambda_0'$. Now, the derivative of P is

$$-\int \frac{u+u''}{(u^2+u'^2)^{\frac{3}{2}}} v\,.$$

The derivative of $\pi\lambda_0$ is the derivative of $\int_{\{u<1\}} \chi_{\Omega\setminus B} + \int_{\{u>1\}} \chi_{B\setminus\Omega}$ which gives

$$\frac{1}{2}\int_{\{u<1\}} \frac{-2v}{u^3} + \frac{1}{2}\int_{\{u>1\}} \frac{2v}{u^3}\,.$$

(2) The derivative of the constraints

$$\frac{1}{2}\int \frac{d\theta}{u^2(\theta)} = \pi\,, 0 = \frac{1}{2}\int_0^{2\pi} \frac{\cos\theta}{u^3} = \frac{1}{2}\int_0^{2\pi} \frac{\sin\theta}{u^3}$$

equals respectively

$$-\int \frac{\hat\mu_0}{u^3}, -3\int \frac{\hat\mu_1\cos\theta}{2u^4}, -3\int \frac{\hat\mu_2\sin\theta}{2u^4}\,.$$

□

We are now able to prove Theorem 5.2:

Proof. We are going to use the notations of the above proposition. On the strictly convex parts of the boundary, $\xi = 0$ and u satisfies a second order ordinary differential equation:

(1) in the exterior of the unit ball $u < 1$, and so u'' is continuous. By a classical bootstrap argument u is C^∞;
(2) in the exterior of the unit ball $u > 1$, and so u'' is continuous. By a bootstrap argument u is C^∞;
(3) on the boundary of the unit ball $u = 1$, u'' is bounded, but not continuous. Thus u is $W^{2,\infty}$ there.

This and the above proposition imply that on strictly convex parts on ∂B, u is $C^{1,1}$.

Now, let us prove that Ω is C^1. If this was not the case, we would have a corner for some θ_0. This implies that the Gauge function satisfies: $u'' + u$ contains a Dirac mass, with a positive weight at θ_0. Thus, the H^1 function ξ appearing in the optimality condition (31) must also satisfy: $\xi'' + \xi$ contains a Dirac mass at θ_0. Now, since $\xi(\theta_0) = 0$ and $\xi \geq 0$, the weight of this Dirac mass must be non-negative, in contradiction with the minus sign appearing in the left-hand side of (31) in front of $u'' + u$.

We are left with the conjunctions between a strictly convex part of the boundary and a non strictly convex part. For that, it is sufficient to remark that any $C^2(\mathbb{R}^+)(C^\infty(\mathbb{R}^+))$ function, nul for $x = 0$, can be extended by 0 on $\mathbb{R}$, getting a $C^{1,1}(\mathbb{R})$ function. This ends the proof that an optimal set is $C^{1,1}$.

□

We are going to write differently the optimality conditions on strictly convex parts. In particular, this will give the explicit expression of the Lagrange multipliers in (31). We can assume that all our sets have area equal to π.

Theorem 5.4. *Let Ω be an optimal set. Let B be the unit ball centered at the origin. Let $\partial\Omega^{IN} = \partial\Omega \cap B$, $\partial\Omega^{OUT} = \partial\Omega \cap B^c$, $\partial B^{IN} = \partial B \cap \Omega$, $\partial B^{OUT} = \partial B \cap \Omega^c$. Then the curvature of Ω satisfies on every strictly convex part:*

$$\mathcal{C} = 1 - 3\delta + \frac{4\delta}{2\pi\lambda_0}\left(|\partial B^{OUT}| - |\partial B^{IN}|\right) \pm \frac{4\delta}{\lambda_0} + \hat{\mu_1}x + \hat{\mu_2}y\,, \tag{32}$$

(+ at the exterior of B and $-$ in the interior of B) where

$$\hat{\mu_1} = \frac{4\delta}{\pi\lambda_0}\left[\int_{\partial B^{OUT}} \cos t dt - \int_{\partial B^{IN}} \cos t dt\right],$$

$$\hat{\mu_2} = \frac{4\delta}{\pi\lambda_0}\left[\int_{\partial B^{OUT}} \sin t dt - \int_{\partial B^{IN}} \sin t dt\right].$$

Proof. We are going to perform shape variations on the strictly convex parts of $\partial\Omega$. The proof is divided into several steps.

(1) Let $\Omega_t = (I + tV)(\Omega)$. Then

$$|\Omega_t| = \pi + t\int_{\partial\Omega} V \cdot n + o(t)\,.$$

(2) The barycenter constraint implies that $\displaystyle\int_{\Omega_t} x dx dy = 0 + t\int_{\partial\Omega} xV \cdot n + o(t)$. Since by definition $x_t = \dfrac{1}{|\Omega_t|}\displaystyle\int_{\Omega_t} x dx dy$, by the above formulas one has

$$x_t = \frac{t}{\pi}\int_{\partial\Omega} xV \cdot n + o(t)\,.$$

A similar formula holds for y_t:

$$y_t = \frac{t}{\pi}\int_{\partial\Omega} yV \cdot n + o(t)\,.$$

(3) Let $B_t = (I + tW)(B)$, where

$$W(x, y) = (a, b) + \alpha(x, y)\,,$$

with

$$(a, b) = \frac{1}{\pi}\left(\int_{\partial\Omega} xV \cdot n, \int_{\partial\Omega} yV \cdot n\right), \alpha := \frac{1}{2\pi}\int_{\partial\Omega} V \cdot n\,.$$

(4) The difference between $|\Omega_t \Delta B_t|$ and $|\Omega \Delta B|$ is given by two terms: one comes from the area from B and B_t, and the other one from the deformation of Ω. More precisely

$$|\Omega_t \Delta B_t| = |\Omega \Delta B| \pm t\int_{\partial B} W \cdot n \pm t\int_{\partial\Omega} V \cdot n\ :$$

for the second term of the right hand side $+$ is on ∂B^{OUT} and $-$ is on ∂B^{IN}; for the last term of the right hand side, $+$ is $\partial\Omega^{OUT}$ and $-$ on $\partial\Omega^{IN}$.

In the next part of the proof we will write $|\Omega_t \Delta B_t| = |\Omega \Delta B| + tR$.

(5) We have

$$\lambda_0(\Omega_t) = \frac{|\Omega_t \Delta B_t|}{|\Omega_t|} = \frac{|\Omega \Delta B| + tR}{|\Omega| + t\int_{\partial\Omega} V \cdot n} = \lambda_0(\Omega) \cdot \frac{1 + t\frac{R}{|\Omega \Delta B|}}{1 + \frac{t}{\pi}\int_{\partial\Omega} V \cdot n} = \lambda_0(\Omega) + t\left[\frac{R}{\pi} - \frac{\lambda_0}{\pi}\int_{\partial\Omega} V \cdot n\right].$$

Therefore

$$d\lambda_0(\Omega, V) = \frac{1}{\pi}\left[\pm\int_{\partial B} W \cdot n \pm \int_{\partial\Omega} V \cdot n - \lambda_0\int_{\partial\Omega} V \cdot n\right],$$

that is,

$$d\lambda_0(\Omega, V) = \frac{1}{\pi}\left[\int_{\partial B^{OUT}} W \cdot n - \int_{\partial B^{IN}} W \cdot n + \int_{\partial\Omega^{OUT}} V \cdot n - \int_{\partial\Omega^{IN}} V \cdot n - \lambda_0\int_{\partial\Omega} V \cdot n\right].$$

(6) If r_t is the radius of the ball having the same area as Ω_t, then $r_t = \sqrt{\frac{\pi + t\int_{\partial\Omega} V\cdot n}{\pi}} = 1 + t\frac{\int_{\partial\Omega} V\cdot n}{2\pi}$. This gives

$$\delta(\Omega_t) = \frac{P(\Omega_t)}{2\pi r_t} - 1 = \frac{P(\Omega_t)}{2\pi + t\int_{\partial\Omega} V\cdot n} - 1 = \frac{P(\Omega) + t\int_{\partial\Omega} \mathcal{C}V\cdot n}{2\pi + t\int_{\partial\Omega} V\cdot n} - 1\,.$$

With the same computations as for λ_0

$$\delta(\Omega_t) = \delta(\Omega) + t\frac{\int_{\partial\Omega} \mathcal{C}V\cdot n}{2\pi} - t\frac{P(\Omega)\int_{\partial\Omega} V\cdot n}{4\pi^2}$$

and so

$$d\delta(\Omega, V) = \int_{\partial\Omega}\left[\frac{\mathcal{C}}{2\pi} - \frac{(\delta+1)2\pi}{4\pi^2}\right] V\cdot n = \int_{\partial\Omega}\frac{\mathcal{C}-\delta-1}{2\pi} V\cdot n\,.$$

The optimality condition $\dfrac{d\delta}{\lambda_0^2} - \dfrac{2\delta}{\lambda_0^3} d\lambda_0 = 0$ can be written as

$$\int_{\partial\Omega}(\mathcal{C}-\delta-1)V\cdot n = \frac{4\delta}{\lambda_0}\left[\int_{\partial B^{OUT}} W\cdot n - \int_{\partial B^{IN}} W\cdot n + \int_{\partial\Omega^{OUT}} V\cdot n - \int_{\partial\Omega^{IN}} V\cdot n - \lambda_0\int_{\partial\Omega} V\cdot n\right].$$

Now, $W\cdot n = a\cos\theta + b\sin\theta + \alpha$ (since $(x,y)\cdot n = 1$ on ∂B), so

$$W\cdot n = \cos\theta\int_{\partial\Omega} xV\cdot n + \sin\theta\int_{\partial\Omega} yV\cdot n + \frac{1}{2\pi}\int_{\partial\Omega} V\cdot n$$

which gives

$$\mathcal{C} = \delta + 1 - 4\delta + \frac{4\delta}{2\pi\lambda_0}\left(|\partial B^{OUT}| - |\partial B^{IN}|\right) \pm \frac{4\delta}{\lambda_0} + \hat{\mu_1}x + \hat{\mu_2}y\,.$$

□

Remark 5.5. *If S denotes the stadium of Theorem 2.8, we can prove that S is the only stadium satisfying the optimality conditions (32).*

To see that, let us consider a stadium centered at 0, given by the union of a rectangle of dimensions $2r\times 2l$ and two half discs of radius $r\le 1$. Let θ be the angle such that $r = \sin\theta$. Assuming without loss of generality that the area is π, one has

$$l = \frac{\pi - \pi\sin^2\theta}{4\sin\theta}\,.$$

The perimeter equals $4l + 2\pi r = \frac{\pi}{\sin\theta} + \pi\sin\theta$ which implies

$$\delta(\theta) = \frac{1}{2\sin\theta} + \frac{\sin\theta}{2} - 1\,.$$

The double of the area of the stadium minus the ball of radius 1 helps us computing λ_0:

$$\lambda_0(\theta) = \frac{2}{\pi}(\pi - 2\theta - \sin(2\theta))\,.$$

On one hand, an optimal stadium is a critical point of the function $\theta\mapsto\delta(\theta)/\lambda_0^2(\theta)$. This leads to solve the nonlinear equation

$$8\sin\theta(1-\sin\theta)^2 - \cos\theta(\pi - 2\theta - \sin(2\theta)) = 0. \tag{33}$$

It is a simple exercise to prove that this equation has a unique solution, providing the stadium S which corresponds to the value $\theta\sim 0.5750$.
On the other hand, writing condition (32) for a stadium yields, with the same notations, to the equation

$$4\sin\theta - \frac{3}{2}\sin^2(\theta) - \frac{5}{2} + 2\frac{(1-\sin\theta)^2(\pi-2\theta)}{\pi - 2\theta - \sin(2\theta)} = 0. \tag{34}$$

It is easy to check that equation (34) has a unique solution in $(0,\pi/2)$ and that solution is the same as the one to equation (33). Therefore an only stadium satisfies (32); this stadium is S, since λ_0 and λ take the same value on any stadium.

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