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To cite this version:

Karsten Bohlen, Jean-Marie Lescure. A GEOMETRIC APPROACH TO K-HOMOLOGY FOR LIE MANIFOLDS. 2019. hal-02090537

HAL Id: hal-02090537

https://hal.science/hal-02090537v1

Preprint submitted on 4 Apr 2019

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A GEOMETRIC APPROACH TO K-HOMOLOGY FOR LIE MANIFOLDS

KARSTEN BOHLEN, JEAN-MARIE LESCURE

Abstract. We prove that the computation of the Fredholm index for fully elliptic pseudodifferential operators on Lie manifolds can be reduced to the computation of the index of Dirac operators perturbed by smoothing operators. To this end we adapt to our framework ideas coming from Baum-Douglas geometric K-homology and in particular we introduce a notion of geometric cycles that can be classified into a variant of the famous geometric K-homology groups, for the specific situation here. We also define comparison maps between this geometric K-homology theory and relative K-theory.

1. Introduction

The Atiyah-Singer index theorem is a celebrated and fundamental result with numerous applications in Geometry and Analysis. A particular approach to index theory is due to P. Baum and R. Douglas, cf. [5] and [6]. They constructed a geometric K-homology and a suitable comparison homomorphism between the geometric and analytic K-homology groups. A complete proof that the comparison map is in fact an isomorphism was published recently [7]. In the Baum-Douglas approach to the index theorem, the computation of the Fredholm index of an elliptic pseudodifferential operator on a compact closed manifold can be reduced to the computation of the index of a suitable geometric Dirac operator, naturally associated to a geometric cycle. The origin of this work is to address the corresponding question for singular manifolds, at least the ones for which a suitable Lie groupoid permits to well pose the index problem. More precisely, we consider Lie manifolds $(M, \mathcal{G})$, that is Lie groupoids $\mathcal{G}$ over compact manifolds with corners such that $M_0 = M \setminus \partial M$ is saturated and $\mathcal{G}_{M_0} = M_0 \times M_0$. This occurs in many cases, for example manifolds with corners or fibered corners and manifolds with foliated boundary. In such a case, there is a well defined notion of full ellipticity for operators in the corresponding calculus, that ensures the Fredholmness of the associated operators on M. The question can now be made more precise. Given a fully elliptic operator P on $(M, \mathcal{G})$, can we construct a Dirac operator D in the same calculus, which is Fredholm and with the same index than P? Contrary to the case of C^∞ compact manifolds without boundary, we are not able to give an affirmative answer to this question. Nevertheless, we are able to solve positively the question by allowing tamed Dirac operators, that is, Dirac operators perturbed by smoothing elements in the calculus.
Along the way, we prove that if there is no obstruction at the level of K-theory for the full ellipticity (and thus Fredholmness) of Dirac operators, then a perturbation into a Fredholm operator using smoothing operators and elementary Dirac operators always exists. This echoes previous works by Bunke [10] and Carrillo Rouse-Lescure [12]. Also, these considerations bring a notion of geometric cycles that mimetizes the original one of Baum-Douglas with the following main variations into the choice of ingredients:

(1) We only accept submersions of manifolds with corners $\varphi : \Sigma \to M$ instead of general continuous maps from Spin^c manifolds to M;

(2) We replace Dirac operators by tame Dirac operators.

The point (1) is apparently rather restrictive but actually sufficient for our purpose, since the geometric cycles constructed by the clutching process are of this kind. Furthermore, the classical operations on geometric cycles: isomorphisms, direct sums, cobordisms and vector bundle modifications have a natural analog here. The resulting abelian group, that we call geometric K-homology of $(M, \mathcal{G})$ can then be compared with the suitable relative K-theory group (also known as stable homotopy group of fully elliptic operators). Our approach is definitely tied to the clutching construction which, in a sense, dictated to us the most convenient notion of geometric cycles for our purpose, the latter being, as explained above, the possibility of representing the index of abstract pseudodifferential operators by the one of Dirac operators, perturbed by smoothing operators. We mention that other and more general approaches of geometric K-homology for groupoids exist, in particular in [13, 15, 24].

Overview. The article is organized as follows.

- In section 2 we study fully elliptic operators contained in the pseudodifferential calculus on the tuple $(M, \mathcal{G})$, where $\mathcal{G}$ is a Lie groupoid over M. We introduce the group of stable homotopy classes of fully elliptic operators ${}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M)$ which is defined to equal the relative K-theory group $K(\mu)$, where μ is the homomorphism of the continuous functions M into the full symbol algebra, given by the action as multiplication operators. We prove a Poincaré duality type result which states that the Fredholm index can be expressed in a precise way in terms of an index map defined in terms of purely geometric data, given by suitable deformation groupoids. More precisely, we recall the geometric model for the full symbol space of a pseudodifferential operator on a Lie groupoid in terms of the deformation groupoid $\mathcal{T}$, referred to as the *noncommutative tangent bundle*. The K-theory group $K_0(C^*(\mathcal{T}))$ turns out to be the natural receptable for the stable homotopy class of the full symbol of a fully elliptic pseudodifferential operator. The Poincaré duality result is the isomorphism of groups ${}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M) \cong K_0(C^*(\mathcal{T}))$ via the non-commutative principal symbol homomorphism σ_{nc}.
- In section 3 we investigate so-called *tamings* of geometric Dirac operators on Lie groupoids. The main result is a *Diracification theorem* which states that any fully elliptic pseudodifferential operator P on a Lie groupoid, whose principal symbol has the same class in K-theory as a given Dirac operator D, has a so-called taming $B = (D \oplus D') + R$, consisting of a Dirac operator D' and a smoothing operator R, such that the classes of the operators B and P agree in the relative K-theory group $K(\mu)$. We introduce a pushforward operation on the level of deformation groupoids and show that this operation commutes with the Fredholm index.
- In section 4 we introduce geometric K-homology groups ${}^{\mathcal{V}}\mathrm{K}^{geo}(M)$ (of even and odd degree) on a given Lie manifold $(M, \mathcal{G})$. Using the pushforward operation, introduced in the previous section, we define the comparison map $\lambda\colon {}^{\mathcal{V}}\mathrm{K}^{geo}(M) \to K(\mu)$. The main result of this section is a proof that the comparison map λ is a well-defined group homomorphism. This result is correctly viewed as a generalization of the cobordism invariance of the analytic index in the standard setting.
- In section 5 we define the so-called *clutching construction* in the setting of Lie manifolds. The resulting quotient map $c\colon K_0(C^*(\mathcal{T})) \to {}^{\mathcal{V}}\mathrm{K}^{geo}(M)$ is induced

by the map associating to a full symbol of a (fully elliptic) pseudodifferential operator a geometric cycle. We show that c is a well-defined homomorphism of groups. Then we finish the proof of the main result of the paper, which states that the computation of the Fredholm index of any fully elliptic pseudodifferential operator on a Lie manifold can be reduced to the computation of the index of a geometric Callias type operator, associated to the geometric data, generating the geometric K-homology group.

The view of index theory on Lie manifolds considered in this work can be summarized in terms of the following commutative diagram:

$$\begin{tikzcd}
 & K_0(C^*(\mathcal{T})) \arrow[rr, "\mathrm{ind}_{\mathcal{F}}"] \arrow[dl, "c"'] & & \mathbb{Z} \\
{}^{\mathcal{V}}\mathrm{K}^{geo}(M) \arrow[rr, "\lambda"] & & K_0(\mu) = {}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M) \arrow[ul, "\simeq", "\mathrm{pd}"'] \arrow[ur, "\mathrm{ind}"'] &
\end{tikzcd} \tag{1}$$

In conjunction with the recent work of Bohlen-Schrohe [9] the reduction of the index problem to first order geometric operators furnishes a corresponding index formula for fully elliptic pseudodifferential operators on Lie manifolds.

2. Poincaré duality

Full ellipticity, groupoids and K-theory. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid denoted by $\mathcal{A}$. The manifold M will always be compact, and may have corners. In that case, we assume that the boundary hypersurfaces of M and $\mathcal{G}$ are embedded [33], and the source and range maps are *submersions between manifolds with corners* [29], also called tame submersions in [39]. We recall that it means, for a C^∞ map $f : M \to N$ between manifolds with corners, that at any point $x \in M$ we have:

$$df_x(T_xM) = T_{f(x)}N \quad \text{and} \quad (df_x)^{-1}(T^+_{f(x)}N) = T^+_xM. \tag{2}$$

Here TM denotes the ordinary tangent vector bundle and T^+M its subspace of inward pointing vectors. Under such assumptions, f preserves the codimension of points and its fibers have no boundary.

If $E_j \to M$ are vector bundles, we denote by $\Psi^*_{\mathcal{G}}(E_0, E_1)$ the space of compactly $\mathcal{G}$-pseudodifferential operators [14, 36, 34, 41]. The principal symbol map is denoted by:

$$\sigma_{\mathrm{pr}} \colon \Psi^m_{\mathcal{G}}(E_0, E_1) \to C^\infty(S(\mathcal{A}^*), \overline{\pi}^*(\mathrm{Hom}(E_0, E_1))).$$

Here $S(\mathcal{A}^*)$ is the sphere bundle of the dual Lie algebroid $\mathcal{A}^*$ of $\mathcal{G}$ and $\overline{\pi}$ denotes any of the projections maps of $\mathcal{A}^*$ and $S(\mathcal{A}^*)$ onto M.

Let $\overline{\Psi^m_{\mathcal{G}}}(E_0, E_1)$ denote the closure of $\Psi^m_{\mathcal{G}}(E_0, E_1))$ into $\mathrm{Mor}(\mathcal{H}^s(\mathcal{G}, E_0), \mathcal{H}^{s-m}(\mathcal{G}, E_1))$ [45] (and we choose once for all, the reduced or maximal completion indifferently). For simplicity we denote $\Psi_{\mathcal{G}}(E) = \overline{\Psi^0_{\mathcal{G}}}(E)$. We get a short exact sequence of C^*-algebras:

$$C^*(\mathcal{G}, \mathrm{End}(E)) \rightarrowtail \Psi_{\mathcal{G}}(E) \overset{\sigma_{pr}}{\twoheadrightarrow} C(S(\mathcal{A}^*), \overline{\pi}^*\mathrm{End}(E)). \tag{3}$$

Now let $F \subset M$ be a closed subspace which is saturated, that is $s^{-1}(F) = r^{-1}(F)$. Then $\mathcal{G}_F$ is a continuous family groupoid [42, 28]. By restricting over F, we get the F-indicial symbol map:

$$I_F \colon \Psi^*_{\mathcal{G}}(E_0, E_1) \to \Psi^*_{\mathcal{G}_F}(E_0|_F, E_1|_F)$$

Gathering both symbol maps, we get the *F-joint symbol map*:

$$\sigma_{F,m} = (\sigma_{\mathrm{pr}}, I_F) : \Psi^m_{\mathcal{G}}(E_0, E_1) \longrightarrow C(S(\mathcal{A}^*), \overline{\pi}^*\mathrm{Hom}(E_0, E_1)) \times \overline{\Psi^m_{\mathcal{G}_F}}(E_0|_F, E_1|_F). \quad (4)$$

The range $\Sigma^m_F(E_0, E_1)$ of $\sigma_{F,m}$ is called the *F-joint symbols space* and its closure is denoted by $\overline{\Sigma^m_F}(E_0, E_1)$. We will write Σ_F for $\overline{\Sigma^0_F}$. This gives the short exact sequence of C^*-algebras [28]:

$$C^*(\mathcal{G}_O, \mathrm{End}(E)) \rightarrowtail \Psi_{\mathcal{G}}(E) \xrightarrow{\sigma_F} \Sigma_F(E). \quad (5)$$

where $O = M \setminus F$ and $C^*(\mathcal{G}_O, \mathrm{End}(E))$ is the closure of $C^\infty_c(G_O, r^*\mathrm{End}(E))$ into $\Psi_{\mathcal{G}}(E)$. We recall that for any elliptic $A \in \Psi^1_{\mathcal{G}}(E, E')$, the unbounded operator $\Lambda_E = (1 + A^*A)^{1/2}$ defined by functional calculus belongs to $\overline{\Psi^1_{\mathcal{G}}}(E)$, realizes isomorphisms $\mathcal{H}^s(G, E) \to \mathcal{H}^{s-1}(G, E)$ and satisfies $\Lambda_E^{-1} \in C^*(\mathcal{G}, \mathrm{End}(E))$ [45].

Definition 2.1. We say that $P \in \overline{\Psi^m_{\mathcal{G}}}(E)$ is *F-fully elliptic* if $\sigma_F(\Lambda_E^{-m}P) \in \Sigma_F(E)^\times$.

We now turn our attention to the natural groups constructed out of F-full elliptic operators [27, 44, 43, 3]. A convenient way to handle them is to use the so-called K-group $K(f)$ of a homomorphism $f : A \to B$ ([27, II.2.13]). We recall the definition: the set of cycles $\Gamma(f)$ is given by triples $(\mathcal{E}_0, \mathcal{E}_1, \alpha)$ where the $\mathcal{E}_i$ are finitely generated projective A-modules and $\alpha : \mathcal{E}_0 \otimes_f B \to \mathcal{E}_1 \otimes_f B$ is an isomorphism. The notions of isomorphisms ($\simeq$) and direct sums ($\oplus$) in $\Gamma(f)$ are the obvious ones. A cycle $(\mathcal{E}_0, \mathcal{E}_1, \alpha)$ is elementary if $\mathcal{E}_0 = \mathcal{E}_1$ and α is homotopic to the identity within the automorphisms of $\mathcal{E}_0 \otimes_f B$. Finally, the equivalence relation giving $K(f) = \Gamma(f)/\sim$ is defined by:

$$\sigma \sim \sigma' \text{ if there exists elementary } \tau, \tau' \text{ such that } \sigma \oplus \tau \simeq \sigma' \oplus \tau'.$$

We always have $K(f) \simeq K_0(C_f)$ where C_f is the mapping cone of f and $K(f) \simeq K_0(\ker f)$ when f is onto [3]. In [43] these groups are introduced in a slightly different way, namely as stable homotopy groups of elliptic operators. The comparison is straightforward and we keep a notation inspired by A. Savin's one.

Definition 2.2. The group $\mathcal{F}\mathrm{Ell}_F(\mathcal{G})$ will be defined as the K-group $K(\mu_F)$ of the natural homomorphism $\mu_F : C(M) \longrightarrow \Psi_{\mathcal{G}}/C^*(\mathcal{G}_O) = \Sigma_F$.

We also consider another natural homomorphism $\mu_{pr} : C(M) \to \Psi_{\mathcal{G}}/C^*(\mathcal{G}) \simeq C(S(\mathcal{A}^*))$. If $P \in \Psi^m_{\mathcal{G}}(E_0, E_1)$ is F-fully elliptic, $m \geq 0$, then it canonically defines classes:

$$[P]_F = \left[\mathcal{E}_0, \mathcal{E}_1, \sigma_F(\Lambda_{E_1}^{-m}P)\right] \in K(\mu_F) = \mathcal{F}\mathrm{Ell}_F(\mathcal{G}),$$

$$[P]_{pr} = \left[\mathcal{E}_0, \mathcal{E}_1, \sigma_{pr}(\Lambda_{E_1}^{-m}P)\right] \in K(\mu_{pr}) \simeq K^0_c(A^*),$$

with $\mathcal{E}_j = C(X, E_j)$. The resulting classes do not depend on the choice of Λ_{E_1}. Indeed, consider $\Lambda_A = (1 + A^*A)^{1/2}$ and $\Lambda_B = (1 + B^*B)^{1/2}$. Since $C(t) = (1 - t)A^*A + tB^*B$ is elliptic self-adjoint and non negative we get a family $\Lambda_t = (1 + C(t))^{1/2} \in \Psi^1_{\mathcal{G}}(E_1)$ of invertible elements connecting Λ_A to Λ_B, and $\sigma_F(\Lambda_t^{-m}P)$ provides the desired homotopy. The same is true for the principal symbol classes.
Note that there is a natural isomorphism $C_{\mu_{pr}} \simeq C_0(\mathcal{A}^*)$ and therefore:

$$K(\mu_{pr}) \simeq K^0_c(\mathcal{A}^*).$$

We will denote by $[P]_{\mathrm{pr,ev}}$ the image of $[P]_{\mathrm{pr}}$ through this isomorphism.

Proposition 2.3. *Let $P_+ \in \Psi_{\mathcal{G}}(E_+, E_-)$ be F-fully elliptic. Then P_+^* is F-fully elliptic and*

$$[P_+]_F = -[P_+^*]_F \in K_0(\mu_F). \quad (6)$$

Proof. If P_+ is F-fully elliptic, then $P_- := P_+^*$ is also F-fully elliptic. Set $P = \begin{pmatrix} 0 & P_- \\ P_+ & 0 \end{pmatrix} \in \Psi_{\mathcal{G}}(E)$, $E = E_+ + E_-$. We get by polar decomposition of the invertible element $p = \sigma_F(P)$ in the C^*-algebra $\Sigma_F(E)$ the existence of $U, T \in \Psi_{\mathcal{G}}(E)$ such that $u = \sigma_F(U)$ is unitary, $|p| = \sigma_F(T)$ and $p = u|p|$. Since $|p|$ is of degree 0, since p of degree 1 with respect to the $\mathbb{Z}_2$ grading and since $|p|$ is invertible, the unitary u is necessarily of degree 1 with respect to the $\mathbb{Z}_2$-grading. Therefore, we can assume that $U = \begin{pmatrix} 0 & U_- \\ U_+ & 0 \end{pmatrix}$ and $T = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}$. Since p is selfajoint, we can also assume that the unitary u is self-adjoint, thus $u_- = u_+^* = u_+^{-1}$. Since p is positive, we get a homotopy $p = u|p| \sim u$, therefore

$$[P_+]_F + [P_+^*]_F = [\mathcal{E}, \mathcal{E}, p] = [\mathcal{E}, \mathcal{E}, u] = [\mathcal{E}_+, \mathcal{E}_-, u_+] + [\mathcal{E}_-, \mathcal{E}_+, u_+^{-1}] = 0 \in K(\mu_F).$$

□

Index and Poincaré duality results. There is an index map coming with $\mathcal{F}\mathrm{Ell}_F(\mathcal{G})$. Indeed, the commutative diagram

$$\begin{array}{ccc} C(M) & \xrightarrow{\mu_F} & \Sigma_F \\ {\scriptstyle \mu_0}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \Psi_{\mathcal{G}} & \xrightarrow{\sigma_F} & \Sigma_F \end{array}$$

gives rise to a homomorphism $K(\mu_F) \to K(\sigma_F)$ and since σ_F is onto, we also have a natural isomorphism $K(\sigma_F) \simeq K_0(\ker \sigma_F) = K_0(C^*(\mathcal{G}_O))$ [3]. Their composition is called the (F-)index map:

$$\mathrm{ind}_F \colon \mathcal{F}\mathrm{Ell}_F(\mathcal{G}) \to K(C^*(\mathcal{G}_O)). \tag{7}$$

It is possible to get a slightly more geometrical description of (7) by establishing a kind of Poincaré duality with the help of deformation groupoids. Let us introduce the necessary objects.

The adiabatic groupoid $\mathcal{G}^{ad} \rightrightarrows M_{ad} := M \times [0,1]$ is the natural Lie groupoid integrating the Lie algebroid $(\mathcal{A}^{ad}, \varrho^{ad})$ given by: $\mathcal{A}^{ad} = \mathcal{A} \times [0,1]$ and $\varrho^{ad} \colon \mathcal{A}^{ad} \to TM \times T[0,1]$, $\mathcal{A}^{ad} \ni (x, v, t) \mapsto (x, tv, t, 0) \in TM \times T[0,1] = TM_{ad}$. More precisely,

$$\mathcal{G}^{ad} = \mathcal{A} \times \{0\} \cup \mathcal{G} \times (0,1] \text{ and } \mathcal{A}(\mathcal{G}^{ad}) \cong \mathcal{A}^{ad}.$$

Out of the adiabatic groupoid we construct the so-called *F-Fredholm groupoid*:

$$\mathcal{G}_F^{\mathcal{F}} := \mathcal{G}^{ad} \setminus (\mathcal{G}_F \times \{1\}) \rightrightarrows \mathrm{M}_F^{\mathcal{F}} = (M \times [0,1]) \setminus (F \times \{1\}) \tag{8}$$

This is again a Lie groupoid (as an open subset of $\mathcal{G}^{ad}$). The *non-commutative tangent bundle* is defined by:

$$\mathcal{T}_F M := \mathcal{G}_F^{\mathcal{F}} \setminus (G_O \times (0,1]) \rightrightarrows M_F = \mathrm{M}_F^{\mathcal{F}} \setminus (O \times (0,1]). \tag{9}$$

It is a $C^{\infty,0}$ groupoid [42]. The exact sequence:

$$C^*(\mathcal{G}_O \times (0,1]) \rightarrowtail C^*(\mathcal{G}_{\mathcal{F}}) \xrightarrow{e_0} C^*(\mathcal{T}_F M) \tag{10}$$

possessing a contractible kernel, we get an isomorphism $e_0 : K_0(C^*(\mathcal{G}_F^{\mathcal{F}})) \to K_0(C^*(\mathcal{T}_F M))$, and if (10) has a completely positive section, then $e_0 \in KK(C^*(\mathcal{G}_F^{\mathcal{F}}), C^*(\mathcal{T}_F M))$ provides a KK-equivalence (that is, is invertible). Considering the restriction $e_1 : C^*(\mathcal{G}_F^{\mathcal{F}}) \to$

$C^*(\mathcal{G}_O)$, we get another index map:

$$\mathrm{ind}_F^{\mathcal{F}} := (e_1)_* \circ (e_0)_*^{-1} \colon K_0(C^*(\mathcal{T}_F M)) \to K_0(C^*(G_O)).$$

We think to Σ_F as a *noncommutative cosphere bundle*, relative to F, and to $\mathcal{T}_F M$ as the *noncommutative tangent bundle*, relative to F, associated with $\mathcal{G} \rightrightarrows M$. We end up with a Poincaré duality like theorem.

Theorem 2.4. *There is a group isomorphism:*

$$\sigma_F^{\mathrm{nc}} \colon \mathcal{F}\mathrm{Ell}_F(\mathcal{G}) \to K_0(C^*(\mathcal{T}_F M)) \tag{11}$$

such that $e_0(\sigma_F^{\mathrm{nc}}[P]_F) = [P]_{\mathrm{pr,ev}} \in K_c^0(\mathcal{A}^*)$ *and* $\mathrm{ind}_F^{\mathcal{F}}(\sigma_F^{\mathrm{nc}}[P]_F) = \mathrm{ind}_F([P]_F)$. *Here* e_0 *is the restriction map* $C^*(\mathcal{T}_F M) \to C^*(\mathcal{A})$.

Proof. By [27, 44, 43] we have a natural isomorphism:

$$\mathcal{F}\mathrm{Ell}_F(\mathcal{G}) = K(\mu_F) \simeq K_0(C_{\mu_F}) \tag{12}$$

where C_f denotes the mapping cone of f. The isomorphism $K_0(C_{\mu_F}) \simeq K_0(C^*(\mathcal{T}_F M))$ can be proved following verbatim the proof of Theorem 10.6 in [21, Theorem 10.6]. Alternatively, we use the following short argument[1]. Denote by $\mathcal{G}_0^{ad}$ the restriction of $\mathcal{G}^{ad}$ to $M \times [0,1)$. Since $\Psi_{\mathcal{G}_0^{ad}}/C^*(\mathcal{G}_0^{ad}) \simeq C_0(S(A^*) \times [0,1))$ is contractible, the inclusion $C^*(\mathcal{G}_0^{ad}) \to \Psi_{\mathcal{G}_0^{ad}}$ is an isomorphism in K-theory, as well as the inclusion

$$C^*(\mathcal{T}_F M) = C^*(\mathcal{G}_0^{ad})/C^*(\mathcal{G}_O \times (0,1)) \subset \Psi_{\mathcal{G}_0^{ad}}/C^*(\mathcal{G}_O \times (0,1))$$

as well. If $\mu_0 : C(M) \to \Psi_{\mathcal{G}}$ denotes the natural homomorphism, then we have a homomorphism $\kappa : C_{\mu_0} \to \Psi_{\mathcal{G}_0^{ad}}$ given by $(f,P) \mapsto P$. Using on the one hand the commutative diagram

$$\begin{array}{ccccc} C_0((0,1),\Psi_{\mathcal{G}}) & \rightarrowtail & C_{\mu_0} & \twoheadrightarrow & C(M) \\ \| & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa_0} \\ C_0((0,1),\Psi_{\mathcal{G}}) & \rightarrowtail & \Psi_{\mathcal{G}_0^{ad}} & \twoheadrightarrow & \Psi_{\mathcal{A}}. \end{array}$$

and on the other hand that $\kappa_0 : C(M) \longrightarrow \Psi_{\mathcal{A}}$ is a KK-equivalence, we get by the five lemma that κ is a KK-equivalence too. Quotienting both algebras C_{μ_0} and $\Psi_{\mathcal{G}_0^{ad}}$ by the ideal $C^*(\mathcal{G}_O \times (0,1))$, we get that the homomorphism

$$C_{\mu_F} \longrightarrow \Psi_{\mathcal{G}_0^{ad}}/C^*(\mathcal{G}_O \times (0,1))$$

is a KK-equivalence. The remaining assertions in Theorem 2.4 are then easy. □

3. From abstract operators to tame Dirac operators

Dirac bundles over groupoids. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathcal{A}$. We equip $\mathcal{A}$ with a euclidean structure and denote by $\mathrm{Cl}(\mathcal{A})$ the corresponding bundle of Clifford algebras over M. We define Dirac bundles as in [10, Definition 3.1] by replacing TM by $\mathcal{A}$ in the definition of c, in other words we deal with a $\mathrm{Cl}(\mathcal{A})$-module complex vector bundle E over M with Clifford multiplication induced by c, compatible hermitian metric h, admissible connection ∇ and a $\mathbb{Z}_2$-grading z when $\mathcal{A}$ has even rank. Now by [29], there is an associated Dirac operator $D \in \mathrm{Diff}_{\mathcal{G}}(E)$. For simplicity, we will denote such a Dirac bundle by (E,D), the necessary data being understood.

[1] communicated to us by G. Skandalis

K-tamings. One can check that, for any F-fully elliptic $P \in \Psi_{\mathcal{G}}(E_0, E_1)$, we have:

$$[P]_{F,\mathrm{ev}} := \sigma_F^{nc}([P]_F) = \left[C^*(\mathcal{T}_F M, E_0 \oplus E_1), \begin{pmatrix} 0 & \mathcal{Q}|_{\mathcal{T}_F M} \\ \mathcal{P}|_{\mathcal{T}_F M} & 0 \end{pmatrix}\right] \in KK(\mathbb{C}, C^*(\mathcal{T}_F M)).$$

Here $\mathcal{P}$ is a $F \times \{1\}$-fully elliptic lift of P, that is $\mathcal{P} \in \Psi_{\mathcal{G}^{ad}}$, $\mathcal{P}|_{t=1} = P$ and $\mathcal{P}|_{t=0} = \sigma_{pr}(P)$; while $\mathcal{Q}$ is a similar lift of a full parametrix Q of P. We refer to this situation as the *even* case.
Now, if $P \in \Psi_{\mathcal{G}}(E)$ is selfadjoint and F-fully elliptic, which we refer to as the *odd* case, we will consider instead of $[P]_F$ and $[P]_{\mathrm{pr}}$, the classes:

$$[P]_{F,\mathrm{odd}} = \left[C^*(\mathcal{T}_F M, E), \mathcal{P}|_{\mathcal{T}_F M}\right] \in KK_1(\mathbb{C}, C^*(\mathcal{T}_F M)),$$

$$[P]_{\mathrm{pr,odd}} = \left[C^*(\mathcal{A}, E), \sigma_{\mathrm{pr}}(P)\right] \in KK_1(\mathbb{C}, C^*(A)),$$

The convention is the same for positive order operators after using suitable *order reduction* operator Λ.

Definition 3.1. We say that an elliptic operator $P \in \Psi_{\mathcal{G}}^*$ is F-tameable if $[P]_{\mathrm{pr},*}$ is in the range of

$$\tau = (\mathrm{ev}_{t=0})_* : K_*(C^*(\mathcal{T}_F M)) \longrightarrow K_*(C^*(\mathcal{A})). \tag{13}$$

Using the K-equivalences $\mathcal{G}_F^{\mathcal{F}} \sim_K \mathcal{T}_F M$ and $\mathcal{G}^{ad} \sim_K \mathcal{A}$, we see that the above condition is equivalent to be in the range of

$$i_* : K_*(C^*(\mathcal{G}_F^{\mathcal{F}})) \longrightarrow K_*(C^*(\mathcal{G}^{ad})). \tag{14}$$

where $i : C^*(\mathcal{G}_F^{\mathcal{F}}) \hookrightarrow C^*(\mathcal{G}^{ad})$ is the inclusion.
Also, in the even case, the natural homomorphism $q_F : \Sigma_F \to C(S\mathcal{A}^*)$ induces a map:

$$q_F : K(\mu_F) \longrightarrow K(\mu_{pr}) \tag{15}$$

which coincides with τ under the suitable isomorphisms. In particular $q_F([P]_F) = [P]_{pr}$ and F-tameability of an elliptic operator $P \in \Psi_{\mathcal{G}}^*(E_0, E_1)$ is equivalent to $[P]_{\mathrm{pr}} \in \operatorname{im} q_F$.

Reduction to Dirac. We call (S, D) an even Dirac bundle for $\mathcal{G}$ if $S = S_+ \oplus S_-$ is a $\mathbb{Z}_2$-graded $\mathrm{Cliff}(\mathcal{A})$-module vector bundle over X and $D = \mathrm{Antidiag}(D_+, D_+^*) \in \Psi_{\mathcal{G}}^1(S)$ is a Dirac operator. Odd Dirac bundles are defined in the same way without gradings.

Definition 3.2. Any F-fully elliptic operator of the form:

$$B = (D \oplus D') + R$$

where

(1) (S', D') is a Dirac bundle of the same parity as (S, D),
(2) $R \in \Psi_{\mathcal{G}}^{-\infty}(S \oplus S')$,
(3) $[B]_{\mathrm{pr},*} = [D]_{\mathrm{pr},*}$,

is called a *F-taming* of D.

Theorem 3.3. *Let (S, D) be a F-tameable Dirac bundle. Then for any F-fully elliptic operator $P \in \Psi_{\mathcal{G}}$ such that $[P]_{\mathrm{pr},*} = [D]_{\mathrm{pr},*} \in K^*(\mathcal{A})$, there exists a F-taming B of (S, D) such that*

$$[B]_{F,*} = [P]_{F,*} \in K_*(C^*(\mathcal{T}_F M)) \simeq K(\mu_F). \tag{16}$$

Proof of the Theorem. We begin with the even case and use for that the relative K-theory description. Far any integer N, we can give $\mathrm{Cliff}(\mathcal{A})$-modules structure to the trivial bundles $X\times\mathbb{C}^k$ (by embedding isometrically the bundle $\mathcal{A}$ into some trivial bundle over X) where $k\geq N$ is chosen in order to get an even Dirac bundle $(M\times\mathbb{C}^k, D_k)$. Then $[D_{k,+}\oplus(-D_{k,+})]_{pr}=0\in K(\mu_{pr})$ and we set $D'=D_k+(-D_k)$.
Let $P_+\in\Psi_{\mathcal{G}}(E_+,E_-)$ be a F-fully elliptic element such that $[P_+]_{pr}=[D_+]_{pr}$. By definition of $K(\mu_{pr})$, there exist elementary elements

$$(\xi,\xi,\alpha),(\xi',\xi',\alpha')\in\Gamma(\mu_{pr})$$

such that

$$(\mathcal{E}_+,\mathcal{E}_-,\sigma_{pr}(P_+))+(\xi,\xi,\alpha)\simeq(\mathcal{S}_+,\mathcal{S}_-,\sigma_{pr}(D_+))+(\xi',\xi',\alpha').$$

Adding to both sides another elementary element, and renaming, we can assume that $\xi'=C(S(\mathcal{A}^*),\mathbb{C}^k)$. We rename $\mathcal{S}_\pm\oplus C(S(\mathcal{A}^*),\mathbb{C}^k)$ into $\mathcal{S}_\pm$ and identify $\mathcal{E}_\pm\oplus\xi\simeq\mathcal{S}_\pm$. Replacing α by 1 and α' by $\sigma_{pr}(D'_+)$ after homotopies, and renaming $P_+\oplus 1$ into P_+, we get:

$$\big(\mathcal{S}_+,\mathcal{S}_-,\sigma_{pr}(P_+)\big)\sim\big(\mathcal{S}_+,\mathcal{S}_-,\sigma_{pr}((D\oplus D')_+)\big),$$

and in particular

$$\sigma_{pr}(P_+)^{-1}\sigma_{pr}((D\oplus D')_+)\sim\mathrm{Id}\ \text{ within }\mathrm{Aut}(\mathcal{S}_+). \tag{17}$$

Now we use the homotopies lifting argument of [23, Proposition 4.3]. Since $\mathcal{S}_\pm=p_\pm C(S(A^*))^N$ for some projectors $p_\pm\in M_N(C(M))$, the surjectivity of $q:\Sigma_F\longrightarrow C(S(\mathcal{A}^*))$ implies the surjectivity of

$$\widetilde{q}:\mathcal{L}(p_+(\Sigma_F^N))\longrightarrow\mathcal{L}(\mathcal{S}_+).$$

By the open mapping theorem, $\widetilde{q}$ is then open. Thus, $\widetilde{q}(\mathrm{Aut}(p_\pm(\Sigma_F))_{(0)})$ is an open, and therefore closed, subgroup of $\mathrm{Aut}(\mathcal{S}_+)$, therefore:

$$\widetilde{q}(\mathrm{Aut}(p_\pm\Sigma_F^N)_{(0)})=\mathrm{Aut}(\mathcal{S}_\pm)_{(0)}.$$

By (17), we can choose $y\in\mathrm{Aut}(p_+\Sigma_F^N)_{(0)}$ such that

$$\sigma_{pr}(P_+)\widetilde{q}(y)=\sigma_{pr}(P_+)y=\sigma_{pr}((D\oplus D')_+). \tag{18}$$

Let $Y\in\Psi_{\mathcal{G}}(\mathcal{S}_+)$ such that $\sigma_F(Y)=y$. Let us set $T=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & (P_+Y)^*\\ P_+Y & 0\end{pmatrix}$. Since y is in the connected component of the identity in $\mathrm{Aut}(\mathcal{S}_\pm)$, we have $[T_\pm]_F=[P_\pm]_F$. If $\Delta\in\Psi^2_{\mathcal{G}}(\mathcal{S})$ denotes a laplacian, Equation (18) gives:

$$T=(1+\Delta)^{-1/2}(D\oplus D')+R\text{ with }R\in C^*(\mathcal{G},\mathrm{End}(\mathcal{S})). \tag{19}$$

We know that the class $\sigma_F(T)$ of T in the C^*-algebra $\Psi_{\mathcal{G}}(\mathcal{S})/C^*(G_O,\mathrm{End}(\mathcal{S}))$ is invertible. By density of $C_c^\infty(\mathcal{G},\mathrm{End}(\mathcal{S}))$ into $C^*(\mathcal{G},\mathrm{End}(\mathcal{S}))$, we can choose $R_1\in C_c^\infty(\mathcal{G},\mathrm{End}(\mathcal{S}))$ such that $\sigma_F(T_1))$ remains invertible, where $T_1=T-R+R_1$. We write $R_1=(1+\Delta)^{-1/2}(1+\Delta)^{1/2}R_1$ and we approximate $(1+\Delta)^{1/2}R_1\in C^*(\mathcal{G},\mathrm{End}(\mathcal{S}))$ by an element $R_2\in C_c^\infty(\mathcal{G},\mathrm{End}(\mathcal{S}))$ close enough so that $\sigma_F(T_2))$ remains invertible, where

$$T_2=(1+\Delta)^{-1/2}(D\oplus D')+(1+\Delta)^{-1/2}R_2.$$

Thus, $B:=(D\oplus D')+R_2$ is a F-fully elliptic operator and $R_2\in\Psi_{\mathcal{G}}^{-\infty}(\mathcal{S})$. Without loss of generality, we can assume that R,R_1,R_2 are close enough so that T and T_2 are homotopic among F-fully elliptic operators. We conclude:

$$[B_+]_F=[T_+]_F=[P_+]_F=-[P_-]_F=-[B_-]_F.$$

In the odd case now, we consider the map $\tau_* : KK_1(\mathbb{C}, C^*(\mathcal{T}_F M)) \longrightarrow KK_1(\mathbb{C}, C^*(\mathcal{A}))$. Let $\mathcal{P}$ be a fully elliptic lift of P and $p \in M_N(C(M))$ a projector such that $E = p(M \times \mathbb{C}^N)$. We have $\tau_*[pC^*(\mathcal{T}_F M)^N, \mathcal{P}|_{\mathcal{T}_F X}] = [pC^*(\mathcal{A})^N, \sigma_{pr}(P)]$, therefore by assumption $[pC^*(\mathcal{A})^N, \sigma_{pr}(P)] = [pC^*(\mathcal{A})^N, \sigma_{pr}(D)] \in KK_1(\mathbb{C}, C^*(\mathcal{A}))$. This means that after the addition of degenerate Kasparov modules we have a homotopy between $\sigma_{pr}(P)$ and $\sigma_{pr}(D)$ within the automorphisms of the C^*-algebra $\mathcal{L}(\tau\mathcal{E})/\mathcal{K}(\tau\mathcal{E})$ where $\mathcal{E} = C^*(\mathcal{T}_F M, E)$ and $\tau E = C^*(\mathcal{A}, E) = \mathcal{E} \otimes_\tau C^*(\mathcal{A})$. Now by lifting homotopies through the epimorphism of C^*-algebras:

$$\widetilde{\tau} : \mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \longrightarrow \mathcal{L}(\tau\mathcal{E})/\mathcal{K}(\tau\mathcal{E})$$

as in the even case, and using again a density argument, we conclude the proof. □

Lie manifolds. If M is a compact manifold with embedded corners, it is understood that a decomposition into closed faces is given, which are themselves compact manifolds with embedded corners inheriting the induced decomposition into faces. The set of codimension k faces of M is denoted by $\mathcal{F}_k M$.

We will consider Lie manifold structures on manifolds with embedded corners [1]. Any such structure on M is given by an almost injective Lie algebroid $\mathcal{A}$, the latter being integrable by [18] into a unique, up to isomorphism, s-connected Lie groupoid $\mathcal{G}$. By a slight abuse of notation, we will continue to call Lie manifold any pair $(M, \mathcal{G})$ such that

(1) $\mathcal{G} \rightrightarrows M$ is a Lie groupoid over M;
(2) $M_0 = M \setminus \partial M$ is saturated and $\mathcal{G}_{M_0} = M_0 \times M_0$.

Let $(M, \mathcal{G})$ be a Lie manifold. We denote by $\Psi^*_{\mathcal{V}}(M; E_0, E_1)$ the algebra of pseudodifferential operators of Lie type on $M_0 = M \setminus \partial M$ acting between the sections of vector bundles $E_j \to M$ [1]. It coincides with the image of $\Psi^*_{\mathcal{G}}(E_0, E_1)$ by the vector representation $r_\#$ when $\mathcal{G}$ is s-connected [1]. By a slight abuse of notation, we continue to set $\Psi^*_{\mathcal{V}} = r_\# \Psi^*_{\mathcal{G}}$ in the general case. Equivalently, $\Psi^*_{\mathcal{V}}$ is isomorphic to the space obtained by restricting elements of $\Psi^*_{\mathcal{G}}$ to the fiber of $\mathcal{G}$ over any arbitrary interior point x. The isomorphism comes then from the diffeomorphism $r : \mathcal{G}_x \to M_0$.

The ∂M-indicial symbols and ∂M-joint symbols can be related with their analogues for any $F \in \mathcal{F}_1(M)$ in the obvious way, and we can summarize this in the following commutative diagram in which the outer rectangle is made of fibered products:

$$\begin{tikzcd}
\Sigma^m_{\partial M}(E) \arrow[rr, "\pi_1"] \arrow[dd, "\pi_2"'] & & C^\infty(S(\mathcal{A}^*), \overline{\pi}^*\mathrm{End}(E)) \arrow[dd, "(r_F)_{F\in\mathcal{F}_1(M)}"] \\
& \Psi^m_{\mathcal{G}}(E) \arrow[ul, two heads, "\sigma_{\mathrm{pr}}\oplus I_{\partial M}"'] \arrow[ur, two heads, "\sigma_{\mathrm{pr}}"] \arrow[dl, two heads, "I_{\partial M}"] \arrow[dr, two heads] & \\
\prod_{F\in\mathcal{F}_1(M)} \Psi^m_{\mathcal{G}_F}(E|_F) \arrow[rr, two heads, "\oplus_F \sigma_{\mathrm{pr},F}"] & & \prod_F C^\infty(S(\mathcal{A}^*_{|F}), \overline{\pi}^*\mathrm{End}(E_{|F}))
\end{tikzcd}$$

We observe that with $F = \partial M$, we have $O = M \setminus \partial M =: M_0$ and thus $\mathcal{G}_O = M_0 \times M_0$. It follows that the ideal in (5) is the ideal of compact operators and that $\mathrm{ind}_{\partial M}$ takes integer values. Since $r_\# : C^*(\mathcal{G}_O, \mathrm{End}(E)) \xrightarrow{\simeq} \mathcal{K}(L^2_{\mathcal{V}}(M, E))$ and $r_\# : \Psi_{\mathcal{G}}(E) \hookrightarrow \mathcal{B}(L^2_{\mathcal{V}}(M, E))$ [1], the sequence (5) also shows that $r_\#(P) : H^s_{\mathcal{V}}(M; E) \to H^{s-m}_{\mathcal{V}}(M; E)$ is Fredholm, when $P \in \Psi^m_{\mathcal{G}}(E)$ is *∂M-fully elliptic*, and that $\mathrm{ind}_{\partial M}$ computes its Fredholm index. Here $H^s_{\mathcal{V}}(M, E)$ are the Sobolev spaces of sections of E associated with the Lie structure and $L^2_{\mathcal{V}} = H^0_{\mathcal{V}}$.

From now on, whenever we take $F = \partial M$ as a closed saturated subspace of a Lie manifold $(M, \mathcal{G})$, we will omit it in the notation, for instance: fully means ∂M-fully, $\mathcal{G}^{\mathcal{F}}$ denotes $\mathcal{G}^{\mathcal{F}}_{\partial M}$, etc ... and we set ${}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M) = \mathcal{F}\mathrm{Ell}_{\partial M}(\mathcal{G})$.

Pullbacks. Let $\varphi : \Sigma \longrightarrow M$ be a surjective tame submersion, that is, a surjective submersion satisfying (2). Then the decomposition into faces $\mathcal{F}_*(M)$ can be lifted to a decomposition into faces of Σ. An interesting example occurs in the so-called clutching spaces and vector bundle modification:

Example 3.4. Let M be a manifold with corners and $\pi : V \to M$ be a real vector bundle. Consider the sphere bundle $\varphi : Z = S(V \oplus \mathbb{R}) \to M$. Then φ is a tame surjective submersion. We say that Z is the clutching space of $\pi : V \to M$.

We recall the notion of pull-backs.

Definition 3.5. Let $(\mathcal{A}, \varrho)$ be a Lie algebroid over M and $\mathcal{G} \rightrightarrows M$ a Lie groupoid over M. Let $\varphi\colon \Sigma \to M$ be a surjective submersion.
(1) The *Lie agebroid pullback* of $(\mathcal{A}, \varrho)$ over Σ is the Lie-algebroid $({}^{\varphi}\mathcal{A}, {}^{\varphi}\varrho)$ given by:

$$ {}^{\varphi}\mathcal{A} = \{(v, w) \in \varphi^*\mathcal{A} \times T\Sigma : \varrho(v) = d\varphi(w)\} \subset \varphi^*\mathcal{A} \oplus T\Sigma \text{ and } {}^{\varphi}\varrho = \mathrm{pr}_2. $$

(2) The *Lie groupoid pullback* of $\mathcal{G}$ over Σ is the Lie groupoid ${}^{\varphi}\mathcal{G} \rightrightarrows \Sigma$ given by

$$ {}^{\varphi}\mathcal{G} = \{(x, \gamma, y) \in \Sigma \times \mathcal{G} \times \Sigma : \varphi(x) = r(\gamma),\ \varphi(y) = s(\gamma)\}, $$

with structure maps: ${}^{\varphi}\mathrm{r}(x, \gamma, y) = x$, ${}^{\varphi}\mathrm{s}(x, \gamma, y) = y$, $(x, \gamma, y)(y, \gamma', z) = (x, \gamma\gamma', z)$. Note that the Lie groupoid pullback of the Lie algebroid $\mathcal{A}$ also makes sense and will be denoted by ${}^{\varphi}\mathbf{A}$ if a risk of confusion occurs.

Both notions are consistent since:

Theorem 3.6. *Let $(M, \mathcal{G})$ be a Lie manifold and $\varphi\colon \Sigma \to M$ be a tame surjective submersion. Then $(\Sigma, {}^{\varphi}\mathcal{G})$ is a Lie manifold such that $\mathcal{A}({}^{\varphi}\mathcal{G}) \cong {}^{\varphi}\mathcal{A}$.*
Moreover, ${}^{\varphi}\mathcal{G}$ is Morita equivalent to $\mathcal{G}$ and the ($\mathcal{V}$-related) Lie structure ${}^{\varphi}\mathcal{V}$ is given by:

$$ {}^{\varphi}\mathcal{V} = \{\widetilde{V} \in \Gamma(T\Sigma)\ ; \exists V \in \mathcal{V},\ d\varphi(\widetilde{V}) = V \circ \varphi\}. $$

This is a compilation of known facts: the isomorphism $\mathcal{A}({}^{\varphi}\mathcal{G}) \cong {}^{\varphi}\mathcal{A}$ is proved in [32], the Morita equivalence is proved in [22], that ${}^{\varphi}\mathcal{G}$ gives a Lie structure on Σ is done in [39] and further developments can be found in [46].

Let $(M, \mathcal{G})$ be a Lie manifold and $\varphi\colon \Sigma \to M$ a tame surjective submersion. The operations consisting of taking the adiabatic deformation and taking a pullback do not commute, however there are natural deformation groupoids relating them and it is exploited in [46] in order to obtain pushforward maps (which are there occurences of wrong way maps). We need to recall the results of [46] in our notation. Firstly, there is a deformation Lie groupoid:

$$ \mathcal{L}_{\varphi} = ({}^{\varphi}\mathcal{G})^{ad} \times \{u = 0\} \cup {}^{\varphi_1}(\mathcal{G}^{ad}) \times (0, 1]_u \rightrightarrows \Sigma \times [0, 1]_t \times [0, 1]_u, \tag{20} $$

where $\varphi_1 = \varphi \times \mathrm{Id} : \Sigma \times [0, 1]_t \to M \times [0, 1]_t$. Let z be the lift by φ of a defining function for the boundary of M and consider the following saturated subgroupoids of $\mathcal{L}_{\varphi}$:

$$ \mathcal{L}^{\mathcal{F}}_{\varphi} := \mathcal{L}_{\varphi}|_{(z,t)\neq(0,1)} = ({}^{\varphi}\mathcal{G})^{\mathcal{F}} \times \{u = 0\} \cup {}^{\varphi_1}(\mathcal{G}^{\mathcal{F}}) \times (0, 1]_u, \tag{21} $$

$$ \mathcal{L}^{nc}_{\varphi} := \mathcal{L}_{\varphi}|_{tz=0} = \mathcal{T}\Sigma \times \{u = 0\} \cup {}^{\varphi_1}\mathcal{T}M \times (0, 1]_u \rightrightarrows \Sigma_{\partial\Sigma} \times [0, 1]_u. \tag{22} $$

$$ \mathcal{L}^{0}_{\varphi} := \mathcal{L}_{\varphi}|_{t=0} = {}^{\varphi}\mathcal{A} \times \{u = 0\} \cup {}^{\varphi}\mathbf{A} \times (0, 1]_u \rightrightarrows M \times [0, 1]_u. \tag{23} $$

Using the restrictions homomorphisms $\mathrm{ev}_{u=i}$, $i=0,1$, and the natural Morita equivalence $\mathcal{M}$ we get homomorphisms (see [46] for more details):

$$\varphi_!^{ad} = \mathcal{M} \circ [\mathrm{ev}_{u=1}] \circ [\mathrm{ev}_{u=0}]^{-1} : K_*(C^*(({}^{\varphi}\mathcal{G})^{ad})) \longrightarrow K_*(C^*(\mathcal{G}^{ad}))). \tag{24}$$

Considering the variations (21), (22) and (23) of (20) leads to similar homomorphisms:

$$\varphi_!^{\mathcal{F}} : K_*(C^*(({}^{\varphi}\mathcal{G})^{\mathcal{F}})) \longrightarrow K_*(C^*(\mathcal{G}^{\mathcal{F}}))), \tag{25}$$

$$\varphi_!^{nc} : K_*(C^*(\mathcal{T}\Sigma)) \longrightarrow K_*(C^*(\mathcal{T}M)). \tag{26}$$

$$\varphi_!^{0} : K_*(C^*({}^{\varphi}\mathcal{A})) \longrightarrow K_*(C^*(\mathcal{A})). \tag{27}$$

Then the following result is essentially a rephrasing of [46]:

Theorem 3.7. *Let $(M,\mathcal{G})$ be a Lie manifold and $\varphi : \Sigma \to M$ a tame surjective submersion. Then the map $\varphi_!^{nc}$ commutes with Fredholm index:*

$$\mathrm{ind}_{\partial M}^{\mathcal{F}} \circ \varphi_!^{nc} = \mathrm{ind}_{\partial\Sigma}^{\mathcal{F}} \tag{28}$$

Here the target group of the index maps $\mathrm{ind}_{\bullet}^{\mathcal{F}}$ is replaced by $\mathbb{Z}$ after applying the obvious Morita equivalences.
In other words, if $B \in \Psi^*_{\varphi\mathcal{G}}(\widetilde{E}_+, \widetilde{E}_-)$ and $P \in \Psi^*_{\mathcal{G}}(E_+, E_-)$ are fully elliptic operators and satisfy $\varphi_!^{nc}[B]_{\partial\Sigma,\mathrm{ev}} = [P]_{\partial\Sigma,\mathrm{ev}}$ then B and P have the same Fredholm index.

Proof. Using the groupoid $\mathcal{L}_\varphi$ and the definition of $\mathrm{ind}^{\mathcal{F}}$ we get a commutative diagram:

$$\begin{tikzcd}
K_0(C^*(\mathcal{T}\Sigma)) \arrow[rrr, bend left, "\mathrm{ind}_{\partial\Sigma}^{\mathcal{F}}"] \arrow[dddd, bend right, "\varphi_!^{\mathrm{nc}}"'] & K_0(C^*(({}^{\varphi}\mathcal{G})^{\mathcal{F}}_{\partial\Sigma})) \arrow[l, "\mathrm{ev}_{tz=0}"'] \arrow[r, "\mathrm{ev}_{t=1}"] & K_0(C^*(\Sigma_0\times\Sigma_0)) \arrow[r, "\mathcal{M}"] & \mathbb{Z} \arrow[d, equal] \\
K_0(C^*(\mathcal{L}^{\mathrm{nc}}_{\varphi})) \arrow[u, "\mathrm{ev}_{u=0}"] \arrow[d, "\mathrm{ev}_{u=1}"] & K_0(C^*(\mathcal{L}^{\mathcal{F}}_{\varphi})) \arrow[l, "\mathrm{ev}_{tz=0}"'] \arrow[r, "\mathrm{ev}_{t=1}"] \arrow[u, "\mathrm{ev}_{u=0}"] \arrow[d, "\mathrm{ev}_{u=1}"] & K_0(C^*(\Sigma_0\times\Sigma_0\times[0,1])) \arrow[r, "\mathcal{M}\times p_*"] \arrow[u, "\mathrm{ev}_{u=0}"] \arrow[d, "\mathrm{ev}_{u=1}"] & \mathbb{Z} \arrow[d, equal] \\
K_0(C^*({}^{\varphi}\mathcal{T}M)) \arrow[d, "\mathcal{M}"] & K_0(C^*({}^{\varphi}(\mathcal{G}^{\mathcal{F}}_{\partial M}))) \arrow[l, "\mathrm{ev}_{tz=0}"'] \arrow[r, "\mathrm{ev}_{t=1}"] \arrow[d, "\mathcal{M}"] & K_0(C^*(\Sigma_0\times\Sigma_0)) \arrow[r, "\mathcal{M}"] \arrow[d, "\mathcal{M}"] & \mathbb{Z} \arrow[d, equal] \\
K_0(C^*(\mathcal{T}M)) & K_0(C^*(\mathcal{G}^{\mathcal{F}}_{\partial M})) \arrow[l, "\mathrm{ev}_{tx=0}"'] \arrow[r, "\mathrm{ev}_{t=1}"] & K_0(C^*(M_0\times M_0)) \arrow[r, "\mathcal{M}"] & \mathbb{Z} \\
& & & & \\
& & \mathrm{ind}_{\partial M}^{\mathcal{F}} \arrow[uuuu, bend left=60] & &
\end{tikzcd} \tag{29}$$

Above $p = \mathrm{ev}_{u=\alpha}$ for arbitrary $\alpha \in [0,1]$. The result follows immediately. □

Theorem 3.8. *Let $(M,\mathcal{G})$ be a Lie manifold, $\pi : V \to M$ be a real vector bundle and denote by $\varphi : \Sigma = S(V\oplus\mathbb{R}) \to M$ the associated clutching and $i : V \hookrightarrow \Sigma$ the embedding as an open hemisphere. Then*

(1) $\pi_!^{nc}$ and $\pi_!^{0}$ are isomorphisms, the latter being the inverse of the Thom isomorphism of the complex bundle ${}^{\pi}\mathcal{A} \to A$.
(2) We have the identities:

$$\varphi_!^{\bullet} \circ \left(i_* \circ (\pi_!^{\bullet})^{-1}\right) = \mathrm{Id}, \qquad \textit{with } \bullet = nc \textit{ or } 0. \tag{30}$$

Proof. Considering the groupoids $\mathcal{L}_\pi$, $\mathcal{L}_\varphi$ and the embeddings provided by i, we get commutative diagrams:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_0(C^*({}^\varphi\mathcal{G}|_{\partial\Sigma}\times(0,1)) & \longrightarrow & K_0(C^*(\mathcal{T}\Sigma)) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*({}^\varphi\mathcal{A})) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\mathcal{M}} & & \downarrow{\scriptstyle\varphi_!^{nc}} & & \downarrow{\scriptstyle\varphi_!^0} & & \\
\cdots & \longrightarrow & K_0(C^*(\mathcal{G}|_{\partial M}\times(0,1))) & \longrightarrow & K_0(C^*(\mathcal{T}M)) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*(\mathcal{A})) & \longrightarrow & \cdots
\end{array} \tag{31}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_0(C^*({}^\pi\mathcal{G}|_{\partial V}\times(0,1)) & \longrightarrow & K_0(C^*(\mathcal{T}V)) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*({}^\pi\mathcal{A})) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\mathcal{M}} & & \downarrow{\scriptstyle\pi_!^{nc}} & & \downarrow{\scriptstyle\pi_!^0} & & \\
\cdots & \longrightarrow & K_0(C^*(\mathcal{G}|_{\partial M}\times(0,1))) & \longrightarrow & K_0(C^*(\mathcal{T}M)) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*(\mathcal{A})) & \longrightarrow & \cdots
\end{array} \tag{32}$$

$$\begin{array}{ccc}
K_0(C^*(\mathcal{T}\Sigma)) & \xrightarrow{\varphi_!^{nc}} & K_0(C^*(\mathcal{T}M)) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \mathrm{Id}} \\
\mathcal{K}_0(C^*(\mathcal{T}V)) & \xrightarrow{\pi_!^0} & K_0(C^*(\mathcal{T}M))
\end{array} \tag{33}$$

$$\begin{array}{ccc}
K_0(C^*({}^\varphi\mathcal{A})) & \xrightarrow{\varphi_!^0} & K_0(C^*(\mathcal{A})) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \mathrm{Id}} \\
\mathcal{K}_0(C^*({}^\pi\mathcal{A})) & \xrightarrow{\pi_!^0} & K_0(C^*(\mathcal{A}))
\end{array} \tag{34}$$

Since $\mathcal{L}_\pi^0$ is the Thom groupoid of the complex bundle ${}^\pi\mathcal{A}\to A$, we get that $\pi_!^0$ is the inverse of the Thom isomorphism of this complex bundle [20]. By Diagram (32) and the five lemma, $\pi_!^{nc}$ is an isomorphism too. Now Diagrams (34) and (33) give:

$$\varphi_!^\bullet\circ\big(i_*\circ(\pi_!^\bullet)^{-1}\big)=\mathrm{Id},\qquad \text{with } \bullet=nc \text{ or } 0. \tag{35}$$

This proves that $\varphi_!^0$ and $\varphi_!^{nc}$ are surjective and gives explicit sections. □

Finally, we denote by map $\varphi_*: K(\mu_{\partial\Sigma})\longrightarrow K(\mu_{\partial M})$ the map induced by $\varphi_!^{nc}$ through the isomorphism of Theorem 2.4. Note that the homomorphisms $q_{\partial\Sigma}$ and $q_{\partial M}$ are then exchanged with the homomorphisms $\mathrm{ev}_{t=0}$.

4. Geometric K-homology

Geometric cycles. Let $(M,\mathcal{G})$ be a Lie manifold.

Definition 4.1. A even (odd) geometric cycle over $(M,\mathcal{G})$ is a 4-tuple $x=(\Sigma,\varphi,E,B)$ consisting of

(1) an even (odd) dimensional compact manifold with corners M;
(2) a tame surjective submersion $\varphi:\Sigma\longrightarrow M$;
(3) an even (odd) Dirac bundle (E,D) on $(\Sigma,{}^\varphi\mathcal{G})$;
(4) an self-adjoint even (odd) Dirac $\partial\Sigma$-taming B of D.

The set of geometric cycles of parity j is denoted by ${}^{\mathcal{V}}\mathrm{E}_j^{geo}(M)$.

If the geometric cycle $x=(\Sigma,\varphi,E,B)$ is even, we get a class

$$[B]_0:=[B_+]_{\partial\Sigma,\mathrm{ev}}\in K_0(C^*(\mathcal{T}\Sigma))$$

and if it is odd, we get a class

$$[B]_1 := [B]_{\partial\Sigma,\mathrm{odd}} \in KK_1(\mathbb{C}, C^*(\mathcal{T}\Sigma)) \simeq K_1(C^*(\mathcal{T}\Sigma)).$$

Applying the pushforward map $\varphi_!^{\mathrm{nc}}$, we get classes in $K_*(C^*(\mathcal{T}M))$ as well.

Definition 4.2. Let $x = (\Sigma, \varphi, E, B)$ be a geometric cycle of parity j. A geometric cycle $x' = (\Sigma', \varphi', E', B')$ is isomorphic to x if there exists a diffeomorphism $\kappa : \Sigma \to \Sigma'$ such that

(1) $\varphi' \circ \kappa = \varphi$ and $(E, D) \simeq \kappa^*(E', D')$ is an isomorphism of Dirac bundles.
(2) $[B]_j = \kappa^*[B']_j \in K_j(C^*(\mathcal{T}\Sigma))$.

Definition 4.3.

(1) The disjoint union of $x_i = (\Sigma_i, \varphi_i, E_i, B_i) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M)$, $i = 0, 1$, is
$$x_0 \cup x_1 = (\Sigma_0 \cup \Sigma_1, \varphi_0 \cup \varphi_1, E_0 \cup E_1, B_0 \cup B_1) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M). \tag{36}$$
(2) The direct sum of $x_i = (\Sigma, \varphi, E_i, B_i) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(X)$, $i = 0, 1$, is
$$x_0 \oplus x_1 = (\Sigma, \varphi, E_0 \oplus E_1, B_0 \oplus B_1) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M). \tag{37}$$
(3) If $x = (\Sigma, \varphi, E, B) \in {}^{\mathcal{V}}\mathrm{E}_i^{geo}(M)$ then the opposite cycle is
$$-x = (\Sigma, \varphi, E^{op}, B^{op}). \tag{38}$$
Here E^{op} is the same bundle, with the opposite grading in the even case, and $B^{op} = -B$. We have $[B^{op}]_j = -[B]_j$ for $j = 0, 1$.

Cobordisms. To define cobordism of geometric cycles, we introduce:

Definition 4.4. A cobordism over M is a triple (W, Y, Φ) such that

(1) $\Phi : W \longrightarrow M$ is a surjective submersion,
(2) $Y = H_1 \cup \cdots \cup H_k$ is a union of boundary faces of W called relative boundary faces, the remaining boundary faces being called absolute and absolute faces are assumed to be pairwise disjoint,
(3) $\Phi^{-1}(\partial M) = Y$ and for all relative boundary faces H, $\Phi(H) \in \mathcal{F}_1(M)$,
(4) If $H \in \mathcal{F}_1(W)$ is absolute, then $\Phi|_H : H \to M$ is tame.

The submersion $\Phi : W \to M$ is no more tame. As a consequence, if $(M, \mathcal{G})$ is a Lie manifold, the absolute faces of W are not saturated for the pullback groupoid ${}^{\Phi}\mathcal{G}$. We will replace the latter by [34, 22] :

$${}^{\Phi_b}\mathcal{G} \rightrightarrows W \tag{39}$$

defined by blowing up successively all the absolute faces $H_0, \ldots, H_\ell$ of W. Firstly, set:

$$\mathcal{G}_0 = {}^{\Phi}\mathcal{G}, \qquad \beta_0 = \mathrm{Id}, \qquad \mathcal{H}_0 = {}^{\Phi|_{H_0}}\mathcal{G} \tag{40}$$

and then for any $i \geq 0$:

$$\mathcal{G}_{i+1} = \mathrm{Sblup}_{r,s}(\mathcal{G}_i; \mathcal{H}_i) \xrightarrow{\beta_{i+1}} \mathcal{G}_i, \quad \mathcal{H}_{i+1} = (\beta_1 \circ \ldots \beta_{i+1})^{-1}({}^{\Phi|_{H_{i+1}}}\mathcal{G}(M)) \tag{41}$$

The required groupoid is ${}^{\Phi_b}\mathcal{G} = \mathcal{G}_\ell$ and it does not depend on the order of blows-up for the initial submanifolds ${}^{\Phi|_{H_i}}\mathcal{G}(M)$ are pairwise disjoint. This is true for any i for the blowups manifolds $\mathrm{Sblup}(\mathcal{G}_i; \mathcal{H}_i) = [\mathcal{G}_i, \mathcal{H}_i]$ by [33], and since the structural maps coincide over the open dense subset $(W \setminus \partial W)^2$, the result follows. We now have by construction:

(1) all faces $H \in \mathcal{F}_1(W)$ are saturated for ${}^{\Phi_b}\mathcal{G}$.

(2) For any absolute $H \in \mathcal{F}_1(W)$, we consider an open neighborhood $H \subset \mathcal{U} \simeq H \times [0, +\infty)$ and then we have by [22, Paragraph 5.3.5] (after rearranging if necessary the order of blowups in order to have H at the end):

$$\begin{aligned}({}^{\Phi_b}\mathcal{G})_{\mathcal{U}} &\simeq \big(\mathrm{Sblup}_{r,s}(W^2, H^2) \times_{W^2} \mathcal{G}_{\ell-1}\big)|_U \\ &\simeq (\mathrm{Sblup}_{r,s}(W^2, H^2)|_{\mathcal{U}} \ \times_{\mathcal{U}^2} \ {}^{\Phi|_{\mathcal{U}}}(\mathcal{G}) \\ &\simeq (H^2 \times [0, +\infty) \rtimes \mathbb{R}) \ \times_{\mathcal{U}^2} \ {}^{\Phi|_{\mathcal{U}}}(\mathcal{G}), \text{ using } \mathcal{U} \simeq H \times [0, +\infty). \end{aligned} \tag{42}$$

In particular

$$({}^{\Phi_b}\mathcal{G})_H \simeq \mathbb{R} \times {}^{\Phi|_H}(\mathcal{G}) \tag{43}$$

The last isomorphism will be called a boundary decomposition of ${}^{\Phi_b}\mathcal{G}$ and the choice of collar diffeomorphism $\mathcal{U} \simeq H \times [0, +\infty)$ (coming from the choice of a defining function) for the absolute faces will be considered as part of the data in the sequel. The identifications above provide an isomorphism

$$({}^{\Phi_b}\mathcal{A})_{\mathcal{U}} \simeq \rho^*(\mathbb{R} \times {}^{\Phi|_H}(\mathcal{A}))$$

where $\rho : \mathcal{U} \to H$ corresponds to the first projection through the collar diffeomorphism. A Dirac bundle (E, D) over $(W, {}^{\Phi_b}\mathcal{G})$ is locally of product type if it satisfies [10, Definition 3.4] with $TH \times TN$ replaced here by $T[0, 1) \times {}^{\Phi|_H}(\mathcal{A})$ and using the isomorphism above. In such a case, we can apply the boundary reduction of [10, Paragraph 3.2], replacing H and TN there respectively by $[0, +\infty)$ and ${}^{\Phi|_H}(\mathcal{A})$ here, and we get a Dirac bundle (or rather an isomorphism class of Dirac bundles) over $(H, {}^{\Phi|_H}(\mathcal{G}))$ of the opposite parity denoted by (E_H, D_H) and called the boundary reduction of (E, D) to the absolute face H.

Definition 4.5. A (even, odd) cobordism over $(M, \mathcal{G})$ is a 5-tuple $w = (W, Y, \Phi, E, B)$ where:

(1) (W, Y, Φ) is a (even, odd dimensional) manifold cobordism over M;
(2) A (even, odd) Dirac bundle (E, D) on $(W, {}^{\Phi_b}\mathcal{G})$ is given;
(3) a self-adjoint (even, odd) Dirac Y-taming B of D.

A null-cobordism is a cobordism with exactly one absolute face.

If j is the parity of the cobordism, we get a class $[B]_j \in K_j(C^*({}^{\Phi_b}\mathcal{G}_Y^{\mathcal{F}}))) \simeq K_j(C^*(\mathcal{T}_Y W))$. Let $w = (W, Y, \Phi, E, B)$ be a odd cobordism over $(M, \mathcal{G})$, let $\varphi : \Sigma \to M$ where Σ is an absolute face of W. Denote by D the underlying Dirac operator of w and pick up a boundary reduction (E_Σ, D_Σ) of (E, D). We consider the commutative diagram:

$$\begin{array}{ccccc} K_1(C^*(\mathcal{T}_Y W)) & \xrightarrow{\rho_\Sigma} & K_1(C^*(\mathbb{R} \times \mathcal{T}\Sigma)) & \xrightarrow{\text{Bott}} & K_0(C^*(\mathcal{T}\Sigma)) \\ \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} \\ K^1({}^{\Phi_b}\mathcal{A}) & \xrightarrow{\rho_\Sigma} & K^1(\mathbb{R} \times {}^{\varphi}\mathcal{A}) & \xrightarrow{\text{Bott}} & K_c^0({}^{\varphi}\mathcal{A}) \end{array} \tag{44}$$

where ρ_Σ is the composition of the natural restriction homomorphism

$$C^*(\mathcal{T}_Y W) \longrightarrow C^*(({}^{\Phi_b}\mathcal{G})^{\mathcal{F}}|_\Sigma)$$

with the isomorphism induced by (43) at the level of adiabatic and Fredholm groupoids:

$$C^*(({}^{\Phi_b}\mathcal{G})^{\mathcal{F}}|_\Sigma) \simeq C^*(\mathbb{R} \times ({}^{\varphi}\mathcal{G})^{\mathcal{F}})$$

and then with the morphism $\mathrm{ev}_{zt=0} : C^*({}^{\varphi}\mathcal{G})^{\mathcal{F}}) \to C^*(\mathcal{T}\Sigma)$ (that is, the one which gives the K-equivalence $C^*(\mathbb{R} \times ({}^{\varphi}\mathcal{G})^{\mathcal{F}}) \overset{K}{\sim} C^*(\mathbb{R} \times \mathcal{T}\Sigma)$). It follows immediately from the diagram (44) that

$$\mathrm{ev}_{t=0}\big(\mathrm{Bott} \circ \mathrm{ev}_{zt=0} \circ \rho_\Sigma([B]_{Y,\mathrm{odd}})\big) = [D_\Sigma]_{\mathrm{pr},\mathrm{ev}}. \tag{45}$$

In particular, D_Σ is $\partial\Sigma$-tameable and we can pick up a taming B_Σ using Theorem 3.3 such that:

$$[B_\Sigma]_{\partial\Sigma,\mathrm{ev}} = \mathrm{Bott} \circ \rho_\Sigma([B]_{Y,\mathrm{odd}}). \tag{46}$$

We get an even geometric cycle

$$\big(\Sigma, \varphi, E_\Sigma, B_\Sigma\big) \in {}^{\mathcal{V}}\mathrm{E}_0^{geo}(M). \tag{47}$$

Picking up a different representative (E'_Σ, D'_Σ) and a different taming B'_Σ satisfying (46) provides isomorphic geometric cycles.

Definition 4.6. With the notation above, the isomorphism class of (47) is called the boundary reduction of w to Σ.

The case of the other parity is similar.

Definition 4.7.
• Two geometric cycles $x_i = (\Sigma_i, \varphi_i, E_i, B_i)$ over $(M, \mathcal{G})$, $i = 0, 1$, are cobordant if there exists a cobordism $w = (W, Y, \Phi, E, B)$ such that

(1) The absolute faces of W are exactly Σ_0 and Σ_1;
(2) $\Phi|_{\Sigma_i} = \varphi_i$
(3) The boundary reduction of w with respect to Σ_i is the isomorphism class of $(-1)^i x_i$.

• A geometric cycle $x = (\Sigma, \varphi, E, \Psi)$ over $(M, \mathcal{G})$ is null-cobordant if there exists a null-cobordism with Σ as unique absolute face.

Example 4.8.

(1) Let $x_i = (\Sigma, \varphi, E, B_i) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M)$, $i = 0, 1$, be two even cycles connected by a *tame homotopy*. This means that there exists a C^∞ homotopy of Dirac bundles (E, D_t) (i.e., C^∞ homotopies of Clifford homomorphisms and of connections) and a family $(B_t)_{t\in[0,1]}$ of tamings of $(D_t)_{t\in[0,1]}$ connecting B_0 to B_1 and such that:

$$\sigma_{\partial\Sigma}(\Lambda^{-1}B_t) \in \Sigma_{\partial\Sigma}(E)^\times$$

is a continuous path. In particular: $[B_{0,+}]_{\partial\Sigma,\mathrm{ev}} = [B_{1,+}]_{\partial\Sigma,\mathrm{ev}} \in \mathcal{K}_0(C^*(\mathcal{T}\Sigma))$. The cycles x_0 and x_1 are then cobordant. A cobordism is given by

$$w = \big(\Sigma \times [0,1]_t, Y, \varphi', E', B'\big)$$

where:
(a) $Y = \partial\Sigma \times [0,1]$, $\varphi' = \mathrm{pr}_1 \circ \varphi$ and $E' = \mathrm{pr}_1^* E \otimes \mathbb{C}$.
(b) (E', D') is the product of (E, D_t), with $([0,1] \times \mathbb{C}, it(1-t)\frac{\partial}{\partial t})$. Thus:

$$D' = D_t + it(1-t)\frac{\partial}{\partial t}. \tag{48}$$

(c) The absolute faces are $\Sigma_0 = \Sigma \times \{0\}$ and $\Sigma_1 = \Sigma \times \{1\}$.

(d) We then consider the commutative diagram:

$$\begin{array}{ccc} K_1(C^*(\mathcal{T}_Y(\Sigma\times[0,1]))) & \xrightarrow{\tau} & K^1({}^{\varphi}\mathcal{A}\times {}^bT[0,1]) \\ \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} \\ K_1(C^*((\mathcal{T}\Sigma)\times\mathbb{R})) & \xrightarrow{\tau} & K^1({}^{\varphi}\mathcal{A}\times\mathbb{R}) \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \beta} \\ K_0(C^*(\mathcal{T}\Sigma)) & \xrightarrow{\tau} & K^0({}^{\varphi}\mathcal{A}) \end{array} \tag{49}$$

The vertical arrows are isomorphisms. Since $\mathrm{ev}_{t=0}[D']_{\mathrm{pr,odd}} = [D_{0,+}]_{\mathrm{pr,ev}}\otimes\beta$ and $[D_{0,+}]_{\mathrm{pr,ev}}$ belongs to $\operatorname{im}\tau_*$, we get that $[D']_{\mathrm{pr,odd}}$ belongs to $\operatorname{im}\tau_*$ too and we apply Theorem 3.3 to choose a Y-taming B' of D' such that $\mathrm{ev}_{t=0}[B']_{Y,\mathrm{odd}} = [B_{0,+}]_{\partial\Sigma,\mathrm{ev}}\otimes\beta$.

(2) For any $x\in {}^{\mathcal{V}}\mathrm{E}^{geo}(M)$, the geometric cycle $x\cup(-x)$ is null-cobordant. A null-cobordism is given by the previous cobordism in which $\Sigma\times\{0,1\}$ is considered as the unique absolute face.

(3) Let $x=(\Sigma,\varphi,E,B)\in {}^{\mathcal{V}}\mathrm{E}^{geo}_0(M)$ with underlying Dirac operator denoted by D. Let (E',D') be the product of (E,D) with the spin Dirac bundle (E_2,D_2) of $\mathbb{D}^2$. Let (E_1,D_1) be the boundary reduction of (E_2,D_2), that is the spin Dirac bundle associated with the spin structure of $\mathbb{S}^1$ that bounds the one of (E_2,D_2) of $\mathbb{D}^2$. Let also (E'',D'') be the product of (E,D) with (E_1,D_1). Consider the commutative diagram:

$$\begin{array}{ccc} K_0(C^*(\mathcal{T}_{(\partial\Sigma)\times\mathbb{D}^2}(\Sigma\times\mathbb{D}^2))) & \xrightarrow{\tau} & K^0({}^{\varphi}\mathcal{A}\times {}^bT\mathbb{D}^2) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ K_0(C^*(\mathcal{T}_{\partial\Sigma\times\mathbb{S}^1}(\Sigma\times\mathbb{S}^1)\times\mathbb{R})) & \xrightarrow{\tau} & K^0({}^{\varphi}\mathcal{A}\times T\mathbb{S}^1\times\mathbb{R}) \\ \uparrow{\scriptstyle S} & & \uparrow{\scriptstyle S} \\ K_1(C^*(\mathcal{T}_{\partial\Sigma\times\mathbb{S}^1}(\Sigma\times\mathbb{S}^1))) & \xrightarrow{\tau} & K^1({}^{\varphi}\mathcal{A}\times T\mathbb{S}^1) \\ \uparrow{\scriptstyle \otimes D_1} & & \uparrow{\scriptstyle \otimes D_1} \\ K_0(C^*(\mathcal{T}_{\partial\Sigma}(\Sigma))) & \xrightarrow{\tau} & K^0({}^{\varphi}\mathcal{A}) \end{array} \tag{50}$$

The map ρ corresponds to the restriction to the boundary of $\mathbb{D}^2$ and S is the suspension isomorphism. All the vertical arrows are isomorphisms.

Using the lower part of the diagram we first see that there exists a $\partial\Sigma\times\mathbb{S}^1$-taming B'' of D'' such that

$$[B'']_{\partial\Sigma\times\mathbb{S}^1,\mathrm{odd}} = [B]_{\partial\Sigma,\mathrm{ev}}\otimes[D_1]_{\mathrm{odd}}.$$

We then set:

$$0_x = \left(\Sigma\times\mathbb{S}^1, \mathrm{pr}_1\circ\varphi, E'', B''\right)\in {}^{\mathcal{V}}\mathrm{E}^{geo}_1(M).$$

Now using the upper part of the diagram, we obtain a $\partial\Sigma\times\mathbb{D}^2$-taming B' of D' such that

$$\rho[B']_{(\partial\Sigma)\times\mathbb{D}^2,\mathrm{ev}} = S[B'']_{\partial\Sigma\times\mathbb{S}^1,\mathrm{odd}}.$$

This provides a null-cobordism for 0_x as follows:

$$w = \left(\Sigma\times\mathbb{D}^2, (\partial\Sigma)\times\mathbb{D}^2, \mathrm{pr}_1\circ\varphi, E', B'\right). \tag{51}$$

Starting with an odd x, we get a null-cobordant cycle 0_x in the same way.

(4) Combining the previous constructions, we see that for any $x, y \in {}^{\mathcal{V}}\mathrm{E}^{geo}(M)$, the geometric cycles y and $y \cup 0_x$ are cobordant.

Vector bundle modification. Let $x = (\Sigma, \varphi, E, B)$ be an even geometric cycle over $(M, \mathcal{G})$ and $\pi : V \to \Sigma$ an even rank Spin^c-vector bundle. Consider the sphere bundle $Z = S(V \oplus 1_{\mathbb{R}})$ of $\mathrm{pr}_1 \circ \pi : V \oplus \mathbb{R} \to \Sigma$. The total space Z is a compact manifold with corners and the projection $\pi_m : Z \to \Sigma$ is tame, as well as $\varphi_m = \varphi \circ \pi_m : Z \longrightarrow M$. Note that

$$ {}^{\varphi_m}(\mathcal{G}) = {}^{\pi_m}({}^{\varphi}\mathcal{G}) \quad \text{and} \quad {}^{\varphi_m}\mathcal{A} \simeq \pi_m^*({}^{\varphi}\mathcal{A}) \oplus VZ \tag{52}$$

where $VZ = \ker d\pi_m$. The Spin^c-structure of V induces a Spin^c-structure on the bundle $VZ \to \Sigma$ and therefore on the bundle

$$\mathcal{A}_Z := {}^{\varphi_m}\mathcal{A} \longrightarrow {}^{\varphi}\mathcal{A} =: \mathcal{A}_\Sigma. \tag{53}$$

Using $\mathcal{L}_{\pi_m}$, we get a commutative diagram:

$$\begin{array}{ccc} K_0(C^*(\mathcal{G}(Z)^{ad})) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*(\mathcal{A}_Z)) \\ \Big\downarrow{\scriptstyle (\pi_m)_!^{ad}} & & \Big\downarrow{\scriptstyle (\pi_m)_!^{0}} \\ K_0(C^*(\mathcal{G}(\Sigma)^{ad})) & \xrightarrow{\mathrm{ev}_{t=0}} & K_0(C^*(\mathcal{A}_\Sigma)) \end{array} \tag{54}$$

Here $\mathcal{G}(Z) = {}^{\varphi_m}(\mathcal{G})$ and $\mathcal{G}(\Sigma) = {}^{\varphi}\mathcal{G}$. The horizontal maps are isomorphisms. Proceeding as in [20], we prove that $(\pi_m)_!^0$ is the Thom isomorphism of the Spin^c-bundle $\mathcal{A}_Z \longrightarrow \mathcal{A}_\Sigma$. Then $(\pi_m)_!^{ad}$ is an isomorphism too.

Let D_Σ be the Dirac operator of the cycle x, let S_Z be the complex spinor bundle of VZ and set $E_Z = E \otimes S_Z$. By [7, Propositions 3.6 and 3.11] we can choose a Dirac bundle (E_Z, D_Z) such that

$$(\pi_m)_!^0[D_Z)]_{\mathrm{pr,ev}} = [D_\Sigma]_{\mathrm{pr,ev}}.$$

We then deduce from the previous diagram that

$$(\pi_m)_!^{ad}([D_Z]_{\mathrm{ad,ev}}) = [D_\Sigma]_{\mathrm{ad,ev}} \in K_0(C^*(\mathcal{G}(\Sigma)^{ad})).$$

Now, since D_Σ is tameable, we obtain by using the commutative diagram:

$$\begin{array}{ccccccc} K_1(C^*(\mathcal{G}(Z)|_{\partial Z})) & \xrightarrow{\partial} & K_0(C^*(\mathcal{G}(Z)^{\mathcal{F}})) & \longrightarrow & K_0(C^*(\mathcal{G}(Z)^{ad})) & \xrightarrow{\mathrm{ev}_{\partial Z\times 1}} & K_0(C^*(\mathcal{G}(Z)|_{\partial Z})) \\ \simeq\Big\downarrow{\scriptstyle \mathcal{M}} & & \simeq\Big\downarrow{\scriptstyle (\pi_m)_!^{\mathcal{F}}} & & \simeq\Big\downarrow{\scriptstyle (\pi_m)_!^{ad}} & & \simeq\Big\downarrow{\scriptstyle \mathcal{M}} \\ K_1(C^*(\mathcal{G}(\Sigma)|_{\partial\Sigma})) & \xrightarrow{\partial} & K_0(C^*(\mathcal{G}(\Sigma)^{\mathcal{F}})) & \longrightarrow & K_0(C^*(\mathcal{G}(\Sigma)^{ad})) & \xrightarrow{\mathrm{ev}_{\partial\Sigma\times 1}} & K_0(C^*(\mathcal{G}(\Sigma)|_{\partial\Sigma})). \end{array} \tag{55}$$

that D_Z is ∂Z-tameable too and we pick up a ∂Z-taming B_Z such that

$$(\pi_m)_!^{\mathcal{F}}[B_Z] = [B]$$

The constructions are similar for odd geometric cycles.

Definition 4.9. A vector bundle modification of a geometric cycle $x = (\Sigma, \varphi, E, B)$ over $(M, \mathcal{G})$ by an even rank Spin^c-vector bundle $\pi : V \to \Sigma$ is a geometric cycle $m(x, V) = (Z, \varphi_m, E_m, B_Z)$ over $\mathcal{G}(M)$ such that

(1) $Z = S(V \oplus 1_{\mathbb{R}})$ is the sphere bundle;
(2) $\varphi_m = \varphi \circ \pi_m : Z \longrightarrow M$;
(3) $\pi_!^{\mathrm{nc}}[B_Z]_{\partial Z,*} = [B]_{\partial\Sigma,*} \in K_*(C^*\mathcal{T}\Sigma))$.

Geometric K-homology.

Definition 4.10. The equivalence relation $\sim$ on ${}^{\mathcal{V}}\mathrm{E}_*^{geo}(M)$ is the one generated by the following operations:

(i) Isomorphisms of geometric cycles;
(ii) Direct sums: if $x_i = (\Sigma, \varphi, E_i, B_i)$, $i = 1, 2$ are geometric cycles then
$$x_1 \cup x_2 \sim (\Sigma, \varphi, E_1 \oplus E_2, B_1 \oplus B_2); \tag{56}$$
(iii) Cobordisms;
(iv) Vector bundle modifications.

The quotient set
$${}^{\mathcal{V}}\mathrm{K}_*^{geo}(M) := {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M)/\sim$$
is called geometric K-homology of $(M, \mathcal{G})$.

Theorem 4.11. *The following formula:*
$$\forall x_0, x_1 \in {}^{\mathcal{V}}\mathrm{E}^{geo}(M), \qquad [x_0] + [x_1] = [x_0 \cup x_1]. \tag{57}$$
turns $({}^{\mathcal{V}}\mathrm{K}^{geo}(M), +)$ *into an abelian group.*

Proof. If the equivalence $x_0 \sim x_0'$ is given by one of the four elementary operations of Definition 4.10 then $x_0 \cup x_1 \sim x_0' \cup x_1$. Therefore (57) is well defined. Commutativity and associativity of $+$ is obvious. The neutral element is represented by 0_x and $-[x] = [-x]$ for any $x \in {}^{\mathcal{V}}\mathrm{E}^{geo}(M)$: details are provided in Example 4.8. □

Comparison map.

Theorem 4.12. *The map* $(\Sigma, \varphi, E, B) \in {}^{\mathcal{V}}\mathrm{E}_*^{geo}(M) \longmapsto \varphi_!^{nc}[B]_{\partial\Sigma,*}$ *gives rise to a well defined homomorphism:*
$$\lambda\colon {}^{\mathcal{V}}\mathrm{K}_*^{geo}(M) \to K_*(C^*(\mathcal{T}M)). \tag{58}$$

Proof. We prove the even case, the odd one is similar.
Invariance under cobordism: Let $w = (W, Y, \Phi, E, B)$ be a cobordism between two even geometric cycles $x_i = (\Sigma_i, \varphi_i, E_i, B_i)$, $i = 0, 1$ over $(M, \mathcal{G})$. We denote by:

(1) $\mathcal{G}(W) = {}^{\Phi_b}\mathcal{G}$ the groupoid associated with the cobordism w;
(2) $\mathcal{G}_i = {}^{\varphi_i}\mathcal{G}$ the groupoid associated with x_i, $i = 0, 1$;
(3) $\rho_i := \rho_{\Sigma_i} : C^*(\mathcal{T}_Y W) \to C^*(\mathcal{T}\Sigma_i \times \mathbb{R})$ the homomorphism defined just after Diagram (44).

• Recall that we have by definition of a cobordism of geometric cycles:
$$\rho_i[B]_{Y,\mathrm{odd}} = (-1)^i [B_i]_{\partial\Sigma_i,\mathrm{ev}} \otimes \beta$$
where β is the chosen Bott generator of $K_1(C^*(\mathbb{R}))$.

• We are going to prove that
$$(\varphi_0)_!^{\mathrm{nc}}([B_0]_{\partial\Sigma_0,\mathrm{ev}}) = (\varphi_1)_!^{\mathrm{nc}}([B_1]_{\partial\Sigma_1,\mathrm{ev}}) \tag{59}$$
where the homomorphisms are defined in (26).
Denote by J_i the kernel of ρ_i and observe that $J_0 + J_1 = C^*(\mathcal{T}_Y W)$. Set $J = J_0 \cap J_1$ and consider the exact sequences:
$$0 \longrightarrow J \longrightarrow J_i \xrightarrow{\rho_i} C^*(\mathcal{T}\Sigma_i \times \mathbb{R}) \longrightarrow 0. \tag{60}$$
for $i = 0, 1$. We denote by $\partial_i \in KK_1(C^*(\mathcal{T}\Sigma_i \times \mathbb{R}), J_i)$ the associated boundary elements. Consider also:
$$0 \longrightarrow J \longrightarrow C^*(\mathcal{T}_Y W) \xrightarrow{\rho_0 \oplus \rho_1} C^*(\mathcal{T}\Sigma_0 \times \mathbb{R}) \times C^*(\mathcal{T}\Sigma_1 \times \mathbb{R}) \longrightarrow 0 \tag{61}$$

whose ideal is given by $J = J_0 \cap J_1$ and boundary element denoted by ∂. By [46, Lemma 3.5]) we have:

$$\partial(x_0, x_1) = \partial_0(x_0) + \partial_1(x_1) \in KK_1(\mathbb{C}, J), \quad x_i \in KK_1(\mathbb{C}, C^*(\mathcal{T}\Sigma_i \times \mathbb{R})). \tag{62}$$

We know that

$$\rho_0 \oplus \rho_1([B]_{Y,\mathrm{odd}}) = ([B_0]_{\partial\Sigma_0,\mathrm{ev}} \otimes \beta, -[B_1]_{\partial\Sigma_1,\mathrm{ev}} \otimes \beta)\,, \tag{63}$$

therefore formula (62) and exactness imply:

$$\partial_0([B_0]_{\partial\Sigma_0,\mathrm{ev}} \otimes \beta) = \partial_1([B_1]_{\partial\Sigma_1,\mathrm{ev}} \otimes \beta) \in K_0(J). \tag{64}$$

It is now time to extend the deformation (20):

$$\mathcal{L}_w = (\mathcal{G}(W))^{ad} \times \{u = 0\} \cup {}^{(\Phi_1)_b}(\mathcal{G}^{ad}) \times (0,1]_u \rightrightarrows W \times [0,1]_t \times [0,1]_u. \tag{65}$$

Here $\Phi_1 = \Phi \circ \mathrm{pr}_1 : W \times [0,1]_t \to M$. We can consider various saturated sub-groupoids of $\mathcal{L}_w$. For instance:

$$\mathcal{L}_Y^{\mathrm{nc}}(w) = \mathcal{L}_w|_{(W \times \{t=0\} \cup Y \times (0,1)_t) \times [0,1]_u}. \tag{66}$$

If we consider in $\mathcal{L}_Y^{\mathrm{nc}}(w)$ the faces corresponding to Σ_i, we recover the groupoids $\mathcal{L}_{\varphi_i}^{\mathrm{nc}} \times \mathbb{R}$. Continuing in this way and denoting by $\Phi' : W \setminus (\Sigma_0 \cup \Sigma_1) \to M$ the restriction of Φ (which is again a surjective submersion), we get the following commutative diagram:

$$\begin{tikzcd}
K_1(C^*(\mathcal{T}\Sigma_0 \times \mathbb{R})) \oplus K_1(C^*(\mathcal{T}\Sigma_1 \times \mathbb{R})) \arrow[r, "\partial"] & K_0(J) \\
K_1(C^*(\mathcal{L}_{\varphi_0}^{\mathrm{nc}} \times \mathbb{R})) \oplus K_1(C^*(\mathcal{L}_{\varphi_1}^{\mathrm{nc}} \times \mathbb{R})) \arrow[r, "\partial'"] \arrow[u, "\simeq"', "u=0"] \arrow[d, "u=1"] & K_0(...) \arrow[u, "\simeq"', "u=0"] \arrow[d, "u=1"] \\
K_1(C^*({}^{\varphi_1}(\mathcal{T}M \times \mathbb{R}))) \oplus K_1(C^*({}^{\varphi_1}(\mathcal{T}M \times \mathbb{R}))) \arrow[r, "\partial''"] \arrow[d, "\simeq"', "\mathcal{M}"] & K_0(C^*({}^{\Phi'_1}(\mathcal{T}M))) \arrow[d, "\simeq"', "\mathcal{M}"] \\
K_1(C^*(\mathcal{T}M \times \mathbb{R})) \oplus K_1(C^*(\mathcal{T}M \times \mathbb{R})) \arrow[r, "\partial'''"] & K_0(C^*\mathcal{T}M))
\end{tikzcd} \tag{67}$$

The map in the bottom line is given by addition and Bott periodicity, in particular, if $\partial'''(u \oplus v) = 0$ then $u = -v$.
The map obtained from top to bottom in the left column is equal to $(\varphi_0)_!^{\mathrm{nc}} \otimes \mathrm{Id} \oplus (\varphi_1)_!^{\mathrm{nc}} \otimes \mathrm{Id}$.
Therefore, using the equality (64) together with the commutativity of the previous diagram and the remarks just above, we conclude that the equality (59) holds true.

Invariance under vector bundle modification:
Consider $x = (\Sigma, \varphi, E, \Psi) \in {}^{\mathcal{V}}\mathrm{E}^{geo}(M)$ and a vector bundle modification $m(x, V) = (Z, \varphi_m, E_m, B_Z)$. One one hand, we know that

$$(\pi_m)_!^{\mathcal{F}}([B_Z]) = [B].$$

On the other hand, we have $\varphi_m = \varphi \circ \pi_m$, therefore:

$$(\varphi_m)_!^{\mathcal{F}}([B_Z]) = (\varphi \circ \pi_m)_!^{\mathcal{F}} = (\varphi_!^{\mathcal{F}} \circ (\pi_m)_!^{\mathcal{F}})([B_Z]) = \varphi_!^{\mathcal{F}}([B])$$

and the equality $(\varphi_m)_!^{\mathcal{F}}([B_Z]) = \varphi_!^{\mathcal{F}}([B])$ proves the invariance under vector bundle modification. □

5. Reduction to Callias

Let $(M, \mathcal{G})$ be a Lie manifold. We are going to define a *clutching map*[2]:

$$\widetilde{c}\colon \Gamma(\mu_{\partial M}) \to {}^{\mathcal{V}}\mathrm{K}_0^{geo}(M) \tag{68}$$

and show that it descends to a *clutching homomorphism* $c\colon {}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M) \to {}^{\mathcal{V}}\mathrm{K}_0^{geo}(M)$. We set $\Sigma_{\mathcal{A}} = S(\mathcal{A} \oplus \mathbb{R})$ for the clutching space of $\mathcal{A}$. We denote by φ the tame submersion $\Sigma_{\mathcal{A}} \to M$. We can write, $\Sigma_{\mathcal{A}} = \mathcal{B}(\mathcal{A})_+ \cup \mathcal{B}(\mathcal{A})_-$, where $\mathcal{B}(\mathcal{A})$ denotes the ball bundle in $\mathcal{A}$ and $\mathcal{B}(\mathcal{A})_\pm$ denote the upper and lower hemispheres respectively. Let $\widehat{\mathcal{A}} = \mathcal{A} \cup S(\mathcal{A})$ be the radial compactification of $\mathcal{A}$ and $\widehat{\pi}\colon \widehat{\mathcal{A}} \to M$ the corresponding projection map.
Let $P \in \Psi_{\mathcal{V}}^m(M; E_0, E_1)$ be a fully elliptic operator. By ellipticity of P we have an isomorphism

$$\sigma_{pr}(P)\colon \pi^* E_0 \xrightarrow{\sim} \pi^* E_1.$$

The clutched bundle $E_\sigma \to \Sigma_{\mathcal{A}}$ is defined by the glueing of pullbacks of E_0 and E_1, along the boundary stratum $S(\mathcal{A})$, using $\sigma_{pr}(P)$:

$$E_\sigma = \widehat{\pi}^* E_0 \cup_{S(\mathcal{A})} \widehat{\pi}^* E_1.$$

To define a geometric cycle $\widetilde{c}(\mathcal{E}_0, \mathcal{E}_1, \sigma_f(\Lambda^{-1}P))$ associated with $(\mathcal{E}_0, \mathcal{E}_1, \sigma_f(\Lambda^{-1}P)) \in \Gamma(\mu_{\partial M})$, we consider the Dirac operator $D_{\Sigma_{\mathcal{A}}}$ on the Lie manifold $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G})$ associated with the Spin^c-structure of ${}^{\varphi}\mathcal{A} \to \Sigma_{\mathcal{A}}$, and we first observe:

Lemma 5.1. *([8]) Let $(M, \mathcal{G})$ be a Lie manifold and $P \in \Psi_{\mathcal{G}}(E_0, E_1)$ be an elliptic operator. Then:*

$$(i \circ \mathrm{Thom})([P]_{\mathrm{pr,ev}}) = [D_{\Sigma_{\mathcal{A}}}]_{\mathrm{pr,ev}} \otimes ([E_\sigma] - [\varphi^* E_1]) \in K^0({}^{\varphi}\mathcal{A}). \tag{69}$$

This lemma is proved in [8] in the case $\mathcal{A} = TM$. The proof is the same here. Secondly:

Proposition 5.2. *Let $(M, \mathcal{G})$ be a Lie manifold and $P \in \Psi_{\mathcal{G}}^m(E_0, E_1)$ be a fully elliptic operator. Then there exists an even Dirac bundle (E, D) on $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G})$ with boundary taming B such that*

$$\varphi_!^{nc}([B]_{\partial\Sigma_{\mathcal{A}},\mathrm{ev}}) = [P]_{\partial M,\mathrm{ev}} \in K_0(C^*(\mathcal{T}M)). \tag{70}$$

In particular, B_+ and P have the same Fredholm index.

Proof. Rewriting the proof of Theorem 3.8, we get a commutative diagram:

$$\begin{array}{ccccc}
K_c^0(\mathcal{A}^*) & \xrightarrow{\ \mathrm{Thom}\ } & K_c^0({}^{\pi}\mathcal{A}^*) & \xrightarrow{\ -\otimes[i]\ } & K_c^0({}^{\varphi}\mathcal{A}^*) \\
\uparrow{\scriptstyle \mathrm{ev}_{t=0}} & & \uparrow{\scriptstyle \mathrm{ev}_{t=0}} & & \uparrow{\scriptstyle \mathrm{ev}_{t=0}} \\
K_0(C^*(\mathcal{T}M)) & \xrightarrow{\ (\pi_!^{nc})^{-1}\ } & K_0(C^*(\mathcal{T}\mathcal{A})) & \xrightarrow{\ -\otimes[i]\ } & K_0(C^*(\mathcal{T}\Sigma_{\mathcal{A}}))
\end{array} \tag{71}$$

[2]We only treat here the clutching map in the even case, the odd case is similar.

By the previous lemma and this diagram, we obtain the existence of a fully elliptic Q on ${}^{\varphi}\mathcal{G}$ such that

$$[Q]_{\mathrm{pr,ev}} = [D_{\Sigma_{\mathcal{A}}}]_{\mathrm{pr,ev}} \otimes ([E_\sigma] - [\varphi^* E_1]) = [D_1]_{\mathrm{pr,ev}} \tag{72}$$

where $(D_1, E_\sigma \oplus \varphi^* E_1)$ is the even Dirac bundle with:

$$D_1 = D_{\Sigma_{\mathcal{A}}} \otimes E_\sigma \oplus D^{op}_{\Sigma_{\mathcal{A}}} \otimes \varphi^* E_1$$

and $D_{\Sigma_{\mathcal{A}}} \otimes E_\sigma$ is the Dirac operator on Σ twisted by E_σ and $D^{op}_{\Sigma_{\mathcal{A}}} = -D_{\Sigma_{\mathcal{A}}}$. Now, using Theorem 3.3, we get an even Dirac bundle $(\widetilde{E}, \widetilde{D})$ with boundary taming $B = \widetilde{D} + R$ such that we still have

$$[B]_{\partial\Sigma_{\mathcal{A}},\mathrm{ev}} = [Q]_{\partial\Sigma_{\mathcal{A}},\mathrm{ev}} \in K_0(C^*(\mathcal{T}\Sigma_{\mathcal{A}})),$$

from which we conclude that (70) holds true using Theorem 3.8, (2). The last assertion comes from Theorem 3.7. □

Now, using Proposition 5.2 to pick up suitable $(\widetilde{E}, \widetilde{D})$ and B, we define:

$$\widetilde{c}(\mathcal{E}_0, \mathcal{E}_1, \sigma_{\partial M}(\Lambda^{-m} P)) = [(\Sigma_{\mathcal{A}}, \widetilde{E}, \varphi, B)]_{iso} \in {}^{\mathcal{V}}\mathrm{K}_0^{geo}(M).$$

The clutching map $\widetilde{c}$ sends a relative cycle to an isomorphism class of geometric cycles (note that two choices of B yield isomorphic cycles).

Theorem 5.3. *The map $\widetilde{c}$ induces a well defined homomorphism*

$$c\colon {}^{\mathcal{V}}\mathcal{F}\mathrm{Ell}(M) \to {}^{\mathcal{V}}\mathrm{K}_0^{geo}(M).$$

Proof. Let $(P_0)_{rel} := (\mathcal{E}_0, \mathcal{F}_0, p_0)$ and $(P_1)_{rel} := (\mathcal{E}_1, \mathcal{F}_1, p_1)$ be elements of $\Gamma_0(\mu_M)$ which are equivalent. Let $\alpha, \beta \in \Gamma_0(\mu_M)$ be elementary elements such that

$$(\mathcal{E}_0, \mathcal{F}_0, p_0) \oplus \alpha \simeq (\mathcal{E}_1, \mathcal{F}_1, p_1) \oplus \beta.$$

Without loss of generality, we can consider that $E = \mathcal{E}_0 = \mathcal{E}_1$, $F = \mathcal{F}_0 = \mathcal{F}_1$ and p_0, p_1 connected by a smooth homotopy p_t of fully elliptic symbols. Denote by $\sigma_t : \pi^* E \longrightarrow \pi^* F$ the map obtained by extending by homogeneity $\sigma_{\mathrm{pr}}(P_t)$ over $\mathcal{A} \setminus \{0\}$ and multiplying it with a function $\chi \in C^\infty(\mathcal{A})$ such that $\chi(\xi) = 0$ in a neighborhood of the 0-section and $\chi(\xi) = 1$ near infinity.
Define $\widehat{E}$ via the glueing diffeomorphism $[0,1] \times S(\mathcal{A}) \times E \to [0,1] \times S(\mathcal{A}) \times F$, $(t, x, v) \mapsto (t, x, \sigma_t(v))$ along the cylinder $[0,1] \times S(\mathcal{A})$. This furnishes the vector bundle $\widehat{E}$ over the cylinder clutching space $\widehat{\Sigma} := [0,1] \times \Sigma$. Denote by $\widehat{\varphi} : \widehat{\Sigma} \longrightarrow M$ the natural projection. Setting $Y = [0,1] \times \partial\Sigma$, we already get a cobordism over M, namely $(\widehat{\Sigma}, \widehat{\varphi}, Y)$.
Let D_j be the Dirac operator underlying the geometric cycle $\widetilde{c}((P_j)_{rel})$. We consider on $\widehat{\Sigma}$ the Spin^{c}-structure that coincides with the one of Σ at $\Sigma \times \{0\}$ and to the opposite one at $\Sigma \times \{1\}$.
Lut us consider a Dirac operator $\widehat{D}$ on the Lie manifold $(\widehat{\Sigma}, {}^{\widehat{\varphi}_b}\mathcal{G})$ whose boundary reductions at $\Sigma \times \{j\}$ coincide with D_j. The corresponding Clifford vector bundle over $\widehat{\Sigma}$ is:

$$\widetilde{E} = S_{\widehat{\Sigma}} \otimes \widehat{E} \oplus (-S_{\widehat{\Sigma}}) \otimes \widehat{\varphi}^* F.$$

Arguing as in Example 4.8, we obtain a Y-taming $\widehat{B}$ such that $(\widehat{\Sigma}, \widehat{\varphi}, Y, \widetilde{E}, \widehat{B})$ is a cobordism between the geometric cycles $\widetilde{c}((P_j)_{rel})$.

□

Theorem 5.4. *The diagram*

$$\begin{tikzcd} K_*(\mu) \arrow[d, "c"] \arrow[dr, "\mathrm{pd}"] & \\ {}^{\mathcal{V}}\mathrm{K}^{geo}(M) \arrow[r, "\lambda"] & K(C^*(\mathcal{T}M)) \end{tikzcd}$$

commutes. In particular, if $P\colon C^\infty(M,E_0)\to C^\infty(M,E_1)$ denotes a fully elliptic pseudodifferential operator on the Lie manifold $(M,\mathcal{G})$, then there is a taming $\mathcal{C}$ on the Lie manifold $(\Sigma_{\mathcal{A}},{}^{\varphi}\mathcal{G})$ such that $\mathrm{ind}(\mathcal{C})=\mathrm{ind}(P)$.

The proof follows from Proposition 5.2.

Appendix A. Cobordisms

Theorem A.1. *The cobordism relation is transitive up to isomorphisms of geometric cycles.*

Proof. Let $x_i=(\Sigma_i,\varphi_i,E_i,B_i)$, $i=1,2,3$, be geometric cycles over $(M,\mathcal{G})$ such that x_1 is cobordant to x_2 and x_2 cobordant to x_3. Let $w_i=(W_i,Y_i,\Phi_i,F_i,C_i)$, $i=1,2$ be respective cobordisms. We set $\Phi=\Phi_1\cup\Phi_2 : W=W_1\underset{\Sigma_2}{\cup}W_2\longrightarrow M$, $Y=Y_1\cup Y_2$. The triple (W,Φ,Y) is a manifold cobordism over M between Σ_1 and Σ_3. We set :

$$\widetilde{\mathcal{G}}(W)=\mathcal{G}(W_1)\bigcup_{\mathcal{G}(\Sigma_2)\times\mathbb{R}}\mathcal{G}(W_2). \tag{73}$$

Also, setting $F=F_1\underset{\Sigma_2}{\cup}F_2$ and using the point (3) in Definition 4.7, we get a Dirac bundle $(F,\widetilde{D})$ over $(W,\widetilde{\mathcal{G}}(W))$ that restricts to the one of w_j on W_j, $j=1,2$. In particular, we have by assumption that $(F,\widetilde{D})$ is Y-tameable.
Note that $\widetilde{\mathcal{G}}(W)=SBlup_{r,s}(\mathcal{G}(W),\mathcal{G}(W)^{\Sigma_2}_{\Sigma_2})$ where

$$\mathcal{G}(W)={}^{b}\Phi\mathcal{G}(M). \tag{74}$$

Choosing a collar decomposition around Σ_2 into W, we get a deformation Lie groupoid:

$$\mathcal{H}(W)=\{t=0\}\times\widetilde{\mathcal{G}}(W)\cup(0,1]_t\times\mathcal{G}(W)\rightrightarrows[0,1]\times W$$

and it is clear that the Dirac bundle $(F,\widetilde{D})$ can be lifted to a Dirac bundle (F,C) over $(W\times[0,1],\mathcal{H}(W))$ and we thus end with a Dirac bundle (F,D) over $(W,\mathcal{G}(W))$ by restriction at $t=1$.
Since $\widetilde{D}$ is Y-tameable, we get using the following diagram (where we use this time Fredholm groupoids rather than their K-equivalent non commutative tangent space counterparts) that C is $Y\times[0,1]$-tameable too and finally D is Y-tameable.

$$\begin{tikzcd} K_0(C^*(\widetilde{\mathcal{G}}(W)^{\mathcal{F}}_Y)) \arrow[r] & K_0(C^*(\widetilde{\mathcal{G}}(W)^{ad})) \\ K_0(C^*(\mathcal{H}(W)^{\mathcal{F}}_{[0,1]\times Y})) \arrow[u, "|_{t=0}"] \arrow[r] \arrow[d, "|_{t=1}"] & K_0(C^*(\mathcal{H}(W)^{ad})) \arrow[u, "|_{t=0}", "\simeq"'] \arrow[d, "|_{t=1}"] \\ K_0(C^*(\mathcal{G}(W)^{\mathcal{F}}_Y)) \arrow[r] & K_0(C^*(\mathcal{G}(W)^{ad})). \end{tikzcd} \tag{75}$$

Fixing a Y-taming B for D then provides the required cobordism (W, Y, Φ, F, B) between x_1 and x_3. □

Acknowledgements

For useful discussions we thank Bernd Ammann, Ulrich Bunke, Paulo Carrillo Rouse, Claire Debord, Magnus Goffeng, Victor Nistor, Elmar Schrohe, Georges Skandalis and Georg Tamme. The first author was supported by the DFG-SPP 2026 'Geometry at Infinity'. The second author was supported by the Grant ANR-14-CE25-0012-01 SINGSTAR.

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