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# A high-gain observer for a class of 2x2 hyperbolic systems with $C^1$ exponential convergence

Constantinos Kitsos, Gildas Besançon, and Christophe Prieur \*

## Abstract

*This paper extends a recently proposed observer design based on high-gain to a more general first order quasilinear hyperbolic system of balance laws. This class of systems is written in an observable form with two states and a distributed measurement. The exponential stability of the related observation error is fully established by means of Lyapunov-based analysis.*

*Keywords: high-gain observers, quasilinear hyperbolic systems of balance laws, Lyapunov analysis,  $C^1$  exponential stability.*

## 1. INTRODUCTION

High-gain observers are well-known and largely used for finite-dimensional nonlinear systems. They apply to a large class of cases corresponding to *uniformly observable* systems [8], [9]. They have been extensively studied in the literature and remain widely considered, see [11] and references therein. In the recent paper [12], we extended this approach to a class of hyperbolic systems, for which first results on high-gain observer design have been proposed for a particular case of uniformly observable systems, written as an  $n \times n$  quasilinear hyperbolic system of balance laws and considering distributed measurements. There are some results on observer design for hyperbolic systems, considering the full state vector on the boundaries as measurement. Amongst others, one can refer to [15] and [10] for the backstepping design, to [2] for a discretization approach, to [4] for direct infinite-dimension-based Lyapunov techniques (see also [3]) or to [16] for optimization methods. For semigroup-based methods see [7], [5] and [17].

The contribution of the present paper is two-fold.

First, while in [12] we consider a non-strictly quasilinear hyperbolic system of balance laws with only one characteristic velocity, which is output-dependent and written in a characteristic and uniformly observable form, here we consider a more general case of quasilinear and observable hyperbolic systems. This particular observer design takes also advantage of the knowledge of the space-derivative of the output, which in our case is known, since we have considered distributed measurement. Second, we prove a stronger result of  $C^1$  exponential stability of the observer error, contrary to the  $C^0$  exponential stability result of [12]. This result is inspired by the methodologies that one can find in [1] and [6], where the problem of  $C^1$  Lyapunov stability, mainly for quasilinear systems of conservation laws, is confronted.

The theoretical method for the observer design that we develop is presented in Section 2, where Theorem 2.1 constitutes the main result. Additionally, a numerical example illustrates the results. Some conclusions and perspectives are presented in Section 3.

*Notation:* For a given  $x \in \mathbb{R}^n$ ,  $|x|$  denotes its usual Euclidean norm. For a given constant matrix  $A \in \mathbb{R}^{m \times n}$ ,  $A^T$  denotes its transpose,  $|A| := \sup\{|Ax|, |x| = 1\}$  is its induced norm and  $\text{Sym}(A) = \frac{A+A^T}{2}$  stands for its symmetric part. By  $\underline{\text{eig}}(A)$  we denote the minimum eigenvalue of a matrix  $A$ . For a function  $f(\cdot, \cdot)$ , we use the difference operator given by  $\Delta_{\xi_2}[f(\xi_1, \xi_2)] := f(\xi_1, \hat{\xi}_2) - f(\xi_1, \xi_2)$ , parametrized by  $\hat{\xi}_2$  and by abuse of notation,  $\Delta_{\xi_2}[f(\xi_1, \xi_2)]$  also stands for  $\Delta_{\xi_2}[f(\xi_1, \xi_2)](t, x)$ . By  $\partial_i$  we denote the partial differentiation with respect to variable  $i$  and  $\partial_{ij}$  stands for  $\partial_i \partial_j$ , where  $i, j = x$  or  $t$ . For a continuous ( $C^0$ ) map  $[0, 1] \ni x \mapsto \xi(x) \in \mathbb{R}^n$  we adopt the notation  $\max\{|\xi(x)|, x \in [0, 1]\}$ . For a continuously differentiable ( $C^1$ ) map  $[0, 1] \ni x \mapsto \xi(x) \in \mathbb{R}^n$  we adopt the notation  $\|\xi\|_1 := \|\xi\|_0 + \|\xi_x\|_0$ . For  $f \in C^1$  by  $f'$  we denote its Jacobian. For a function  $\xi \in L^p([0, 1]; \mathbb{R}^n)$ ,  $p \in \mathbb{N}$ , we consider the norm  $\|\xi\|_{L^p} := \left(\int_0^1 |\xi|^p dx\right)^{1/p}$ . For a function

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$\xi \in H^2([0, 1]; \mathbb{R}^n)$  the definition of the  $H^2$ -norm is  $\|\xi\|_{H^2} := \left( \int_0^1 (|\xi|^2 + |\xi_x|^2 + |\xi_{xx}|^2) dx \right)^{1/2}$ .  $\mathcal{B}(\delta)$  denotes the set  $\mathcal{B}(\delta) := \{\xi \in C^0([0, +\infty); H^2([0, 1]; \mathbb{R}^2)) : \|\xi(t, \cdot)\|_{H^2} \leq \delta, \forall t \geq 0\}$ .

## 2. Class of Systems and Observer

Let us consider the 2x2 first-order quasilinear hyperbolic system described by the following equations on the domain  $\Pi := [0, +\infty) \times [0, 1]$ :

$$\begin{aligned} \partial_t \xi_1 + \lambda_{11}(\xi_1) \partial_x \xi_1 &= \xi_2 + f_1(\xi_1) \\ \partial_t \xi_2 + \lambda_{21}(\xi) \partial_x \xi_1 + \lambda_{22}(\xi_1) \partial_x \xi_2 &= f_2(\xi) \end{aligned} \quad (2.1a)$$

$$y = \xi_1 \quad (2.1b)$$

where  $\mathbb{R}^2 \ni \xi := [\xi_1 \quad \xi_2]^T$  is the state and  $y : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$  is the distributed output (measurement). Assume that, in addition to  $\xi_1$ , space derivative  $\partial_x \xi_1$  is also available as a measurement (which is not very restrictive since  $\xi_1$  is available on the full  $x$ -domain). Assume also that  $\lambda_{11}(\xi_1), \lambda_{22}(\xi_1) > 0, \forall \xi_1 \in \mathbb{R}$ .

We consider initial and boundary conditions as

$$\xi(0, x) =: \xi^0(x), x \in [0, 1] \quad (2.2a)$$

$$\xi(t, 0) = H(\xi(t, 1)), t \in [0, +\infty) \quad (2.2b)$$

where  $H = [H_1 \quad H_2]^T : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

We make the following regularity assumption.

**A0.** Functions  $\lambda_{11}(\cdot), \lambda_{22}(\cdot), \lambda_{21}(\cdot), f_1(\cdot), f_2(\cdot), H_i(\cdot), i = 1, 2$  are of class  $C^1$ .

The following assumption is essential for the well-posedness of our system, along with the minimal observer design requirement of "forward completeness" and, furthermore, it imposes boundedness of the solutions in the  $H^2$ -norm, which is essential in the design of our nonlinear observer. For further details, the reader can refer to [1], [14] and references therein, where sufficient conditions for the well-posedness of quasilinear hyperbolic systems of balance laws are presented.

**A1.** There exists  $\mathcal{M} \subset H^2([0, 1]; \mathbb{R}^2)$ , such that for any initial conditions  $\xi^0 \in \mathcal{M}$ , satisfying zero-order and one-order compatibility conditions, problem (2.1a), (2.2) admits a unique solution  $C^0([0, +\infty); H^2([0, 1]; \mathbb{R}^2))$ . Moreover, for any  $\xi^0$  in the above-mentioned class, there exists  $\delta > 0$ , such that  $\|\xi(t, \cdot)\|_{H^2} \leq \delta, \forall t \in [0, +\infty)$ .

Note, that by simple inclusion arguments, Assumption A1 implies that solutions  $\xi \in C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$  (classical solutions).

The following assumption is crucial for the stability analysis of the observer error equation.

**A2.** There exist Lipschitz constants  $L_{f_2}, L_{\lambda_{21}}, L_H, L_{f_2'}, L_{\lambda_{21}'}, L_{H'} > 0$ , such that  $\forall \xi \in \mathbb{R}^2, \hat{\xi}_2 \in \mathbb{R}, \left| \frac{\partial f_2}{\partial \xi_2} \right| \leq L_{f_2}, \left| \frac{\partial \lambda_{21}}{\partial \xi_2} \right| \leq L_{\lambda_{21}}, \left| \frac{\partial H}{\partial \xi_2} \right| \leq L_H, \left| \Delta_{\hat{\xi}_2} \left[ \frac{\partial f_2}{\partial \xi} \right] \right| \leq L_{f_2'} |\hat{\xi}_2 - \xi_2|, \left| \Delta_{\hat{\xi}_2} \left[ \frac{\partial \lambda_{21}}{\partial \xi} \right] \right| \leq L_{\lambda_{21}'} |\hat{\xi}_2 - \xi_2|, \left| \Delta_{\hat{\xi}_2} \left[ \frac{\partial H}{\partial \xi} \right] \right| \leq L_{H'} |\hat{\xi}_2 - \xi_2|$ .

Prior to our main result, we must emphasize the fact that according to a well-known Sobolev inequality, for all  $\xi \in H^2([0, 1]; \mathbb{R}^2)$  there exists  $c_0 > 0$ , such that

$$\|\xi\|_0 + \|\xi_x\|_0 \leq c_0 \|\xi\|_{H^2} \quad (2.3)$$

At this point, let us introduce some operators parametrized by  $\hat{\xi}_2 \in \mathbb{R}$ :

$$\begin{aligned} \eta_0, \eta_1^{\hat{\xi}_2}, \eta_2^{\hat{\xi}_2} &: C^0([0, +\infty); H^2([0, 1]; \mathbb{R}^2)) \\ &\rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^2), \\ \eta_3^{\hat{\xi}_2} &: C^0([0, +\infty); H^2([0, 1]; \mathbb{R}^2)) \\ &\rightarrow C^0([0, +\infty); L^2([0, 1]; \mathbb{R}^2)), \\ \Lambda_1, \Lambda_2^{\hat{\xi}_2} &: C^0([0, +\infty); H^2([0, 1]; \mathbb{R}^2)) \\ &\rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^{2 \times 2}) \end{aligned}$$

acting on  $\xi$  (where  $\xi_1$  is substituted by  $y$ ) and defined by

$$\eta_0[\xi] := \lambda_{22}'(y) \lambda_{22}^{-1}(y) y_t \quad (2.4a)$$

$$\eta_1^{\hat{\xi}_2}[\xi] := \frac{\partial f_2(y, \hat{\xi}_2)}{\partial \hat{\xi}_2} - y_x \frac{\partial \lambda_{21}(y, \hat{\xi}_2)}{\partial \hat{\xi}_2} + \eta_0[\xi] \quad (2.4b)$$

$$\eta_2^{\hat{\xi}_2}[\xi] := \Delta_{\hat{\xi}_2} [f_2(y, \hat{\xi}_2)] - y_x \Delta_{\hat{\xi}_2} [\lambda_{21}(y, \hat{\xi}_2)] \quad (2.4c)$$

$$\begin{aligned} \eta_3^{\hat{\xi}_2}[\xi] &:= \Delta_{\hat{\xi}_2} \left[ \frac{\partial f_2(y, \hat{\xi}_2)}{\partial (y, \hat{\xi}_2)} \right] \partial_t \xi - \Delta_{\hat{\xi}_2} [\lambda_{21}(y, \hat{\xi}_2)] y_{xt} \\ &\quad - y_x \Delta_{\hat{\xi}_2} \left[ \frac{\partial \lambda_{21}(y, \hat{\xi}_2)}{\partial (y, \hat{\xi}_2)} \right] \xi_t - \eta_0[\xi] \eta_2^{\hat{\xi}_2}[\xi] \end{aligned} \quad (2.4d)$$

$$\Lambda_1(\xi) := \begin{bmatrix} \lambda_{11}(\xi_1) & 0 \\ \lambda_{21}(\xi) & \lambda_{22}(\xi_1) \end{bmatrix} \quad (2.4e)$$

$$\Lambda_2^{\hat{\xi}_2}[\xi] := \text{diag} \left( \eta_0[\xi], \eta_1^{\hat{\xi}_2}[\xi] \right) \quad (2.4f)$$

In the subsequent part of the proof, by abuse of notation, we use the above-mentioned notations for operators  $\eta_i^{\hat{\xi}_2}[\xi], i = 1, \dots, 3, \eta_0[\xi]$  and  $\Lambda_2^{\hat{\xi}_2}[\xi]$  implying that they are functions of  $(t, x)$ .

Now, notice that whenever  $y \in \mathcal{B}(\delta)$ , due to continuity and positiveness of  $\lambda_{ii}, i = 1, 2$  and further the fact that  $\|\xi(t, \cdot)\|_0 \leq c_0 \delta, \forall t \geq 0$ , as a result of (2.3), the quantities  $\max_{\xi_1 \in \mathcal{B}(\delta)} (\lambda_{ii}(\xi_1)), \min_{\xi_1 \in \mathcal{B}(\delta)} (\lambda_{ii}(\xi_1))$  are well-defined and positive. In addition, note that whenever  $\xi \in \mathcal{B}(\delta)$ , as a result of (2.3) and the use

of hyperbolic dynamics (2.1a), we easily calculate constants  $\delta_1, \delta_2 > 0$ , such that

$$\|\hat{\xi}_t(t, \cdot)\|_0 \leq \delta_1, \|y_{tx}(t, \cdot)\|_{L^2} \leq \delta_2, \forall t \geq 0; \quad (2.5)$$

Note that to obtain  $\delta_2$ , we use the fact that  $\|y_{tx}(t, \cdot)\|_{L^2} = \|\lambda'_{11}(y(t, \cdot))y_x^2(t, \cdot) - \lambda_{11}(y(t, \cdot))y_{xx}(t, \cdot) + \partial_x \xi_2(t, \cdot) + f'_1(y(t, \cdot))y_x(t, \cdot)\|_{L^2}$  and we subsequently apply trivial inequalities. By virtue of (2.3), (2.5), continuity and global Lipschitzness (Assumption A2) of the involved mappings, we can easily calculate positive constants  $\gamma_i, i = 1, \dots, 5$ , such that whenever  $\xi \in \mathcal{B}(\delta)$ , the following inequalities are satisfied for all  $t \geq 0, x \in [0, 1]$ :

$$\begin{aligned} |\eta_1^{\hat{\xi}_2(t,x)}[\xi](t,x)| &\leq \gamma_1, |\Lambda_2[\xi](t,x)| \leq \gamma_2, \\ |\eta_0[\xi](t,x)| &\leq \gamma_3, |\eta_2^{\hat{\xi}_2(t,x)}[\xi](t,x)| \\ &\leq \gamma_4 |\hat{\xi}_2(t,x) - \xi_2(t,x)|, |\eta_3^{\hat{\xi}_2(t,x)}[\xi](t,x)| \\ &\leq \gamma_5 |\hat{\xi}_2(t,x) - \xi_2(t,x)|, \hat{\xi}_2(t,x) \in \mathbb{R} \end{aligned} \quad (2.6)$$

Let us now introduce our candidate observer dynamics on the domain  $\Pi$ , as follows:

$$\begin{aligned} \partial_t \hat{\xi}_1 + \lambda_{22}(y) \partial_x \hat{\xi}_1 &= \hat{\xi}_2 + (\lambda_{22}(y) - \lambda_{11}(y))y_x + f_1(y) \\ &\quad + \theta k_1 (\hat{\xi}_1 - y) \end{aligned} \quad (2.7a)$$

$$\begin{aligned} \partial_t \hat{\xi}_2 + \lambda_{22}(y) \partial_x \hat{\xi}_2 &= f_2(y, \hat{\xi}_2) - \lambda_{21}(y, \hat{\xi}_2)y_x \\ &\quad + \theta^2 k_2 (\hat{\xi}_1 - y) \end{aligned} \quad (2.7b)$$

with high-gain constant  $\theta > 1$  and boundary conditions for  $t \geq 0$  satisfying

$$\hat{\xi}(t, 0) = H(y(t, 1), \hat{\xi}_2(t, 1)) \quad (2.8)$$

The following lemma guarantees the existence of a unique global classical solution for our candidate observer. We invoke paper [13], where an analogous result is proven under Lipschitzness properties of the dynamics. It is easy to check that our candidate observer satisfies semilinear hyperbolic laws and is written in a well-posed characteristic form. Assumptions A0-A2 in conjunction with the previously mentioned comments (details are left to the reader) are compatible with the sufficient conditions of Theorem 2.1 in [13] and, thereby similar global existence result is established for our observer system. This yields the following result.

**Lemma 2.1** *Under Assumptions A0 - A2, the problem described by equations (2.7), (2.8) on the domain  $\Pi$  and initial conditions  $\hat{\xi}^0 := \hat{\xi}(0, x), \forall x \in [0, 1]$  satisfying zero-order and one-order compatibility conditions<sup>1</sup> admits a unique classical solution in  $\Pi$ , i.e., there exists a unique solution  $\hat{\xi} \in C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$ .*

<sup>1</sup>For the definition of compatibility conditions, refer to [1], Appendix B.

We now present our main result on the observer design.

**Theorem 2.1** *Consider system (2.1a)-(2.2), defined on  $\Pi$  with output (2.1b) and suppose that Assumptions A0, A1 and A2 hold. Let also  $P \in \mathbb{R}^{2 \times 2}$  be a positive definite symmetric matrix satisfying<sup>2</sup>*

$$2\text{Sym}(P(A + KC)) = -qI_{2 \times 2} \quad (2.9)$$

$$\text{where } A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\mathbb{R}^2 \ni K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T, q > 0 \quad (2.10)$$

*Then, for  $\theta > 1$ , system (2.7), (2.8) with initial condition  $\hat{\xi}^0 \in C^1([0, 1]; \mathbb{R}^2)$ , with  $\hat{\xi}(0, x) = \hat{\xi}^0(x)$ , satisfying zero-order and one-order compatibility conditions, is a well-posed high-gain observer, in the sense that it admits a unique classical solution in  $\Pi$  on one hand, providing an estimate for the state of the system for  $\theta$  large enough on the other hand. More precisely, there exists a constant  $\theta_0 \geq 1$ , such that for every  $\theta > \theta_0$ , there exist constants  $l, \kappa > 0$ , such that for all  $t \geq 0$ ,  $\|\xi(t, \cdot) - \hat{\xi}(t, \cdot)\|_1 \leq l e^{-\kappa t} \|\xi^0(\cdot) - \hat{\xi}^0(\cdot)\|_1$  holds for the solutions of (2.1a), (2.2) and (2.7), (2.8). The observer convergence rate  $\kappa$  is adjustable by the choice of the high-gain constant  $\theta$  and can be arbitrarily large.*

*Proof:* We define the linearly transformed observer error  $\varepsilon = (\varepsilon_1, \varepsilon_2)^T$  by

$$\varepsilon_1 = \theta^{-1}(\hat{\xi}_1 - \xi_1); \varepsilon_2 = \theta^{-2}(\hat{\xi}_2 - \xi_2) \quad (2.11)$$

By use of (2.1a) and (2.7), the error components  $\varepsilon_1, \varepsilon_2$  satisfy the following equations in  $\Pi$ :

$$\partial_t \varepsilon_1 + \lambda_{22}(y) \partial_x \varepsilon_1 = \theta(k_1 \varepsilon_1 + \varepsilon_2) \quad (2.12a)$$

$$\partial_t \varepsilon_2 + \lambda_{22}(y) \partial_x \varepsilon_2 = \theta k_2 \varepsilon_1 + \theta^{-2} \eta_2^{\hat{\xi}_2}[\xi] \quad (2.12b)$$

Furthermore, the following equation is satisfied on the boundaries for  $t \geq 0$ , as a consequence of (2.2b) and (2.8):

$$\varepsilon_i(t, 0) = \theta^{-i} \Delta_{\hat{\xi}_2} [H_i(y, \xi_2)](t, 1), i = 1, 2 \quad (2.13)$$

Next, by temporarily assuming that  $\varepsilon$  is of class  $C^2$ , we perform time differentiation of equations (2.12), (2.13) and we can easily obtain the following hyperbolic equations for  $\varepsilon_t$  defined on  $\Pi$ :

$$\begin{aligned} \partial_{tt} \varepsilon_1 + \lambda_{22}(y) \partial_{tx} \varepsilon_1 &= \theta(k_1 \partial_t \varepsilon_1 + \partial_t \varepsilon_2) \\ &\quad + \eta_0[\xi] \partial_t \varepsilon_1 - \theta \eta_0[\xi](k_1 \varepsilon_1 + \varepsilon_2) \end{aligned} \quad (2.14a)$$

$$\begin{aligned} \partial_{tt} \varepsilon_2 + \lambda_{22}(y) \partial_{tx} \varepsilon_2 &= \theta k_2 \partial_t \varepsilon_1 + \eta_1^{\hat{\xi}_2}[\xi] \partial_t \varepsilon_2 \\ &\quad - \theta \eta_0[\xi] k_2 \varepsilon_1 + \theta^{-2} \eta_3^{\hat{\xi}_2}[\xi] \end{aligned} \quad (2.14b)$$

<sup>2</sup>This is always possible since  $(A, C)$  is observable.

and boundary conditions for  $t \geq 0$  given by

$$\begin{aligned} \partial_t \varepsilon_i(t, 0) &= \theta^{-i} \Delta_{\xi_2} \left[ \frac{\partial H_i(y, \xi_2)}{\partial (y, \xi_2)} \right] (t, 1) \xi_i(t, 1) \\ &+ \theta^{2-i} \frac{\partial H_i(y, \xi_2)}{\partial \xi_2} (t, 1) \partial_t \varepsilon_2(t, 1), i = 1, 2 \end{aligned} \quad (2.15)$$

To proceed to the stability analysis, we need to define functionals  $w_p, \mathcal{W}_p : C^1([0, 1]; \mathbb{R}^2) \rightarrow \mathbb{R}$  by

$$w_p[\varepsilon] := (\varepsilon^T P \varepsilon + \rho_0 \varepsilon_t^T P \varepsilon_t)^p \quad (2.16a)$$

$$\mathcal{W}_p[\varepsilon] := \left( \int_0^1 \pi(x) e^{p\mu x} w_p[\varepsilon] dx \right)^{1/p} \quad (2.16b)$$

where  $\rho_0 \in (0, 1]$  is a constant (to be chosen appropriately),  $p \in \mathbb{N}$ ,  $P \in \mathbb{R}^{2 \times 2}$  is a positive definite symmetric matrix satisfying (2.9),  $\pi : [0, 1] \rightarrow \mathbb{R}$  is given by

$$\pi(x) := \left( \frac{\max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y))}{\min_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y))} - 1 \right) x + 1, x \in [0, 1] \quad (2.17)$$

and  $\mu \in \mathbb{R}$  is given by

$$\mu := \ln(\mu_0 \theta^2); \quad (2.18a)$$

$$\mu_0 := \frac{|P|}{\underline{\text{eig}}(P)} \max \left( (L_{H'} + L_H \delta_1)^2, L_H (L_H + L_{H'} \delta_1) \right) \quad (2.18b)$$

where the above-used constants are defined in (2.5) and Assumption A2. Notice that

$$\pi(x) \in \left[ 1, \frac{\max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y))}{\min_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y))} \right], \forall x \in [0, 1] \quad (2.19)$$

By invoking Lemma 2.1 and Assumption A1, which establish global unique classical solutions for observer system (2.7), (2.8) and system (2.1a), (2.2) respectively, we are now in a position to define  $w_p, W_p : [0, +\infty) \rightarrow \mathbb{R}$  by

$$w_p(t) := w_p[\varepsilon](t), W_p(t) := \mathcal{W}_p[\varepsilon](t), t \geq 0 \quad (2.20)$$

(we use the notation  $\varepsilon(t)(x) := \varepsilon(t, x)$ ). Calculating the time-derivative  $\dot{W}_p$  along the classical solutions  $\varepsilon, \varepsilon_t$  of (2.12) - (2.13), (2.14) - (2.15), we get:

$$\begin{aligned} \dot{W}_p &= \frac{1}{p} W_p^{1-p} \int_0^1 p \pi(x) e^{p\mu x} w_{p-1} \\ &\times (\varepsilon_t^T P \varepsilon + \varepsilon^T P \varepsilon_t + \rho_0 \varepsilon_{tt}^T P \varepsilon_t + \rho_0 \varepsilon_t^T P \varepsilon_{tt}) dx \end{aligned} \quad (2.21)$$

After substituting the dynamical equations (2.12) and (2.14) into the above equation and applying integration by parts,  $\dot{W}_p$  can be written in the following form:

$$\dot{W}_p = W_p^{1-p} \left( \frac{1}{p} T_{1,p} + \frac{1}{p} T_{2,p} + T_{3,p} + T_{4,p} \right) \quad (2.22)$$

where

$$T_{1,p} := -\pi(1) \lambda_{22}(y(1)) e^{p\mu} w_p(1) + \pi(0) \lambda_{22}(y(0)) w_p(0) \quad (2.23a)$$

$$T_{2,p} := \int_0^1 \partial_x (\pi(x) e^{p\mu x} \lambda_{22}(y)) w_p dx \quad (2.23b)$$

$$\begin{aligned} T_{3,p} &:= 2 \int_0^1 \pi(x) e^{p\mu x} w_{p-1} \left( \varepsilon^T P \theta^{-2} \begin{bmatrix} 0 \\ \eta_2^{\xi_2}[\xi] \end{bmatrix} \right. \\ &\left. + \rho_0 \varepsilon_t^T P \theta^{-2} \begin{bmatrix} 0 \\ \eta_3^{\xi_2}[\xi] \end{bmatrix} + \rho_0 \varepsilon_t^T \text{Sym}(P \Lambda_2^{\xi_2}[\xi]) \varepsilon_t \right) dx \end{aligned} \quad (2.23c)$$

$$T_{4,p} := \theta \int_0^1 \pi(x) e^{p\mu x} w_{p-1}$$

$$\begin{aligned} &\times (2 \varepsilon^T \text{Sym}(P(A + KC)) \varepsilon + 2 \rho_0 \varepsilon_t^T \text{Sym}(P(A + KC)) \varepsilon_t \\ &- \rho_0 \eta_0[\xi] \varepsilon_t^T P(A + KC) \varepsilon - \rho_0 \eta_0[\xi] \varepsilon^T (A + KC)^T P \varepsilon_t) dx \end{aligned} \quad (2.23d)$$

After substituting boundary equations (2.13) and (2.15) in  $T_{1,p}$  and by virtue of Lipschitz properties (Assumption A2), equations (2.5), (2.17) and trivial inequalities, we can easily obtain the following bound:

$$T_{1,p} \leq \max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y)) w_p(1) \left( -e^{p\mu} + (\theta^2 \mu_0)^p \right) \quad (2.24)$$

As a result of (2.18), we get:

$$T_{1,p} \leq 0 \quad (2.25)$$

Next, observe that term  $T_{2,p}$  is written as follows:

$$\begin{aligned} T_{2,p} &= \int_0^1 ((\pi(1) - 1) \lambda_{22}(y) + p \mu \pi(x) \lambda_{22}(y) \\ &+ \pi(x) \lambda'_{22}(y) y_x) e^{p\mu x} w_p dx \end{aligned} \quad (2.26)$$

By the fact that  $\pi(x) \geq 1, \forall x \in [0, 1]$ , we obtain the following inequality:

$$T_{2,p} \leq \left( \alpha + p |\mu| \max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y)) \right) W_p^p \quad (2.27)$$

where  $\alpha := (\pi(1) - 1) \max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y)) + c_0 \delta \max_{y \in \mathcal{B}(\delta)} (\lambda'_{22}(y))$ .

By exploiting bounds given in (2.6),  $T_{3,p}$  can be bounded as follows:

$$\begin{aligned} T_{3,p} &\leq \int_0^1 \pi(x) e^{p\mu x} w_{p-1} \\ &\times (2|P|(\gamma_4 |\varepsilon|^2 + \rho_0 \gamma_5 |\varepsilon| |\varepsilon_t| + \rho_0 \gamma_2 |\varepsilon_t|^2)) dx \\ &\leq \int_0^1 \pi(x) e^{p\mu x} w_{p-1} \frac{|P|}{\underline{\text{eig}}(P)} (2\gamma_4 \varepsilon^T P \varepsilon + 2\rho_0 \gamma_2 \varepsilon_t^T P \varepsilon_t \\ &+ \gamma_5 w_1) dx \leq (\gamma_5 + 2 \max(\gamma_2, \gamma_4)) \frac{|P|}{\underline{\text{eig}}(P)} W_p^p \end{aligned} \quad (2.28)$$

$T_{4,p}$  can be written in the following form:

$$T_{4,p} := -\theta \int_0^1 \pi(x) e^{\rho \mu x} w_{p-1} \left( \begin{bmatrix} \varepsilon^T & \varepsilon_t^T \end{bmatrix} \Sigma \begin{bmatrix} \varepsilon \\ \varepsilon_t \end{bmatrix} \right) dx \quad (2.29)$$

where, after utilizing (2.9),  $\Sigma$  is given by

$$\Sigma := \begin{bmatrix} qI_{2 \times 2} & -\rho_0 \eta_0 [\xi] (A + KC)^T P \\ -\rho_0 \eta_0 [\xi] P (A + KC) & \rho_0 q I_{2 \times 2} \end{bmatrix} \quad (2.30)$$

Now, we can easily verify that matrix  $\Sigma$  is positive definite if and only if

$$q > \sqrt{\rho_0} \gamma_3 |P(A + KC)| \quad (2.31)$$

where  $\gamma_3$  is given by (2.6). It turns out that for every choice of matrices  $P$  and  $K$  and constant  $q$  satisfying equation (2.9), there always exists a  $\rho_0$  (sufficiently small), such that (2.31) is satisfied and this fact renders  $\Sigma$  positive definite. Consequently, for appropriate choice of  $\rho_0$ , there exists  $\sigma > 0$ , such that

$$\begin{aligned} T_{4,p} &\leq -\sigma \theta \int_0^1 \pi(x) e^{\rho \mu x} w_{p-1} (|\varepsilon|^2 + |\varepsilon_t|^2) dx \\ &\leq -\sigma \theta \int_0^1 \pi(x) e^{\rho \mu x} w_{p-1} (|\varepsilon|^2 + \rho_0 |\varepsilon_t|^2) dx \\ &\leq -\theta \frac{\sigma}{|P|} W_p^p \end{aligned} \quad (2.32)$$

Combining equations (2.25), (2.27), (2.28), (2.32) with (2.22) and taking into account that  $p \geq 1$ , we obtain

$$\dot{W}_p \leq (-\theta A + B \ln(\theta) + \Gamma) W_p \quad (2.33)$$

where  $A := \frac{\sigma}{|P|}$ ,  $B := 2 \max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y))$ ,  $\Gamma := a + \ln(\mu_0) \max_{y \in \mathcal{B}(\delta)} (\lambda_{22}(y)) + (\gamma_5 + 2 \max(\gamma_2, \gamma_4)) \frac{|P|}{\text{eig}(P)}$ .

We obtained the estimate (2.33) of  $\dot{W}_p$  for  $\varepsilon$  of class  $C^2$ , but the proof so far implies that the result does not depend on the  $C^2$ -norms. Therefore, by invoking density arguments, the results remain valid with  $\varepsilon$  only of class  $C^1$  (see [6] for further details).

Applying the comparison lemma to (2.33), we obtain an estimate for  $W_p$  of the following form:

$$W_p(t) \leq e^{-(A\theta - B \ln(\theta) - \Gamma)t} W_p(0), \forall t \geq 0 \quad (2.34)$$

Now, one can select the high gain  $\theta$ , such that

$$\theta > \max(1, \Gamma/A) := \theta_0 \quad (2.35)$$

and, therefore, for sufficiently large  $\theta$  we achieve to obtain  $A\theta - B \ln(\theta) - \Gamma > 0$ .

Next, by taking into account (2.19) we obtain the following property:

$$\begin{aligned} \lim_{p \rightarrow \infty} W_p &= \lim_{p \rightarrow \infty} \|\pi(\cdot)^{\frac{1}{p}} e^{\mu x} (\varepsilon^T P \varepsilon + \rho_0 \varepsilon_t^T P \varepsilon_t)\|_{L^p} \\ &= \|e^{\mu x} \varepsilon^T P \varepsilon\|_0 + \|\rho_0 e^{\mu x} \varepsilon_t^T P \varepsilon_t\|_0 \end{aligned} \quad (2.36)$$

which holds for continuous  $\varepsilon$  and  $\varepsilon_t$ . We are now in a position to define functional  $V : C^1([0, 1]; \mathbb{R}^2)$  by

$$\mathcal{V}[\varepsilon] := \|e^{\mu x} \varepsilon^T P \varepsilon\|_1 \quad (2.37)$$

and as in (2.20),

$$V(t) := \mathcal{V}[\varepsilon](t), t \geq 0 \quad (2.38)$$

In conjunction with property (2.36), (2.34) yields:

$$V(t) \leq c \rho_0^{-1} e^{-2\kappa t} V(0), \forall t \geq 0 \quad (2.39)$$

where  $\kappa$  is a positive constant, such that  $A\theta - B \ln(\theta) - \Gamma \geq 2\kappa$  and  $0 < c \sim \theta^2$  (where  $c$  is obtained after exploiting (2.12) and bounds derived from the Assumptions and (2.6), in order to calculate the bound in the sup-norm for  $\varepsilon_x$  with respect to the bounds of  $\varepsilon_t$  and  $\varepsilon$ . Details are left to the reader). By use of the inequality  $e^{-\frac{\mu+\mu|}{2}} \text{eig}(P) \|\varepsilon(t, \cdot)\|_1^2 \leq V(t) \leq e^{-\frac{\mu-\mu|}{2}} |P| \|\varepsilon(t, \cdot)\|_1^2$ , we easily obtain for all  $t \geq 0$

$$\|\varepsilon(t, \cdot)\|_1 \leq c^{1/2} \rho_0^{-1/2} e^{|\mu|/2} \sqrt{|P| \text{eig}^{-1}(P)} e^{-\kappa t} \|\varepsilon^0\|_1 \quad (2.40)$$

where  $\varepsilon^0(x) := \varepsilon(0, x)$ . By virtue of (2.11), we derive the following estimate, which holds for every  $t \geq 0$ :

$$\|\xi(t, \cdot) - \hat{\xi}(t, \cdot)\|_1 \leq l e^{-\kappa t} \|\xi^0 - \hat{\xi}^0\|_1 \quad (2.41)$$

where  $l := \theta c^{1/2} \rho_0^{-1/2} e^{|\mu|/2} \sqrt{|P| \text{eig}^{-1}(P)}$ . Concluding, we designed an exponential in the  $C^1$ -norm high-gain observer of adjustable convergence rate  $\kappa$ , dependent on the selection of  $\theta$ , namely, for every  $\theta > \theta_0$  given in (2.35), there exist  $l, \kappa > 0$ , such that (2.41) is satisfied. The higher the values  $\theta$  attains, the faster the observation error converges to zero.  $\square$

We are now in a position to illustrate our result by an example.

**Example 2.1** Consider the following system on  $\Pi$ :

$$\partial_t \xi_1 + (2 + \cos(\xi_1)) \partial_x \xi_1 = \xi_2 + \sin(\xi_1) \quad (2.42a)$$

$$\partial_t \xi_2 - \sin(\xi_1 \xi_2) \partial_x \xi_1 + (2 + \sin(\xi_1)) \partial_x \xi_2 = \sin(\xi_2 - \xi_1) \quad (2.42b)$$

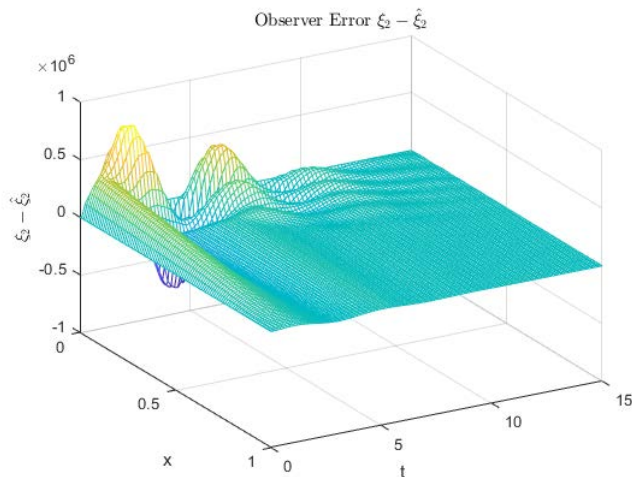
$$y = \xi_1 \quad (2.42c)$$

and boundary conditions satisfying

$$\xi_1(t, 0) = \frac{3}{2} \xi_1(t, 1) - \xi_2(t, 1) \quad (2.42d)$$

$$\xi_2(t, 0) = -\xi_1(t, 1) \quad (2.42e)$$

Consider the initial condition  $\xi_1^0(x) = \pi(1-x)$ ,  $\xi_2^0(x) = -\pi x$ . All our Assumptions A0-A2 are satisfied for the



**Figure 1. Space and time evolution of the observer error of the second state**

above system and, thus, we can design the high-gain observer (2.7) in  $\Pi$  for a choice of high gain  $\theta = 50$  and  $k_1 = -1, k_2 = -2$ : We choose observer initial conditions (in accordance with the compatibility conditions)  $\hat{\xi}_1^0(x) = 2\pi(1-x), \hat{\xi}_2^0(x) = -2\pi x$ . Fig. 1 illustrates the estimation error function for the second state, which exhibits exponential convergence to zero, as predicted by Theorem 2.1.

### 3. Conclusion

In this paper a high-gain observer for a class of observable hyperbolic systems with distributed measurement has been designed. This result constituted an extension of the high-gain observer design for finite-dimensional systems and at the same time an extension of [12]. The exponential decay of the observer error in the  $C^1$ -norm by the choice of an appropriate Lyapunov functional has been proven and an example illustrates the main result. The extension of this methodology to more general hyperbolic systems and weakening of the assumptions are future perspectives.

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