Magnetic Ginzburg-Landau energy with a periodic rapidly oscillating and diluted pinning term

Mickaël dos Santos

MAGNETIC GINZBURG-LANDAU ENERGY WITH A PERIODIC RAPIDLY OSCILLATING AND DILUTED PINNING TERM

MICKAËL DOS SANTOS

Abstract. We study the $2D$ full Ginzburg-Landau energy with a periodic rapidly oscillating, discontinuous and [strongly] diluted pinning term using a perturbative argument. This energy models the state of an heterogeneous type II superconductor submitted to a magnetic field. We calculate the value of the first critical field which links the presence of vorticity defects with the intensity of the applied magnetic field. Then we prove a standard dependance of the quantized vorticity defects with the intensity of the applied field. Our study includes the case of a London solution having several *minima*. The macroscopic location of the vorticity defects is understood with the famous Bethuel-Brezis-Hélein renormalized energy. The mesoscopic location, *i.e.*, the arrangement of the vorticity defects around the *minima* of the London solution, is the same than in the homogenous case. The microscopic location is exactly the same than in the heterogeneous case without magnetic field. We also compute the value of secondary critical fields that increment the quantized vorticity.

1. Introdution

This article studies the pinning phenomenon in type-II superconducting composites.

Superconductivity is a property that appears in certain materials cooled below a critical temperature. These materials are called superconductors. Superconductivity is characterized by a total absence of electrical resistance and a perfect diamagnetism. Unfortunately, when the imposed conditions are too intense, superconductivity is destroyed in certain areas of the material called *vorticity defects*.

We are interested in type II superconductors which are characterized by the fact that the vorticity defects first appear in small areas. Their number increases with the intensity of the conditions imposed until filling the material. For example, when the intensity h_{ex} of an applied magnetic field exceeds a first threshold, the first vorticity defects appear: the magnetic field begins to penetrate the superconductor. The penetration is done along thin wires and may move resulting an energy dissipation. These motions may be limited by trapping the vorticity defects in small areas.

The behavior of a superconductor is modeled by minimizers of a Ginzburg-Landau type energy. In order to study the presence of traps for the vorticity defects we consider an energy including a pinning term that models impurities in the superconductor. These impurities would play the role of traps for the vorticity defects. We are thus lead to the subject of this article: the type-II superconducting composites with impurities.

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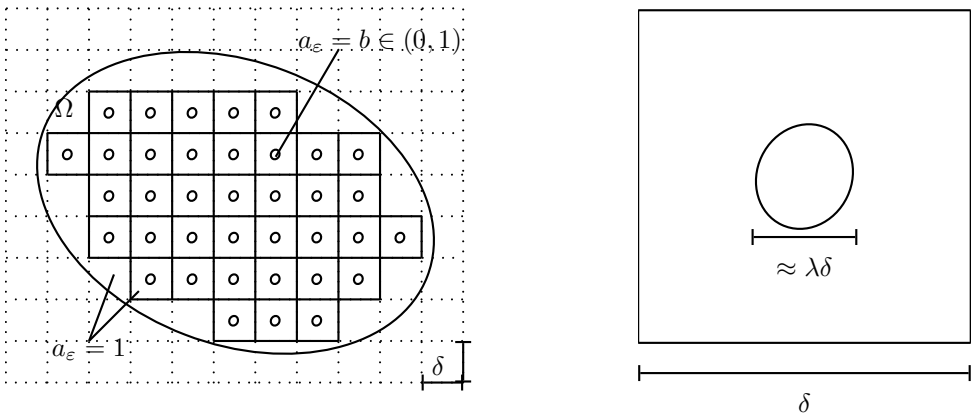
The case of an infinite long homogenous type II superconducting cylinder was intensively studied in mathematics by various authors since the 90's [see [16] for a guide to the litterature]. Namely, the present work deals with a cylindrical superconductor $\mathcal{S} = \Omega \times \mathbb{R}$ [whose section is $\Omega \subset \mathbb{R}^2$] submitted to a vertical magnetic field $(0, 0, h_{\text{ex}})$. Under these considerations, the vorticity defects are thin vertical cylinder. Thus their study may be done *via* a 2D problem formulated on $\Omega \subset \mathbb{R}^2$. Following the works of various authors [see [14], [1], [11]], for a small parameter $\varepsilon > 0$ [$\varepsilon \to 0$ in this article] and $h_{\text{ex}} = h_{\text{ex}}(\varepsilon) \geq 0$, we are interested in the description of the [global] minimizers of the functional

$$\begin{array}{rccc} \mathcal{E}_{\varepsilon,h_{\text{ex}}} : & \mathscr{H} & \to & \mathbb{R}^+ \\ & (u,A) & \mapsto & \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u - \imath A u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2 + |\text{curl}(A) - h_{\text{ex}}|^2 \end{array},$$

where [see Section 2 for more detailed notation]

- $\Omega \subset \mathbb{R}^2$ is a smooth bounded simply connected open set,
- $\mathscr{H} := H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$,
- $a_\varepsilon : \Omega \to \{1, b\}$ [$b \in (0,1)$ is independent of ε] is a periodic diluted pinning term [see Figure 1 and Section 2.3 for a construction of a_ε]. The impurities are the connected components of $\omega_\varepsilon := a_\varepsilon^{-1}(\{b\})$. In the definition of a_ε, $\delta = \delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$ is the parameter of period, $\lambda = \lambda(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$ is the parameter of dilution and $0 \in \omega \subset \mathbb{R}^2$ is a smooth bounded simply connected open set which gives the form of the impurities.



(a) The pining term is periodic on a $\delta \times \delta$-grid

(b) The parameter λ controls the size of an inclusion in the cell

FIGURE 1. The periodic pinning term

We focus on a strongly diluted case [$\lambda^{1/4} |\ln \varepsilon| \to 0$] with not too small connected components of ω_ε [$|\ln(\lambda\delta)| = \mathcal{O}(\ln |\ln \varepsilon|)$] but with a sufficiently small parameter of the period [see (4)].

Under these considerations, if $(u_\varepsilon, A_\varepsilon)$ minimizes $\mathcal{E}_{\varepsilon,h_{\text{ex}}}$, then the vorticity defects may be interpreted as the set $\{|u_\varepsilon| < b/2\}$.

As said above, our study takes place in the extrem type II case $\varepsilon \to 0$ and we also assume a divergent upper bound for $h_{\rm ex}$. Vorticity defects appear for minimizers above a critical valued $H_{c_1} = [b^2|\ln\varepsilon| + (1-b^2)|\ln(\lambda\delta)|]/(2\|\xi_0\|_{L^\infty(\Omega)}) + \mathcal{O}(1)$ [see Corollary 64 and (75)]. Here $\xi_0 \in H^1_0 \cap H^2$ is called the *London solution* and is the unique solution of the *London equation*

$$\begin{cases} -\Delta^2\xi_0 + \Delta\xi_0 = 0 & \text{in } \Omega \\ \Delta\xi_0 = 1 & \text{on } \partial\Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{cases}. \tag{1}$$

The value H_{c_1} is calculated by a standard balance of the energetic costs of a configuration without vorticity defects [$|u| \geq b/2$] with well prepared competitors having an arbitrary number of quantized vorticity defects. Here quantization as to be interpreted by the degree of u around a vorticity defect. It is an observable quantity related with the circulation of the superconducting current.

In order to lead the study, the set $\Lambda := \{z \in \Omega \,|\, \xi_0(z) = \min\xi_0\} \subset \Omega$ is of major interest [it is standard to prove that, in Ω, $-1 < \xi_0 < 0$]. From Lemma 4.4 in [17] and Lemma 4 in [15] we have the following :

Lemma 1. *The set Λ is finite. Moreover there exist $\eta > 0$ and $M \geq 1$ s.t. for $a \in \Omega$ we have $\xi_0(a) \geq \min\xi_0 + \eta{\rm dist}(a,\Lambda)^M$* [(1)] .

We write $N_0 := {\rm Card}(\Lambda)$ and $\Lambda = \{p_1, ..., p_{N_0}\}$.

We may give a simple picture of the emergence of the vorticity defects. The first vorticity defects appear close to H_{c_1}. If $N_0 = 1$ then there is first a unique vorticity defect and it is close to Λ. If $N_0 \geq 2$ the situation is less clear: we first have $d_1^\star \in \{1, ..., N_0\}$ vorticity defect and each of them is located close to $d_1^\star$ elements of Λ. By increasing the intensity of the applied field $h_{\rm ex}$ by a bounded quantity we increment the number of vorticity defects until filling Λ.

Once each elements of Λ is close to a vorticity defect, then by increasing $h_{\rm ex}$ of a $\mathcal{O}(\ln|\ln\varepsilon|)$, additional defects appear one by one.

We may now state the main theorems of the present work. For simplicity of the presentation the theorems are not stated on their most general form [see Theorem 4].

These main results are obtained assuming that λ, δ and $h_{\rm ex}$ satisfy

$$\lambda^{1/4}|\ln\varepsilon| \to 0 \text{ and } |\ln(\lambda\delta)| = \mathcal{O}(\ln|\ln\varepsilon|), \tag{2}$$

$$\text{There is } K \geq 1 \text{ s.t. } h_{\rm ex} \leq \frac{b^2|\ln\varepsilon|}{2\|\xi_0\|_{L^\infty(\Omega)}} + K\ln|\ln\varepsilon| \tag{3}$$

and when $h_{\rm ex} \to \infty$ we need

$$\frac{\ln(\delta\sqrt{h_{\rm ex}})}{\ln(\ln h_{\rm ex})} \to -\infty. \tag{4}$$

Namely, in order to meet Hypothesis (2), (3) and (4), we may think $\lambda \simeq |\ln\varepsilon|^{-s}, \delta \simeq |\ln\varepsilon|^{-t}$ with $s > 4$ and $t > 1/2$.

We need also assume that

(5) the minimal points of ξ_0, $\Lambda = \{p_1, ..., p_{N_0}\}$, are non degenerate critical points

[1] In Lemma 4 in [15], M is just a positive number, but $\xi \in C^0(\overline{\Omega})$, and then, up to consider $\eta > 0$ sufficiently small, we may assume $M \geq 1$.

in the sense that for $p \in \Lambda$, letting $\mathrm{Hess}_{\xi_0}(p)$ be the Hessian matrix of ξ_0 at p, the quadratic form $Q_p(z) = z \cdot \mathrm{Hess}_{\xi_0}(p) z$ is a definite positive quadratic form. Note that if (5) holds then we may take $M = 2$ in Lemma 1.

The strategy of this work is based on a perturbative argument. This argument applies for families of *quasi-minimizers* of the energy with some regularity assumptions [see Theorem 4]. In particular, we cannot have a sharp profil near a zero of a quasi-minimizer since such profil does not make any sense for quasi-minimizer. Therefore we cannot speak about an *ad-hoc* notion of *vortices* s.t. "isolated zeros". However with a natural L^∞-bound on the gradient of quasi-minimizers, the notion of vorticity defects is sufficiently robust to give them a nice description.

For simplicity of the presentation we first state the main results for a family $\{(u_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\} \subset \mathscr{H}$ s.t.

$$(u_\varepsilon, A_\varepsilon) \text{ minimizes } \mathcal{E}_{\varepsilon, h_{\mathrm{ex}}} \text{ in } \mathscr{H}. \tag{6}$$

Theorem 1. *Assume that (5) holds and $\lambda, \delta, h_{\mathrm{ex}}, K$ satisfy (2), (3) and (4). There exists $\mathcal{D}_{K,b} > 1$ s.t. for $\{(u_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\} \subset \mathscr{H}$ satisfying (6), for sufficiently small ε, there exits $d_\varepsilon \in \mathbb{N}$ s.t. if $d_\varepsilon = 0$ then $|u_\varepsilon| \geq b/2$ in Ω, and if $d_\varepsilon \in \mathbb{N}^*$ then there exists a set of d_ε points, $\mathcal{Z}_\varepsilon = \{z_1^\varepsilon, ..., z_{d_\varepsilon}^\varepsilon\} \subset \Omega$, s.t. for $\mu > 0$ sufficiently small and independent of ε we have:*

(1) $d_\varepsilon \leq \mathcal{D}_{K,b}$
(2) $\{|u_\varepsilon| < b/2\} \subset \cup B(z_i^\varepsilon, \varepsilon^\mu) \subset \Omega$,
(3) $|z_i^\varepsilon - z_j^\varepsilon| \geq h_{\mathrm{ex}}^{-1} \ln h_{\mathrm{ex}}$ *for* $i \neq j$,
(4) $\mathrm{dist}(z_i^\varepsilon, \Lambda) \leq h_{\mathrm{ex}}^{-1/2} \ln h_{\mathrm{ex}}$ *for all* i,
(5) $\deg_{\partial B(z_i^\varepsilon, \varepsilon^\mu)}(u_\varepsilon) = 1$ *for all* i.

Moreover:

(1) There is $\eta_{\omega,b} > 0$ depending only on ω and b s.t. for all i we have $B(z_i^\varepsilon, \eta_{\omega,b}\lambda\delta) \subset \omega_\varepsilon$.
(2) If for a sequence $\varepsilon = \varepsilon_n \downarrow 0$ we have $h_{\mathrm{ex}} = \mathcal{O}(1)$ then $d_\varepsilon = 0$ for small ε.

From Theorem 1 we know that, for small ε, if $\{|u_\varepsilon| < b/2\} \neq \emptyset$, then the vorticity defects are contained in small disks which are well separated, trapped by the impurities and located near Λ. The second theorem gives sharper informations related with the location of these disks. We divide the second theorem in three parts:

- Macroscopic location: We know that the disks are near Λ, for some $p \in \Lambda$, how many disks are near p ?
- Mesoscopic location: For $p \in \Lambda$, how the disks near p are they organized ? What is their inter-distance ?
- Microscopic location: We know that the disks are trapped by the inclusion ω_ε, what is their location inside ω_ε.

These questions are related with the crucial notion of *renormalized energy* [see Section 6].

Theorem 2. [Direct part]
Assume that (5) holds and $\lambda, \delta, h_{\mathrm{ex}}, K$ satisfy (2), (3) and (4). Assume also $h_{\mathrm{ex}} \to \infty$.

Let $\{(u_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\} \subset \mathscr{H}$ satisfying (6) and let $\varepsilon = \varepsilon_n \downarrow 0$ be a sequence. Since $d = d_\varepsilon \leq \mathcal{D}_{K,b}$, up to pass to a subsequence, we may assume that d is independent of ε. Assume $d > 0$.

Macroscopic location. *Recall that* $\Lambda = \{p_1, ..., p_{N_0}\}$ *and for* $k \in \{1, ..., N_0\}$ *we let* $D_k := \deg_{\partial B(p_k, 2\ln(h_{\rm ex})/\sqrt{h_{\rm ex}})}(u_\varepsilon)$. *Write* $\mathbf{D} = (D_1, ..., D_{N_0})$. *Up to pass to a subsequence we may assume that* $\mathbf{D}$ *is independent of* ε. *We then have:*

- *The distribution of the disks* $B(z_i^\varepsilon, \varepsilon^\mu)$ *around the elements of* Λ *is the most homogenous possible :*

$$\mathbf{D} \in \Lambda_d := \left\{ \mathbf{D}' \in \left\{ \left\lceil \frac{d}{N_0} \right\rceil ; \left\lfloor \frac{d}{N_0} \right\rfloor \right\}^{N_0} \,\middle|\, \sum_{k=1}^{N_0} D'_k = d \right\}.$$

  Here, for $x \in \mathbb{R}$, *we wrote* $\lceil x \rceil$ *for the ceiling of* x *and* $\lfloor x \rfloor$ *for the floor of* x.
- *There exists a renormalized energy* $\mathcal{W}_d : \Lambda_d \to \mathbb{R}$ *[see (106)] s.t.* $\mathbf{D}$ *minimizes* $\mathcal{W}_d$.

Mesoscopic location. *The mesoscopic location is the same than in the homogenous case. Namely, for* $p \in \Lambda$ *s.t.* $\deg_{\partial B(p, 2\ln(h_{\rm ex})/\sqrt{h_{\rm ex}})}(u_\varepsilon) = D > 0$, *there exists a renormalized energy [see Section 6.2]*

$$W^{\rm meso}_{p,D} : \{(a_1, ..., a_D) \in (\mathbb{R}^2)^D \,|\, a_i \neq a_j \text{ for } i \neq j\} \to \mathbb{R}$$

s.t., denoting $\ell := \sqrt{\dfrac{D}{h_{\rm ex}}}$ *and for* $z_i^\varepsilon \in B(p, 2\ln(h_{\rm ex})/\sqrt{h_{\rm ex}})$ *letting* $\breve{z}_i^\varepsilon := \dfrac{z_i^\varepsilon - p}{\ell}$, *we have* $\breve{\mathbf{z}}^\varepsilon = (\breve{z}_1^\varepsilon, ..., \breve{z}_D^\varepsilon)$ *[assuming* $z_i^\varepsilon \in B(p, 2\ln(h_{\rm ex})/\sqrt{h_{\rm ex}}) \Leftrightarrow i \in \{1, ..., D\}$*] which converges to a minimizer of* $W^{\rm meso}_{p,D}$. *In particular* ℓ *is the typical interdistance between two close* $z_i^\varepsilon, z_j^\varepsilon$.

Microscopic location. *We know that, for* $i \in \{1, ..., d\}$, $B(z_i^\varepsilon, \eta_{\omega,b}\lambda\delta) \subset \omega_\varepsilon$. *Moreover for* $i \neq j$ *we have* $|z_i^\varepsilon - z_j^\varepsilon| \geq \ln(h_{\rm ex})h_{\rm ex}^{-1} \gg \lambda\delta$. *Then each connected component of* ω_ε *contains at most one disk* $B(z_i^\varepsilon, \varepsilon^\mu)$.

There exists a renormalized energy $W^{\rm micro} : \omega \to \mathbb{R}$ *[see Section 6.3] s.t. for* $i \in \{1, ..., d\}$, *letting* $y_i^\varepsilon \in \delta \cdot \mathbb{Z}^2$ *be s.t.* $B(z_i^\varepsilon, \eta_{\omega,b}\lambda\delta) \subset y_i^\varepsilon + \lambda\delta\omega$ *and* $\hat{z}_i^\varepsilon := \dfrac{z_i^\varepsilon - y_i^\varepsilon}{\lambda\delta} \in \omega$ *we have*

- $W^{\rm micro}(\hat{z}_i^\varepsilon) \to \min_\omega W^{\rm micro}$,
- *Up to pass to a subsequence, there is* $a_i \in \omega$ *s.t.* $\hat{z}_i^\varepsilon \to a_i$ *and* a_i *minimizes* $W^{\rm micro}$. [2]

[Optimality of the renormalized energies]

Consider a sequence $\varepsilon = \varepsilon_n \downarrow 0$ *previously fixed [in order to have* $\mathbf{D}$ *independent of* ε*] and assume* $d \neq 0$. *We let*

- $\mathbf{D}' \in \Lambda_d$ *be a minimizer of* $\mathcal{W}_d$,
- *for* $k \in \{1, ..., N_0\}$ *s.t.* $D'_k \geq 1$, $\mathbf{a}'_k$ *be a minimizer of* $W^{\rm meso}_{p_k, D'_k}$,
- a_0 *be a minimizer of* $W^{\rm micro}$.

Then, for $\varepsilon = \varepsilon_n$, *there exist* $(u'_\varepsilon, A'_\varepsilon) \in \mathscr{H}$ *and* d *distinct points of* Ω, $\{z'_1, ..., z'_d\} = \{z_1^{\varepsilon\prime}, ..., z_d^{\varepsilon\prime}\} \subset \omega_\varepsilon$, *s.t.*

- $\mathcal{E}_{\varepsilon, h_{\rm ex}}(u'_\varepsilon, A'_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{E}_{\varepsilon, h_{\rm ex}} + o(1)$,
- $\{|u'_\varepsilon| \leq b/2\} \subset \cup B(z'_i, \sqrt{\varepsilon}) \subset \cup_{p \in \Lambda} B(p, \ln(h_{\rm ex})/\sqrt{h_{\rm ex}})$,
- *for* $k \in \{1, ..., N_0\}$, $D'_k = \deg_{\partial B(p_k, 2\ln(h_{\rm ex})/\sqrt{h_{\rm ex}})}(u'_\varepsilon)$,
- $\deg_{\partial B(z'_i, \sqrt{\varepsilon})}(u'_\varepsilon) = 1$ *for all* i,
- *writing for* $p_k \in \Lambda$ *[s.t.* $D'_k \geq 1$*] and* $z'_i \in B(p_k, \ln(h_{\rm ex})/\sqrt{h_{\rm ex}})$, $\breve{z}'_i := (z_i - p_k)/\sqrt{D_k/h_{\rm ex}}$ *and* $\breve{\mathbf{z}}'_{p_k} := \{\breve{z}'_i \,|\, z'_i \to p_k\}$ [3] , *we have* $\breve{\mathbf{z}}'_{p_k} \to \mathbf{a}'_k$,

[2] For example if ω is a disk then a_i is the center of the disk [7] .

[3] We used a little abuse of notation for the simplicity of the presentation.

- *For $i \in \{1,...,d\}$, letting $y_i^\varepsilon \in \delta\cdot\mathbb{Z}^2$ be s.t. $z_i' \in y_i^\varepsilon + \lambda\delta\cdot\omega$ and $\hat{z}_i' := \dfrac{z_i' - y_i^\varepsilon}{\lambda\delta} \in \omega$ we have $\hat{z}_i' \to a_0$.*

The third theorem underline the link between the number d and h_{ex}. In this theorem we write, for $x \in \mathbb{R}$, $[x]^+ = \max(x,0)$ and $[x]^- = \min(x,0)$.

Theorem 3. *Assume that Ω satisfies (5), $\lambda, \delta, h_{\mathrm{ex}}, K$ satisfy (2), (3) and (4).*

There are integers $L \in \{1,...,N_0\}$, $0 = d_0^\star < d_1^\star < \cdots < d_L^\star = N_0$ $\lfloor d_k^\star \in \mathbb{N}$ is independent of $\varepsilon\rfloor$ and critical fields $\lfloor$depending on $\varepsilon\rfloor$ $\mathtt{K}_1^{(\mathtt{I})} < \cdots < \mathtt{K}_L^{(\mathtt{I})} < \mathtt{K}_1^{(\mathtt{II})} < \mathtt{K}_2^{(\mathtt{II})} < \cdots$ $\lfloor$see (126) and (127) for the expressions of $\mathtt{K}_k^{(\mathtt{I})}$ and $\mathtt{K}_k^{(\mathtt{II})}\rfloor$ s.t. for $\{(u_\varepsilon, A_\varepsilon)\,|\,0 < \varepsilon < 1\} \subset \mathscr{H}$ a family satisfying (6) and for a sequence $\varepsilon = \varepsilon_n \downarrow 0$:

- *If $d_\varepsilon = 0$ for small ε, then $[h_{\mathrm{ex}} - \mathtt{K}_1^{(\mathtt{I})}]^+ \to 0$.*
- *If $d_\varepsilon > 0$ for small ε, then $[h_{\mathrm{ex}} - \mathtt{K}_1^{(\mathtt{I})}]^- \to 0$.*
- *Assume $L \geq 2$. For $k \in \{1,...,L-1\}$, if for small ε we have $d_{k-1}^\star < d_\varepsilon \leq d_k^\star$, then*
$$\left[h_{\mathrm{ex}} - \mathtt{K}_k^{(\mathtt{I})}\right]^- \to 0 \text{ and } \left[h_{\mathrm{ex}} - \mathtt{K}_{k+1}^{(\mathtt{I})}\right]^+ \to 0.$$
- *For $L \geq 1$, if for small ε we have $d_{L-1}^\star < d_\varepsilon \leq d_L^\star = N_0$, then*
$$\left[h_{\mathrm{ex}} - \mathtt{K}_L^{(\mathtt{I})}\right]^- \to 0 \text{ and } \left[h_{\mathrm{ex}} - \mathtt{K}_1^{(\mathtt{II})}\right]^+ \to 0.$$
- *Let $l \in \mathbb{N}^*$. If for small ε we have $d_\varepsilon = N_0 + l$, then*
$$\left[h_{\mathrm{ex}} - \mathtt{K}_l^{(\mathtt{II})}\right]^- \to 0 \text{ and } \left[h_{\mathrm{ex}} - \mathtt{K}_{l+1}^{(\mathtt{II})}\right]^+ \to 0.$$

Remark 2. A more complete statement for $d_\varepsilon \in \{0,...,N_0\}$ may be found in Proposition 68.

2. Notation

2.1. Sets, vectors and numbers.

- We identify the real plan $\mathbb{R}^2$ with $\mathbb{C}$ and we denote by $\mathbb{S}^1$ the unit circle in $\mathbb{C}$.
- For $\mathscr{U} \subset \mathbb{R}^2$, $N \in \mathbb{N}\backslash\{0;1\}$, $(\mathscr{U}^N)^* := \{(z_1,...,z_N) \in \mathscr{U}^N \,|\, z_i \neq z_j \text{ for } i \neq j\}$.
- For $k \in \{1;2\}$, $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.
- If $(a_1,a_2),(b_1,b_2) \in \mathbb{R}^2$, then $|(a_1,a_2)| = \sqrt{a_1^2+a_2^2}$, $(a_1,a_2)^\perp = (-a_2,a_1)$, $(a_1,a_2)\cdot(b_1,b_2) = a_1b_1 + a_2b_2$ and $(a_1,a_2)\wedge(b_1,b_2) = a_1b_2 - a_2b_1$.
- For $\mathscr{U} \subset \mathbb{R}^2$, $\overline{\mathscr{U}}$ is the closure of $\mathscr{U}$ w.r.t. $|\cdot|$
- For $\emptyset \neq \mathscr{U}, \mathscr{V} \subset \mathbb{R}^2$ and $x_0 \in \mathbb{R}^2$ we write $\mathrm{dist}(\mathscr{U},\mathscr{V}) := \inf\{|x-y|\,|\,x \in \mathscr{U},\, y \in \mathscr{V}\}$ and $\mathrm{dist}(x_0,\mathscr{V}) := \mathrm{dist}(\{x_0\},\mathscr{V})$.
- For $\Gamma \subset \mathbb{R}^2$ a Jordan curve we let:
  - $\mathrm{int}(\Gamma)$, the interior of Γ, be the bounded open set $\mathscr{U} \subset \mathbb{R}^2$ s.t. $\Gamma = \partial\mathscr{U}$ where $\partial\mathscr{U}$ is the boundary of $\mathscr{U}$.
  - ν be the outward normal unit vector of $\mathrm{int}(\Gamma)$
  - τ be the direct unit tangent vector of Γ ($\tau = \nu^\perp$)
- If S is a finite set then $\mathrm{Card}(S)$ is the cardinal of S.
- If $x \in \mathbb{R}$, then we write $\lceil x\rceil := \min\{m \in \mathbb{Z}\,|\,m \geq x\}$, the ceiling of x, and $\lfloor x\rfloor := \max\{m \in \mathbb{Z}\,|\,m \leq x\}$, the floor of x.
- If $x \in \mathbb{R}$, then we write $[x]^+ = \max(x,0)$ and $[x]^- = \min(x,0)$.

2.2. **Functions.**

- For $\mathscr{U} \subset \mathbb{R}^2$ a smooth open set and $K \subset \mathbb{C}$, $H^1(\mathscr{U}, K) = \{u \in H^1(\mathscr{U}, \mathbb{C}) \,|\, u(x) \in K \text{ for a.e. } x \in \mathscr{U}\}$ where $H^1(\mathscr{U}, \mathbb{C})$ is the Classical Sobolev space of the first order modeled on the Lebesgue space L^2.

  For $k \in \mathbb{N}^*$ and $p \in [1; \infty]$ we use the standard notation for the higher order Sobolev space $H^k(\mathscr{U}, K)$ modeled on L^2 and $W^{k,p}(\mathscr{U}, K)$ for the Sobolev space of order k modeled on L^p.
- We use the standard notation for the differential operators: "∇" for the gradient, "curl" for the curl, "div" for the divergence, "$\partial_\tau = \tau \cdot \nabla$" for the tangential derivative, "$\partial_\nu = \nu \cdot \nabla$" for the normal derivative...
- For $\mathscr{U} \subset \mathbb{R}^2$ a smooth bounded open set we let $\mathrm{tr}_{\partial\mathscr{U}} : H^1(\mathscr{U}, \mathbb{C}) \to H^{1/2}(\partial\mathscr{U}, \mathbb{C})$ be the [surjective] trace operator. For Γ a connected component of $\partial\mathscr{U}$ and $u \in H^1(\mathscr{U}, \mathbb{C})$, we let $\mathrm{tr}_\Gamma(u)$ be the restriction of $\mathrm{tr}_{\partial\mathscr{U}}(u)$ to Γ.

  We write $H^1_0(\mathscr{U}, \mathbb{C}) := \{u \in H^1(\mathscr{U}, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\mathscr{U}}(u) = 0\}$.
- For $u : \Omega \to \mathbb{C}$ a function we let $\underline{u} := \begin{cases} u & \text{if } |u| \leq 1 \\ u/|u| & \text{if } |u| > 1 \end{cases}$.
- For $\Gamma \subset \mathbb{R}^2$ a Jordan curve and $g \in H^{1/2}(\Gamma, \mathbb{S}^1)$, the degree of g is defined as

$$\deg_\Gamma(g) := \frac{1}{2\pi} \int_\Gamma g \wedge \partial_\tau g \in \mathbb{Z}.$$

  For a smooth and bounded open set $\mathscr{U} \subset \mathbb{R}^2$, Γ a connected component of $\partial\mathscr{U}$ and $u \in H^1(\mathscr{U}, \mathbb{C})$, if there exists $\eta > 0$ s.t. $g := \mathrm{tr}_\Gamma(u)$ satisfies $|g| \geq \eta$, then $g/|g| \in H^{1/2}(\Gamma, \mathbb{S}^1)$ and we write $\deg_\Gamma(u) := \deg_\Gamma(g/|g|)$.

  When $\mathscr{U}, \mathscr{V} \subset \mathbb{R}^2$ are smooth bounded simply connected open sets s.t. $\overline{\mathscr{V}} \subset \mathscr{U}$ and $u \in H^1(\mathscr{U} \setminus \overline{\mathscr{V}}, \mathbb{S}^1)$, then we write [without ambiguity] $\deg(u)$ instead of $\deg_\Gamma(u)$ for any Jordan curve $\Gamma \subset \overline{\mathscr{U}} \setminus \mathscr{V}$ s.t. $\mathscr{V} \subset \mathrm{int}(\Gamma)$.

2.3. **Construction of the pinning term.** Let

- $\delta = \delta(\varepsilon) \in (0, 1)$, $\lambda = \lambda(\varepsilon) \in (0, 1)$;
- $\omega \subset \mathbb{R}^2$ be a smooth bounded and simply connected open set s.t. $(0, 0) \in \omega$ and $\overline{\omega} \subset Y := (-1/2, 1/2)^2$.

For $m \in \mathbb{Z}^2$ we denote $Y^\delta_m := \delta m + \delta \cdot Y$ and $\omega_\varepsilon = \bigcup_{\substack{m \in \mathbb{Z}^2 \text{ s.t.} \\ Y^\delta_m \subset \Omega}} [\delta m + \lambda\delta \cdot \omega]$. For $b \in (0, 1)$ we define

$$a_\varepsilon : \begin{array}{ccc} \mathbb{R}^2 & \to & \{b, 1\} \\ x & \mapsto & \begin{cases} b & \text{if } x \in \omega_\varepsilon \\ 1 & \text{otherwise} \end{cases} \end{array}.$$

2.4. **Asymptotic.**

- In this article $\varepsilon \in (0; 1)$ is a small number. We are essentially interested in the asymptotic $\varepsilon \to 0$.
- The notation $o(1)$ means a quantity depending on ε which tends to 0 when $\varepsilon \to 0$.
- The notation $o[f(\varepsilon)]$ means a quantity $g(\varepsilon)$ s.t. $\dfrac{g(\varepsilon)}{f(\varepsilon)} = o(1)$.
- The notation $\mathcal{O}[f(\varepsilon)]$ means a quantity $g(\varepsilon)$ s.t. $\dfrac{g(\varepsilon)}{f(\varepsilon)}$ is bounded for small ε.

3. Classical facts and the strongest theorem

Gauge invariance and Coulomb Gauge

It is standard to quote the *gauge invariance* of the energy $\mathcal{E}_{\varepsilon,h_{\rm ex}}$. Namely, two configurations $(u,A),(u',A')\in\mathscr{H}$ are gauge equivalent, denoted by $(u,A)\overset{\text{gauge}}{\sim}(u',A')$, if there exists a gauge transformation from (u,A) to (u',A'):

$$(u,A)\overset{\text{gauge}}{\sim}(u',A')\iff\begin{cases}\exists\,\varphi\in H^2(\Omega,\mathbb{R})\text{ s.t.}\\ u'=u\mathrm{e}^{\imath\varphi}\text{ and }A'=A+\nabla\varphi\end{cases}.$$

Two gauge equivalent configurations describe the same physical state. Then, physical quantities are those which are gauge invariant. For example, if $(u,A)\in\mathscr{H}$, then $|u|$, $|\nabla u-\imath Au|$, $\mathrm{curl}(A)$ and then $\mathcal{E}_{\varepsilon,h_{\rm ex}}(u,A)$, $\{|u|<b/2\}$ also are gauge invariants.

In the context the Ginzburg-Landau energy, a classical choice of gauge is the *Coulomb gauge*. We say that (u,A) is in the Coulomb gauge if

$$\begin{cases}\mathrm{div}(A)=0 & \text{in }\Omega\\ A\cdot\nu=0 & \text{on }\partial\Omega\end{cases}. \tag{7}$$

One may prove [see Proposition 3.2 in [16]] that, for $(u,A)\in\mathscr{H}$, there exists $\varphi\in H^2(\Omega,\mathbb{R})$ s.t. $A':=A+\nabla\varphi$ satisfies (7). Then, letting $u'=u\mathrm{e}^{\imath\varphi}$, we have (u',A') which is in the Coulomb gauge and $(u,A)\overset{\text{gauge}}{\sim}(u',A')$.

One of the main motivations in using the Coulomb gauge comes from the fact that $\|\mathrm{curl}(A)\|_{L^2}$ controls $\|A\|_{H^1}$. Namely there exists $C\geq 1$ [which depends only on Ω] s.t. if A satisfies (7) then [see Proposition 3.3 in [16]]

$$\|A\|_{H^1(\Omega,\mathbb{R}^2)}\leq C\|\mathrm{curl}(A)\|_{L^2(\Omega)} \tag{8}$$

and

$$\|A\|_{H^2(\Omega,\mathbb{R}^2)}\leq C\|\mathrm{curl}(A)\|_{H^1(\Omega)}. \tag{9}$$

Moreover we have an easy representation of $A\in H^1(\Omega,\mathbb{R}^2)$ satisfying (7)

$$A\in H^1(\Omega,\mathbb{R}^2)\text{ is a solution of (7)}\iff\exists\,\xi\in H^1_0\cap H^2(\Omega,\mathbb{R})\text{ s.t. }A=\nabla^\perp\xi. \tag{10}$$

Basic description of a minimizer

We first note that, by direct minimization, for all $a_\varepsilon\in L^\infty(\Omega,[b;1])$, $\varepsilon,h_{\rm ex}>0$, the minimization problem of $\mathcal{E}_{\varepsilon,h_{\rm ex}}$ in $\mathscr{H}$ admits [at least] a solution $(u_\varepsilon,A_\varepsilon)\in\mathscr{H}$.

Writing $h_\varepsilon:=\mathrm{curl}(A_\varepsilon)$, it is standard to check that a such minimizer solves:

$$\begin{cases}-(\nabla-\imath A_\varepsilon)^2u_\varepsilon=\dfrac{u_\varepsilon}{\varepsilon^2}(a_\varepsilon^2-|u_\varepsilon|^2)^2 & \text{in }\Omega\\ (\nabla-\imath A)u_\varepsilon\cdot\nu=0 & \text{on }\Omega\\ -\nabla^\perp h_\varepsilon=u_\varepsilon\wedge(\nabla-\imath A_\varepsilon)u_\varepsilon & \text{in }\Omega\\ h_\varepsilon=h_{\rm ex} & \text{on }\partial\Omega\end{cases}. \tag{11}$$

Using a maximum principle, we may get the following proposition:

Proposition 3. *Let $\varepsilon,h_{\rm ex}>0$ and $a\in L^\infty(\Omega,[b,1])$. If $(u_\varepsilon,A_\varepsilon)$ is a minimizer of $\mathcal{E}(u,A)=\dfrac{1}{2}\displaystyle\int_\Omega|\nabla u-\imath Au|^2+\frac{1}{2\varepsilon^2}(a^2-|u|^2)^2+|\mathrm{curl}(A)-h_{\rm ex}|^2$ in $\mathscr{H}$ then $|u_\varepsilon|\leq 1$ in Ω.*

On the other hand, if $(u_\varepsilon, A_\varepsilon)$ is a minimizer of $\mathcal{E}_{\varepsilon,h_{\text{ex}}}$ in the Coulomb gauge, then it solves

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{u_\varepsilon}{\varepsilon^2}(a_\varepsilon^2 - |u_\varepsilon|^2)^2 - 2\imath(A_\varepsilon u_\varepsilon \cdot \nabla u_\varepsilon) - |A_\varepsilon|^2 u_\varepsilon & \text{in } \Omega \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Omega \end{cases}. \tag{12}$$

A fundamental bound in the study concerns $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)}$. We have the following lemma which is a Gagliardo-Nirenberg type inequality with homogenous Neumann boundary condition.

Lemma 4. [4] *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected open set. There exists $C_\Omega \geq 1$ s.t. if $u \in H^2(\Omega)$ is s.t. $\partial_\nu u = 0$ on $\partial\Omega$ then*

$$\|\nabla u\|^2_{L^\infty(\Omega)} \leq C_\Omega \left(\|\Delta u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\right) \|u\|_{L^\infty(\Omega)}.$$

Consequently, with Lemma 4 [up to change the value of C_Ω], for $\varepsilon, h_{\text{ex}} > 0$ and $a_\varepsilon \in L^\infty(\Omega, [b^2, 1])$, if $(u_\varepsilon, A_\varepsilon) \in \mathscr{H}$ minimizes $\mathcal{E}_{\varepsilon,h_{\text{ex}}}$ is in the Coulomb gauge and is s.t. $\|A_\varepsilon\|_{L^\infty(\Omega)} \leq 1/\varepsilon$ [which is the case in the present work] then

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C_\Omega}{\varepsilon}. \tag{13}$$

In the homogenous case as well as in the case without magnetic field, Estimate (13) is crucial to describe vorticity defects. It is the same in the present work. More precisely, the main result [Theorem 4] states that the three above theorems are true replacing $(u_\varepsilon, A_\varepsilon)$ that minimizes $\mathcal{E}_{\varepsilon,h_{\text{ex}}}$ in $\mathscr{H}$ by any configuration $(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon)$ s.t. $E_\varepsilon(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon) = \inf_{\mathscr{H}} \mathcal{E}_{\varepsilon,h_{\text{ex}}} + o(1)$ with two extra hypotheses on $|\tilde{u}_\varepsilon|$: $\|\nabla|\tilde{u}_\varepsilon|\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1})$ and $|\tilde{u}_\varepsilon| \in W^{2,1}(\Omega)$ [see (17)]

Lassoued-Mironescu decoupling

In order to study pinned Ginzburg-Landau type energies, a nice trick was initiated by Lassoued and Mironescu in [12]. Before explaining this trick we have to do a direct calculation for $(u, A) \in \mathscr{H}$:

$$\mathcal{E}_{\varepsilon,h_{\text{ex}}}(u, A) = E_\varepsilon(u) + \frac{1}{2}\int_\Omega -2(u \wedge \nabla u) \cdot A + |u|^2|A|^2 + |\text{curl}(A) - h_{\text{ex}}|^2 \tag{14}$$

with

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2.$$

The Lassoued-Mironescu decoupling is obtained by first minimizing E_ε in $H^1(\Omega, \mathbb{C})$. It is clear that E_ε admits minimizers and if U minimizes E_ε then it satisfies

$$\begin{cases} -\Delta U = \dfrac{U}{\varepsilon^2}(a_\varepsilon^2 - |U|^2) & \text{in } \Omega \\ \partial_\nu U = 0 & \text{on } \partial\Omega \end{cases}. \tag{15}$$

By an energetic argument it is easy to prove that, if U minimizes E_ε in $H^1(\Omega, \mathbb{C})$, then $b \leq |U| \leq 1$. Moreover from (15), $U \wedge \nabla U = 0$, *i.e.* $U = |U|\mathrm{e}^{\imath\theta}$ with $\theta \in \mathbb{R}$.

Then one may consider a scalar minimizer $U_\varepsilon : \Omega \to [b, 1]$. This scalar minimizer may be seen as a regularization of a_ε [see Proposition 7].

[4] The proof of Lemma 4 is done by first using $\Phi : \mathbb{D} \to \Omega$, a conformal representation of Ω on the unit disk $\mathbb{D}$. Then we extend $\tilde{u} := u \circ \Phi$ in the disk $B(0, 2)$ by letting $u'(x) = \tilde{u}(x/|x|)$ for $x \in B(0, 2) \setminus \mathbb{D}$. By using the boundary condition we have $u' \in H^2(B(0, 2), \mathbb{C})$. And finally one may conclude by using an interior version of Lemma 4 [Lemma A.1 in [3]].

Using this scalar minimizer one may get the well known Lassoued-Mironescu decoupling: for $v \in H^1(\Omega, \mathbb{R})$ we have

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v) \tag{16}$$

with

$$F_\varepsilon(v) := \frac{1}{2}\int_\Omega U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1-|v|^2)^2.$$

Using this decoupling, one may prove that, for $\varepsilon > 0$, there exists a unique positive minimizer $U_\varepsilon : \Omega \to [b,1]$ of E_ε in $H^1(\Omega, \mathbb{R})$.

On the other hand, from (14) and (16), for $(u, A) \in \mathscr{H}$ and $v = u/U_\varepsilon$ we have:

$$\begin{aligned}\mathcal{F}_{\varepsilon,h_{\rm ex}}(v,A) &:= \mathcal{E}_{\varepsilon,h_{\rm ex}}(U_\varepsilon v, A) - E_\varepsilon(U_\varepsilon)\\ &= \frac{1}{2}\int_\Omega U_\varepsilon^2|\nabla v - \imath A v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1-|v|^2)^2 + |\mathrm{curl}(A) - h_{\rm ex}|^2.\end{aligned}$$

It is easy to check that $\mathcal{F}_{\varepsilon,h_{\rm ex}}(v,A)$ is gauge invariant. This functional is of major interest in the study since (v,A) minimizes $\mathcal{F}_{\varepsilon,h_{\rm ex}}$ in $\mathscr{H}$ if and only if $(U_\varepsilon v, A)$ minimizes $\mathcal{E}_{\varepsilon,h_{\rm ex}}$ in $\mathscr{H}$.

An easy comparaison argument implies that if $(v_\varepsilon, A_\varepsilon)$ minimizes $\mathcal{F}_{\varepsilon,h_{\rm ex}}$ then $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq 1$.

From now on we focus on the study of the minimizer of $\mathcal{F}_{\varepsilon,h_{\rm ex}}$. Namely we have the following theorem.

Theorem 4. *Assume that (5) holds and $\lambda, \delta, h_{\rm ex}, K$ satisfy (2), (3) and (4).*

Let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ be s.t. $\mathcal{F}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{F} + o(1)$. Assume also that

$$\begin{cases} |v_\varepsilon| \in W^{2,1}(\Omega, \mathbb{C}) \\ \||\nabla |v_\varepsilon|\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1}) \end{cases}. \tag{17}$$

Then Theorems 1, 2 and 3 hold for $u_\varepsilon = U_\varepsilon v_\varepsilon$.

Remark 5. Theorem 4 may be rephrased in term of U_ε. Let $(h_{\rm ex})_{0<\varepsilon<1} \subset (0,\infty)$, $\{(u_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ and let $v_\varepsilon := u_\varepsilon/U_\varepsilon \in H^1(\Omega,\mathbb{C})$. On the one hand, from the decoupling (16), we have $\{(u_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ is s.t. $\mathcal{E}_{\varepsilon,h_{\rm ex}}(u_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{E}_{\varepsilon,h_{\rm ex}} + o(1)$ if and only $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\}$ is s.t. $\mathcal{F}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{F}_{\varepsilon,h_{\rm ex}} + o(1)$. On the other hand v_ε satisfies (17) if and only if we have $|u_\varepsilon| \in W^{2,1}(\Omega,\mathbb{C})$ and $\||\nabla|u_\varepsilon|\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1})$.

4. Plan of the article and proof of Theorem 4

The proof of Theorem 4 is done in several steps. It is based on a perturbative argument by replacing the energy $\mathcal{F}_{\varepsilon,h_{\rm ex}}$ with an energy $\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}$. This step is called the energetic cleaning [Section 5.1]. The functional $\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}$ is a perturbation of $\mathcal{F}_{\varepsilon,h_{\rm ex}}$: for $(v_\varepsilon, A_\varepsilon) \in \mathscr{H}$ which is in the Coulomb gauge and s.t. $\mathcal{F}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) = \mathcal{O}(h_{\rm ex}^2)$ we have $\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) - \mathcal{F}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) = o(1)$ [see Proposition 8]. In particular we have $\mathcal{F}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{F}_{\varepsilon,h_{\rm ex}} + o(1)$ if and only if $\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}} + o(1)$.

In section 5.2 we apply a vortex ball construction of Sandier-Serfaty [Proposition 10] and we follow the strategy of Sandier-Serfaty developed in [15] to prove that the vorticity of a reasonable configuration is bounded [see Theorem 5].

Once the bound on the vorticity yields, we adapt a result of Serfaty [17] which gives a decomposition of $\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}(v_\varepsilon, A_\varepsilon)$ in term of $F_\varepsilon(v_\varepsilon)$ and the location of the vorticity

defects [Proposition 11].

The decomposition obtained in Proposition 11 allows to focus the study on the energy F_ε which ignores the magnetic field. From this point, the study of a configuration $(v_\varepsilon, A_\varepsilon)$ is done for a major part *via* classical results based on the case without magnetic field [as in [4]]. To this end we adapt to our case some standard estimates ignoring the magnetic field, in particular the crucial notion of Renormalized energies is presented Section 6.

With these preliminary results, in Section 7, for $d \in \mathbb{N}^*$, we construct competitors $(v_\varepsilon, A_\varepsilon) \in \mathscr{H}$ with d quantized vorticity defects and then we get a sharp upper bound [see Proposition 39]:

$$\inf_{\mathscr{H}} \mathcal{F}_{\varepsilon,h_{\rm ex}} \leq h_{\rm ex}^2 \mathbf{J_0} + dM_\Omega \left[-h_{\rm ex} + H^0_{c_1}\right] + \mathscr{L}_1(d) \ln h_{\rm ex} + \mathscr{L}_2(d) + o(1).$$

Here $\mathbf{J_0}\&M_\Omega$ are independent of ε and d, $\mathscr{L}_1(d)\&\mathscr{L}_2(d)$ are independent of ε and $H^0_{c_1}$ is the leading term in the expression of the first critical field.

With the above upper bound for the minimal energy, the heart of the work consists in getting lower bounds for quasi-minimizers. Before getting such lowers bounds we adapt to our case some tools in Section 8: an η-ellipticity result is proved [Proposition 40], a construction of *ad-hoc* bad-discs is done [Proposition 42] and the strong effect of the dilution is expressed by various result in Section 8.3.

In Section 9 we begin the proof of the theorems. The part of Theorem 4 related with Theorem 1 is a direct consequence of Propositions 52, 53, 55 and 56 [and also Corollary 65].

The part of Theorem 4 related with Theorem 2 is given by Corollary 62 and Proposition 39.

The part of Theorem 4 related with Theorem 3 is a direct consequence of Corollary 65 and Propositions 68&69.

5. Some preliminaries

5.1. **Energetic cleaning.** In order to do the cleaning step, we have to get some estimates. Our goal is to study *quasi-minimizer* of $\mathcal{F}_{\varepsilon,h_{\rm ex}}$. To keep a simple presentation, we write $\mathcal{F}$ instead of $\mathcal{F}_{\varepsilon,h_{\rm ex}}$ and F instead of F_ε when there is no ambiguity.

From (8), (9) and classical elliptic regularity arguments we have the following proposition.

Proposition 6. *Let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ be a family of configuration in the Coulomb gauge. Then there is $\xi_\varepsilon \in H^1_0 \cap H^2(\Omega,\mathbb{R})$ s.t. $A_\varepsilon = \nabla^\perp \xi_\varepsilon$. Moreover, if for some $h_{\rm ex} = h_{\rm ex}(\varepsilon)$ we have*

$$\mathcal{F}(v_\varepsilon, A_\varepsilon) = \mathcal{O}(h_{\rm ex}^2), \tag{18}$$

then there exists C [independent of ε] s.t.

$$\|\xi_\varepsilon\|_{H^2(\Omega)} \quad \leq \quad C h_{\rm ex}. \tag{19}$$

Consequently, for $p \in [1,\infty)$, there exists $C_p > 1$ [independent of ε] s.t.

$$\|\nabla \xi_\varepsilon\|_{L^p(\Omega)} = \|A_\varepsilon\|_{L^p(\Omega)} \leq C_p h_{\rm ex}. \tag{20}$$

Moreover, up to increase the value of $C > 1$ [independently of ε], we have

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)} \leq C h_{\rm ex}. \tag{21}$$

And if $\mathrm{curl}(A_\varepsilon) \in H^1(\Omega)$ *then*

$$\|\xi_\varepsilon\|_{H^3(\Omega)} \leq C \|\mathrm{curl}(A_\varepsilon)\|_{H^1(\Omega)}. \tag{22}$$

In particular, for further use, note that if $\mathrm{curl}(A_\varepsilon) \in H^1(\Omega)$ *then* $\xi_\varepsilon \in H^1_0 \cap H^2 \cap W^{1,\infty}(\Omega)$ *and*

$$\|\nabla \xi_\varepsilon\|_{L^\infty(\Omega)} \leq C \|\mathrm{curl}(A_\varepsilon)\|_{H^1(\Omega)}. \tag{23}$$

In order to do the cleaning step we need to underline the fact that U_ε may be seen as a regularization of a_ε in $W^{1,\infty}$ with estimates that become bad when approaching $\partial\omega_\varepsilon$.

Proposition 7. *There exist $C_b, s_b > 0$ depending only on b and Ω s.t. for $\varepsilon, r > 0$ we have:*

$$\|\nabla U_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C_b}{\varepsilon}, \tag{24}$$

$$|U_\varepsilon - a_\varepsilon| \leq C_b \mathrm{e}^{-\frac{s_b r}{\varepsilon}} \text{ in } \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\varepsilon) \geq r\}, \tag{25}$$

$$|\nabla U_\varepsilon| \leq \frac{C_b \mathrm{e}^{-\frac{s_b r}{\varepsilon}}}{\varepsilon} \text{ in } \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\varepsilon) \geq r\}. \tag{26}$$

Proof. Estimate (24) is a consequence of Lemma 4. The proof of (25) is the same than Proposition 2 in [9]. Estimate (26) is proved in Appendix A. □

Since the 2-dimensional Hausdorff measure of ω_ε satisfies $\mathcal{H}^2(\omega_\varepsilon) = \mathcal{O}(\lambda^2)$, from (25), for $p \in [1, \infty[$, we have the following crucial estimate

$$\|U_\varepsilon^2 - 1\|_{L^p(\Omega)} = \mathcal{O}(\lambda^{2/p}). \tag{27}$$

We are now in position to do the cleaning step. We assume that $\{(v_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\} \subset \mathscr{H}$ is a family of configuration in the Coulomb gauge which satisfies (18). We denote $\alpha_\varepsilon = U_\varepsilon^2$ and $\rho_\varepsilon = |v_\varepsilon|$. From direct computations, by splitting the integrals with the identity $\alpha_\varepsilon = (\alpha_\varepsilon - 1) + 1$ and using $(1 - \rho_\varepsilon)^4 \leq (1 - \rho_\varepsilon^2)^2$, we have the existence of $C \geq 1$ [independent of ε] s.t.

$$\left| \int_\Omega \alpha_\varepsilon (v_\varepsilon \wedge \nabla v_\varepsilon) \cdot A_\varepsilon - \int_\Omega (v_\varepsilon \wedge \nabla v_\varepsilon) \cdot A_\varepsilon \right| \leq \frac{C}{2} \left[\sqrt{\lambda} h_{\rm ex}^2 + \lambda^{1/4} h_{\rm ex}^3 \varepsilon \right] \leq C \sqrt{\lambda} h_{\rm ex}^2 \tag{28}$$

and

$$\left| \int_\Omega \alpha_\varepsilon \rho_\varepsilon^2 |A_\varepsilon|^2 - \int_\Omega |A_\varepsilon|^2 \right| \leq C h_{\rm ex}^2 (\varepsilon h_{\rm ex} + \lambda). \tag{29}$$

By combining (28) and (29) we immediately get the following proposition.

Proposition 8. *If $(v_\varepsilon, A_\varepsilon)$ is in the Coulomb gauge and satisfies (18) then*

$$|\tilde{\mathcal{F}}(v_\varepsilon, A_\varepsilon) - \mathcal{F}(v_\varepsilon, A_\varepsilon)| \leq C h_{\rm ex}^2 (\varepsilon h_{\rm ex} + \sqrt{\lambda})$$

with C which is independent of ε and

$$\tilde{\mathcal{F}}(v, A) = \tilde{\mathcal{F}}_{\varepsilon, h_{\rm ex}}(v, A) := F(v) + \frac{1}{2} \int_\Omega -2(v \wedge \nabla v) \cdot A + |A|^2 + |\mathrm{curl}(A) - h_{\rm ex}|^2. \tag{30}$$

Remark 9. (1) One may claim that $\tilde{\mathcal{F}}$ is not gauge invariant if $\alpha_\varepsilon \not\equiv 1$.

(2) Note that if $\lambda^{1/4}|\ln\varepsilon|\to 0$ and if $h_{\rm ex}=\mathcal{O}(|\ln\varepsilon|)$ then for $(v_\varepsilon,A_\varepsilon)\in\mathscr{H}$ which is in the Coulomb gauge and satisfies (18) we have $\tilde{\mathcal{F}}(v_\varepsilon,A_\varepsilon)-\mathcal{F}(v_\varepsilon,A_\varepsilon)=o(1)$ without hypothesis on $\delta\in(0;1)$.

5.2. **Bound on the vorticity and energetic decomposition.** By applying Proposition 1 in [15] with $U_\varepsilon\geq b$ we immediately get the following proposition which does not need any assumption for $\lambda,\delta\in(0;1)$.

Proposition 10. *Assume $h_{\rm ex}\leq C_0|\ln\varepsilon|$ with $C_0\geq 1$ which is independent of ε. Let $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}$ be a family s.t. $\mathcal{F}(v_\varepsilon,A_\varepsilon)\leq C_0|\ln\varepsilon|^2$.*

Then there exist $C,\varepsilon_0>0$ [depending only on Ω, b and C_0] s.t. for $\varepsilon<\varepsilon_0$ we have either $|v_\varepsilon|\geq 1-|\ln\varepsilon|^{-2}$ in Ω or there exists a finite family of disjoint disks $\{B_i\,|\,i\in\mathcal{J}\}$ with $\mathcal{J}\subset\mathbb{N}^*$ [$\mathcal{J}$ depends on ε] and $B_i:=B(a_i,r_i)$ satisfying :

(1) $\{|v_\varepsilon|<1-|\ln\varepsilon|^{-2}\}\subset\cup B_i$
(2) $\sum r_i<|\ln\varepsilon|^{-10}$,
(3) writing $h_\varepsilon=\mathrm{curl}(A_\varepsilon)$, $\rho_\varepsilon=|v_\varepsilon|$ and $v_\varepsilon=\rho_\varepsilon \mathrm{e}^{\imath\varphi_\varepsilon}$ [φ_ε is locally defined] we have

$$\frac{1}{2}\int_{B_i}\rho^2|\nabla\varphi_\varepsilon-A_\varepsilon|^2+|h_\varepsilon-h_{\rm ex}|^2\geq\pi|d_i|(|\ln\varepsilon|-C\ln|\ln\varepsilon|), \tag{31}$$

with $d_i=\deg_{\partial B_i}(v)$ if $B_i\subset\Omega$ and 0 otherwise.

By following the argument of Sandier and Serfaty [15], we get the main result of this section.

Theorem 5. *Assume that λ,δ satisfy (2) and $\delta^2|\ln\varepsilon|\leq 1$. Assume also Hypothesis (3) holds for $h_{\rm ex}$ with some $K\geq 1$.*

Then there exist $\varepsilon_K>0$ and $\mathcal{M}_K\geq 1$ [independent of ε] s.t. if $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}\subset\mathscr{H}$ is a family in the Coulomb gauge satisfying $\mathcal{F}(v_\varepsilon,A_\varepsilon)\leq\inf_{\mathscr{H}}\mathcal{F}+K\ln|\ln\varepsilon|$ then for $0<\varepsilon<\varepsilon_K$ we have

$$\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\leq\mathcal{M}_K|\ln\varepsilon|. \tag{32}$$

Moreover, if $|v_\varepsilon|\not>1-|\ln\varepsilon|^{-2}$ in Ω, then letting $\{B_i\,|\,i\in\mathcal{J}\}$ be a family of disks given by Proposition 10, for $0<\varepsilon<\varepsilon_K$, we have $d_i\geq 0$ for all $i\in\mathcal{J}$ and there is $s_0>0$ [depending only on Ω] s.t. if $i\in\mathcal{J}$ is s.t. $d_i\neq 0$ then $\mathrm{dist}(B_i,\Lambda)\leq\mathcal{M}_K|\ln\varepsilon|^{-s_0}$.

The proof of this theorem is postponed in Appendix B.

We let

$$\mathbf{J_0}:=\tilde{\mathcal{F}}_{1,1}(1,\nabla^\perp\xi_0)=\frac{\tilde{\mathcal{F}}_{\varepsilon,h_{\rm ex}}(1,h_{\rm ex}\nabla^\perp\xi_0)}{h_{\rm ex}^2}. \tag{33}$$

Note that if $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}$ is a family of quasi-minimizers then

$$\mathcal{F}_{\varepsilon,h_{\rm ex}}(v_\varepsilon,A_\varepsilon)\leq\mathcal{F}_{\varepsilon,h_{\rm ex}}(1,\nabla^\perp\xi_0)+o(1)=h_{\rm ex}^2\mathbf{J_0}+o(1)=\mathcal{O}(h_{\rm ex}^2).$$

The discs given by Proposition 10 are "too large" for our strategy. Indeed one of the main argument is a construction of *bad discs* in the spirit of [4] which links $x_\varepsilon\in\{|v_\varepsilon|\leq 1/2\}$ with the energetic cost in a ball $B(x_\varepsilon,\varepsilon^\mu)$ with small $\mu>0$. Namely if $x_\varepsilon\in\{|v_\varepsilon|<1-|\ln\varepsilon|^{-2}\}\subset\cup B_i$ then the energetic cost in a ball $B(x_\varepsilon,\varepsilon^\mu)$ is not sufficiently large comparing to our error term.

In the next proposition we present the good framework of vortex balls required in the study. The first step in the study is an energetic decomposition valid under some assumptions [no assumption on $\delta\in(0;1)$ is required].

Proposition 11. *Let $C_0 > 1$, $(v_\varepsilon)_{0<\varepsilon<1} \subset H^1(\Omega,\mathbb{C})$ and $h_{\rm ex} > 0$ be s.t.*

$$F(v_\varepsilon) \leq C_0|\ln\varepsilon|^2,\ h_{\rm ex} \leq C_0|\ln\varepsilon|. \tag{34}$$

Assume furthermore that $\lambda^{1/4}|\ln\varepsilon| \to 0$ and, for $\varepsilon \in (0;1)$, either $|v_\varepsilon| > 1/2$ in Ω or v_ε admits a family of valued disks $\{(B(a_i,r_i),d_i)\,|\,i\in\mathcal{J}\}$ [$\mathcal{J}$ is finite] s.t. :

- *the disks $B_i = B(a_i,r_i)$ are pairwise disjoint*
- *$\{|v_\varepsilon| \leq 1/2\} \subset \cup_{i\in\mathcal{J}} B_i$*
- *$\sum_{i\in\mathcal{J}} r_i < |\ln\varepsilon|^{-10}$*
- *For $i\in\mathcal{J}$, letting $d_i = \begin{cases}\deg_{\partial B_i}(v) & \text{if } B_i\subset\Omega\\ 0 & \text{otherwise}\end{cases}$, we assume $\sum_{i\in\mathcal{J}}|d_i| \leq C_0$.*

Then, if $(\xi_\varepsilon)_\varepsilon \subset H^1_0\cap H^2\cap W^{1,\infty}(\Omega,\mathbb{R})$ is s.t.

$$\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega)} \leq C_0|\ln\varepsilon|, \tag{35}$$

writing $\zeta_\varepsilon := \xi_\varepsilon - h_{\rm ex}\xi_0$ we have in the case $|v_\varepsilon| \not> 1/2$ in Ω:

$$\mathcal{F}(v_\varepsilon,\nabla^\perp\xi_\varepsilon) - h_{\rm ex}^2\mathbf{J_0} = F(v_\varepsilon) + 2\pi h_{\rm ex}\sum_{i\in\mathcal{J}} d_i\xi_0(a_i) + \tilde{V}_{(\mathbf{a},\mathbf{d})}(\zeta_\varepsilon) + o(1) \tag{36}$$

where for $\zeta\in H^1_0\cap H^2(\Omega)$ we denoted

$$\tilde{V}_{(\mathbf{a},\mathbf{d})}(\zeta) := 2\pi\sum_{i\in\mathcal{J}} d_i\zeta(a_i) + \frac{1}{2}\int_\Omega (\Delta\zeta)^2 + |\nabla\zeta|^2. \tag{37}$$

And if $|v| > 1/2$ in Ω then

$$\mathcal{F}(v_\varepsilon,\nabla^\perp\xi_\varepsilon) - h_{\rm ex}^2\mathbf{J_0} = F(v_\varepsilon) + \frac{1}{2}\int_\Omega (\Delta\zeta_\varepsilon)^2 + |\nabla\zeta_\varepsilon|^2 + o(1) \tag{38}$$

The proof of Proposition 11 is an adaptation of an argument of Serfaty [17] [section 4]. The proof is presented Appendix C

Before going further, we state a result which will be useful in this article and whose proof is left to the reader.

Lemma 12. *For $v\in H^1(\Omega,\mathbb{C})$, $0<\varepsilon<1$ and $h_{\rm ex}>0$, there exists a unique potential $A_{v,\varepsilon,h_{\rm ex}} = A_v \in H^1(\Omega,\mathbb{R}^2)$ s.t. (v,A_v) is in the Coulomb gauge and satisfies*

$$\begin{cases} -\nabla^\perp{\rm curl}(A_v) = \alpha(\imath v)\cdot(\nabla v - \imath A_v v) & \text{in } \Omega\\ {\rm curl}(A_v) = h_{\rm ex} & \text{on } \partial\Omega\end{cases}. \tag{39}$$

Moreover A_v is the unique solution of the minimization problem

$$\inf_{A \text{ satisfies } (7)} \mathcal{F}_{\varepsilon,h_{\rm ex}}(v,A) \tag{40}$$

and from (9) and (10) we have $A_v = \nabla^\perp\xi_v$ with $\xi_v\in H^1_0\cap H^2\cap W^{1,\infty}(\Omega,\mathbb{R})$.

Remark 13. Assume λ,δ satisfy (2), $\delta^2|\ln\varepsilon| \leq 1$ and Hypothesis (3) holds. Consider $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}\subset\mathscr{H}$ a family in the Coulomb gauge satisfying $\mathcal{F}(v_\varepsilon,A_\varepsilon) \leq \inf_{\mathscr{H}}\mathcal{F} + \mathcal{O}(\ln|\ln\varepsilon|)$.

- From Theorem 5, either $|v_\varepsilon| > 1 - |\ln\varepsilon|^{-2}$ in Ω or the family of disjoint disks given by Proposition 10 satisfies the properties of the family of discs used in Proposition 11.
- Let $A_{v_\varepsilon} = \nabla^\perp\xi_{v_\varepsilon} \in H^1(\Omega,\mathbb{R}^2)$ be given by Lemma 12. Then with (9)&(39) we have $A_{v_\varepsilon}\in L^\infty(\Omega)$ and $\|A_{v_\varepsilon}\|_{L^\infty(\Omega)} \leq C|\ln\varepsilon|$ where C depends only on Ω.

As noted by Serfaty [17], with the help of the decomposition given by Proposition 11, we may prove that $h_{\rm ex}^2\mathbf{J_0}$ is almost the minimal energy of a vortex less configuration.

Corollary 14. *Let* $\mathscr{H}^0 := \big\{(\rho \mathrm{e}^{\imath\varphi}, A)\,|\,\rho \in H^1(\Omega,[0,\infty)),\ \varphi \in H^1(\Omega,\mathbb{R})$ *and* $A \in H^1(\Omega,\mathbb{R}^2)\big\}$. *Note that* $\mathscr{H}^0$ *is gauge invariant. Assume* $\lambda^{1/4}|\ln\varepsilon| \to 0$.

(1) Let $\varepsilon = \varepsilon_n \downarrow 0$. *Assume* $h_{\rm ex} = \mathcal{O}(|\ln\varepsilon|)$ *and for each* ε *let* $(v_\varepsilon, \nabla^\perp\xi_\varepsilon) \in \mathscr{H}^0$ *be s.t.* $\xi_\varepsilon \in H_0^1 \cap H^2 \cap W^{1,\infty}(\Omega,\mathbb{R})$ *with* $\|\nabla\xi_\varepsilon\|_{L^\infty(\Omega)} = \mathcal{O}(|\ln\varepsilon|)$. *Writing* $\zeta_\varepsilon := \xi_\varepsilon - h_{\rm ex}\xi_0$ *we have:*

$$\mathcal{F}(v_\varepsilon, \nabla^\perp\xi_\varepsilon) = h_{\rm ex}^2\mathbf{J_0} + F(v_\varepsilon) + \frac{1}{2}\int_\Omega (\Delta\zeta_\varepsilon)^2 + |\nabla\zeta_\varepsilon|^2 + o(1). \tag{41}$$

Thus, if $\mathcal{F}(v_\varepsilon, \nabla^\perp\xi_\varepsilon) \le h_{\rm ex}^2\mathbf{J_0} + o(1)$ *then* $\zeta_\varepsilon \to 0$ *in* $H^2(\Omega)$, $|v_\varepsilon| \to 1$ *in* $H^1(\Omega)$ *and, up to pass to a subsequence, there exists* $\mathtt{v} \in \mathbb{S}^1$ *s.t.* $v_\varepsilon \to \mathtt{v}$ *in* $H^1(\Omega)$.

(2) We have $\inf_{\mathscr{H}^0}\mathcal{F} = h_{\rm ex}^2\mathbf{J_0} + o(1)$.

Proof. We prove the first assertion. Estimate (41) is a direct consequence of Proposition 11.

For sake of simplicity of the presentation we drop the subscript ε. If $\mathcal{F}(v,\nabla^\perp\xi) \le h_{\rm ex}^2\mathbf{J_0} + o(1)$, then $F(v) + \|\zeta\|_{H^2(\Omega)} = o(1)$ and then $\zeta \to 0$ in $H^2(\Omega)$, $|v| \to 1$ in $H^1(\Omega)$. Moreover $\|\nabla v\|_{L^2(\Omega)} = o(1)$ and $\|v\|_{L^2(\Omega)} = \mathcal{O}(1)$. This clearly implies the remaining part of the assertion.

We prove the second assertion. We first claim, by the definition of $\mathbf{J_0}$, that using the configuration $(1, h_{\rm ex}\nabla^\perp\xi_0) \in \mathscr{H}^0$ we have $\inf_{\mathscr{H}^0}\mathcal{F} \le h_{\rm ex}^2\mathbf{J_0} + o(1)$.

By the gauge invariance of $\mathscr{H}^0$ we may consider a family of quasi-minimizer $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}^0$ which is in the Coulomb gauge. We write $(v_\varepsilon, A_\varepsilon) = (v, A)$. Let $(\tilde v, \tilde A) \in \mathscr{H}^0$ be defined by $\tilde v = \underline{v}$ and $\tilde A$ is the unique solution of (40) associated to $\tilde v$.

By direct calculations we have: $\mathcal{F}(\tilde v, \tilde A) \le \mathcal{F}(\tilde v, A) \le \mathcal{F}(v, A) \le h_{\rm ex}^2\mathbf{J_0} + o(1)$.

Moreover, by denoting $h := \mathrm{curl}(\tilde A)$, we have $\nabla h = \alpha\tilde v \wedge (\nabla^\perp\tilde v - \tilde A^\perp\tilde v)$ in Ω and $h = h_{\rm ex}$ on $\partial\Omega$. Then $\|h\|_{H^1(\Omega)} = \mathcal{O}(|\ln\varepsilon|)$ and using (22) we get $\|\tilde A\|_{H^2(\Omega)} = \mathcal{O}(|\ln\varepsilon|)$.

We are then able to apply the first assertion to get $\mathcal{F}(\tilde v, \tilde A) \ge h_{\rm ex}^2\mathbf{J_0} + o(1)$.

□

5.3. **Pseudo vortex structure.** We assume $\lambda^{1/4}|\ln\varepsilon| \to 0$. Let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ be a family of configurations in the Coulomb gauge satisfying (34). We assume that $|v_\varepsilon| \not> 1/2$ in Ω and that there exists $\{(B(a_i,r_i), d_i)\,|\,i \in \mathcal{J}\}$ as in Proposition 11. Then Proposition 11 gives a decomposition of $\mathcal{F}(v,A)$. Except in the crucial hypothesis $\sum r_i < |\ln\varepsilon|^{-10}$, the radii r_i do not play any role as well as the disks "$B(a_i,r_i)$" associated to a zero degree. We thus introduce an *ad-hoc* notion of *pseudo vortex*.

Definition 15. We assume that we have either $\varepsilon = \varepsilon_n \downarrow 0$ or $0 < \varepsilon < 1$. We consider $(v_\varepsilon)_\varepsilon \subset H^1(\Omega,\mathbb{C})$, $(h_{\rm ex})_\varepsilon \subset (1,\infty)$ satisfying (34).

Let $\{B_i = B(a_i,r_i)\,|\,i \in \mathcal{J}\}$ be a family of disks as in Proposition 11 and let $d_i = d_i^{(\varepsilon)} \in \mathbb{Z}$ be the associated "degrees" defined in Proposition 11. We denote $\mathcal{J}' = \mathcal{J}'_\varepsilon := \{i \in \mathcal{J}\,|\,d_i \neq 0\}$ [note that we have $\mathrm{Card}(\mathcal{J}'_\varepsilon) \le \sum|d_i| = \mathcal{O}(1)$].

If $\mathcal{J}' \neq \emptyset$, then we say that $\{(\mathbf{a},\mathbf{d})\} = \{(a_i,d_i)\,|\,i \in \mathcal{J}'\}$ is a set of *pseudo vortices* of v_ε.

For a fixed configuration $(\mathbf{a},\mathbf{d})$ of pseudo vortices, Serfaty studied in [17] the minimization problem of $\tilde{V}_{(\mathbf{a},\mathbf{d})}$ [defined in (37)]. We have the following result [Proposition 4.2 in [17]].

Proposition 16. *Let $(\mathbf{a},\mathbf{d}) = \{(a_i,d_i)\,|\,i\in\mathcal{J}'\}\subset\Omega\times\mathbb{Z}^*$ be a configuration s.t. $1\leq \mathrm{Card}(\mathcal{J}')<\infty$ and $a_i\neq a_j$ for $i\neq j$. Then $\tilde{V}_{(\mathbf{a},\mathbf{d})}(\zeta)$ is minimal for $\zeta=\zeta_{(\mathbf{a},\mathbf{d})}$ which satisfies*

$$\begin{cases} -\Delta^2\zeta_{(\mathbf{a},\mathbf{d})}+\Delta\zeta_{(\mathbf{a},\mathbf{d})} = 2\pi\sum_{i\in\mathcal{J}'} d_i\delta_{a_i} & \text{in } \Omega \\ \zeta_{(\mathbf{a},\mathbf{d})} = \Delta\zeta_{(\mathbf{a},\mathbf{d})} = 0 & \text{on } \partial\Omega \end{cases}. \tag{42}$$

[Here δ_a is the Dirac mass at $a\in\mathbb{R}^2$]

And we have $\tilde{V}[\zeta_{(\mathbf{a},\mathbf{d})}] = \pi\sum_{i\in\mathcal{J}'} d_i\zeta_{(\mathbf{a},\mathbf{d})}(a_i)$.

In order to prove the above proposition, Serfaty introduced for $a\in\Omega$ the function $\zeta^a\in H^1_0\cap H^2(\Omega)$ which is the unique solution of

$$\begin{cases} -\Delta^2\zeta^a+\Delta\zeta^a = 2\pi\delta_a & \text{in } \Omega \\ \zeta^a = \Delta\zeta^a = 0 & \text{on } \partial\Omega \end{cases}.$$

In particular we have $\zeta^a\leq 0$ in Ω. It is easy to see that $\zeta_{(\mathbf{a},\mathbf{d})}=\sum_{i\in\mathcal{J}'} d_i\zeta^{a_i}$ is the unique solution of (42).

Lemma 4.6 in [17] gives important properties related with ζ^a and $\zeta_{(\mathbf{a},\mathbf{d})}$:

Proposition 17. *For $s\in(0,1)$, there exists $C_s>0$ s.t. for $a,b\in\Omega$*

$$\|\zeta^a\|_{L^\infty(\Omega)}\leq C_s\mathrm{dist}(a,\partial\Omega)^s$$

and

$$\|\zeta^a-\zeta^b\|_{H^2(\Omega)}\leq C_s|a-b|^s.$$

Consequently there exists $C>0$ depending only on Ω s.t., if $\zeta_{(\mathbf{a},\mathbf{d})}$ is the unique solution of (42), then

$$\tilde{V}[\zeta_{(\mathbf{a},\mathbf{d})}] = \pi\sum_{i,j\in\mathcal{J}'} d_id_j\zeta^{a_i}(a_j)\leq C\left(\sum_{i\in\mathcal{J}'}|d_i|\right)^2.$$

For a further use we need the following lemma.

Lemma 18. *Let $(\mathbf{a},\mathbf{d})$ as in Proposition 16 then $\zeta_{(\mathbf{a},\mathbf{d})}\in H^1_0\cap H^2\cap W^{1,\infty}(\Omega,\mathbb{R})$ and there is $C\geq 1$ depending only on Ω s.t.*

$$\|\nabla\zeta_{(\mathbf{a},\mathbf{d})}\|_{L^\infty(\Omega)}\leq\frac{C\sum|d_i|}{\min\mathrm{dist}(a_i,\partial\Omega)}.$$

Proof. Let $(\mathbf{a},\mathbf{d})$ be as in Proposition 16, with Proposition 17 we have $\zeta_{(\mathbf{a},\mathbf{d})}=\sum d_i\zeta^{a_i}\in H^1_0\cap H^2$ and $\|\zeta_{(\mathbf{a},\mathbf{d})}\|_{H^2(\Omega)}\leq C\sum_i|d_i|$ where C depends only on Ω.

Moreover, for $a\in\Omega$, from (42), we have $\Delta\zeta_{(\mathbf{a},\mathbf{d})}=\zeta_{(\mathbf{a},\mathbf{d})}-\sum d_i\ln|x-a_i|-R_{(\mathbf{a},\mathbf{d})}$ where $R_{(\mathbf{a},\mathbf{d})}$ is the harmonic extension of $\mathrm{tr}_{\partial\Omega}(-\sum d_i\ln|x-a_i|)$ in Ω.

Consequently there exists $C\geq 1$ depending only on Ω s.t.

$$\|\Delta\zeta_{(\mathbf{a},\mathbf{d})}\|_{L^3(\Omega)}\leq\frac{C\sum|d_i|}{\min\mathrm{dist}(a_i,\partial\Omega)}$$

and therefore by elliptic regularity and a Sobolev embedding we get the result. □

Until now, the only way to get a nice magnetic potential associated to a function v was to consider $A_v = A_{v,\varepsilon,\alpha} \in H^2(\Omega, \mathbb{R}^2)$, the unique solution of (40). The previous results give that, after the cleaning step, we can do asymptotically as well by using a magnetic potential depending on a pseudo vortices structure of v instead of v itself [see Remark 20].

Definition 19. Let $N \geq 1$ and $(\mathbf{a}, \mathbf{d}) \in (\Omega^N)^* \times (\mathbb{Z}^*)^N$, $h_{\text{ex}} > 0$. Then we define $A_{(\mathbf{a},\mathbf{d})} := h_{\text{ex}} \nabla^\perp \xi_0 + \nabla^\perp \zeta_{(\mathbf{a},\mathbf{d})}$ where $\zeta_{(\mathbf{a},\mathbf{d})}$ is the unique solution of (42), *the potential associated to* $(\mathbf{a}, \mathbf{d})$.

Remark 20. Let $C_0 > 1$ and $(v_\varepsilon)_{0<\varepsilon<1} \subset H^1(\Omega, \mathbb{C})$, $h_{\text{ex}} > 0$ satisfying (34) be s.t. $(v_\varepsilon)_{0<\varepsilon<1}$ admits a set of pseudo vortices $((\mathbf{a}, \mathbf{d})_\varepsilon)_{0<\varepsilon<1}$ with $\sum |d_i| \leq C_0$. We write $v\&(\mathbf{a}, \mathbf{d})$ instead of $v_\varepsilon\&(\mathbf{a}, \mathbf{d})_\varepsilon$.

Assume $\min \text{dist}(a_i, \partial\Omega) > |\ln \varepsilon|^{-1}$ in order to have $\|\nabla \zeta_{(\mathbf{a},\mathbf{d})}\|_{L^\infty(\Omega)} = \mathcal{O}(|\ln \varepsilon|)$ [with Lemma 18] and $\lambda^{1/4} |\ln \varepsilon| \to 0$.

For $0 < \varepsilon < 1$, let $A_v \in H^1(\Omega, \mathbb{R}^2)$ be the unique solution of (40) and $A_{(\mathbf{a},\mathbf{d})}$ be defined in Definition 19. Then we have $A_{(\mathbf{a},\mathbf{d})} = \nabla^\perp \xi_{(\mathbf{a},\mathbf{d})}$ and $A_v = \nabla^\perp \xi_v$ where $\xi_{(\mathbf{a},\mathbf{d})}, \xi_v \in H^1_0 \cap H^2 \cap W^{1,\infty}(\Omega, \mathbb{R})$ satisfy the hypotheses of Proposition 11 [here we used (9)&(39)]. Therefore we have the following inequalities

$$\mathcal{F}(v, 0) \geq \mathcal{F}(v, A_v) = \tilde{\mathcal{F}}(v, A_v) + o(1) \geq \tilde{\mathcal{F}}(v, A_{(\mathbf{a},\mathbf{d})}) + o(1),$$

$$\mathcal{F}(v, A_v) \leq \mathcal{F}(v, A_{(\mathbf{a},\mathbf{d})}) = \tilde{\mathcal{F}}(v, A_{(\mathbf{a},\mathbf{d})}) + o(1).$$

In particular we have $\mathcal{F}(v, A_v) = \mathcal{O}(|\ln \varepsilon|^2)$ and $\mathcal{F}(v, A_{(\mathbf{a},\mathbf{d})}) = \mathcal{O}(|\ln \varepsilon|^2)$.

5.4. **Cluster of pseudo vortices.** From a standard result for the homogenous case, it is expected that, for a reasonable magnetic field, the asymptotic location of pseudo vortices of a studied configuration is a subset of Λ. This problem is related to the *macroscopic location* of the pseudo vortices. To treat this problem we use an *ad-hoc* notion of *cluster of pseudo vortices*.

Definition 21. Let $N, \tilde{N}_0 \in \mathbb{N}^*$, $\tilde{N}_0 \leq N$, $(\mathbf{p}, \mathbf{D}) \in (\overline{\Omega}^{\tilde{N}_0})^* \times \mathbb{Z}^{\tilde{N}_0}$, $\varepsilon = \varepsilon_n \downarrow 0$ and $(\mathbf{a}, \mathbf{d})_\varepsilon \in (\Omega^N)^* \times \mathbb{Z}^N$ s.t. $\mathbf{d}$ is independent of ε. We say that $((\mathbf{a}, \mathbf{d})_\varepsilon)_\varepsilon$ admits a *cluster structure* on $(\mathbf{p}, \mathbf{D})$ if

- for $i \in \{1, ..., N\}$, $\lim a_i$ exists, $\lim a_i \in \{p_1, ..., p_{\tilde{N}_0}\}$ and we write for $k \in \{1, ..., \tilde{N}_0\}$, $S_k := \{i \in \{1, ..., N\} \,|\, a_i \to p_k\}$
- for $k \in \{1, ..., \tilde{N}_0\}$ $S_k \neq \emptyset$,
- for $k \in \{1, ..., \tilde{N}_0\}$, $D_k = \sum_{i \in S_k} d_i$.

Remark 22. In this article we will use the notion of cluster structure with $(\mathbf{a}, \mathbf{d})$ as in Proposition 11 and $\mathbf{p} \subset \Lambda$.

Proposition 23. *Let $N \geq 1$, $\varepsilon = \varepsilon_n \downarrow 0$, $(\mathbf{a}, \mathbf{d})_\varepsilon \in (\Omega^N)^* \times \mathbb{Z}^N$ s.t. $\sum |d_i|$ is bounded independently of ε.*

(1) If $((\mathbf{a}, \mathbf{d})_\varepsilon)_\varepsilon$ admits a cluster structure on $(\mathbf{p}, \mathbf{D})$ [and then $\mathbf{d}$ is independent of ε] then $(\mathbf{p}, \mathbf{D})$ is unique [up to change the order]. We say that $(\mathbf{p}, \mathbf{D})$ is the cluster of $((\mathbf{a}, \mathbf{d})_\varepsilon)_\varepsilon$.

(2) Up to pass to a subsequence, there exist $1 \leq \tilde{N}_0 \leq N$ and $(\mathbf{p}, \mathbf{D}) \in (\overline{\Omega}^{\tilde{N}_0})^* \times \mathbb{Z}^{\tilde{N}_0}$ s.t. $(\mathbf{p}, \mathbf{D})$ is the cluster of $((\mathbf{a}, \mathbf{d})_\varepsilon)_\varepsilon$.

(3) If $(\mathbf{p},\mathbf{D})$ is the cluster of $((\mathbf{a},\mathbf{d})_\varepsilon)_\varepsilon$ then, denoting $\chi := \max_k \max_{i\in S_k} |a_i^\varepsilon - p_k|$, we have

$$\left|\sum_{k=1}^{\tilde{N}_0}\sum_{i\in S_k} |d_i||\xi_0(a_i^\varepsilon)-\xi_0(p_k)|\right| \leq C\chi \tag{43}$$

and

$$\left|\tilde{V}[\zeta_{(\mathbf{a},\mathbf{d})_\varepsilon}] - \tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]\right| \leq C\sqrt{\chi} \tag{44}$$

where C depends only on N, $\sum|d_i|$ and Ω.

Proof. The two first assertions are obvious. Estimate (43) is direct by noting that ξ_0 a Lipschitzian function in Ω. Estimate (44) is a direct consequence of Proposition 17. □

We then have:

Corollary 24. *Assume that $\lambda,\delta,h_{\mathrm{ex}}$ satisfy (2) and (3) for some $K\geq 0$ independent of ε. Assume also $\delta^2|\ln\varepsilon|\leq 1$.*

Let $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}\subset\mathscr{H}$ be a family s.t. $\mathcal{F}(v_\varepsilon,A_\varepsilon)\leq \inf_{\mathscr{H}}\mathcal{F}+K\ln|\ln\varepsilon|$ which is in the Coulomb gauge and let $\{(\mathbf{a}_\varepsilon,\mathbf{d}_\varepsilon)=(\mathbf{a},\mathbf{d})\,|\,0<\varepsilon<1\}$ be a family of pseudo vortices associated to $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}$ [indexed on $\mathcal{J}=\mathcal{J}_\varepsilon$ possibly empty].

(1) Letting $A_{v_\varepsilon}\in H^1(\Omega,\mathbb{R}^2)$ be defined by Lemma 12 we have

$$\mathcal{F}(v_\varepsilon,A_\varepsilon)\geq \mathcal{F}(v_\varepsilon,A_{v_\varepsilon})\geq h_{\mathrm{ex}}^2\mathbf{J_0}+2\pi h_{\mathrm{ex}}\sum_{i\in\mathcal{J}} d_i\xi_0(a_i)+F(v_\varepsilon)+\tilde{V}[\zeta_{(\mathbf{a},\mathbf{d})}]+o(1). \tag{45}$$

And then

$$\mathcal{F}(v_\varepsilon,A_\varepsilon)\geq h_{\mathrm{ex}}^2\mathbf{J_0}+2\pi h_{\mathrm{ex}}\sum_{i\in\mathcal{J}} d_i\xi_0(a_i)+F(v_\varepsilon)+\mathcal{O}(1). \tag{46}$$

(2) Assume furthermore that $(\mathbf{a},\mathbf{d})$ admits a cluster structure on $(\mathbf{p},\mathbf{D})$. Then we have

$$\mathcal{F}(v_\varepsilon,A_\varepsilon)\geq h_{\mathrm{ex}}^2\mathbf{J_0}+2\pi h_{\mathrm{ex}}\sum_{i\in\mathcal{J}} d_i\xi_0(a_i)+F(v_\varepsilon)+\tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]+o(1). \tag{47}$$

Proof. The lower bounds (45) and (46) are direct consequences of Theorem 5, Lemma 12, Remark 13 and Propositions 6&11&16.

Estimate (47) is a direct consequence of Proposition 23 and (45). □

We then have the following corollary.

Corollary 25. *Assume that $\lambda,\delta,h_{\mathrm{ex}}$ satisfy (2) and (3). Assume also $\delta^2|\ln\varepsilon|\leq 1$.*

Let $(v_\varepsilon)_{0<\varepsilon<1}\subset H^1(\Omega,\mathbb{C})$ be s.t. $|v_\varepsilon|\not\to 1/2$ in Ω and assume the existence of $(B_\varepsilon)_{0<\varepsilon<1}\subset H^1(\Omega,\mathbb{R}^2)$ s.t. $(v_\varepsilon,B_\varepsilon)$ is in the Coulomb gauge and $\mathcal{F}(v_\varepsilon,B_\varepsilon)\leq \inf_{\mathscr{H}}\mathcal{F}+\mathcal{O}(\ln|\ln\varepsilon|)$. Assume also that $(\mathbf{a}_\varepsilon,\mathbf{d}_\varepsilon)=(\mathbf{a},\mathbf{d})$ are pseudo-vortices as in Definition 15 for v_ε [note that we thus have $\sum|d_i|=\mathcal{O}(1)$], then

$$\mathcal{F}(v_\varepsilon,A_{(\mathbf{a},\mathbf{d})})=h_{\mathrm{ex}}^2\mathbf{J_0}+2\pi h_{\mathrm{ex}}\sum d_i\xi_0(a_i)+F(v_\varepsilon)+\tilde{V}[\zeta_{(\mathbf{a},\mathbf{d})}]+o(1). \tag{48}$$

where $A_{(\mathbf{a},\mathbf{d})}:=h_{\mathrm{ex}}\nabla^\perp\xi_0+\nabla^\perp\zeta_{(\mathbf{a},\mathbf{d})}$.

Consequently we get

$$F(v_\varepsilon)\leq 2\pi h_{\mathrm{ex}}\sum d_i|\xi_0(a_i)|+\mathcal{O}(\ln|\ln\varepsilon|)\leq \pi b^2\sum|d_i||\ln\varepsilon|+\mathcal{O}(\ln|\ln\varepsilon|). \tag{49}$$

Proof. Corollary 25 is a direct consequence of $\inf_{\mathscr{H}} \mathcal{F} \leq h_{\rm ex}^2 \mathbf{J_0}$, Corollary 24 and Propositions 11&17. □

Remark 26. We may state an analog of Corollary 25 if $(\mathbf{a}, \mathbf{d})$ admits a structure of cluster.

6. Renormalized energies

6.1. **Macroscopic renormalized energy [at scale 1].** We consider in this section:

- $N \in \mathbb{N}^*$, $\mathbf{z} = \mathbf{z}^{(n)} \in (\Omega^N)^* := \{(z_1, ..., z_N) \subset \Omega \,|\, z_i \neq z_j \text{ pour } i \neq j\}$,
- $\mathbf{d} = (d_1, ..., d_N) \in \mathbb{Z}^N$.
- $\hbar = \hbar(\mathbf{z}) := \min_i \operatorname{dist}(z_i, \partial\Omega)$

We are going to deal with functions defined in the set Ω perforated by disks with radius $\tilde{r} = \tilde{r}_n \downarrow 0$:

$$\Omega_{\tilde{r}} = \Omega_{\tilde{r}}(\mathbf{z}) := \Omega \setminus \cup_i \overline{B(z_i, \tilde{r})}.$$

We assume

$$\tilde{r} < \frac{1}{8} \min \left\{ \min_{i \neq j} |z_i - z_j| \,;\, \hbar \right\}. \tag{50}$$

For a radius $\tilde{r} > 0$ s.t. (50) is satisfied, we consider the set of functions

$$\mathcal{I}_{\tilde{r}}^{\rm deg} := \left\{ w \in H^1(\Omega_{\tilde{r}}, \mathbb{S}^1) \,|\, \deg_{\partial B(z_i, \tilde{r})}(w) = d_i \text{ for } i \in \{1, ..., N\} \right\}$$

and

$$\mathcal{I}_{\tilde{r}}^{\rm Dir} := \left\{ w \in H^1(\Omega_{\tilde{r}}, \mathbb{S}^1) \;\middle|\; \begin{array}{c} w(z_i + \tilde{r}\mathrm{e}^{\imath\theta}) = C_i \mathrm{e}^{\imath d_i \theta} \text{ for } i \in \{1, ..., N\}, \\ (C_1, ..., C_N) \in (\mathbb{S}^1)^N \end{array} \right\}.$$

In this section we are interested in the minimization of the Dirichlet functional in $\mathcal{I}_{\tilde{r}}^{\rm deg}$ and $\mathcal{I}_{\tilde{r}}^{\rm Dir}$.

Before beginning we state an easy result proved by direct minimization [the proof is left to the reader, see [4]].

Proposition 27. *For $N \geq 1$, $(\mathbf{z}, \mathbf{d}) \in (\Omega^N)^* \times \mathbb{Z}^N$ and $\tilde{r} > 0$ s.t. (50) is satisfied, the following minimization problems admit solutions:*

$$I_{\tilde{r}}^{\rm deg} = I_{\tilde{r}}^{\rm deg}(\mathbf{z}, \mathbf{d}) := \inf_{w \in \mathcal{I}_{\tilde{r}}^{\rm deg}} \frac{1}{2} \int_{\Omega_{\tilde{r}}} |\nabla w|^2 \tag{51}$$

and

$$I_{\tilde{r}}^{\rm Dir} = I_{\tilde{r}}^{\rm Dir}(\mathbf{z}, \mathbf{d}) := \inf_{w \in \mathcal{I}_{\tilde{r}}^{\rm Dir}} \frac{1}{2} \int_{\Omega_{\tilde{r}}} |\nabla w|^2. \tag{52}$$

Moreover, these solutions are unique up to the multiplication by an $\mathbb{S}^1$ constant.

6.1.1. *Study of $I_{\tilde{r}}^{\rm deg}$ and $I_{\tilde{r}}^{\rm Dir}$.* Following [4], it is standard to define the *canonical harmonic map associated to* $(\mathbf{z}, \mathbf{d})$.

Definition 28. Let $N \in \mathbb{N}^*$ and $(\mathbf{z}, \mathbf{d}) \in (\Omega^N)^* \times \mathbb{Z}^N$. A function $w_\star^{(\mathbf{z},\mathbf{d})} \in \cap_{0<p<2} W^{1,p}(\Omega, \mathbb{S}^1) \cap C^\infty(\Omega \setminus \{z_1, ..., z_N\}, \mathbb{S}^1)$ is the *canonical harmonic map associated to the singularities* $(\mathbf{z}, \mathbf{d})$ if

$$w_\star^{(\mathbf{z},\mathbf{d})}(z) = \mathrm{e}^{\imath\varphi_\star(z)} \prod_{i=1}^N \left(\frac{z - z_i}{|z - z_i|} \right)^{d_i} \text{ with } \begin{cases} \varphi_\star \text{ is harmonic in } \Omega \\ \partial_\nu w_\star^{(\mathbf{z},\mathbf{d})} = 0 \text{ on } \partial\Omega, \int_{\partial\Omega} \varphi_\star = 0 \end{cases}. \tag{53}$$

Remark 29. In this framework, it is classic to define $\Phi_\star^{(\mathbf{z},\mathbf{d})}$ [with the notation of Definition 28], the unique solution of

$$\begin{cases} \Delta \Phi_\star^{(\mathbf{z},\mathbf{d})} = 2\pi \sum_{i=1}^N d_i \delta_{z_i} & \text{in } \Omega \\ \Phi_\star^{(\mathbf{z},\mathbf{d})} = 0 & \text{on } \partial\Omega \end{cases}.$$

This function satisfies $\nabla^\perp \Phi_\star^{(\mathbf{z},\mathbf{d})} = w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})}$. Moreover, by denoting $R_{(\mathbf{z},\mathbf{d})}$ the unique solution of

$$\begin{cases} \Delta R_{(\mathbf{z},\mathbf{d})} = 0 & \text{in } \Omega \\ R_{(\mathbf{z},\mathbf{d})}(z) = -\sum_i d_i \ln|z - z_i| & \text{on } \partial\Omega \end{cases},$$

we have $\Phi_\star^{(\mathbf{z},\mathbf{d})}(z) = \sum_i d_i \ln|z - z_i| + R_{(\mathbf{z},\mathbf{d})}(z)$.

We first study the asymptotic behavior of minimizers of $I_{\tilde r}^{\deg}(\mathbf{z},\mathbf{d})$ when $\tilde r \to 0$.

Proposition 30. *Let $N \in \mathbb{N}^*$, $(\mathbf{z},\mathbf{d}) = (\mathbf{z},\mathbf{d})^{(n)} \subset (\Omega^N)^* \times \mathbb{Z}^N$ and $\hbar := \min_i \operatorname{dist}(z_i, \partial\Omega)$. We assume that $\sum_i |d_i| = \mathcal{O}(1)$.*

For $\tilde r > 0$ s.t. (50) is satisfied, we may consider $w_{\tilde r}^{(\mathbf{z},\mathbf{d})}$, the unique solution of the problem

$$I_{\tilde r}^{\deg}(\mathbf{z},\mathbf{d}) := \inf_{w \in \mathcal{I}_{\tilde r}^{\deg}} \frac{1}{2} \int_{\Omega_{\tilde r}} |\nabla w|^2, \tag{54}$$

of the form

$$w_{\tilde r}^{(\mathbf{z},\mathbf{d})}(z) = \mathrm{e}^{\imath \varphi_{\tilde r}(z)} \prod_{i=1}^N \left(\frac{z - z_i}{|z - z_i|} \right)^{d_i} \quad \text{with} \quad \begin{cases} \varphi_{\tilde r} \in H^1 \cap C^\infty(\Omega_{\tilde r}, \mathbb{R}) \\ \int_{\partial\Omega} \varphi_{\tilde r} = 0 \end{cases}. \tag{55}$$

We thus have the existence of $C > 0$ [depending only on Ω, N and the bound of $\sum_i |d_i|$] s.t.

$$\|\nabla w_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde r})} \leq \frac{C(1 + |\ln \tilde r|)}{\tilde r}. \tag{56}$$

We denote

$$X := \begin{cases} \dfrac{\tilde r(1 + |\ln(\hbar)|)}{\hbar} \left(1 + \dfrac{\tilde r(1 + |\ln(\hbar)|)}{\hbar} \right) & \text{if } N = 1 \\ \left(\dfrac{\tilde r}{\min_{i \neq j} |z_i - z_j|} + \dfrac{\tilde r(1 + |\ln(\hbar)|)}{\hbar} \right) \left(1 + \dfrac{\tilde r(1 + |\ln(\hbar)|)}{\hbar} \right) & \text{if } N \geq 2 \end{cases} \tag{57}$$

and we have

$$\|\varphi_{\tilde r} - \varphi_\star\|_{H^1(\Omega_{\tilde r})}^2 \leq CX, \tag{58}$$

$$0 \leq \frac{1}{2} \int_{\Omega_{\tilde r}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 - \inf_{w \in \mathcal{I}_{\tilde r}^{\deg}} \frac{1}{2} \int_{\Omega_{\tilde r}} |\nabla w|^2 \leq CX. \tag{59}$$

Moreover, if there exists $\eta > 0$ [independent of n] s.t. $\hbar > \eta$ then (56) may be refined into

$$\|\nabla w_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde r})} \leq \frac{C}{\tilde r}. \tag{60}$$

The proof of Proposition 30 is in Appendix D.1.

By adapting the proof of Proposition 5.1 in [17] we have

Proposition 31. *For $N \geq 1$, there exists an application $W_N^{\mathtt{macro}} = W^{\mathtt{macro}} : (\Omega^N)^* \times \mathbb{Z}^N \to \mathbb{R}$ s.t. for sequences $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}, \mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$ and $\tilde{r} = \tilde{r}_n \to 0$ satisfying (50) and s.t. $\mathbf{d}$ is independent of n, there exists $C \geq 1$ [depending only on N, $\sum |d_i|$ and Ω] s.t.*

$$\left| \frac{1}{2} \int_{\Omega_{\tilde{r}}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 - \pi \sum_i d_i^2 |\ln \tilde{r}| - W^{\mathtt{macro}}(\mathbf{z}, \mathbf{d}) \right| \leq CX$$

with

$$W^{\mathtt{macro}}(\mathbf{z}, \mathbf{d}) = -\pi \sum_{i \neq j} d_i d_j \ln |z_i - z_j| - \pi \sum_i d_i R_{(\mathbf{z},\mathbf{d})}(z_i),$$

$$R_{(\mathbf{z},\mathbf{d})} \in C^\infty(\Omega, \mathbb{R}) \text{ satisfies } \|R_{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega)} \leq C(1 + |\ln \hbar|).$$

Proposition 31 is proved in D.2. We immediately obtain from Proposition 31 the following corollary.

Corollary 32. *Under the hypotheses of Proposition 31 and assuming that there exists $C_1 > 0$ [independent of r] s.t. $\dfrac{\tilde{r}(1 + |\ln \hbar|)}{\hbar} \leq C_1$, there is $C > 1$ [depending only on Ω, N, $\sum_i |d_i|$ and C_1] s.t. $\displaystyle\int_{\Omega_{\tilde{r}}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 \leq C |\ln \tilde{r}|$.*

We end this section by linking $I_{\tilde{r}}^{\deg}$ and $I_{\tilde{r}}^{\mathrm{Dir}}$.

Proposition 33. *Let $N \geq 1$, $\mathbf{z} \in (\Omega^N)^*$ and $\tilde{r} = \tilde{r}_n \downarrow 0$ satisfying (50). Assume $\dfrac{\tilde{r}}{\hbar} \to 0$ and if $N \geq 2$, we also assume $\dfrac{\tilde{r}}{\min_{i \neq j} |z_i - z_j|} \to 0$.*

Let

$$\eta := \begin{cases} 10^{-1}\hbar & \text{if } N = 1 \\ 10^{-1} \min\{\hbar\,;\, \min_{i \neq j} |z_i - z_j|\} & \text{if } N \geq 2 \end{cases}.$$

Assume furthermore

$$Z := \frac{1}{\ln(\eta/\tilde{r})} \left[\frac{\eta(1 + |\ln(\hbar)|)}{\hbar} + 1 \right]^2 \to 0.$$

Then for $\mathbf{d} \in \mathbb{Z}^N$ [independent of n], there exists $C > 1$ [depending only on Ω, N and $\sum |d_i|$] s.t.

$$0 \leq \inf_{w \in \mathcal{I}_{\tilde{r}}^{\mathrm{Dir}}} \frac{1}{2} \int_{\Omega_{\tilde{r}}} |\nabla w|^2 - \inf_{w \in \mathcal{I}_{\tilde{r}}^{\deg}} \frac{1}{2} \int_{\Omega_{\tilde{r}}} |\nabla w|^2 \leq C(X + Z).$$

Proposition 33 is proved Appendix D.3.

6.1.2. *Macroscopic renormalized energy and cluster of vortices.* We first state an easy lemma.

Lemma 34. *(1) Let $N \in \mathbb{N}^*$ and $\mathbf{d} \in \mathbb{Z}^N$. Let $\chi > 0$ and $\mathbf{z}, \mathbf{z}' \in (\Omega^N)^*$ be s.t. for $i \in \{1, ..., N\}$ we have $|z_i - z_i'| \leq \chi$. Then we have*

$$\|R_{(\mathbf{z},\mathbf{d})} - R_{(\mathbf{z}',\mathbf{d})}\|_{L^\infty(\Omega)} \leq \sum_i |d_i| \frac{\chi}{\max\{\hbar(\mathbf{z}), \hbar(\mathbf{z}')\}}.$$

(2) Let $1 \leq \tilde{N}_0 \leq N$, $\mathbf{p} \in (\Omega^{\tilde{N}_0})^*$, $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}, \mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$ be s.t. $\mathbf{d}$ is independent of n and for $i \in \{1, ..., N\}$ there exists $k \in \{1, ..., \tilde{N}_0\}$ s.t. $z_i \to p_k$. We let $\chi := \max_i \operatorname{dist}(z_i, \{p_1, ..., p_{\tilde{N}_0}\})$.

For $k \in \{1,...,\tilde{N}_0\}$ we let $D_k := \sum_{z_i \to p_k} d_i$ and $\mathbf{D} = (D_1,...,D_{\tilde{N}_0})$. Then we have

$$\|R_{(\mathbf{z},\mathbf{d})} - R_{(\mathbf{p},\mathbf{D})}\|_{L^\infty(\Omega)} \leq \sum_i |d_i| \frac{\chi}{\hbar(\mathbf{p})}.$$

Proof. The first assertion is obtained with the help of the maximum principle and the bound $|R_{(\mathbf{z},\mathbf{d})} - R_{(\mathbf{z}',\mathbf{d})}| \leq \sum_i |d_i| \frac{\chi}{\max\{\hbar(\mathbf{z}),\hbar(\mathbf{z}')\}}$ on $\partial\Omega$. The second assertion assertion follows by the same way. □

With Lemma 34 we may exploit a structure of cluster for $W^{\rm macro}$.

Proposition 35. *Let $1 \leq \tilde{N}_0 \leq N$, $\mathbf{p} \in (\Omega^{\tilde{N}_0})^*$ [independent of n] and write*

$$\gamma_{\mathbf{p}} := \begin{cases} 1 & \text{if } \tilde{N}_0 = 1 \\ \min_{k\neq l} |p_k - p_l| & \text{otherwise} \end{cases}.$$

Let $(\mathbf{z},\mathbf{d}) = (\mathbf{z},\mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$ be s.t. $\mathbf{d}$ is independent of n and for $i \in \{1,...,N\}$ there exists $k \in \{1,...,\tilde{N}_0\}$ s.t. $z_i \to p_k$. We denote $\chi := \max_i \operatorname{dist}(z_i, \{p_1,...,p_{\tilde{N}_0}\})$.

For $k \in \{1,...,\tilde{N}_0\}$ we denote $D_k := \sum_{z_i \to p_k} d_i$ and $\mathbf{D} = (D_1,...,D_{\tilde{N}_0})$. Then there exists $C \geq 1$ [depending only on Ω, N and $\sum |d_i|$] s.t.

$$\left| W_N^{\rm macro}(\mathbf{z},\mathbf{d}) - \left(W_{\tilde{N}_0}^{\rm macro}(\mathbf{p},\mathbf{D}) - \pi \sum_{k=1}^{\tilde{N}_0} \sum_{\substack{z_i,z_j \to p_k \\ i\neq j}} d_i d_j \ln|z_i - z_j| \right) \right|$$
$$\leq C\chi \left(\frac{1 + |\ln[\hbar(\mathbf{p})]|}{\hbar(\mathbf{p})} + \frac{1}{\gamma_{\mathbf{p}}} \right).$$

Proof. We have

$$W^{\rm macro}(\mathbf{z},\mathbf{d}) = -\pi \sum_{k=1}^{\tilde{N}_0} \sum_{\substack{z_i,z_j \to p_k \\ i\neq j}} d_i d_j \ln|z_i - z_j| - \pi \sum_{\substack{z_i \to p_k \\ z_j \to p_l \\ k\neq l}} d_i d_j \ln|z_i - z_j| - \pi \sum_i d_i R_{(\mathbf{z},\mathbf{d})}(z_i).$$

It is easy to check that

$$\sum_{\substack{z_i \to p_k \\ z_j \to p_l \\ k\neq l}} d_i d_j \ln|z_i - z_j| = \sum_{k\neq l} D_k D_l \ln|p_k - p_l| + H \tag{61}$$

with $H \leq 4\left(\sum_i |d_i|\right)^2 \frac{\chi}{\gamma_{\mathbf{p}}}$ for sufficiently large n.

On the other hand, from Lemma 34 [second assertion], we have $\|R_{(\mathbf{z},\mathbf{d})} - R_{(\mathbf{p},\mathbf{D})}\|_{L^\infty(\Omega)} \leq \sum_i |d_i| \frac{\chi}{\max\{\hbar(\mathbf{z}),\hbar(\mathbf{p})\}}$. From standard pointwise estimates for the gradient of harmonic functions [see (166)] there exists $C \geq 1$ depending only on Ω, $\sum |D_k|$ and N [here we used $1 \leq \tilde{N}_0 \leq N$] s.t. for $z_i \to p_k$ we have $\left|R_{(\mathbf{p},\mathbf{D})}(z_i) - R_{(\mathbf{p},\mathbf{D})}(p_k)\right| \leq C\chi \frac{1 + |\ln[\hbar(\mathbf{p})]|}{\hbar(\mathbf{p})}$.

Then, up to change the value of C, we have

$$\left| \sum_i d_i R_{(\mathbf{z},\mathbf{d})}(z_i) - \sum_k D_k R_{(\mathbf{p},\mathbf{D})}(p_k) \right| \leq C\chi \frac{1 + |\ln[\hbar(\mathbf{p})]|}{\hbar(\mathbf{p})}. \tag{62}$$

By combining (61) and (62) we get the result. □

6.2. **Mesoscopic renormalized energy [at scale $h_{\rm ex}^{-1/2}$].** From the work of Sandier and Serfaty we may obtain mesoscopic informations. To this end we need to assume a non degeneracy assumption for minimal points of ξ_0. So we assume in this section that Hypothesis (5) holds.

Let

$$\eta_\Omega := \begin{cases} 10^{-3}\min\{1;\mathrm{dist}(\Lambda,\partial\Omega)\} & \text{if } N_0=1 \\ 10^{-3}\min\{1;\mathrm{dist}(\Lambda,\partial\Omega);\min_{k\neq l}|p_k-p_l|\} & \text{if } N_0\geq 2 \end{cases}. \tag{63}$$

For $p\in\Lambda$, by applying Lemma 11.1 in [16] in the disk $B(p,\eta_\Omega)$, we get the following proposition.

Proposition 36. *Assume that Hypothesis (5) holds. Let $D\in\mathbb{N}^*$ and $h_{\rm ex}\uparrow\infty$ when $\varepsilon\to 0$. Then for $p\in\Lambda$ and $R=R(\varepsilon)\to 0$ s.t. $R\sqrt{h_{\rm ex}}\to\infty$ we have*

$$\begin{aligned} &\inf_{\mathbf{z}\in[B(p,R)^D]^*}\left\{-\pi\sum_{i\neq j}\ln|z_i-z_j|+2\pi h_{\rm ex}\sum_i[\xi_0(z_i)-\xi_0(p)]\right\} \\ &= \frac{\pi}{2}(D^2-D)\ln\left(\frac{h_{\rm ex}}{D}\right)+C_{p,D}+o(1) \end{aligned} \tag{64}$$

with

$$C_{p,D}:=\min_{[(\mathbb{R}^2)^D]^*}W_{p,D}^{\rm meso} \tag{65}$$

and

$$\begin{array}{rccc} W_{p,D}^{\rm meso}: & [(\mathbb{R}^2)^D]^* & \to & \mathbb{R} \\ & \mathbf{x}=(x_1,...,x_D) & \mapsto & -\pi\displaystyle\sum_{i\neq j}\ln|x_i-x_j|+\pi D\sum_{i=1}^D Q_p(x_i). \end{array} \tag{66}$$

where $Q_p(x):=x\cdot\mathrm{Hess}_{\xi_0}(p)x$, $\mathrm{Hess}_{\xi_0}(p)$ is the Hessian matrix of ξ_0 at p.

Moreover the infimum in (64) is reached and if $\mathbf{z}^\varepsilon\in[B(p,R)^D]^*$ is s.t.

$$-\pi\sum_{i\neq j}\ln|z_i^\varepsilon-z_j^\varepsilon|+2\pi h_{\rm ex}\sum_i[\xi_0(z_i^\varepsilon)-\xi_0(p)]=\frac{\pi}{2}(D^2-D)\ln\left(\frac{h_{\rm ex}}{D}\right)+C_{p,D}+o(1)$$

then for all sequence $\varepsilon=\varepsilon_n\downarrow 0$, up to pass to a subsequence, denoting $\ell=\sqrt{\dfrac{D}{h_{\rm ex}}}$ and $\breve{z}_i^\varepsilon=\dfrac{z_i^\varepsilon-p}{\ell}$, we have $\breve{\mathbf{z}}^\varepsilon=(\breve{z}_1^\varepsilon,...,\breve{z}_D^\varepsilon)$ which converges to a minimizer of $W_{p,D}^{\rm meso}$. In particular $|\breve{z}_i^\varepsilon|\leq C_{\Omega,D}$ with $C_{\Omega,D}>0$ which depends only on Ω and D.

6.3. **Microscopic renormalized energy [at scale $\lambda\delta$].** The location of the vorticity defects at scale $\lambda\delta$ [inside a connected component of ω_ε] is given by the microscopic renormalized energy exactly as in the case without magnetic field. In order to define the microscopic renormalized energy we need some notation. Recall that the pinning term $a_\varepsilon:\Omega\to\{b,1\}$ is obtained [see Section 2.3] from a smooth bounded simply connected set ω s.t. $0\in\omega\subset\overline{\omega}\subset Y:=(-1/2,1/2)^2$. The construction of the pinning term uses two parameters $\delta=\delta(\varepsilon)$ [the parameter of period] and $\lambda=\lambda(\varepsilon)$ [the parameter of dilution]. For $x_0\in\omega$ and a sequence $\varepsilon=\varepsilon_n\downarrow 0$, we consider $\hat{x}_\varepsilon\in\omega$ s.t. $\hat{x}_\varepsilon\to x_0\in\omega$.

Let $m_\varepsilon\in\mathbb{Z}^2$ be s.t. the cell $Y_\varepsilon=\delta(m_\varepsilon+Y)$ satisfies $\overline{Y}_\varepsilon\subset\Omega$. We then denote $z_\varepsilon=\delta[m_\varepsilon+\lambda\hat{x}_\varepsilon]$. It is proved in [7] [see Estimates (9) and (10)] that for $R=R_\varepsilon\gg\lambda\delta$

and $r = r_\varepsilon \ll \lambda\delta$, denoting $\hat{R} = R/(\lambda\delta)$, $\hat{r} = r/(\lambda\delta)$, $\mathcal{D}_\varepsilon = B(\delta m_\varepsilon, R) \setminus \overline{B(z_\varepsilon, r)}$, $\hat{\mathcal{D}}_\varepsilon = B(0,\hat{R}) \setminus \overline{B(\hat{x}_\varepsilon, \hat{r})}$ and $\hat{\mathcal{D}} = B(0,\hat{R}) \setminus \overline{B(x_0, \hat{r})}$:

$$\inf_{\substack{w\in H^1(\mathcal{D}_\varepsilon,\mathbb{S}^1)\\ \deg(w)=1}} \frac{1}{2}\int_{\mathcal{D}_\varepsilon} U_\varepsilon^2 |\nabla w|^2 = \inf_{\substack{w\in H^1(\mathcal{D}_\varepsilon,\mathbb{S}^1)\\ w(z_\varepsilon+R\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta}\\ w(x_\varepsilon+r\mathrm{e}^{\imath\theta})=\mathrm{Cst}\,\mathrm{e}^{\imath\theta}}} \frac{1}{2}\int_{\mathcal{D}_\varepsilon} U_\varepsilon^2|\nabla w|^2 + o_\varepsilon(1) \tag{67}$$

$$= \inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}}_\varepsilon,\mathbb{S}^1)\\ \deg(w)=1}} \frac{1}{2}\int_{\hat{\mathcal{D}}_\varepsilon} a^2|\nabla \hat{w}|^2 + o_\varepsilon(1). \tag{68}$$

Moreover from the main result in [8], we have the existence of an application $\tilde{W}^{\mathrm{micro}} : \omega \to \mathbb{R}$ [depending only on ω and b] s.t.

$$\inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}}_\varepsilon,\mathbb{S}^1)\\ \deg(w)=1}} \frac{1}{2}\int_{\hat{\mathcal{D}}_\varepsilon} a^2|\nabla \hat{w}|^2 = f_\omega(\hat{R}) + b^2\pi|\ln(\hat{r})| + \tilde{W}^{\mathrm{micro}}(x_0) + o(1). \tag{69}$$

where $f_\omega(\hat{R}) := \inf_{\substack{w\in H^1[B(0,\hat{R})\setminus\overline{\omega},\mathbb{S}^1]\\ \deg(w)=1}} \frac{1}{2}\int_{B(0,\hat{R})\setminus\overline{\omega}} |\nabla w|^2$.

It is clear that there exists $C_\omega \in \mathbb{R}$ [depending only on ω] s.t. when $\hat{R}\to\infty$ we have $f_\omega(\hat{R}) = \pi\ln(\hat{R}) + C_\omega$.

Then, by denoting $W^{\mathrm{micro}}(x_0) := \tilde{W}^{\mathrm{micro}}(x_0) + C_\omega$, we get from (69) :

$$\inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}},\mathbb{S}^1)\\ \deg(w)=1}} \frac{1}{2}\int_{\hat{\mathcal{D}}} a^2|\nabla \hat{w}|^2 = \pi\ln(\hat{R}) + b^2\pi|\ln(\hat{r})| + W^{\mathrm{micro}}(x_0) + o(1). \tag{70}$$

Moreover, from [9] we know that W^{micro} admits minimizers in ω.

7. Sharp upper bound: construction of a test function

From now on we assume that Hypothesis (5) holds. We thus may use for $p\in\Lambda$ and $D\in\mathbb{N}^*$ the constant $C_{p,D}$ defined in (65). We denote also $C_{p,0} := 0$.

We let for $d\in\mathbb{N}^*$:

$$\Lambda_d := \left\{ \mathbf{D} \in \left\{ \left\lceil \frac{d}{N_0} \right\rceil ; \left\lfloor \frac{d}{N_0} \right\rfloor \right\}^{N_0} \,\middle|\, \sum_{k=1}^{N_0} D_k = d \right\}, \tag{71}$$

$$\overline{\mathcal{W}}_{d,\Omega} = \overline{\mathcal{W}}_d := \min_{\mathbf{D}\in\Lambda_d} \left\{ W^{\mathrm{macro}}(\mathbf{p},\mathbf{D}) + \sum_{k=1}^{N_0} C_{p_k,D_k} + \tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}] \right\} \tag{72}$$

where, for $x\in\mathbb{R}$, $\lceil x\rceil$ is the ceiling of x, $\lfloor x\rfloor$ is the floor of x, $W^{\mathrm{macro}}(\cdot)$ is defined in Proposition 31 and $\tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]$ is defined in Proposition 17.

We now state an easy lemma whose proof is left to the reader.

Lemma 37. *Let $d\in\mathbb{N}^*$ and $\mathbf{D}\in\Lambda_d$. Then the following quantities are independent of $\mathbf{D}$:*

$$\mathscr{L}_1(d) := \frac{\pi}{2}\left[\left(\sum_{k=1}^{N_0} D_k^2 \right) - d \right],$$

$$\mathscr{L}_2(d) := \overline{\mathcal{W}}_d + \frac{\pi}{2} \sum_{\substack{k=1\\ s.t.\ D_k\geq 1}}^{N_0} (D_k - D_k^2)\ln(D_k).$$

Moreover: $d \leq N_0 \iff \mathscr{L}_1(d) = 0 \iff \mathscr{L}_2(d) = \overline{\mathcal{W}}_d$.

Notation 38. We let $\mathscr{L}_1(0) = \mathscr{L}_2(0) = 0$.

The main result of this section is the following proposition.

Proposition 39. *Assume that* $h_{\rm ex} = \mathcal{O}(|\ln\varepsilon|)$, $h_{\rm ex} \to +\infty$,

$$\lambda^{1/4}|\ln\varepsilon| \to 0 \text{ and } \delta\sqrt{h_{\rm ex}} \to 0 \tag{73}$$

and assume that Hypothesis (5) *holds.*

Let $d \in \mathbb{N}^*$ *and let* $\mathbf{D} \in \Lambda_d$ *be a minimizer of the minimizing problem* (72).

For $0 < \varepsilon < 1$, *there exists* $(v_\varepsilon, A_\varepsilon) \in \mathscr{H}$ *which is in the Coulomb gauge with* d *vortices of degree* 1 *s.t.*

$$\mathcal{F}(v_\varepsilon, A_\varepsilon) = h_{\rm ex}^2 \mathbf{J_0} + dM_\Omega\left[-h_{\rm ex} + H^0_{c_1}\right] + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathscr{L}_2(d) + o(1) \tag{74}$$

with $M_\Omega := 2\pi\|\xi_0\|_{L^\infty(\Omega)}$ *and*

$$H^0_{c_1} := \frac{b^2|\ln\varepsilon| + (1-b^2)|\ln(\lambda\delta)|}{2\|\xi_0\|_{L^\infty(\Omega)}} + \tilde{\gamma}_{b,\omega} \tag{75}$$

where

$$\tilde{\gamma}_{b,\omega} := \frac{\min\limits_{\omega} W^{\rm micro} + b^2[\gamma + \pi\ln b]}{2\pi\|\xi_0\|_{L^\infty(\Omega)}}, \tag{76}$$

γ *is a universal constant defined in Lemma IX.1* [4] *and* $W^{\rm micro}$ *is defined in Section 6.3.*

Proposition 39 is proved in Appendix E.

8. Tool box

The proof of the main theorems of this article is done in a classic way: by matching upper and lower bounds. A [sharp] upper bound is obtained by Proposition 39. Getting a sharp lower bound is the most challenging part of the proof. It needs the proof of several facts related with the vorticity defects of a family of quasi-minimizers [quantization, localization, size ...].

In this section we present some technical and quite classical results adapted to our situation.

8.1. An η-ellipticity property.

In this section we focus on quasi-minimizers. We let $h_{\rm ex} = \mathcal{O}(|\ln\varepsilon|)$ and we consider $\{(v_\varepsilon, A_\varepsilon)\,|\,0 < \varepsilon < 1\}$ be a family of quasi-minimizers for $\mathcal{F}$, *i.e.*,

$$\mathcal{F}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{F} + o(1). \tag{77}$$

We assume that for all $\varepsilon \in (0;1)$, $(v_\varepsilon, A_\varepsilon)$ is in the Coulomb gauge and that $v_\varepsilon \in H^1(\Omega, \mathbb{C})$ is s.t.

$$\||\nabla| v_\varepsilon|\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1}). \tag{78}$$

The major result of this section is a key tool in this article: an η ellipticity property.

Proposition 40. *Let* $h_{\rm ex} = \mathcal{O}(|\ln\varepsilon|)$ *and let* $\{(v_\varepsilon, A_\varepsilon)\,|\,0 < \varepsilon < 1\} \subset \mathscr{H}$ *be a family in the Coulomb gauge satisfying* (77) *and* (78).

For $\eta \in (0,1)$ there exist $\varepsilon_\eta > 0$ and $C_\eta > 0$ [depending on the bound of $\varepsilon \|\nabla |v_\varepsilon|\|_{L^\infty(\Omega)}$] s.t. for $0 < \varepsilon < \varepsilon_\eta$, if $z \in \Omega$ is s.t.

$$b^2 \int_{B(z,\sqrt{\varepsilon})\cap\Omega} |\nabla v_\varepsilon|^2 + \frac{b^2}{\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_\eta |\ln \varepsilon|,$$

then $|v_\varepsilon(z)| > \eta$.

Proposition 40 is proved in Appendix F.

By combining Proposition 40 with Theorem 5 we get immediately a first step in the [macroscopic] localization of the vorticity defects. In order to apply Theorem 5 we need assume

$$\begin{cases} \lambda, \delta \text{ satisfy (2)}, \ \delta^2 |\ln \varepsilon| \to 0, \ h_{\rm ex} \to \infty \\ \text{(3) holds for } h_{\rm ex} \text{ with some } K \geq 0 \text{ independent of } \varepsilon \end{cases} . \tag{79}$$

Corollary 41. *Assume that λ, δ and $h_{\rm ex}$ satisfy (79) and let $\{(v_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\} \subset \mathscr{H}$ be s.t. (77) and (78) hold. There exist $0 < \varepsilon_0 \leq \varepsilon_K$ and $M \geq 1$ s.t. for $0 < \varepsilon < \varepsilon_0$, letting $\tilde{\Lambda}_\varepsilon := \Lambda \cap \cup_{d_i \neq 0} B(a_i, 2\mathcal{M}_K |\ln \varepsilon|^{-s_0})$ where the (a_i, d_i)'s [depend on ε] are given by Proposition 10 and $\varepsilon_K \& \mathcal{M}_K \& s_0$ are given by Theorem 5, we have*

$$\{|v_\varepsilon| \leq 1/2\} \subset \bigcup_{p\in\tilde{\Lambda}_\varepsilon} B(p, M|\ln\varepsilon|^{-\tilde{s}_0}) \text{where } \tilde{s}_0 := \min\{s_0, 10\}.$$

Proof. We argue by contradiction and we assume that there exist $\varepsilon = \varepsilon_n \downarrow 0$ and a sequence $((v_\varepsilon, A_\varepsilon))_\varepsilon \subset \mathscr{H}$ s.t. (77) and (78) hold and s.t. for all n there exists

$$z_0 = z_0^n \in \{|v| \leq 1/2\} \setminus \bigcup_{p\in\tilde{\Lambda}_\varepsilon} B(p, n|\ln\varepsilon|^{-\tilde{s}_0}).$$

Since (77) and (78) are gauge invariant we may assume that, for all ε, $(v_\varepsilon, A_\varepsilon)$ is in the Coulomb gauge.

Let $\mathcal{B} := \{(B(a_i, r_i), d_i) \,|\, i \in \mathcal{J}\}$ be given by Proposition 10. Write $B_i := B(a_i, r_i)$ for $i \in \mathcal{J}$. Note that by Theorem 5, from the quasi-minimality of $(v_\varepsilon, A_\varepsilon)$, for ε sufficiently small, we have $d_i \geq 0$ for all i and $d := \sum |d_i| = \sum d_i = \mathcal{O}(1)$. Up to pass to a subsequence, we may thus assume that d is independent of ε.

From the definition of $\tilde{\Lambda}_\varepsilon$, we have

$$\bigcup_{d_i>0} B_i \subset \bigcup_{p\in\tilde{\Lambda}} B(p, 2\mathcal{M}_K |\ln\varepsilon|^{-s_0}).$$

Note that from Theorem 5 we have $\mathcal{F}(v_\varepsilon, 0) = \mathcal{O}(|\ln\varepsilon|^2)$. Then we may use Proposition 10 for the configuration $(v_\varepsilon, 0) \in \mathscr{H}$ to get a covering $\cup_{i\in\tilde{\mathcal{J}}} \tilde{B}_i$ of $\{|v_\varepsilon| < 1 - |\ln\varepsilon|^{-2}\}$ with disjoint disks $\tilde{B}_i = B(\tilde{a}_i, \tilde{r}_i)$, $\sum \tilde{r}_i < |\ln\varepsilon|^{-10}$.

Therefore there is $\rho \in [2\mathcal{M}_K |\ln\varepsilon|^{-\tilde{s}_0}; (2\mathcal{M}_K + 6)|\ln\varepsilon|^{-\tilde{s}_0}]$ s.t.

$$\left[\bigcup_{p\in\tilde{\Lambda}_\varepsilon} \partial B(p,\rho)\right] \cap \left[\bigcup_{i\in\mathcal{J}} B_i \cup \bigcup_{i\in\tilde{\mathcal{J}}} \tilde{B}_i\right] = \emptyset.$$

In particular $|v_\varepsilon| \geq 1 - |\ln\varepsilon|^{-2}$ on $\bigcup_{p\in\tilde{\Lambda}_\varepsilon} \partial B(p,\rho)$. Thus, writing $\tilde{d}_i := \deg_{\partial\tilde{B}_i}(v_\varepsilon)$ when $\tilde{B}_i \subset \Omega$, we get for $p \in \tilde{\Lambda}_\varepsilon$

$$\sum_{\tilde{B}_i \subset B(p,\rho)} |\tilde{d}_i| \geq \left| \sum_{\tilde{B}_i \subset B(p,\rho)} \tilde{d}_i \right| = \deg_{\partial B(p,\rho)}(v_\varepsilon) = \sum_{B_i \subset B(p,\rho)} d_i.$$

Note that for sufficiently large n we have $B(z_0,\sqrt{\varepsilon})\cap\bigcup_{p\in\tilde\Lambda_\varepsilon}B(p,\rho)=\emptyset$.

On the other hand, since $\sum\tilde r_i<|\ln\varepsilon|^{-10}$, we have for $\tilde B_i\subset\Omega$

$$F(v,\tilde B_i)\geq\pi b^2|\tilde d_i|(|\ln\varepsilon|-C\ln|\ln\varepsilon|).$$

Using Proposition 40 we obtain

$$F(v)\geq(\pi b^2 d+C_{1/2})|\ln\varepsilon|-\mathcal{O}(\ln|\ln\varepsilon|) \tag{80}$$

where $C_{1/2}>0$ is given by Proposition 40 with $\eta=1/2$. Estimate (80) is in contradiction with (49).

□

8.2. **Construction of the ε^s-bad discs.** As in the previous section we assume that λ,δ and $h_{\rm ex}$ satisfy (79). In this section we establish the existence of *ε^s-bad discs associated to a quasi-minimizing sequence*. The construction of the bad discs requires the hypotheses: $|v_\varepsilon|\in W^{2,1}(\Omega)$.

An ε^s-bad discs family associated to a familly $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}\subset\mathscr{H}$ consists in sets of discs that have small diameters [a roots of ε] s.t. for fix ε the discs are "well separated", the union of the discs is a covering of $\{|v|\leq 1/2\}$ and each "heart" of a disc intersects $\{|v|\leq 1/2\}$. Such sets of discs give thus a nice visualization of $\{|v|\leq 1/2\}$.

In the next section [Section 9], adding an extra hypothesis on λ,δ and $h_{\rm ex}$ we get some informations in terms of location and quantification of the ε^s-bad discs.

Proposition 42. *Assume that λ,δ and $h_{\rm ex}$ satisfy (79). There exists $M_0\in\mathbb{N}^*$ s.t. for $\mu\in(0,1/2)$, if $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}$ is in the Coulomb gauge and agrees (17)&(77), then there exist $\varepsilon_\mu>0$ and $C_\mu\geq 1$ [independent of ε] s.t. for $0<\varepsilon<\varepsilon_\mu$, there is $J_\mu=J_{\mu,\varepsilon}\subset\{1,...,M_0\}$ [possibly empty] s.t. if $J_\mu=\emptyset$ then $|v|>1/2$ in Ω and if $J_\mu\neq\emptyset$ then there are $\{z_i\,|\,i\in J_\mu\}\subset\Omega$, a set of mutually distinct points, and $r\in[\varepsilon^\mu,\varepsilon^{\mu_*}]$ with $\mu_*:=2^{-L_0^2}\mu$ verifying:*

(1) $|z_i-z_j|\geq r^{3/4}$ for $i,j\in J_\mu$, $i\neq j$,

(2) $\{|v_\varepsilon|\leq 1/2\}\subset\cup_{J_\mu}B(z_i,r)\subset\Omega$ and, for $i\in J_\mu$, $B(z_i,r/4)\cap\{|v_\varepsilon|\leq 1/2\}\neq\emptyset$,

(3) For $i\in J_\mu$ we have $r\displaystyle\int_{\partial B(z_i,r)}|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\leq C_\mu$ and $|v|\geq 1-|\ln\varepsilon|^{-2}$ on $\partial B(z_i,r)$.

Proposition 42 is proved in Appendix G. We have the following standard estimate.

Proposition 43. *Assume (79) and let $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}$ be as in Proposition 42. Fix $\mu\in(0,1/2)$ and let ε_μ, C_μ be given by Proposition 42. For $0<\varepsilon<\varepsilon_\mu$ we consider J_μ, $\{z_i\,|\,i\in J_\mu\}\subset\Omega$ and r obtained in Proposition 42. We denote $d_i:=\deg_{\partial B(z_i,r)}(v_\varepsilon)$.*

There exists $c_{\mu,b}\geq 1$ independent of ε s.t. for $\varepsilon<\varepsilon_\mu$ we have

$$|d_i|\leq 4\sqrt{C_\mu}, \tag{81}$$

$$\frac12\int_{B(z_i,r)}|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\geq\pi|d_i|\ln\left(\frac{r}{\varepsilon}\right)-c_{\mu,b} \tag{82}$$

and then

$$F(v_\varepsilon,B(z_i,r))\geq\pi|d_i|\inf_{B(z_i,r)}\alpha\left[\ln\left(\frac{r}{\varepsilon}\right)-c_{\mu,b}\right]\geq\pi\inf_{B(z_i,r)}\alpha\,|d_i|[(1-\mu)\ln\varepsilon-c_{\mu,b}]. \tag{83}$$

Moreover there is $0 < \tilde{\varepsilon}_\mu \leq \varepsilon_\mu$ *s.t. for* $0 < \varepsilon < \tilde{\varepsilon}_\mu$ *we have*

$$d_i \neq 0 \text{ for all } i \tag{84}$$

and

$$\sum_{i \in J_\mu} |d_i| \leq \mathcal{D}_{K,b} := \frac{3\mathcal{M}_K}{b^2} \tag{85}$$

Proof. It is classical to get (81) from Proposition 42.3 and the Cauchy Schwartz inequality. Estimate (82) follows from Proposition 42 & Lemma VI.1 in [2] and (83) is a consequence of (82).

The proof of (84) is done arguing by contradiction with the construction of a comparaison function $\tilde{v} := \begin{cases} v & \text{in } \Omega \setminus B(z_{i_0}, r) \\ \tilde{\rho} e^{\imath \tilde{\phi}} & \text{in } B(z_{i_0}, r) \end{cases}$ s.t. $\tilde{v} \in H^1(\Omega, \mathbb{C})$ and $F(\tilde{v}, B(z_{i_0}, r)) = \mathcal{O}(1)$ where we assumed $d_{i_0} = 0$.

Since (v, A) is a quasi-minimizer of $\mathcal{F}$ we have $\mathcal{F}(v, A) \leq \mathcal{F}(\tilde{v}, A) + o(1)$.

On the other hand, by direct calculations $\mathcal{F}(v, A) - \mathcal{F}(\tilde{v}, A) = F(v, B(z_{i_0}, r)) - F(\tilde{v}, B(z_{i_0}, r)) + o(1)$. Consequently $F(v, B(z_{i_0}, r)) = \mathcal{O}(1)$ which is in contradiction with $F(v, B(z_{i_0}, r)) \geq C_{1/2} |\ln \varepsilon|$ [given by Proposition 40] for small ε.

We now prove (85). From (83) we have $\sum_{J_\mu} |d_i| \left[\pi(1-\mu)|\ln \varepsilon| - c_{\mu,b}\right] \leq \dfrac{\mathcal{M}_K |\ln \varepsilon|}{b^2}$. Since $\mu \in (0, 1/2)$, the last estimate gives the result for $\varepsilon > 0$ sufficiently small. □

8.3. **Lower bounds in perforated disks.** The goal of this section is to get lower bounds for $\frac{1}{2} \int_{\mathcal{D}} \alpha |\nabla v|^2$ where $\mathcal{D}$ is a perforated disk s.t. $\mathcal{D} \subset \Omega$ and $|v| \geq 1/2$ in $\mathcal{D}$.

The starting point of the argument is an estimate on circles. Let $\tilde{b} \in (0, 1)$, $\beta \in L^\infty((0, 2\pi), [\tilde{b}, 1])$. With Lemma D.7 in [6], for $\varphi \in H^1((0, 2\pi), \mathbb{R})$ s.t. $\varphi(2\pi) - \varphi(0) = 2\pi$, we have the following lower bound:

$$\frac{1}{2} \int_0^{2\pi} \beta |\partial_\theta \varphi|^2 \geq \frac{2\pi^2}{\displaystyle\int_0^{2\pi} \frac{1}{\beta}}. \tag{86}$$

In order to use (86) we need to do a preliminary analysis.

For $\alpha = U_\varepsilon^2 \in L^\infty(\Omega, [b^2, 1])$, using Lemma E.1 in [6], we have the existence of $C \geq 1$ [independent of ε] s.t.

$$\begin{cases} \text{For almost all } s \geq \delta/3, \text{ letting } \mathscr{C}_s \text{ be a circle with radius } s, \\ \text{we have } \int_{\mathscr{C}_s \cap \Omega} (1 - \alpha) \leq C \lambda s. \end{cases} \tag{87}$$

From now on, in all this section, we consider a sequence $\varepsilon = \varepsilon_n \downarrow 0$, $\lambda, \delta, h_{\text{ex}}$ and $((v_\varepsilon, A_\varepsilon))_\varepsilon \subset \mathscr{H}$ satisfying the hypotheses of Proposition 42 [namely (17), (77) and (79)]. We drop the subscript ε writing (v, A) instead of $(v_\varepsilon, A_\varepsilon)$

Recall that η_Ω is defined in (63) and consider

$$x_\varepsilon \in \Omega \text{ and } 0 < r = r_\varepsilon < R = R_\varepsilon < \eta_\Omega \text{ s.t. } \operatorname{dist}(x_\varepsilon, \partial\Omega) > \eta_\Omega > 0. \tag{88}$$

We then denote $\mathscr{R} := B(x_\varepsilon, R) \setminus \overline{B(x_\varepsilon, r)} \subset \Omega$.

Assume $|v| \geq 1/2$ in $\mathscr{R}$ and let $d := \deg_{\partial\mathscr{R}}(v)$. From the proof of Proposition 42 [see (189) in Appendix G], there exists $1/2 < t_\varepsilon < 1$, $t_\varepsilon = 1 + o(1)$ s.t. $t_\varepsilon \in$

$\mathrm{Im}(|v|) \cap [1-2/|\ln\varepsilon|; 1-1/|\ln\varepsilon|]$ and

$$\begin{cases} V(t_\varepsilon) := \{|v| = t_\varepsilon\} \text{ is a finite union of Jordan curves included in } \Omega \text{ and} \\ \quad \text{of simple curves whose endpoints are on } \partial\Omega \text{ and } \mathcal{H}^1[V(t_\varepsilon)] = o(1). \end{cases} \tag{89}$$

and since $\mathcal{H}^2(\{|v| \le t_\varepsilon\}) = o(1)$ we then have

$$\begin{cases} \text{if } U \text{ is a connected component of } \{|v| \le t_\varepsilon\} \text{ s.t. } \overline{U} \subset \Omega \text{ then there is } \Gamma, \\ \text{a connected component of } V(t_\varepsilon), \text{ which is a Jordan curve s.t. } U \subset \mathrm{int}(\Gamma). \end{cases} \tag{90}$$

Remark 44. Since $\mathcal{H}^1[V(t_\varepsilon)] = o(1)$, for sufficiently small ε, if Γ [resp. U] is a connected component of $V(t_\varepsilon)$ [resp. $\{|v| \le t_\varepsilon\}$] which intersects $\mathscr{R}$ then Γ is a Jordan curve [resp. ∂U is a union of connected components of $V(t_\varepsilon)$].

We have the following lemma:

Lemma 45. *Assume x_ε, r, R satisfy (88) and we assume $|v| \ge 1/2$ in $\mathscr{R}$. Then, for $s \in (r,R)$, letting*

$$K_s := \{\theta \in [0, 2\pi) \,|\, |v(x_\varepsilon + s\mathrm{e}^{\imath\theta})| \le t_\varepsilon\}$$

we have

$$\mathcal{H}^1(K_s) \le \pi \frac{\mathcal{H}^1[V(t_\varepsilon)]}{s}.$$

Proof. Let $s \in (r,R)$ be s.t. $\mathcal{H}^1(K_s) > 0$ and denote $\widehat{\mathcal{K}}_s := \{x_\varepsilon + s\mathrm{e}^{\imath\theta} \,|\, \theta \in K_s\} \subset \partial B(x_\varepsilon, s)$. Then $\mathcal{H}^1(\widehat{\mathcal{K}}_s) = s\mathcal{H}^1(K_s)$.

On the one hand, letting $\mathcal{V}_{\mathscr{R}}(t_\varepsilon)$ be the union of the connected components of $\{|v| \le t_\varepsilon\}$ which intersect $\mathscr{R}$, we have $\widehat{\mathcal{K}}_s = \mathcal{V}_{\mathscr{R}}(t_\varepsilon) \cap \partial B(x_\varepsilon, s)$.

On the other hand, by Remark 44, $\partial\mathcal{V}_{\mathscr{R}}(t_\varepsilon)$ is a union of connected components of $V(t_\varepsilon)$ which are Jordan curves. Among these Jordan curves, we may select the maximal curves w.r.t. the inclusion of their interior. We denote these maximal curves by $\Gamma_1, ..., \Gamma_N$ and we let for $i \in \{1, ..., N\}$, $\mathcal{V}_i := \overline{\mathrm{int}(\Gamma_i)}$. We then obtain $\mathcal{V}_{\mathscr{R}}(t_\varepsilon) \subset \cup_{i=1}^N \mathcal{V}_i$ and thus $\widehat{\mathcal{K}}_s \subset \cup_{i=1}^N [\partial B(x_\varepsilon, s) \cap \mathcal{V}_i]$.

For $i \in \{1, ..., N\}$, we fix $x_i \in \mathcal{V}_i$ and we define the disk $B_i := \overline{B(x_i, \mathrm{diam}(\mathcal{V}_i))}$. It is clear that $\mathcal{V}_i \subset B_i$. Consequently

$$\mathcal{H}^1[\partial B(x_\varepsilon, s) \cap \mathcal{V}_i] \le \mathcal{H}^1[\partial B(x_\varepsilon, s) \cap B_i] \le 2\pi\, \mathrm{diam}(\mathcal{V}_i).$$

We claim that $2\mathrm{diam}(\mathcal{V}_i) \le \mathcal{H}^1(\Gamma_i)$. Since the curves Γ_i are pairwise disjoint, we have $\sum_{i=1}^N \mathcal{H}^1(\Gamma_i) \le \mathcal{H}^1[V(t_\varepsilon)]$.

We may now conclude:

$$s\mathcal{H}^1(K_s) = \mathcal{H}^1(\widehat{\mathcal{K}}_s) \le \sum_{i=1}^N \mathcal{H}^1[\partial B(x_\varepsilon, s) \cap \mathcal{V}_i] \le \pi \sum_{i=1}^N 2\mathrm{diam}(\mathcal{V}_i) \le \pi\mathcal{H}^1[V(t_\varepsilon)].$$

□

The next proposition is one of the major use of the dilution $[\lambda \to 0]$.

Proposition 46. *Let x_ε, r, R satisfying (88) and assume $|v| \ge 1/2$ in $\mathscr{R}$. We write $d := \deg_{\mathscr{R}}(v)$ and, in $\mathscr{R}$, we let $w := v/|v|$ & $\rho := |v|$.*

(1) If $r \ge \delta/3$ and if $\mathcal{H}^1[V(t_\varepsilon)]/r + (1 - t_\varepsilon^2) + \lambda = o[\ln(R/r)]$ then

$$\frac{1}{2}\int_{\mathscr{R}} \alpha|\nabla v|^2 \ge \frac{1}{2}\int_{\mathscr{R}} \alpha\rho^2|\nabla w|^2 \ge \pi d^2 \left[\ln\left(\frac{R}{r}\right) - o(1)\right].$$

(2) If $r = o(1)$ and if $\mathcal{H}^1[V(t_\varepsilon)]/r + (1-t_\varepsilon^2) = o[\ln(R/r)]$ then

$$\frac{1}{2}\int_{\mathscr{R}} |\nabla v|^2 \geq \frac{1}{2}\int_{\mathscr{R}} \rho^2 |\nabla w|^2 \geq \pi d^2 \left[\ln\left(\frac{R}{r}\right) - o(1)\right].$$

Proof. We prove the first assertion. We claim that, up to replace v with $\underline{v}$, we may assume $|v| \leq 1$ in Ω. Moreover, if $d = 0$ then there is nothing to prove. We then assume $d \neq 0$.

We write $v = \rho \mathrm{e}^{\imath d\varphi}$ where φ is locally defined and its gradient is globally defined. Letting $x_\varepsilon + \mathbb{R}^+ := \{x_\varepsilon + s \,|\, s \geq 0\}$, we may assume $\varphi \in H^1(\mathscr{R} \setminus (x_\varepsilon + \mathbb{R}^+), \mathbb{R})$. For $s \in (r, R)$, we let $\varphi_s(\theta) = \varphi(x_\varepsilon + s\mathrm{e}^{\imath\theta})$, $\rho_s(\theta) = |v(x_\varepsilon + s\mathrm{e}^{\imath\theta})|$ and $\alpha_s(\theta) = \alpha(x_\varepsilon + s\mathrm{e}^{\imath\theta})$. Then $\varphi_s \in H^1((0, 2\pi), \mathbb{R})$ is s.t. $\varphi_s(2\pi) - \varphi_s(0) = 2\pi$ and we immediately get

$$\frac{1}{2}\int_{\mathscr{R}} \alpha\rho^2 |\nabla w|^2 \geq \frac{d^2}{2}\int_r^R \frac{\mathrm{d}s}{s}\int_0^{2\pi} \alpha_s \rho_s^2 |\partial_\theta \varphi_s|^2 \mathrm{d}\theta.$$

From (86) with $\beta := \alpha_s \rho_s^2$ we get

$$\frac{1}{2}\int_0^{2\pi} \alpha_s \rho_s^2 |\partial_\theta \varphi_s|^2 \geq \frac{2\pi^2}{\displaystyle\int_0^{2\pi} \frac{1}{\alpha_s \rho_s^2}}.$$

Since $b^2/4 \leq \alpha_s \rho_s^2 \leq 1$ we have

$$0 \leq \left(\int_0^{2\pi} \frac{1}{\alpha_s \rho_s^2}\right) - 2\pi = \int_0^{2\pi} \frac{1 - \alpha_s \rho_s^2}{\alpha_s \rho_s^2} \leq \frac{4}{b^2}\left(\int_0^{2\pi} 1 - \rho_s^2 + \int_0^{2\pi} 1 - \alpha_s\right).$$

On the one hand, from Lemma 45 we have

$$\int_0^{2\pi} 1 - \rho_s^2 \leq \mathcal{H}^1(K_s) + \left[2\pi - \mathcal{H}^1(K_s)\right](1 - t_\varepsilon^2) \leq \frac{\pi \mathcal{H}^1[V(t_\varepsilon)]}{s} + 2\pi(1 - t_\varepsilon^2).$$

On the other hand, using (87), there is $C \geq 1$ [independent of ε] s.t. $\displaystyle\int_0^{2\pi} 1 - \alpha_s \leq C\lambda$. Then

$$\int_0^{2\pi} \frac{1}{\alpha_s \rho_s^2} \leq 2\pi + \frac{4}{b^2}\left[\frac{\pi \mathcal{H}^1[V(t_\varepsilon)]}{s} + 2\pi(1 - t_\varepsilon^2) + C\lambda\right].$$

We thus get

$$\begin{aligned}\frac{1}{2}\int_{\mathscr{R}} \alpha\rho^2 |\nabla w|^2 &\geq d^2 \int_r^R \frac{\mathrm{d}s}{s} \frac{2\pi^2}{2\pi - \frac{4}{b^2}\left[\pi \mathcal{H}^1[V(t_\varepsilon)]/s + 2\pi(1 - t_\varepsilon^2) + C\lambda\right]} \\ &= \pi d^2 \left[\ln\left(\frac{R}{r}\right) + o(1)\right].\end{aligned}$$

The second assertion is obtain exactly in the same way than the first one. Indeed, since α plays no role in the statement, we may use the same argumentation with $\lambda = 0$ and $\delta > 0$ an arbitrary small number. □

We now state the reformulation of Proposition 46 by replacing the annular $\mathscr{R}$ with a perforated disk.

Corollary 47. *Let $D_0 \in \mathbb{N}^*$ be independent of ε, $0 < r = r_\varepsilon < R = R_\varepsilon$ be s.t. $r = o(R)$, $N = N_\varepsilon \in \mathbb{N}^*$ be s.t. $N \leq D_0$ and $z_1 = z_1^\varepsilon, ..., z_N = z_N^\varepsilon$ be s.t. $|z_i - z_j| \geq 8r$ for $i \neq j$.*

Let $y = y_\varepsilon \in \Omega$ and assume $z_1, ..., z_N \in B(y, R) \subset B(y, 4R) \subset B(y, \eta_\Omega) \subset \Omega$. We let $\mathcal{D} := B(y, 2R) \setminus \cup_{i=1}^N \overline{B(z_i, r)}$.

Assume $\rho = |v| \geq 1/2$ in $\mathcal{D}$. For $i \in \{1, ..., N\}$, we let $d_i := \deg_{\partial B(z_i,r)}(v)$. We also assume $d_i > 0$ for all $i \in \{1, ..., N\}$ and $\sum_{i=1}^N d_i \leq D_0$. Write $v = \rho w$ in $\mathcal{D}$.

Then there exists $C_0 > 0$ depending only on D_0 s.t. :

(1) If $r \geq \delta/3$ and $\mathcal{H}^1[V(t_\varepsilon)]/r + (1 - t_\varepsilon^2) + \lambda = o[\ln(R/r)]$ then, for sufficiently small ε, we have

$$\frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla v|^2 \geq \frac{1}{2}\int_{\mathcal{D}} \alpha\rho^2|\nabla w|^2 \geq \pi\sum_{i=1}^N d_i^2 \ln(R/r) - C_0.$$

(2) If $\mathcal{H}^1[V(t_\varepsilon)]/r + (1 - t_\varepsilon^2) = o[\ln(R/r)]$ then, for sufficiently small ε, we have

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla v|^2 \geq \frac{1}{2}\int_{\mathcal{D}} \rho^2|\nabla w|^2 \geq \pi\sum_{i=1}^N d_i^2 \ln(R/r) - C_0.$$

Proof. We claim that, up to replace v with $\underline{v}$, we may assume $|v| \leq 1$ in Ω.

We first proceed to a scaling with the conformal mapping:

$$\begin{array}{rccc} \Phi: & B(y,4R) & \to & B(0,4) \\ & x & \mapsto & \dfrac{x-y}{R} \end{array}.$$

We then let $\hat{z}_i := \Phi(z_i)$, $\hat{r} := r/R$, $\hat{\mathcal{D}} := \Phi[\mathcal{D}] = B(0,2) \setminus \cup_{i=1}^N \overline{B(\hat{z}_i, \hat{r})}$, $\hat{\alpha} := \alpha \circ \Phi^{-1}$ and $\hat{v} := v \circ \Phi^{-1}$.

If $N = 1$ or $N \geq 2$ and $|\hat{z}_i - \hat{z}_j| \geq 4 \times 10^{-2D_0}$ for $i \neq j$ then, letting $\tilde{\Omega} := B(0,4)$, $\eta_{\tilde{\Omega}} = 10^{-1}$, we may apply Proposition 46.1

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla v|^2 = \frac{1}{2}\int_{\hat{\mathcal{D}}} \hat{\alpha}|\nabla\hat{v}|^2 &\geq \sum_{i=1}^N \frac{1}{2}\int_{B(\hat{z}_i, 2\times 10^{-2D_0})\setminus\overline{B(\hat{z}_i,\hat{r})}} \hat{\alpha}|\nabla\hat{v}|^2 \\ &\geq \pi\sum_{i=1}^N d_i^2 \left(|\ln(R/r)| - |\ln(2\times 10^{-2D_0})|\right) + o(1). \end{aligned}$$

This estimate is the desired result with $C_0 = \pi D_0^2 |\ln(2 \times 10^{-2D_0})| + 1$.

If we are not in the previous case, *i.e.* $N \geq 2$ and there exists $i \neq j$ s.t. $|\hat{z}_i - \hat{z}_j| < 4 \times 10^{-2D_0}$, then we apply the separation process presented Appendix C [Section C.3.1] in [6] to the domain $\hat{\mathcal{D}}$ with $\eta_{\text{stop}} := 10^{-2D_0}$.

The key ingredient in the separation process is a variant of Theorem IV.1 in [4] [stated with $P = 9$, the general case $P \in \mathbb{N} \setminus \{0, 1\}$ is left to the reader]:

Lemma 48. *Let $N \geq 2$, $P \in \mathbb{N} \setminus \{0, 1\}$, $x_1, ..., x_N \in \mathbb{R}^2$ and $\eta > 0$. There are $\kappa \in \{P^0, ..., P^{N-1}\}$ and $\emptyset \neq J \subset \{1, ..., N\}$ s.t.*

$$\cup_{i=1}^N B(x_i, \eta) \subset \cup_{i\in J} B(x_i, \kappa\eta) \text{ and } |x_i - x_j| \geq (P-1)\kappa\eta \text{ for } i, j \in J,\ i \neq j.$$

The separation process is an iterative selection of points in $\{\hat{z}_1, ..., \hat{z}_N\}$ associated to the construction of a good radius.

We initialize the process by letting $\eta_0 := \hat{r}$, $M_0 := N$ and $J_0 = \{1, ..., M_0\}$.

For $k \geq 1$ [where k is the index in the iterative process] we construct a set $\emptyset \neq J_k \subsetneq J_{k-1}$, $M_k := \text{Card}(J_k)$ and 3 numbers

$$\kappa_k \in \{9^1, ..., 9^{M_{k-1}-1}\},\ \eta_k' := \frac{1}{4}\min_{\substack{i,j\in J_{k-1} \\ i\neq j}} |\hat{z}_i - \hat{z}_j| \text{ and } \eta_k := 2\kappa_k\eta_k'.$$

These objects are obtained with Lemma 48 with $P = 9$, $N = M_{k-1} = \mathrm{Card}(J_{k-1})$, $\{x_1, ..., x_N\} = \{z_i \,|\, i \in J_{k-1}\}$, $J = J_k$, $\eta = \eta_k$, $\kappa = \kappa_k$ The process stops at the end of Step $K_0 \geq 1$ if $M_{K_0} = 1$ or $M_{K_0} \geq 2$ and $\min_{\substack{i,j\in J_{K_0}\\ i\neq j}} |\hat{z}_i - \hat{z}_j| > 4\eta_{\mathrm{stop}}$.

By construction, we have for $1 \leq k \leq K_0$, $\emptyset \neq J_k \subsetneq J_{k-1}$ and $\eta_{k-1} \leq \eta'_k < \eta_k$. In particular, since $\mathrm{Card}(J_0) \leq D_0$, we get $K_0 \leq D_0 - 1$.

By definition, for $k \in \{1, ..., K_0\}$ we have $2 \cdot 9\eta'_k \leq \eta_k \leq 9^{D_0}\eta'_k$. We let

$$\eta_0 := \begin{cases} 9^{D_0} \cdot \eta_{\mathrm{stop}} & \text{if } M_{K_0} = 1 \\ \min\{9^{D_0} \cdot \eta_{\mathrm{stop}}, \dfrac{1}{4} \min\limits_{\substack{i,j\in J_{K_0}\\ i\neq j}} |\hat{z}_i - \hat{z}_j|\} & \text{if } M_{K_0} \geq 2 \end{cases}$$

and then $\eta_0 \geq \eta_{\mathrm{stop}} = 10^{-2D_0}$. For $k \in \{0, ..., K_0 - 1\}$ and $i \in J_k$ we denote $\mathscr{R}_{i,k} := B(\hat{z}_i, \eta'_{k+1}) \setminus \overline{B(\hat{z}_i, \eta_k)}$, and, for $i \in J_{K_0}$, $\mathscr{R}_i := B(\hat{z}_i, \eta_0) \setminus \overline{B(\hat{z}_i, \eta_{K_0})}$. By construction, the previous rings are pairwise disjoint. From Proposition 46.1 we have for $k \in \{0, ..., K_0 - 1\}$ and $i \in J_k$:

$$\begin{aligned} \frac{1}{2}\int_{\mathscr{R}_{i,k}} \hat{\alpha}|\nabla\hat{v}|^2 &\geq \pi \mathrm{deg}_{\mathscr{R}_{i,k}}(\hat{v})^2 \left[\ln(\eta_{k+1}/\eta_k) - \ln(9^{D_0})\right] - o(1) \\ &\geq \pi \sum_{\hat{z}_j \in B(\hat{z}_i, \eta'_{k+1})} d_j^2 \ln(\eta_{k+1}/\eta_k) - \pi D_0^2 \ln(9^{D_0}) - o(1). \end{aligned}$$

And for $i \in J_{K_0}$:

$$\begin{aligned} \frac{1}{2}\int_{\mathscr{R}_i} \hat{\alpha}|\nabla\hat{v}|^2 &\geq \pi \mathrm{deg}_{\mathscr{R}_i}(\hat{v})^2 \ln(\eta_0/\eta_{K_0}) - o(1) \\ &\geq \pi \sum_{\hat{z}_j \in B(\hat{z}_i, \eta_0)} d_j^2 \ln(\eta/\eta_{K_0}) - o(1). \end{aligned}$$

By summing the previous lower bound we get the result. As for Proposition 46, the second assertion is obtained in a similar way than the first assertion. □

8.4. **Lower bounds in a perforated domain.** In this section we state a lower bound for a weighted Dirichlet energy in the domain Ω perforated by small [but not too small] disks. The philosophy of this lower bound is that in the case which interest us we may ignore the weight if the perforations are not too small ; it is an effect of the dilution $\lambda \to 0$.

Proposition 49. *Let $\beta \in (0, 1)$, $(\tilde{\alpha}_n)_n \subset L^\infty(\Omega, [\beta^2, 1])$ be s.t.*

$$K_n := \sqrt{\int_\Omega (1 - \tilde{\alpha}_n)^2} \to 0.$$

Let $N \in \mathbb{N}^*$ and $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}, \mathbf{d})^{(n)} \subset (\Omega^N)^* \times \mathbb{Z}^N$ be s.t. $\mathbf{d}$ is independant of n. We denote $\hbar := \min_i \mathrm{dist}(z_i, \partial\Omega)$.

Assume the existence of $\tilde{r} > 0$ s.t. $\tilde{r} = o(1)$, (50) holds and s.t. there is $C_1 > 0$ [independent of n] satisfying $\dfrac{\tilde{r}|\ln\tilde{r}|}{\hbar} \leq C_1$. Write $\Omega_{\tilde{r}} := \Omega \setminus \cup\overline{B(z_i, \tilde{r})}$.

Let $(u_n)_n \subset H^1(\Omega, \mathbb{C})$ satisfying $|u_n| \geq \dfrac{1}{2}$ in $\Omega_{\tilde{r}}$ and $\mathrm{deg}_{\partial B(z_i,\tilde{r})}(u_n) = d_i$ for all i.

Assume also

$$L_n := \sqrt{\int_{\Omega_{\tilde{r}}} (1 - |u_n|^2)^2} \to 0.$$

Then

$$\int_{\Omega_{\tilde r}} \tilde\alpha_n |\nabla u_n|^2 \geq \int_{\Omega_{\tilde r}} |\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}|^2 - (4\beta^{-1}+3)\|\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde r})}\|\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^2(\Omega_{\tilde r})}\,(K_n+L_n) - \mathcal{O}(X)$$

with $\Phi_\star^{(\mathbf{z},\mathbf{d})}$ *is defined in Remark 29 and* X *is defined in* (57).

Proposition 49 is proved Appendix H.

9. Study of the ε^s-bad discs

In this section, in addition to the assumption (79) on λ, δ and $h_{\rm ex}$, we assume that (4) holds. This [technical] hypothesis (4) is a little bit more restrictive than (73) [$\delta\sqrt{h_{\rm ex}} \to 0$] used to get a nice upper bound.

Let $\varepsilon = \varepsilon_n \downarrow 0$ and let $((v,A))_\varepsilon = ((v_\varepsilon, A_\varepsilon))_\varepsilon$ be a sequence that agrees (17) and (77). Let also $\mu \in (0,1/2)$.

Since (17) and (77) are gauge invariant we may assume that (v,A) is in the Coulomb gauge.

The goal of this section is to prove that, for sufficiently small $\varepsilon\&\mu$, if $J_\mu \neq \emptyset$ then $d_i = 1$ & $\mathrm{dist}(z_i,\Lambda) \leq \ln(h_{\rm ex})/\sqrt{h_{\rm ex}}$ & $z_i \in \omega_\varepsilon$ for all $i \in J_\mu$ and for $i \neq j$, $|z_i - z_j| \geq \ln(h_{\rm ex})/h_{\rm ex}$ with a "uniform" distribution of the z_i's around Λ.

With the notation of Proposition 42 we let $\Omega_r := \Omega \setminus \cup_{i\in J_\mu} \overline{B(z_i,r)}$ and $d := \sum_{i\in J_\mu} |d_i|$.

In view of the goal of this section we may argue on subsequences. First note that from (84) we have $d_i \neq 0$ for all i. Up to pass to a subsequence, from (85), we may assume that $J_\mu \neq \emptyset$ and independent of ε as well as the d_i's.

Since we are interested here only on informations related with $|v|$ and the d_i's, we may consider that (v,A) is in the Coulomb gauge and we may also change the potential vector. Namely, we may assume that $A = \nabla^\perp \xi$ with $\xi = \xi_\varepsilon \in H^1_0 \cap H^2(\Omega,\mathbb{R})$ is the unique solution of (40). Note that (77) still holds.

Consequently, $\mathrm{curl}(A) \in H^1$ and then with (11)&(22): $\|\xi\|_{H^3(\Omega)} \leq C\|\mathrm{curl}(A_\varepsilon)\|_{H^1(\Omega)} \leq C|\ln\varepsilon|$.

From Proposition 11 and letting $\zeta = \zeta_\varepsilon := \xi - h_{\rm ex}\xi_0$

$$\mathcal{F}(v,\nabla^\perp\xi) = h_{\rm ex}^2 \mathbf{J_0} + F(v) + 2\pi h_{\rm ex}\sum d_i\xi_0(z_i) + \tilde V_{(\mathbf{z},\mathbf{d})}(\zeta) + o(1).$$

Proposition 17 infers $\tilde V_{(\mathbf{z},\mathbf{d})}(\zeta) = \mathcal{O}(1)$. Consequently

$$\mathcal{F}(v,\nabla^\perp\xi) = h_{\rm ex}^2 \mathbf{J_0} + F(v) + 2\pi h_{\rm ex}\sum d_i\xi_0(z_i) + \mathcal{O}(1). \tag{91}$$

In particular we have $\mathcal{F}(v,\nabla^\perp\xi) \leq h_{\rm ex}^2\mathbf{J_0} + o(1)$, thus with (91) we get

$$F(v) \leq -2\pi h_{\rm ex}\sum d_i\xi_0(z_i) + \mathcal{O}(1). \tag{92}$$

From Corollary 41 and Propositions 42&43 we deduce $-\sum d_i\xi_0(z_i) = \|\xi_0\|_{L^\infty(\Omega)}\sum d_i + o(1)$ and we immediately obtained

$$\sum d_i \geq 0. \tag{93}$$

On the other hand, from Proposition 39, we have

$$\mathcal{F}(v,\nabla^\perp\xi) \leq h_{\rm ex}^2\mathbf{J_0} + dM_\Omega\left[-h_{\rm ex} + H^0_{c_1}\right] + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathcal{O}(1). \tag{94}$$

By combining (91) and (94) we get

$$F(v) \leq d\pi\left[b^2|\ln\varepsilon| + (1-b^2)|\ln(\lambda\delta)|\right] + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathcal{O}(1). \tag{95}$$

In conclusion, from (82) in conjunction with (95) we obtain

$$\frac{1}{2}\int_{\Omega_r}\alpha|\nabla v|^2 \leq d\pi\left[b^2|\ln r| + (1-b^2)|\ln(\lambda\delta)|\right] + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathcal{O}(1). \tag{96}$$

We first have the following proposition.

Proposition 50. *Assume*

$$0 < \mu < \min\left\{\frac{1}{\mathcal{D}_{K,b}+1}, \frac{1-b^2}{2(\mathcal{D}_{K,b}+1)}\right\} \tag{97}$$

where $\mathcal{D}_{K,b} = \dfrac{3\mathcal{M}_K}{b^2}$ *and* $\mathcal{M}_K$ *is as in Theorem 5.*

Then there exists $\tilde{\varepsilon}'_\mu > 0$ *s.t. for* $0 < \varepsilon < \tilde{\varepsilon}'_\mu$ *if* $J_\mu \neq \emptyset$ *then*

(1) $d_i > 0$ *for all* i,

(2) $\mathrm{dist}(z_i, \omega_\varepsilon) < \sqrt{\varepsilon}$.

Proof. **Step 1. We prove that** $d_i > 0$ **for all** i

We argue by contradiction and we assume the existence of an extraction still denoted by $\varepsilon = \varepsilon_n \downarrow 0$ s.t. $J_- := \{i \in J_\mu \,|\, d_i < 0\} \neq \emptyset$ [from (84), for $0 < \varepsilon < \tilde{\varepsilon}_\mu$, we have $d_i \neq 0$ for all $i \in J_\mu$].

From (93) we thus obtain: $\sum_{i\in J_\mu\setminus J_-} d_i \geq d+1$. Then, with the help of (83), we obtain

$$F(v) \geq b^2(1-\mu)\pi|\ln\varepsilon|\left(\sum_{i\in J_-}|d_i| + \sum_{i\in J_\mu\setminus J_-} d_i\right) \geq (d+2)\pi(1-\mu)b^2|\ln\varepsilon| + \mathcal{O}(1).$$

Consequently (95) implies $d(1+o(1)) \geq (d+2)(1-\mu) - o(1)$. This inequality gives $\mu \geq \dfrac{2}{d+2} - o(1)$ which is in contradiction with $0 < \mu < (\mathcal{D}_{K,b}+1)^{-1}$ for sufficiently small $\varepsilon > 0$ [here we used $\mathcal{D}_{K,b} \geq \mathcal{M}_K \geq d$].

Step 2. We prove that $\mathrm{dist}(z_i, \omega_\varepsilon) < \sqrt{\varepsilon}$ **for all** i

We argue by contradiction and we assume the existence of a subsequence still denoted by $\varepsilon = \varepsilon_n \downarrow 0$ and $i_0 \in J_\mu$ s.t. $\mathrm{dist}(z_{i_0}, \omega_\varepsilon) \geq \sqrt{\varepsilon}$. From (25) we have $\inf_{B(z_{i_0},r)}\alpha \geq 1 - o(|\ln\varepsilon|^{-2})$. Consequently using (83) we get $F(v, B(z_{i_0}, r)) \geq d_{i_0}\pi(1-\mu)|\ln\varepsilon| - \mathcal{O}(1)$. Then $F(v) \geq \pi b^2(1-\mu)d|\ln\varepsilon| + \pi(1-b^2)(1-\mu)d_{i_0}|\ln\varepsilon| - \mathcal{O}(1)$.

From (95) we obtain

$$db^2|\ln\varepsilon| + \mathcal{O}(\ln|\ln\varepsilon|) \geq b^2(1-\mu)d|\ln\varepsilon| + (1-b^2)(1-\mu)|\ln\varepsilon| - \mathcal{O}(1).$$

The last estimate implies $\mu \geq \dfrac{1-b^2}{b^2d+1-b^2} + o(1)$ which is in contradiction with $\mu \leq \dfrac{1-b^2}{2(\mathcal{D}_{K,b}+1)}$ for $\varepsilon > 0$ sufficiently small. □

Definition 51.

- For $i \in J_\mu$we let $y_i \in \delta\cdot\mathbb{Z}^2$ be the unique point s.t. $z_i \in B(y_i, \delta/2)$. Since $\mathrm{dist}(z_i, \omega_\varepsilon) < \sqrt{\varepsilon}$ for all i, y_i is well defined.
- We denote also $\tilde{J} \subseteq J_\mu$ a set of indices s.t. $\cup_{i\in J_\mu} B(z_i, r) \subset \cup_{k\in\tilde{J}} B(y_k, 2\lambda\delta)$ and for $k, l \in \tilde{J}$ s.t. $k \neq l$ we have $y_k \neq y_l$. We then let for $k \in \tilde{J}$, $\tilde{J}_k := \{i \in J_\mu \,|\, z_i \in B(y_k, 2\lambda\delta)\}$.

- We may also select "good indices" in order to get well separated centers y_k's. Using Lemma 48 with $P = 17, \eta = \delta$, there exists a set $\emptyset \neq J^{(y)} \subset J_\mu$ and a number $\kappa \in \{1, 17, ..., 17^{\mathrm{Card}(J_\mu)-1}\}$ [dependent on ε] s.t.

$\cup_{k\in\tilde{J}} B(y_k, \delta) \subset \cup_{k\in J^{(y)}} B(y_k, \kappa\delta)$ and for $k, l \in J^{(y)}$ with $k \neq l$ we have $|y_k - y_l| \geq 16\kappa\delta$.

  We denote, for $k \in J^{(y)}$, $\tilde{d}_k := \deg_{\partial B(y_k,\kappa\delta)}(v)$.
- There exists also $\{J_k \,|\, k \in J^{(y)}\}$, a partition of J_μ with non empty sets [dependent on ε], s.t.

$$B(z_i, \delta/2) \subset B(y_k, \kappa\delta) \Longleftrightarrow i \in J_k \text{ for } k \in J^{(y)}.$$

We are going to prove that $\tilde{J} = J_\mu$ and for all $k \in J^{(y)}$ we have $J_k = \tilde{J}_k$.

Proposition 52. *Assume (97), for $\varepsilon > 0$ sufficiently small, if $J_\mu \neq \emptyset$ then $d_i = 1$ for all $i \in J_\mu$.*

Proof. We argue by contradiction and we assume the existence of a subsequence [still denoted by $\varepsilon = \varepsilon_n \downarrow 0$] s.t. for all ε there exits $i_0 \in J_\mu$ s.t. $d_{i_0} \geq 2$.

From Corollary 47.2 applied in $B(y_k, 2\lambda\delta) \setminus \cup_{i\in\tilde{J}_k} \overline{B(z_i, r)}$:

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_r} \alpha|\nabla v|^2 &\geq \sum_{k\in\tilde{J}} \frac{b^2}{2} \int_{B(y_k,2\lambda\delta)\setminus\cup_{i\in\tilde{J}_k}\overline{B(z_i,r)}} |\nabla v|^2 \\
&\geq \pi b^2 \sum_{k\in\tilde{J}} \sum_{i\in J_k} d_i^2 \ln\left(\frac{\lambda\delta}{r}\right) - \mathcal{O}(1) \\
&\geq \pi b^2 \left(1 + \sum_{i\in J_\mu} d_i\right) \ln\left(\frac{\lambda\delta}{r}\right) - \mathcal{O}(1).
\end{aligned}$$

We then get $F(v) \geq \pi b^2(d|\ln\varepsilon| + |\ln r|) + \mathcal{O}(|\ln(\lambda\delta)|)$. Since $|\ln\varepsilon| = \mathcal{O}(|\ln r|)$ and $|\ln(\lambda\delta)| + \ln h_{\mathrm{ex}} = o(|\ln\varepsilon|)$, this estimate is in contradiction with (95) for sufficiently small ε. □

Proposition 53. *Assume μ satisfies (97) and $J_\mu \neq \emptyset$. Then for sufficiently small $\varepsilon > 0$ we have* $\mathrm{dist}(\mathbf{z}, \Lambda) \leq \dfrac{\ln h_{\mathrm{ex}}}{\sqrt{h_{\mathrm{ex}}}}$.

The proof of the proposition uses the following obvious lemma whose proof is left to the reader.

Lemma 54. *(1) Let $N \in \mathbb{N}^*$, $\mathbf{D} \in \mathbb{N}^N$ and for $k \in \{1, ..., N\}$ let $N_k \in \mathbb{N}^*$ and $\mathbf{d}^{(k)} \in \mathbb{N}^{N_k}$ be s.t. $D_k = \sum_i d_i^{(k)}$. Then we have*

$$\sum_{k=1}^N D_k^2 \geq \sum_{k=1}^N \sum_{i=1}^{N_k} (d_i^{(k)})^2.$$

Moreover the equality holds if and only if for all $k \in \{1, ..., N\}$ and for all $i \in \{1, ..., N_k\}$ we have $d_i^{(k)} \in \{0, D_k\}$.

(2) Let $N, d \in \mathbb{N}^*$ and denote $E_d := \min_{\mathbf{D}\in\mathbb{N}^N, \sum D_k = d} \sum_{k=1}^N D_k^2$. Then we have for $\mathbf{D} \in \mathbb{N}^N$ s.t. $\sum D_k = d$:

$$\sum_{k=1}^N D_k^2 = E_d \Longleftrightarrow \mathbf{D} \in \{\lfloor d/N \rfloor; \lceil d/N \rceil\}^N.$$

Proof of Proposition 53. We argue by contradiction and we assume the existence of a subsequence [still denoted by $\varepsilon = \varepsilon_n \downarrow 0$] and $i_0 \in J_\mu$ s.t. $\mathrm{dist}(z_{i_0}, \Lambda) > \dfrac{\ln h_{\mathrm{ex}}}{\sqrt{h_{\mathrm{ex}}}}$.

Then there exists $\eta > 0$ [independent of ε] s.t. $h_{\mathrm{ex}}\xi_0(z_{i_0}) \geq -h_{\mathrm{ex}}\|\xi_0\|_{L^\infty(\Omega)} + 4\eta(\ln h_{\mathrm{ex}})^2$. Consequently: $-2\pi h_{\mathrm{ex}} \sum \xi_0(z_i) \leq 2\pi d h_{\mathrm{ex}}\|\xi_0\|_{L^\infty(\Omega)} - 4\eta(\ln h_{\mathrm{ex}})^2$.

From (92) we get [for small ε]

$$\begin{array}{rcl} F(v) & \leq & 2\pi d h_{\mathrm{ex}}\|\xi_0\|_{L^\infty(\Omega)} - 3\eta(\ln h_{\mathrm{ex}})^2 \\ \text{[Hyp. (3)]} & \leq & \pi d|\ln\varepsilon| - 2\eta(\ln h_{\mathrm{ex}})^2. \end{array}$$

Using (82) we get

$$\frac{1}{2}\int_{\Omega_r} \alpha|\nabla v|^2 \leq d\pi\left[b^2|\ln r| + (1-b^2)|\ln(\lambda\delta)|\right] - \eta(\ln h_{\mathrm{ex}})^2. \tag{98}$$

We let $\chi := 10\max_{k\in\tilde{J}} \mathrm{dist}(y_k, \Lambda)$ and for $p \in \Lambda$, $D_p := \deg_{\partial B(p,\chi)}(v)$, $J_p := \{k \in J^{(y)} \,|\, y_k \in B(p,\chi)\}$. For a latter use we claim that $\chi \geq \ln(h_{\mathrm{ex}})/\sqrt{h_{\mathrm{ex}}}$ and then

$$\lambda|\ln\chi|/\chi \to 0. \tag{99}$$

We have [see Definition 51 for notation]

$$\begin{array}{rl} & \dfrac{1}{2}\displaystyle\int_{\Omega_r} \alpha|\nabla v|^2 \\ \geq & \displaystyle\sum_{k\in\tilde{J}} \frac{1}{2}\int_{B(y_k,2\lambda\delta)\setminus\cup_{i\in\tilde{J}_k}\overline{B(z_i,r)}} \alpha|\nabla v|^2 + \sum_{k\in\tilde{J}} \frac{1}{2}\int_{B(y_k,\delta/3)\setminus\overline{B(y_k,2\lambda\delta)}} \alpha|\nabla v|^2 + \\ & + \displaystyle\sum_{p\in\Lambda} \frac{1}{2}\int_{B(p,\chi)\setminus\cup_{k\in J_p}\overline{B(y_k,\kappa\delta)}} \alpha|\nabla v|^2 + \frac{1}{2}\int_{\Omega\setminus\cup_{p\in\Lambda}\overline{B(p,\chi)}} \alpha|\nabla v|^2. \end{array} \tag{100}$$

It is clear that, for $k \in \tilde{J}$, we may use Corollary 47.2 in $B(y_k, 2\lambda\delta) \setminus \cup_{i\in\tilde{J}_k}\overline{B(z_i,r)}$ in order to get

$$\sum_{k\in\tilde{J}} \frac{1}{2}\int_{B(y_k,2\lambda\delta)\setminus\cup_{i\in\tilde{J}_k}\overline{B(z_i,r)}} \alpha|\nabla v|^2 \geq b^2 d\pi \ln\left(\frac{\lambda\delta}{r}\right) + \mathcal{O}(1). \tag{101}$$

Let $k \in \tilde{J}$, from (25) and Proposition 46.2 we obtain

$$\frac{1}{2}\int_{B(y_k,\delta/3)\setminus\overline{B(y_k,2\lambda\delta)}} \alpha|\nabla v|^2 \geq \pi\deg_{\partial B(y_k,2\lambda\delta)}(v)^2|\ln\lambda| + \mathcal{O}(1). \tag{102}$$

Let $p \in \Lambda$ be s.t. $D_p \neq 0$, Corollary 47.1 gives

$$\frac{1}{2}\int_{B(p,\chi)\setminus\cup_{k\in J_p}\overline{B(y_k,\kappa\delta)}} \alpha|\nabla v|^2 \geq \pi\sum_{k\in J_p} \tilde{d}_k^2 \ln\left(\frac{\chi}{\delta}\right) + \mathcal{O}(1).$$

From Propositions 30&31&49 [with (99)] we deduce

$$\frac{1}{2}\int_{\Omega\setminus\cup_{p\in\Lambda}\overline{B(p,\chi)}} \alpha|\nabla v|^2 \geq \pi\sum_{p\in\Lambda} D_p^2|\ln\chi| + \mathcal{O}(1).$$

From Lemma 54.1 we have $d \leq \sum_{k\in\tilde{J}} \deg_{\partial B(y_k,2\lambda\delta)}(v)^2 \leq \sum_{p\in\Lambda}\sum_{k\in J_p} \tilde{d}_k^2 \leq \sum_{p\in\Lambda} D_p^2$. Then we get

$$\frac{1}{2}\int_{\Omega_r} \alpha|\nabla v|^2 \geq d\pi\left[b^2|\ln r| + (1-b^2)|\ln(\lambda\delta)|\right] + \mathcal{O}(1).$$

This estimate is in contradiction with (98) for sufficiently small ε. □

Proposition 55. *Assume μ satisfies (97) and let $\varepsilon = \varepsilon_n \downarrow 0$ be a sequence.*

(1) If $\mathrm{Card}(J_\mu) \geq 2$ then for $\varepsilon > 0$ sufficiently small and for $i \neq j$, $|z_i - z_j| \geq h_{\mathrm{ex}}^{-1} \ln h_{\mathrm{ex}}$.

(2) For $\varepsilon > 0$ sufficiently small we have for $p \in \Lambda$, $\deg_{\partial B(p, h_{\mathrm{ex}}^{-1/2} \ln h_{\mathrm{ex}})}(v) \in \{\lfloor d/N_0 \rfloor; \lceil d/N_0 \rceil\}$.

The proof of Proposition 55 is postponed to Appendix I.

Since $\lambda\delta h_{\mathrm{ex}} \to 0$, Proposition 55 implies that each cell of period contains at most a disc $B(z_i, r)$ with $i \in J_\mu$.

Following the argument in [6] [proof of the third part in Proposition 3.6, see Appendix D-Section 4.5], we may refined Proposition 50.2.

Proposition 56. *Assume μ satisfies (97), then there is $\eta_{\omega,b} > 0$ depending only on ω and b s.t. for $i \in J_\mu$ we have $B(z_i, 2\eta_{\omega,b}\lambda\delta) \subset \omega_\varepsilon$.*

Corollary 57. *Assume μ satisfies (97). Then we have*

$$\int_{\Omega \setminus \cup_{i \in J_\mu} B(z_i, \lambda^2\delta^2)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 = \mathcal{O}(|\ln(\lambda\delta)|). \tag{103}$$

Moreover

$$|v| = 1 + o(1) \text{ in } \Omega \setminus \cup_{i \in J_\mu} B(z_i, 2\lambda^2\delta^2). \tag{104}$$

Proof. We have

$$\frac{b^4}{4} \int_{\Omega \setminus \cup_{i \in J_\mu} B(z_i, \lambda^2\delta^2)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 \leq F(v) - \sum_{i \in J_\mu} F(v, B(z_i, \lambda^2\delta^2)).$$

For $i \in J_\mu$, from Corollary 46.2 :

$$\begin{aligned} F(v, B(z_i, \lambda^2\delta^2)) &\geq \frac{b^2}{2} \int_{B(z_i, \lambda^2\delta^2) \setminus \overline{B(z_i, r)}} |\nabla v|^2 + F(v, B(z_i, r)) \\ &\geq 2b^2\pi \ln(\lambda\delta) + b^2\pi |\ln \varepsilon| + \mathcal{O}(1). \end{aligned}$$

Since, by Proposition 55, the discs $B(z_i, \lambda^2\delta^2)$ are pairwise disjoint, we obtain with (95):

$$\frac{b^4}{4} \int_{\Omega \setminus \cup_{i \in J_\mu} B(z_i, \lambda^2\delta^2)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 \leq \mathcal{O}(|\ln(\lambda\delta)|).$$

This estimate is equivalent to (103).

We are going to prove (104). We argue by contradiction and we assume the existence of an extraction still denoted $\varepsilon = \varepsilon_n \downarrow 0$, $t \in (0, 1)$ and $(x_n)_n \subset \Omega \setminus \cup_{i \in J_\mu} B(z_i, 2\lambda^2\delta^2)$ s.t. $|v_{\varepsilon_n}(x_n)| < t$.

By Proposition 40, there exists $C_t > 0$ s.t. for sufficiently large n:

$$\int_{B(x_n, \sqrt{\varepsilon}_n) \cap \Omega} |\nabla v_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 > C_t |\ln \varepsilon_n|. \tag{105}$$

Moreover, for n sufficiently large to get $\sqrt{\varepsilon}_n < \lambda^2\delta^2$, we have $[B(x_n, \sqrt{\varepsilon}_n) \cap \Omega] \subset \Omega \setminus \cup_{i \in J_\mu} B(z_i, \lambda^2\delta^2)$. This inclusion is in contradiction with (103) and (105). □

From Proposition 56, for $i \in J_\mu$, we have $\hat{z}_i := \dfrac{z_i - y_i}{\lambda\delta} \in \omega$ where $y_i \in \delta\mathbb{Z}^2$ is s.t. $z_i \in B(y_i, \lambda\delta)$. Moreover, up to consider an extraction, we may assume that, for $i \in J_\mu$, there exits $\hat{z}_i^0 \in \omega$ s.t. $\hat{z}_i \to \hat{z}_i^0$.

We start with the following proposition.

Proposition 58. *We have the following sharp lower bound:*

$$\mathcal{F}(v,A) \geq h_{\rm ex}^2\mathbf{J_0} + dM_\Omega\left[-h_{\rm ex}+H^0_{c_1}\right] + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathscr{L}_2(d) + \\ + \sum_{i\in J_\mu}[W^{\rm micro}(\hat{z}_i^0) - \min_\omega W^{\rm micro}] + [\mathcal{W}_d(\mathbf{D}) - \overline{\mathcal{W}}_d] + o(1)$$

where $\overline{\mathcal{W}}_d = \min_{\Lambda_d}\mathcal{W}_d$ *is defined in* (72) *and*

$$\mathcal{W}_d(\mathbf{D}) := W^{\rm macro}_{N_0}(\mathbf{p},\mathbf{D}) + \sum_{p\in\Lambda} C_{p,D_p} + \tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}] \tag{106}$$

where for $p\in\Lambda$, $D\in\mathbb{N}^*$, $C_{p,D}$ *is defined in* (65), $C_{p,0}:=0$ *and* $\tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]$ *is defined in Proposition 17.*

We split the proof of Proposition 58 in several lemmas.

The first step is the following lemma consisting in a "macroscopic/mesoscopic" version of Proposition 58.

Lemma 59. *Let* $\rho=|v|$ *and* $w=v/\rho$ *in* $\Omega\setminus\cup_{i\in J_\mu}\overline{B(y_i,\delta/3)}$ *. We then have*

$$\frac{1}{2}\int_{\Omega\setminus\cup_{i\in J_\mu}\overline{B(y_i,\delta/3)}}\alpha\rho^2|\nabla w|^2 \geq d\pi|\ln(\delta/3)| - \pi\sum_{\substack{p\in\Lambda\\ D_p\geq 2}}\sum_{\substack{i,j\in J_p\\ i\neq j}}\ln|z_i-z_j| + \\ + W^{\rm macro}_{N_0}(\mathbf{p},\mathbf{D}) + o(1).$$

Proof. On the one hand, from Proposition 53 and letting $\chi := h_{\rm ex}^{-1/4}$ we have $|v|\geq 1/2$ in $\Omega\setminus\cup_{p\in\Lambda}\overline{B(p,\chi)}$. Then, from Proposition 49, we have

$$\frac{1}{2}\int_{\Omega\setminus\cup_{p\in\Lambda}\overline{B(p,\chi)}}\alpha|\nabla v|^2 \geq \pi\sum_{p\in\Lambda}D_p^2|\ln\chi| + W^{\rm macro}_{N_0}(\mathbf{p},\mathbf{D}) + o(1). \tag{107}$$

On the other hand, from Proposition 55, if $\mathrm{Card}(J_\mu)\geq 2$ then, for $i,j\in J_\mu$ with $i\neq j$, we have $|y_i-y_j|\geq h_{\rm ex}^{-1}\ln(h_{\rm ex}) - 2\lambda\delta$.

Consequently, if $D_p = \deg_{\partial B(p,\eta_\Omega)}(v)\neq 0$ [η_Ω is defined in (63)], letting $J_p := \{i\in J_\mu \,|\, z_i\in B(p,\eta_\Omega)\}$, $\mathcal{D}_p := B(p,\chi)\setminus\cup_{i\in J_p}\overline{B(y_i,h_{\rm ex}^{-1})}$,

$$\begin{array}{rccc}\Phi: & B(p,\chi) & \to & \mathbb{D}=B(0,1)\\ & x & \mapsto & \dfrac{x-p}{\chi}\end{array},$$

$\hat{v}=v\circ\Phi^{-1}$, $\hat{\alpha}=\alpha\circ\Phi^{-1}$, $\hat{\mathcal{D}}_p := \Phi(\mathcal{D}_p)$ and $\hat{y}_i := \Phi(y_i)$ for $y_i\in B(p,\chi)$, then we may apply Proposition 49. Writing $(\hat{\mathbf{y}}_p,\mathbf{1}) := \{(\hat{y}_i,1)\,|\,i\in J_p\}$, Proposition 49 gives:

$$\frac{1}{2}\int_{\mathcal{D}_p}\alpha|\nabla v|^2 = \frac{1}{2}\int_{\hat{\mathcal{D}}_p}\hat{\alpha}|\nabla\hat{v}|^2 \geq \pi D_p\ln(\chi h_{\rm ex}) + W^{\rm macro}_{D_p,\mathbb{D}}(\hat{\mathbf{y}}_p,\mathbf{1}) + o(1) \tag{108}$$

where $W^{\rm macro}_{D_p,\mathbb{D}}$ is the macroscopic renormalized energy in the unit disc $\mathbb{D}$ with D_p points.

From Proposition 1 in [13] we have

$$W^{\rm macro}_{D_p,\mathbb{D}}(\hat{\mathbf{y}}_p,\mathbf{1}) = -\pi\sum_{\substack{i,j\in J_p\\ i\neq j}}\left[\ln|\hat{y}_i-\hat{y}_j| - \ln|1-\hat{y}_i\overline{\hat{y}}_j|\right] + \pi\sum_{i\in J_p}\ln(1-|\hat{y}_i|^2).$$

Using Proposition 53, we get for $i \in J_p$, $|\hat{y}_i| \leq \dfrac{h_{\rm ex}^{-1/2} \ln h_{\rm ex}}{\chi} = o(1)$ and then

$$W^{\rm macro}_{D_p,\mathbb{D}}(\hat{\mathbf{y}}_p, \mathbf{1}) = -\pi \sum_{\substack{i,j \in J_p \\ i \neq j}} \ln|y_i - y_j| - \pi(D_p^2 - D_p)|\ln\chi| + o(1). \tag{109}$$

For $i \in J_\mu$, we let $\mathscr{R}_i := B(y_i, h_{\rm ex}^{-1}) \setminus \overline{B(y_i, \delta/3)}$. With Proposition 46.1 we obtain

$$\frac{1}{2}\int_{\mathscr{R}_i} \alpha|\nabla v|^2 \geq \pi|\ln(\delta h_{\rm ex}/3)|. \tag{110}$$

By combining (107), (108), (109) and (110) the result is proved. □

The second step is a "microscopic" version of Proposition 58.

Lemma 60. *If $r \leq \tilde{r} \leq \lambda^2\delta^2$, then :*

$$\sum_{i\in J_\mu} F(v, \mathscr{R}_i) \geq d\pi\left(|\ln(3\lambda)| + b^2|\ln(\lambda\delta/\tilde{r})|\right) + \sum_{i\in J_\mu} W^{\rm micro}(\hat{z}_i^0) + o(1)$$

where, for $i \in J_\mu$, $\mathscr{R}_i := B(y_i, \delta/3) \setminus \overline{B(z_i, \tilde{r})}$.

Proof. We first note that in order to prove Lemma 60 [up to replace v by $\underline{v}$] we may assume $\rho = |v| \leq 1$. We may also assume

$$\sum_{i\in J_\mu} F_\varepsilon(v, \mathscr{R}_i) = \mathcal{O}(|\ln(\lambda\delta)|) \tag{111}$$

since in the contrary case there is nothing to prove.

Fix $i \in J_\mu$ and let $v_\star$ be a minimizer of $F_\varepsilon(\cdot, \mathscr{R}_i)$ in $H^1(\mathscr{R}_i, \mathbb{C})$ with the Dirichlet boundary condition ${\rm tr}_{\partial\mathscr{R}_i}(\cdot) = {\rm tr}_{\partial\mathscr{R}_i}(v)$. Note that such minimizers exist and we have $F_\varepsilon(v_\star, \mathscr{R}_i) \leq F_\varepsilon(v, \mathscr{R}_i) = \mathcal{O}(|\ln(\lambda\delta)|)$.

The key ingredient consists in noting that since $v_\star$ is a minimizer of a weighted Ginzburg-Landau type energy we may thus use a sharp interior η-ellipticity result. Namely, following the strategy of [9] to prove Lemma 1 [see Appendix C in [9]], by using the first part of the proof [the interior argument which does not required any information on ${\rm tr}_{\partial\mathscr{R}_i}(v_\star)$], we get

$$\rho_\star := |v_\star| \geq 1 - \mathcal{O}\left(\sqrt{|\ln(\lambda\delta)|/|\ln\varepsilon|}\right) \text{ in } \tilde{\mathscr{R}}_i := B(y_i, \delta/3 - \varepsilon^{1/4}) \setminus \overline{B(z_i, \tilde{r} + \varepsilon^{1/4})}. \tag{112}$$

Write in $\tilde{\mathscr{R}}_i$: $v_\star = \rho_\star w_\star$ where $w_\star \in H^1(\tilde{\mathscr{R}}, \mathbb{S}^1)$.

Note that by (2) [namely $|\ln(\lambda\delta)| = \mathcal{O}(\ln|\ln\varepsilon|)$] we have $|\ln(\lambda\delta)|^3/|\ln\varepsilon| = o(1)$ and then from (111) & (112) [and aslo $\rho_\star \leq 1$] we have

$$\int_{\tilde{\mathscr{R}}_i} \alpha\rho_\star^2|\nabla w_\star|^2 = \int_{\tilde{\mathscr{R}}_i} \alpha|\nabla w_\star|^2 + o(1).$$

We then immediately get:

$$F(v, \mathscr{R}_i) \geq F(v_\star, \mathscr{R}_i) \geq \frac{1}{2}\int_{\tilde{\mathscr{R}}_i} \alpha|\nabla w_\star|^2 + o(1) \geq \inf_{\substack{\tilde{w}\in H^1(\tilde{\mathscr{R}}_i, \mathbb{S}^1) \\ \deg(\tilde{w})=1}} \frac{1}{2}\int_{\tilde{\mathscr{R}}_i} \alpha|\nabla\tilde{w}|^2 + o(1).$$

It suffices now to claim that from (70) we have

$$\inf_{\substack{\tilde{w}\in H^1(\tilde{\mathscr{R}}_i, \mathbb{S}^1) \\ \deg(\tilde{w})=1}} \frac{1}{2}\int_{\tilde{\mathscr{R}}_i} \alpha|\nabla\tilde{w}|^2 = \pi\left(|\ln(3\lambda)| + b^2|\ln(\lambda\delta/\tilde{r})|\right) + W^{\rm micro}(\hat{z}_i^0) + o(1)$$

in order to get $F(v,\mathscr{R}_i) \geq \pi\left(|\ln(3\lambda)| + b^2|\ln(\lambda\delta/\tilde{r})|\right) + W^{\rm micro}(\hat{z}_i^0) + o(1)$. By summing these lower bounds we get the result. □

Lemma 61. *There exits $r \leq \tilde{r} = o(\lambda^2\delta^2)$ s.t. for $i \in J_\mu$ we have*

$$F[v, B(z_i,\tilde{r})] \geq b^2[\pi\ln(\tilde{r}/\varepsilon) + \ln b + \gamma] + o(1).$$

Proof. We first note that we have

$$\sum_{i\in J_\mu} F[v, B(z_i,\lambda^2\delta^2)\setminus\overline{B(z_i,r)}] \leq db^2\pi\ln(\lambda^2\delta^2/r) + \mathscr{L}_1(d)\ln h_{\rm ex} + \mathcal{O}(1). \tag{113}$$

The above estimate is proved by contradiction and assuming the existences of an extraction [still denoted by $\varepsilon = \varepsilon_n \downarrow 0$] and of a sequence $R_n \uparrow \infty$ s.t.

$$\sum_{i\in J_\mu} F[v, B(z_i,\lambda^2\delta^2)\setminus\overline{B(z_i,r)}] \geq db^2\pi\ln(\lambda^2\delta^2/r) + \mathscr{L}_1(d)\ln h_{\rm ex} + R_n.$$

From (83) we get

$$\sum_{i\in J_\mu} F[v, B(z_i,\lambda^2\delta^2)] \geq db^2\pi\ln(\lambda^2\delta^2/\varepsilon) + \mathscr{L}_1(d)\ln h_{\rm ex} + R_n + \mathcal{O}(1).$$

Using Lemmas 59 and 60 we get an estimate which contradicts (95).

By a classical argument, for sufficiently small ε, there exists $\sqrt{r} \leq \tilde{r} \leq r^{1/4}$ s.t. for $i \in J_\mu$

$$\frac{\tilde{r}}{2}\int_{\partial B(z_i,\tilde{r})} |\nabla v|^2 + \frac{b^2}{2\varepsilon^2}(1-|v|^2)^2 \leq \pi + \frac{4\mathscr{L}_1(d)\ln h_{\rm ex} + \mathcal{O}(1)}{|\ln r|}$$

Arguing as in the proof of Proposition 42 [Step 3 in Appendix G] it is clear that we may assume $|v| \geq 1 - |\ln\varepsilon|^{-2}$ on $\partial B(z_i,\tilde{r})$ for $i \in J_\mu$.

We now define for $i \in J_\mu$, $\rho_i := {\rm tr}_{\partial B(z_i,\tilde{r})}(|v|)$, $w_i := {\rm tr}_{\partial B(z_i,\tilde{r})}(v/|v|)$. We immediately get

$$\frac{\tilde{r}}{2}\int_{\partial B(z_i,\tilde{r})} |\nabla w_i|^2 = \pi + o(1),\ \frac{\tilde{r}}{2}\int_{\partial B(z_i,\tilde{r})} |\nabla\rho_i|^2 + \frac{b^2}{2\varepsilon^2}(1-\rho_i^2)^2 = o(1).$$

On the other hand, since $\deg(w_i) = 1$, there exists $\phi_i = \phi_{i,\varepsilon} \in H^1((0,2\pi),\mathbb{R})$ s.t. $\phi_i(0) = \phi_i(2\pi) \in [0,2\pi)$ and $w_i\left(z_i + \tilde{r}{\rm e}^{\imath\theta}\right) = {\rm e}^{-\imath(\theta+\phi_i(\theta))}$. A direct calculation gives:

$$2\pi + o(1) = \tilde{r}\int_{\partial B(z_i,\tilde{r})} |\partial_\tau w_i|^2 = \int_0^{2\pi} |(\phi_i+\theta)'|^2 = 2\pi + \int_0^{2\pi} |\phi_i'|^2.$$

The last equalities imply $\phi_i' \to 0$ in $L^2(0,2\pi)$ and then $\phi_i - \phi_i(0) \to 0$ in $L^2(0,2\pi)$. Hence, up to pass to a subsequence, we get the existence of $\theta_i \in [0,2\pi]$ s.t. $\phi_i \to \theta_i$ in $H^1(0,2\pi)$.

We now define $\tilde{w}_i \in H^1(B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})},\mathbb{S}^1)$ by

$$\tilde{w}_i(z_i + s{\rm e}^{\imath\theta}) = {\rm e}^{\imath[\theta+\tilde{\phi}_i(z_i+s{\rm e}^{\imath\theta})]} \text{ with } \tilde{\phi}_i(z_i+s{\rm e}^{\imath\theta}) = [\phi_i(\theta)-\theta_i]\,\frac{2\tilde{r}-s}{\tilde{r}} + \theta_i.$$

A direct calculation gives $\int_{B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}} |\nabla\tilde{\phi}_i|^2 = o(1)$ and then

$$\frac{1}{2}\int_{B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}} |\nabla\tilde{w}_i|^2 = \frac{1}{2}\int_{B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}} |\nabla[\theta+\tilde{\phi}_i(z_i+s{\rm e}^{\imath\theta})]|^2 + o(1) = \pi\ln(2) + o(1).$$

Let $\tilde{\rho}_i \in H^1[B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})},\mathbb{R}^+]$ be s.t. $\tilde{\rho}_i(z_i+s{\rm e}^{\imath\theta}) := \tilde{\rho}_i(z_i+\tilde{r}{\rm e}^{\imath\theta})\frac{2\tilde{r}-s}{\tilde{r}} + \frac{s-\tilde{r}}{\tilde{r}}$.

We then have $F[\tilde{\rho}_i, B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}]=o(1)$. Consequently, letting $v_i:=\tilde{\rho}_i\tilde{w}_i\in H^1[B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})},\mathbb{C}]$ we have

$$F[v_i,B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}]=\frac{b^2}{2}\int_{B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}}|\nabla\tilde{w}_i|^2+o(1).$$

In order to conclude we let $u_i:=\begin{cases}v_i & \text{in } B(z_i,2\tilde{r})\setminus\overline{B(z_i,\tilde{r})}\\ v & \text{in } B(z_i,\tilde{r})\end{cases}$.

It is clear that $u_i(z_i+2\tilde{r}\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta_i}\mathrm{e}^{\imath\theta}$ and then, using Lemma IX.1 in [4], we get

$$F[u_i,B(z_i,2\tilde{r})]\geq b^2[\pi\ln(2\tilde{r}/\varepsilon)+\gamma+\pi\ln b]+o(1).$$

The last estimate ends the proof of the lemma. □

Proof of Proposition 58. From the three previous lemmas we have

$$\begin{aligned}F(v)\quad\geq\quad & d\pi\left[b^2|\ln\varepsilon|+(1-b^2)|\ln(\lambda\delta)|\right]-\pi\sum_{\substack{p\in\Lambda\\ D_p\geq 2}}\sum_{\substack{i,j\in J_p\\ i\neq j}}\ln|z_i-z_j|+\\ & +W^{\mathrm{macro}}_{N_0}(\mathbf{p},\mathbf{D})+\sum_{i\in J_\mu}W^{\mathrm{micro}}(\hat{z}_i^0)+db^2[\pi\ln b+\gamma]+o(1).\end{aligned}\tag{114}$$

On the other hand, with Corollary 24 [estimate (47)] we get

$$\mathcal{F}(v,A)\geq h_{\mathrm{ex}}^2\mathbf{J_0}+2\pi h_{\mathrm{ex}}\sum_{i\in J_\mu}\xi_0(z_i)+F(v)+\tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]+o(1)\tag{115}$$

where $\zeta_{(\mathbf{p},\mathbf{D})}$ is defined in Proposition 16.

From Proposition 36 [estimate (64)], for $p\in\Lambda$ s.t. $D_p\geq 2$, we have:

$$-\pi\sum_{\substack{i,j\in J_p\\ i\neq j}}\ln|z_i-z_j|+2\pi h_{\mathrm{ex}}\sum_i[\xi_0(z_i)-\xi_0(p)]\geq\frac{\pi}{2}(D_p^2-D_p)\ln\left(\frac{h_{\mathrm{ex}}}{D_p}\right)+C_{p,D_p}+o(1).\tag{116}$$

By combining (114), (115) and (116) [and also $\xi_0\leq 0$] we obtain

$$\begin{aligned}\mathcal{F}(v,A)\quad\geq\quad & h_{\mathrm{ex}}^2\mathbf{J_0}+d\pi\left[b^2|\ln\varepsilon|+(1-b^2)|\ln(\lambda\delta)|\right]-2\pi dh_{\mathrm{ex}}\|\xi_0\|_{L^\infty(\Omega)}+\\ & +\frac{\pi}{2}\sum_{\substack{p\in\Lambda\\ D_p\geq 2}}\left[(D_p^2-D_p)\ln\left(\frac{h_{\mathrm{ex}}}{D_p}\right)+C_{p,D_p}\right]+W^{\mathrm{macro}}_{N_0}(\mathbf{p},\mathbf{D})+\\ & +\sum_{i\in J_\mu}W^{\mathrm{micro}}(\hat{z}_i^0)+\tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}]+db^2[\pi\ln b+\gamma]+o(1).\end{aligned}\tag{117}$$

It suffices to see that, since $\mathbf{D}\in\Lambda_d$, from the definition of $\mathscr{L}_1(d)$ we have

$$\frac{\pi}{2}\sum_{\substack{p\in\Lambda\\ D_p\geq 2}}(D_p^2-D_p)\ln\left(\frac{h_{\mathrm{ex}}}{D_p}\right)=\mathscr{L}_1(d)\ln h_{\mathrm{ex}}+\frac{\pi}{2}\sum_{p\in\Lambda}(D_p-D_p^2)\ln(D_p)$$

in order to deduce from (117) that

$$\begin{aligned}\mathcal{F}(v,A)\quad\geq\quad & h_{\mathrm{ex}}^2\mathbf{J_0}+d\pi\left[-2h_{\mathrm{ex}}\|\xi_0\|_{L^\infty(\Omega)}+b^2|\ln\varepsilon|+(1-b^2)|\ln(\lambda\delta)|\right]+\\ & +\mathscr{L}_1(d)\ln h_{\mathrm{ex}}+\sum_{i\in J_\mu}W^{\mathrm{micro}}(\hat{z}_i^0)+\mathcal{W}_d(\mathbf{D})+\\ & +\frac{\pi}{2}\sum_{p\in\Lambda}(D_p-D_p^2)\ln(D_p)+db^2[\pi\ln b+\gamma]+o(1)\end{aligned}$$

where $\mathcal{W}_d(\mathbf{D})$ is defined in (106). This estimate with the definition of $H^0_{c_1}$ and $\overline{\mathcal{W}}_d$ [see (72)&(75)&(76)] ends the proof of the proposition. □

10. The first critical field and the location of the vorticity defects

We assume that $\lambda, \delta, h_{\rm ex}$ satisfy (2) and (3) for some $K \geq 0$ independent of ε. We assume also (4). We consider a sequence $\varepsilon = \varepsilon_n \downarrow 0$.

As in the previous section we focus on sequences of quasi-minimizers of $\mathcal{F}$. For simplicity we write (v, A) instead of $(v_\varepsilon, A_\varepsilon)$. We assume that (17)&(77) holds and since (17)&(77) are gauge invariant we may also assume that (v, A) is in the Coulomb gauge.

From above results, for a fixed $\mu > 0$ sufficiently small [satisfying (97)] and for $\varepsilon > 0$ sufficiently small, there exists a [finite] set $\mathcal{Z} \subset \Omega$, depending on ε and possibly empty s.t. letting $d := \mathrm{Card}(\mathcal{Z})$ [we write $\mathcal{Z} = \{z_1, ..., z_2\}$]:

- If $d = 0$, then $|v| \geq 1/2$ in Ω.
- If $d > 0$, then $|z_i - z_j| \gtrsim h_{\rm ex}^{-1} \ln h_{\rm ex}$ if $i \neq j$, $|v| \geq 1/2$ in $\Omega \setminus \cup_{i=1}^d \overline{B(z_i, \varepsilon^\mu)}$ and $\deg_{\partial B(z, \varepsilon^\mu)}(v) = 1$ for $z \in \mathcal{Z}$.

Moreover $d = \mathcal{O}(1)$. Then if needed, up to pass to a subsequence, we may assume that d is independent of ε.

By combining Corollary 14, Propositions 36, 39, 55 and 58 we get the following corollary.

Corollary 62. *Assume $\lambda, \delta, h_{\rm ex}$ satisfy (2) and (3) for some $K \geq 0$ independent of ε. Let $\varepsilon = \varepsilon_n \downarrow 0$ and and let $((v_\varepsilon, A_\varepsilon))_\varepsilon \subset \mathscr{H}$ be a sequence satisfying (17)&(77). Assume that d is independent of ε. Without loss of generality we may assume that $(v_\varepsilon, A_\varepsilon)$ is in the Coulomb gauge. We have*

$$\mathcal{F}(v_\varepsilon, A_\varepsilon) = h_{\rm ex}^2 \mathbf{J_0} + d M_\Omega \left[-h_{\rm ex} + H^0_{c_1} \right] + \mathscr{L}_1(d) \ln h_{\rm ex} + \mathscr{L}_2(d) + o(1). \tag{118}$$

Moreover, if $d \neq 0$ then:

- *We have $\mathbf{D} \in \Lambda_d$ [see (71)] and $\mathbf{D}$ minimises $\mathcal{W}_d$ in Λ_d where $\mathcal{W}_d$ is defined in (106).*
- *For $p \in \Lambda$ s.t. $D_p > 0$ and $i \in J_p$, we denote $\breve{z}_i := (z_i - p)\sqrt{D_p / h_{\rm ex}}$ and $\breve{\mathbf{z}}_p := \{\breve{z}_i \,|\, i \in J_p\}$. Then, up to pass to a subsequence, $\breve{\mathbf{z}}_p$ converges to a minimizer of $W^{\rm meso}_{p, D_p}$ defined in (66).*
- *For $i \in \{1, ..., d\}$, we write $\hat{z}_i := (z_i - y_i)/(\lambda\delta) \in \omega$ where $y_i \in \delta\mathbb{Z}^2$ is s.t. $z_i \in B(y_i, \lambda\delta)$. Then, up to pass to a subsequence, $\hat{z}_i$ converges to a minimizer of $W^{\rm micro}$.*

For a further use, we claim that for $d_0 \geq 0$, from Proposition 39, there exits a configuration $(v^0, A^0) \in \mathscr{H}$ which is in the Coulomb gauge s.t.

$$\mathcal{F}(v^0, A^0) - h_{\rm ex}^2 \mathbf{J_0} = d_0 M_\Omega \left[-h_{\rm ex} + H^0_{c_1} \right] + \mathscr{L}_1(d_0) \ln h_{\rm ex} + \mathscr{L}_2(d_0) + o(1). \tag{119}$$

Recall that, from Lemma 37, for $d \neq 0$, we have $d \in \{1, ..., N_0\}$ if and only if $\mathscr{L}_1(d) = 0$ and $\mathscr{L}_2(d) = \overline{\mathcal{W}}_d$. For further use we state another lemma whose proof is left to the reader:

Lemma 63. *For $0 \leq d < d'$ we let :*

(1) $\displaystyle \Delta^{(1)}_d := \frac{\mathscr{L}_1(d+1) - \mathscr{L}_1(d)}{M_\Omega} = \frac{\pi}{M_\Omega} \left\lfloor \frac{d}{N_0} \right\rfloor.$

(2) $\displaystyle \Delta^{(1)}_{d', d} := \frac{\mathscr{L}_1(d') - \mathscr{L}_1(d)}{M_\Omega (d' - d)} = \frac{\pi}{M_\Omega (d' - d)} \sum_{k=d}^{d'-1} \left\lfloor \frac{k}{N_0} \right\rfloor.$

(3) $\Delta^{(2)}_d := \dfrac{\mathscr{L}_2(d+1)-\mathscr{L}_2(d)}{M_\Omega}$ *and* $\Delta^{(2)}_d - \dfrac{\overline{\mathcal{W}}_{d+1}-\overline{\mathcal{W}}_d}{M_\Omega} =$

$$= \begin{vmatrix} 0 \text{ if } d \le N_0-1 \\ -\dfrac{\pi}{2M_\Omega}\left\lfloor\dfrac{d}{N_0}\right\rfloor\left[\left(1+\left\lfloor\dfrac{d}{N_0}\right\rfloor\right)\ln\left(1+\left\lfloor\dfrac{d}{N_0}\right\rfloor\right)+\left(1-\left\lfloor\dfrac{d}{N_0}\right\rfloor\right)\ln\left\lfloor\dfrac{d}{N_0}\right\rfloor\right] \text{ if } d\ge N_0 \end{vmatrix}.$$

(4) $\Delta^{(2)}_{d',d} := \dfrac{\mathscr{L}_2(d')-\mathscr{L}_2(d)}{M_\Omega(d'-d)}$ *thus, if* $d'\le N_0$, *then* $\Delta^{(2)}_{d',d} = \dfrac{\overline{\mathcal{W}}_{d'}-\overline{\mathcal{W}}_d}{M_\Omega(d'-d)}$.

By using (118) and (119) we easily get the following corollary.

Corollary 64. *Let* $\varepsilon=\varepsilon_n\downarrow 0$, λ, δ, $h_{\rm ex}$ *and* $((v_\varepsilon,A_\varepsilon))_\varepsilon\subset\mathscr{H}$ *be as in Corollary 62. Assume that* d *is independent of* ε. *Then we have for* $d'>d$

$$h_{\rm ex}\le H^0_{c_1}+\Delta^{(1)}_{d',d}\times\ln h_{\rm ex}+\Delta^{(2)}_{d',d}+o(1).$$

Then, letting χ *be s.t.* $h_{\rm ex}=H^0_{c_1}(1+\chi)$ *[*$\chi=o(1)$ *from (3)], we have thus*

$$h_{\rm ex}\le H^0_{c_1}+\Delta^{(1)}_{d',d}\times\ln H^0_{c_1}+\Delta^{(2)}_{d',d}+o(1). \tag{120}$$

If $d>d'\ge 0$ *then*

$$h_{\rm ex}\ge H^0_{c_1}+\Delta^{(1)}_{d,d'}\times\ln H^0_{c_1}+\Delta^{(2)}_{d,d'}+o(1). \tag{121}$$

We are now in position to give an asymptotic value for the first critical field. Indeed with Corollary 64 [(120) with $d=0\&d'\in\{1,...,N_0\}$ and (121) with $d\ge 1\&d'=0$].

Recall that we write, for $x\in\mathbb{R}$, $[x]^+=\max(x,0)$ and $[x]^-=\min(x,0)$

Corollary 65. *Denote* $H_{c_1} := H^0_{c_1}+\min_{d\in\{1,...,N_0\}}\dfrac{\overline{\mathcal{W}}_d}{dM_\Omega}$. *Let* $\{(v_\varepsilon,A_\varepsilon)\,|\,0<\varepsilon<1\}\subset\mathscr{H}$ *be a family of quasi-minimizers satisfying (17).*

(1) If for sufficiently small ε *we have* $d=0$ *then* $[h_{\rm ex}-H_{c_1}]^+\to 0$.

(2) If for sufficiently small ε *we have* $d>0$ *then* $[h_{\rm ex}-H_{c_1}]^-\to 0$.

Proof. The corollary is a direct consequence of Corollary 64 taking $d'\in\{1,...,N_0\}$ which minimizes $\Delta^{(2)}_{d',0}=W_{d'}/(M_\Omega d')$ in (120) for the first assertion and $d'=0$ in (121) for the second. □

10.1. **Secondary critical fields for** $d\in\{1,...,N_0\}$**.** If $N_0=1$, if $h_{\rm ex}$ is near H_{c_1} and if $d>0$, then it is standard to prove that $d=1$. If $N_0\ge 2$ and $d\in\{1,...,N_0\}$, then the situation is more involved: we have no *a priori* sharp informations about the number of vorticity defects and their [macroscopic] location. The goal of this section is to get such informations.

10.1.1. *Preliminaries.* Note that for $0\le d<d'\le N_0$ we have $\Delta^{(1)}_{d',d}=0$ and $\Delta^{(2)}_{d',d}=\dfrac{\overline{\mathcal{W}}_{d'}-\overline{\mathcal{W}}_d}{M_\Omega(d'-d)}$.

Rephrasing Corollary 64 for $d,d'\in\{0,...,N_0\}$ we have the following key lemma.

Lemma 66. *Let* $\varepsilon=\varepsilon_n\downarrow 0$, λ, δ, $h_{\rm ex}$ *and* $((v_\varepsilon,A_\varepsilon))_\varepsilon\subset\mathscr{H}$ *be as in Corollary 62. Assume* $\mathrm{Card}(\mathcal{Z})=d$ *is independent of* ε *then the following properties hold:*

(1) If $0\le d'<d$ *then, letting* $\overline{\mathcal{W}}_0:=0$, *we have* $h_{\rm ex}\ge H^0_{c_1}+\dfrac{\overline{\mathcal{W}}_d-\overline{\mathcal{W}}_{d'}}{M_\Omega(d-d')}+o(1)$.

In particular taking $d'=0$ *we get* $h_{\rm ex}\ge H^0_{c_1}+\dfrac{\overline{\mathcal{W}}_d}{M_\Omega d}+o(1)$.

(2) If $d < N_0$ and $d < d' \leq N_0$ then $h_{\rm ex} \leq H^0_{c_1} + \dfrac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{M_\Omega(d'-d)} + o(1)$.

(3) If $N_0 \geq 2$, $N_0 \geq d' > d \geq 1$ then

$$\frac{\overline{\mathcal{W}}_{d'}}{d'} < \frac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{d'-d} \iff \frac{\overline{\mathcal{W}}_d}{d} < \frac{\overline{\mathcal{W}}_{d'}}{d'} \text{ and } \frac{\overline{\mathcal{W}}_{d'}}{d'} > \frac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{d'-d} \iff \frac{\overline{\mathcal{W}}_d}{d} > \frac{\overline{\mathcal{W}}_{d'}}{d'}.$$

(4) If $N_0 \geq 2$ and $N_0 \geq d' > d \geq 1$ then

$$\frac{\overline{\mathcal{W}}_{d'}}{d'} = \frac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{d'-d} \iff \frac{\overline{\mathcal{W}}_d}{d} = \frac{\overline{\mathcal{W}}_{d'}}{d'}.$$

(5) If $N_0 \geq 2$ and $0 \leq d < d' < d'' \leq N_0$ then we have the following convex combination

$$\frac{\overline{\mathcal{W}}_{d''} - \overline{\mathcal{W}}_d}{d''-d} = \frac{d''-d'}{d''-d}\frac{\overline{\mathcal{W}}_{d''} - \overline{\mathcal{W}}_{d'}}{d''-d'} + \frac{d'-d}{d''-d}\frac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{d'-d}.$$

Consequenlty $\dfrac{\overline{\mathcal{W}}_{d''} - \overline{\mathcal{W}}_d}{d''-d}$ *is between* $\dfrac{\overline{\mathcal{W}}_{d''} - \overline{\mathcal{W}}_{d'}}{d''-d'}$ *and* $\dfrac{\overline{\mathcal{W}}_{d'} - \overline{\mathcal{W}}_d}{d'-d}$.

Proof. The two first assertions are obtained with Corollary 64. The remaining part of the lemma consists in basic calculations. □

10.1.2. *First step in the definition of the critical fields.* Assume $N_0 \geq 2$. We are going to define some energetic levels [in term of $\overline{\mathcal{W}}_d$] related with the number of vorticity defects and their [macroscopic] location.

We denote $d_0^\star := 0$, $\mathscr{S}_1 := \{1, ..., N_0\}$, $\mathscr{K}_1^\star := \min_{d\in\mathscr{S}_1} \dfrac{\overline{\mathcal{W}}_d}{d} = \min_{d\in\mathscr{S}_1} \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_0^\star}}{d - d_0^\star}$, $\mathscr{S}_1^\star := \{d \in \mathscr{S}_1 \,|\, \overline{\mathcal{W}}_d/d = \mathscr{K}_1^\star\}$ and $\mathscr{D}_1 := \{\mathbf{D} \in \Lambda_d \,|\, d \in \mathscr{S}_1^\star \text{ and } \mathbf{D} \text{ minimizes } \mathcal{W}_d\}$. We let also $d_1^\star := \max \mathscr{S}_1^\star$ and $\mathscr{D}_1^\star := \mathscr{D}_1 \cap \tilde{\Lambda}_{d_1^\star}$.

If $d_1^\star = N_0$ we are going to prove that for $h_{\rm ex} \geq H_{c_1} + o(1)$ [but $h_{\rm ex}$ not too large], then there is exactly one vorticity defect close to each point of Λ. In the contrary case [$1 \leq d_1^\star < N_0$], then there are other critical fields which govern the number of vorticity defects.

If $d_1^\star < N_0$, then $\mathscr{S}_2 := \{d_1^\star + 1, ..., N_0\} \neq \emptyset$. For $d \in \mathscr{S}_2$ we let $\mathscr{K}_2(d) := \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_1^\star}}{d - d_1^\star}$, $\mathscr{S}_2^\star := \{d \in \mathscr{S}_2 \,|\, \mathscr{K}_2(d) = \min_{\mathscr{S}_2} \mathscr{K}_2\}$, $d_2^\star := \max \mathscr{S}_2^\star$ and $\mathscr{K}_2^\star := \mathscr{K}_2(d_2^\star)$.

We denote $\mathscr{D}_2 := \{\mathbf{D} \in \Lambda_d \,|\, d \in \mathscr{S}_2^\star \text{ and } \mathbf{D} \text{ minimizes } \mathcal{W}_d\}$ and $\mathscr{D}_2^\star := \mathscr{D}_2 \cap \Lambda_{d_2^\star}$.

We claim that for $d \in \mathscr{S}_2$ we have $\overline{\mathcal{W}}_d/d > \overline{\mathcal{W}}_{d_1^\star}/d_1^\star$. Then, with Lemma 66.3, we get $\mathscr{K}_2(d) > \overline{\mathcal{W}}_{d_1^\star}/d_1^\star$. In particular

$$\mathscr{K}_2^\star = \frac{\overline{\mathcal{W}}_{d_2^\star} - \overline{\mathcal{W}}_{d_1^\star}}{d_2^\star - d_1^\star} > \frac{\overline{\mathcal{W}}_{d_1^\star}}{d_1^\star} = \mathscr{K}_1^\star. \tag{122}$$

If $d_2^\star = N_0$ then we stop the construction. In the contrary case, for $d \in \mathscr{S}_3 := \{d_2^\star + 1, ..., N_0\} \neq \emptyset$ we have $\mathscr{K}_2(d) > \mathscr{K}_2(d_2^\star)$.

We continue the iterative construction. For $k \geq 2$, assume that we have $1 < d_{k-1}^\star < d_k^\star < N_0$, we let $\mathscr{S}_{k+1} := \{d_k^\star + 1, ..., N_0\} \neq \emptyset$ and we assume that for $d \in \mathscr{S}_{k+1}$:

$$\mathscr{K}_k(d) := \frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star} > \frac{\overline{\mathcal{W}}_{d_k^\star} - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d_k^\star - d_{k-1}^\star} = \mathscr{K}_k^\star. \tag{123}$$

For $d \in \mathscr{S}_{k+1}$ we let $\mathscr{K}_{k+1}(d) := \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_k^\star}}{d - d_k^\star}$,

$$\mathscr{S}_{k+1}^\star := \left\{ d \in \mathscr{S}_{k+1} \,|\, \mathscr{K}_{k+1}(d) = \min_{\mathscr{S}_{k+1}} \mathscr{K}_{k+1} \right\},$$

$d_{k+1}^\star := \max \mathscr{S}_{k+1}^\star$ and $\mathscr{K}_{k+1}^\star := \mathscr{K}_{k+1}(d_{k+1}^\star)$.

We define also

$$\mathscr{D}_{k+1} := \{\mathbf{D} \,|\, \mathbf{D} \in \Lambda_d,\, d \in \mathscr{S}_{k+1}^\star \text{ and } \mathbf{D} \text{ minimizes } \mathcal{W}_d\} \text{ and } \mathscr{D}_{k+1}^\star := \mathscr{D}_{k+1} \cap \Lambda_{d_{k+1}^\star}.$$

From (123) we have

$$\mathscr{K}_k(d_{k+1}^\star) = \frac{\overline{\mathcal{W}}_{d_{k+1}^\star} - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d_{k+1}^\star - d_{k-1}^\star} > \frac{\overline{\mathcal{W}}_{d_k^\star} - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d_k^\star - d_{k-1}^\star} = \mathscr{K}_k^\star. \tag{124}$$

Then, from Lemma 66.5 with $d = d_{k-1}^\star$, $d' = d_k^\star$ and $d'' = d_{k+1}^\star$, we get that $\mathscr{K}_k(d_{k+1}^\star)$ is between $\mathscr{K}_k^\star$ and $\mathscr{K}_{k+1}^\star$. Consequently, with (124) we get

$$\mathscr{K}_{k+1}^\star > \mathscr{K}_k^\star. \tag{125}$$

We stop the construction at Step L s.t. $d_L^\star = N_0$. Since $1 \le d_k^\star < d_{k+1}^\star \le N_0$, it is clear that a such L exists and $1 \le L \le N_0$.

We then have two possibilities: $L = 1$ or $L \in \{2, ..., N_0\}$. If $L \ge 2$ then, for $k \in \{1, ..., L-1\}$, (125) holds. We also claim that $(1, ..., 1) \in \mathscr{D}_L$.

Lemma 67. *Let $k \in \{1, ..., L\}$, assume that $d_k^\star - d_{k-1}^\star \ge 2$ and fix $d_{k-1}^\star < d < d_k^\star$. We have*

$$\frac{\overline{\mathcal{W}}_{d_k^\star} - \overline{\mathcal{W}}_d}{d_k^\star - d} \le \mathscr{K}_k^\star \le \frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}.$$

Moreover, if $d \notin \mathscr{S}_k^\star$, then

$$\frac{\overline{\mathcal{W}}_{d_k^\star} - \overline{\mathcal{W}}_d}{d_k^\star - d} \le \mathscr{K}_k^\star < \frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}.$$

Proof. From Lemma 66.5, $\mathscr{K}_k^\star$ is between $\dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}$ and $\dfrac{\overline{\mathcal{W}}_{d_k^\star} - \overline{\mathcal{W}}_d}{d_k^\star - d}$. On the other hand, from the definition of $d_k^\star$, $\mathscr{K}_k^\star \le \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}$. Clearly the first part of the lemma holds. If $d \notin \mathscr{S}_k^\star$ then, by definition, $\mathscr{K}_k^\star < \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}$. □

10.1.3. *Main result.* For $k \in \{1, ..., L\}$ we let

$$\mathtt{K}_k^{(\mathtt{I})} := H_{c_1}^0 + \frac{\mathscr{K}_k^\star}{M_\Omega} \tag{126}$$

and we let also

$$\mathtt{K}_1^{(\mathtt{II})} := H_{c_1}^0 + \Delta_{N_0}^{(1)} \times \ln H_{c_1}^0 + \Delta_{N_0}^{(2)}. \tag{127}$$

Recall that the $\mathscr{K}_k^\star$'s are defined in Section 10.1.2 and $\Delta_{N_0}^{(1)}$&$\Delta_{N_0}^{(2)}$ in Lemma 63. Note that $H_{c_1} = \mathtt{K}_1^{(\mathtt{I})}$.

Proposition 68. *Assume that (5) holds and $\lambda, \delta, h_{\rm ex}, K$ satisfy (2), (3) and (4).*

Let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ be a family satisfying (17)&(77) which is in the Coulomb gauge. Assume $d_\varepsilon = {\rm Card}(\mathcal{Z}_\varepsilon) \in \{1, ..., N_0\}$.

We denote $\mathbf{D} = (D_1, ..., D_{N_0})$ with $D_l = \deg_{\partial B(p_l, \eta_\Omega)}(v)$ [η_Ω is defined in (63)].

(1) Assume $L = 1$. For sufficiently small $\varepsilon > 0$ we have $\mathbf{D} \in \mathscr{D}_1$.

Moreover, if $\varepsilon = \varepsilon_n \downarrow 0$ is a sequence s.t. d_ε is independent of ε and $d_\varepsilon \neq N_0$ [i.e. $\mathbf{D} \neq (1, ..., 1)$] then $\left[h_{\rm ex} - \mathtt{K}_1^{(\mathtt{I})}\right]^+ \to 0$.

(2) Assume $L \geq 2$. For $k \in \{1, ..., L-1\}$, if $d_{k-1}^\star < d_\varepsilon \leq d_k^\star$ for small ε or for a sequence indexed by $\varepsilon = \varepsilon_n \downarrow 0$, then

$$\left[h_{\rm ex} - \mathtt{K}_k^{(\mathtt{I})}\right]^- \to 0 \text{ and } \left[h_{\rm ex} - \mathtt{K}_{k+1}^{(\mathtt{I})}\right]^+ \to 0. \tag{128}$$

Moreover, for sufficiently small ε, $\mathbf{D} \in \mathscr{D}_k$. And if $\mathbf{D} \in \mathscr{D}_k \setminus \mathscr{D}_k^\star$ [i.e. $d_{k-1}^\star < d_\varepsilon < d_k^\star$] then

$$\left[h_{\rm ex} - \mathtt{K}_k^{(\mathtt{I})}\right]^+ \to 0. \tag{129}$$

(3) If $d_{L-1}^\star < d_\varepsilon \leq d_L^\star = N_0$ for small ε or for a sequence indexed by $\varepsilon = \varepsilon_n \downarrow 0$, then

$$\left[h_{\rm ex} - \mathtt{K}_L^{(\mathtt{I})}\right]^- \to 0 \text{ and } \left[h_{\rm ex} - \mathtt{K}_1^{(\mathtt{II})}\right]^+ \to 0. \tag{130}$$

Moreover, for sufficiently small ε, $\mathbf{D} \in \mathscr{D}_L$. And if $d_\varepsilon < N_0$ [i.e $\mathbf{D} \neq (1, ..., 1)$] then

$$\left[h_{\rm ex} - \mathtt{K}_L^{(\mathtt{I})}\right]^+ \to 0. \tag{131}$$

In particular, for sufficiently small ε, we have $\mathbf{D} \in \cup_{l=1}^L \mathscr{D}_l$.

Proof. We prove the first item arguing by contradiction. First note that if $N_0 = 1$ then there is nothing to prove. Assume thus $N_0 \geq 2$ & $L = 1$ and let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\}$ be as in the proposition. Assume there exists $\varepsilon = \varepsilon_n \downarrow 0$ s.t. $\mathbf{D} \notin \mathscr{D}_1$. Up to pass to a subsequence we may assume that $\mathbf{D}$ is independent of ε.

From Corollary 62, for sufficiently small ε, $\mathbf{D}$ minimizes $\mathcal{W}_d$ and then, from the definition of $\mathscr{D}_1$, we get $d \notin \mathscr{S}_1^\star$. Consequently $\overline{\mathcal{W}}_{N_0}/N_0 < \overline{\mathcal{W}}_d/d$ and thus, from Lemma 66.2&66.3 [with $d' = N_0$], we get the existence of $t > 0$ s.t. $h_{\rm ex} \leq H_{c_1} - t$. This last estimate is in contradiction with Corollary 65.2. Thus $\mathbf{D} \in \mathscr{D}_1$ for sufficiently small ε. The rest of the first assertion is a direct consequence of $d \in \mathscr{S}_1^\star \setminus \{N_0\}$ and Lemma 66.2&66.4 [with $d' = N_0$].

We now prove the second assertion. Assume $L \geq 2$. For $k \in \{1, ..., L-1\}$, if $d_{k-1}^\star < d \leq d_k^\star$, then, from Lemma 66.1 [with $d' = d_{k-1}^\star$] and Lemma 66.2 [with $d' = d_{k+1}^\star$], we get

$$\frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{M_\Omega(d - d_{k-1}^\star)} + o(1) \leq h_{\rm ex} - H_{c_1}^0 \leq \frac{\overline{\mathcal{W}}_{d_{k+1}^\star} - \overline{\mathcal{W}}_d}{M_\Omega(d_{k+1}^\star - d)} + o(1). \tag{132}$$

From the definition of $d_k^\star$ we have $\mathscr{K}_k^\star \leq \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d_{k-1}^\star}}{d - d_{k-1}^\star}$ and then the lower bound in (132) gives the first convergence in (128).

On the other hand, if $d = d_k^\star$ then, from the definition of $\mathscr{K}_{k+1}^\star$, the upper bound in (132) gives the second convergence in (128).

If $d \neq d^\star_k$, using Lemma 66.5 [with $d < d^\star_k < d^\star_{k+1}$] we obtain that $\dfrac{\overline{\mathcal{W}}_{d^\star_{k+1}} - \overline{\mathcal{W}}_d}{d^\star_{k+1} - d}$ is between $\dfrac{\overline{\mathcal{W}}_{d^\star_k} - \overline{\mathcal{W}}_d}{d^\star_k - d}$ and $\mathscr{K}^\star_{k+1}$. But, from Lemma 67, we get $\dfrac{\overline{\mathcal{W}}_{d^\star_k} - \overline{\mathcal{W}}_d}{d^\star_k - d} \leq \mathscr{K}^\star_k$. Since from (125) we have $\mathscr{K}^\star_{k+1} > \mathscr{K}^\star_k$, we obtain $\dfrac{\overline{\mathcal{W}}_{d^\star_{k+1}} - \overline{\mathcal{W}}_d}{d^\star_{k+1} - d} \leq \mathscr{K}^\star_{k+1}$. Therefore the upper bound of (132) gives the second convergence in (128).

We now demonstrate that, for sufficiently small ε, $\mathbf{D} \in \mathscr{D}_k$ arguing by contradiction. We assume the existence of sequence $\varepsilon = \varepsilon_n \downarrow 0$ s.t. $d^\star_{k-1} < d \leq d^\star_k$ with $k \in \{1, ..., L-1\}$, $\mathbf{D}$ is independent of ε and $\mathbf{D} \notin \mathscr{D}_k$. From Corollary 62, $\mathbf{D}$ minimizes $\mathcal{W}_d$ and then, from the definition of $\mathscr{D}_k$, we get $d \notin \mathscr{S}^\star_k$ [then $d < d^\star_k$].

On the one hand, with Lemma 66.1 [with $d' = d^\star_{k-1}$] and Lemma 66.2 [with $d' = d^\star_k$] we have

$$\frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d^\star_{k-1}}}{M_\Omega(d - d^\star_{k-1})} + o(1) \leq h_{\rm ex} - H^0_{c_1} \leq \frac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d^\star_k}}{M_\Omega(d - d^\star_k)} + o(1).$$

On the other hand, with Lemma 67, we have $\dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d^\star_k}}{d - d^\star_k} < \dfrac{\overline{\mathcal{W}}_d - \overline{\mathcal{W}}_{d^\star_{k-1}}}{d - d^\star_{k-1}}$. This inequality gives a contradiction.

Lemma 66.2 [with $d' = d^\star_k$] and Lemma 67 give immediately (129).

We now treat the last item of the proposition and we assume $d^\star_{L-1} < d \leq d^\star_L = N_0$. From (121) [with $d' = d^\star_{L-1}$] we get $h_{\rm ex} - H^0_{c_1} \geq \Delta^{(2)}_{d,d^\star_{L-1}} + o(1)$. On the other hand, from the definition of $\mathscr{K}^\star_L$, we get

$$h_{\rm ex} - H^0_{c_1} \geq \frac{\mathscr{K}^\star_L}{M_\Omega} + o(1). \tag{133}$$

Before ending the proof of (130) we prove that (131) holds and, for sufficiently small ε, $\mathbf{D} \in \mathscr{D}_L$. Assume that there exists $\varepsilon = \varepsilon_n \downarrow 0$ s.t. $\mathbf{D}$ is independent of ε and $d^\star_{L-1} < d < N_0$.

From Lemma 66.2 [with $d' = N_0$] we have

$$h_{\rm ex} - H^0_{c_1} \leq \frac{\overline{\mathcal{W}}_{N_0} - \overline{\mathcal{W}}_d}{M_\Omega(N_0 - d)} + o(1). \tag{134}$$

Using (133) with (134) we get $\mathscr{K}^\star_L \leq (\overline{\mathcal{W}}_{N_0} - \overline{\mathcal{W}}_d)/(N_0 - d)$. Lemma 67 [with $d^\star_{L-1} < d < N_0$] gives $(\overline{\mathcal{W}}_{N_0} - \overline{\mathcal{W}}_d)/(N_0 - d) \leq \mathscr{K}^\star_L$. Therefore, $(\overline{\mathcal{W}}_{N_0} - \overline{\mathcal{W}}_d)/(N_0 - d) = \mathscr{K}^\star_L$ and then by combining (133) and (134) we deduce that, if for some sequence $\varepsilon = \varepsilon_n \downarrow 0$ we have $d^\star_{L-1} < d < N_0$, then (131) holds.

Arguing as above, [using (119) with $d_0 = N_0$], one may prove that for sufficiently small ε we have $d \in \mathscr{S}^\star_L$ and thus $\mathbf{D} \in \mathscr{D}_L$.

We complete the proof of (130). Assume that $h_{\rm ex}$ is sufficiently large in order to have $d = N_0$ [here we used (131)]. It suffices to use (120) [with $d = N_0$ and $d' = N_0 + 1$] in order to get the remaining part of (130).

□

10.2. **Secondary critical fields for $d \geq N_0 + 1$.** The case $d \geq N_0 + 1$ is easier to handle than the case $1 \leq d \leq N_0$.

For $k \in \mathbb{N}^*$, we let

$$\mathtt{K}^{(\mathtt{II})}_k := H^0_{c_1} + \Delta^{(1)}_{N_0+k} \times \ln H^0_{c_1} + \Delta^{(2)}_{N_0+k}$$

where $\Delta^{(1)}_{N_0+k}\&\Delta^{(2)}_{N_0+k}$ are defined in Lemma 63. We have the following proposition.

Proposition 69. *Assume that (5) holds and $\lambda, \delta, h_{\rm ex}, K$ satisfy (2), (3) and (4).*

Let $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ be a family satisfying (17)&(77) which is in the Coulomb gauge.

Let $k \in \mathbb{N}^*$. If for a sequence $\varepsilon = \varepsilon_n \downarrow 0$ we have $d_\varepsilon = N_0 + k$ then

$$\left[h_{\rm ex} - \mathtt{K}^{(\mathtt{II})}_k\right]^- \to 0 \text{ and } \left[h_{\rm ex} - \mathtt{K}^{(\mathtt{II})}_{k+1}\right]^+ \to 0.$$

Proof. The proposition is a direct consequence of (120) [with $d = N_0 + k$ and $d' = N_0 + k + 1$] and (121) [with $d = N_0 + k$ and $d' = N_0 + k - 1$]. □

Appendix A. Proof of Estimate (26)

Consider a conformal mapping $\Phi : \mathbb{D} \to \Omega$. From a result of Painlevé [see Footnote 4 page 9], the maps Φ and Φ^{-1} may be extended in $\overline{\Omega}$ and $\overline{\mathbb{D}}$ by smooth maps. Then there exists $C_\star \geq 1$ s.t.

$$\|\nabla\Phi\|_{L^\infty(\mathbb{D})}, \|\nabla\Phi^{-1}\|_{L^\infty(\Omega)} \leq C_\star. \tag{135}$$

Write $\tilde{a}_\varepsilon := a_\varepsilon \circ \Phi$ and $\tilde{U}_\varepsilon := U_\varepsilon \circ \Phi$. Since the function $\tilde{U}_\varepsilon$ is a minimizers of $\tilde{E}_\varepsilon$, the analog of E_ε in $\mathbb{D}$, $\tilde{U}_\varepsilon$ is a solution of

$$\begin{cases} -\Delta\tilde{U} = w\dfrac{\tilde{U}}{\varepsilon^2}(\tilde{a}_\varepsilon^2 - |\tilde{U}|^2) & \text{in } \mathbb{D} \\ \partial_\nu \tilde{U} = 0 & \text{on } \mathbb{S}^1 \end{cases}$$

with $w = \operatorname{Jac}\Phi$ is the Jacobian of Φ.

Define $V_\varepsilon : B(0,2) \to [b^2, 1]$ by

$$V_\varepsilon(x) = \begin{cases} \tilde{U}_\varepsilon(x) & \text{if } x \in \mathbb{D} \\ \tilde{U}_\varepsilon(x/|x|^2) & \text{if } x \in B(0,2)\setminus\mathbb{D} \end{cases}.$$

Then $-\Delta V_\varepsilon = -\Delta\tilde{U}_\varepsilon$ in $\mathbb{D}$ and $-\Delta V_\varepsilon(x) = -|x|^{-4}\Delta\tilde{U}_\varepsilon(x/|x|^2)$ in $B(0,2)\setminus\mathbb{D}$. Thus $V_\varepsilon \in H^2(B(0,2),\mathbb{C})$.

First note that if $r \leq \varepsilon$, then (26) is given by (24).

Let $r > \varepsilon$ and $x_0 \in \Omega$ be s.t. $\operatorname{dist}(x_0, \partial\omega_\varepsilon) > r$. Let $\eta := a_\varepsilon(x_0) - V_\varepsilon$ in $B(x_0, r/2)$. From Lemma A.1 in [3] and (25) we get for $x \in B(x_0, r/4)$:

$$\begin{aligned} |\nabla V_\varepsilon(x)|^2 = |\nabla\eta(x)|^2 &\leq C\left(\|\Delta\eta\|_{L^\infty(B(x_0,r/2))} + \frac{4}{r^2}\|\eta\|_{L^\infty(B(x_0,r/2))}\right)\|\eta\|_{L^\infty(B(x_0,r/2))} \\ &\leq \frac{C\mathrm{e}^{-\frac{s_b r}{2\varepsilon}}}{\varepsilon^2}. \end{aligned}$$

In the previous estimate the constants are independent of ε, r and x_0. From (135) we then get (26).

Appendix B. Proof of Theorem 5

Assume that (5) holds and $\lambda, \delta, h_{\rm ex}, K$ satisfy (2), (3) and $\delta^2|\ln\varepsilon| \leq 1$.

Consider a family of configurations $\{(v_\varepsilon, A_\varepsilon)\,|\,0<\varepsilon<1\} \subset \mathscr{H}$ which is in the Coulomb gauge and s.t.

$$\mathcal{F}(v_\varepsilon, A_\varepsilon) \leq \inf_{\mathscr{H}} \mathcal{F} + \mathcal{O}(\ln|\ln\varepsilon|).$$

We drop the subscript ε. From Lemma 12, we may consider $A_v \in H^1(\Omega,\mathbb{R}^2)$ s.t. (v, A_v) is in the Coulomb gauge and (39) holds.

We then have

$$\mathcal{F}(v, A_v) \leq \mathcal{F}(v, A) \leq \inf_{\mathscr{H}} \mathcal{F} + \mathcal{O}(\ln|\ln\varepsilon|) = \mathcal{O}(h_{\rm ex}^2). \tag{136}$$

Proposition 10 gives the existence of $C, \varepsilon_0 > 0$ [independent of ε] s.t., for $\varepsilon < \varepsilon_0$, there exists a family of disjoint disks $\{B_i \,|\, i \in \mathcal{J}\}$ with $B_i = B(a_i, r_i)$ satisfying :

(1) $\{|v| < 1 - |\ln\varepsilon|^{-2}\} \subset \cup B_i$
(2) $\sum r_i < |\ln\varepsilon|^{-10}$,
(3) writing $\rho = |v|$ and $v = \rho e^{\imath\varphi}$ we have

$$\frac{1}{2}\int_{B_i} \rho^2|\nabla\varphi - A|^2 + |\mathrm{curl}(A) - h_{\rm ex}|^2 \geq \pi|d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|), \tag{31}$$

where $d_i = \deg_{\partial B_i}(v)$ if $B_i \subset \Omega$ and 0 otherwise.

From now on, the notation C stands for a positive constant independent of ε whose value may change from one line to another.

B.1. **A substitution lemma.** As in [15], we first state a substitution lemma.

Lemma 70. *There exists $(\tilde v, \tilde A) \in \mathscr{H}$ which is in the Coulomb gauge and s.t., writing, $\rho = |v|$, $v = \rho e^{\imath\varphi}$ and $\tilde\rho = |\tilde v|$, $\tilde v = \tilde\rho e^{\imath\tilde\varphi}$ we have*

(1) $(\tilde v, \tilde A)$ satisfies (39) and $\tilde\rho \leq 1$,
(2) $\tilde\rho = 1$ and $\varphi = \tilde\varphi$ in $\Omega \setminus \cup B_i$,
(3) $\|\rho(\nabla\varphi - A_v) - \tilde\rho(\nabla\tilde\varphi - \tilde A)\|^2_{L^2(\Omega)} = o(1)$,
(4) $\|\mathrm{curl}(A_v) - \mathrm{curl}(\tilde A)\|^2_{L^2(\Omega)} \leq C|\ln\varepsilon|^{-2}$,
(5) $\mathcal{F}(\tilde v, \tilde A) \leq \mathcal{F}(v, A_v) + o(1)$.

Lemma 70 is proved in [15] [Lemma 1] for $\alpha \equiv 1$. The adaptation to our case is presented below.

Proof of Lemma 70. The proof of the lemma follows the same lines than in [15].

We define a continuous function $\chi_\varepsilon = \chi : [0,1] \to [0,1]$ by letting

$$\begin{cases} \chi(x) = x & \text{if } 0 \leq x \leq 1/2 \\ \chi(x) = 1 & \text{if } x \geq 1 - |\ln\varepsilon|^{-2} \\ \chi \text{ is affine} & \text{if } 1/2 \leq x \leq 1 - |\ln\varepsilon|^{-2} \end{cases}.$$

We then let $\tilde v := \dfrac{\chi(\rho)}{\rho} v \in H^1(\Omega, \mathbb{C})$ and we let $\tilde A = A_{\tilde v}$ given by Lemma 12. Letting $\tilde h = \mathrm{curl}(\tilde A)$ we then get

$$-\nabla^\perp \tilde h = \alpha(\imath\tilde v)\cdot(\nabla\tilde v - \imath\tilde A\tilde v). \tag{137}$$

Exactly as in [15] we have

$$\|v \wedge \nabla v - \tilde v \wedge \nabla\tilde v\|^2_{L^2(\Omega)} \leq C|\ln\varepsilon|^{-2}. \tag{138}$$

As in [15], from (7), (39) and (137) we obtain PDE of the second order satisfied by A and $\tilde A$.

By considering the difference of these PDE we get

$$-\Delta(\tilde A - A) + \alpha(\tilde A - A) = \alpha(\tilde v \wedge \nabla\tilde v - v \wedge \nabla v) + \alpha(1-\rho^2)A + \alpha(1-\tilde\rho^2)\tilde A. \tag{139}$$

From (20), (136) and (138), the RHS of (139) is bounded in $L^2(\Omega)$ by $\dfrac{C}{|\ln\varepsilon|}$.

Since $(\tilde A - A)\cdot\nu = 0$ on $\partial\Omega$, by elliptic regularity, we deduce Assertions 3&4 of the lemma.

The end of the proof is exactly as in [15] □

From now on we replace (v, A_v) with $(\tilde{v}, \tilde{A})$ and we claim that the valued disks given by Proposition 10 is valid for (v, A_v) and $(\tilde{v}, \tilde{A})$ and getting the conclusions of Theorem 5 for $(\tilde{v}, \tilde{A})$ implies the same for (v, A).

In order to simplify the presentation we write (v, A) instead of $(\tilde{v}, \tilde{A})$.

B.2. **Energetic Decomposition.** We have the following lower bound:

Proposition 71. *Let $h := \mathrm{curl}(A)$, $h_0 := \Delta\xi_0 = 1 + \xi_0$, $f := h - h_{\mathrm{ex}}h_0$ and let $\{B_i = B(a_i, r_i) \,|\, i \in \mathcal{J}\}$ be the disks given by Proposition 10. We have:*

(140)
$$\mathcal{F}(v, A) \geq h_{\mathrm{ex}}^2\mathbf{J_0} + \sum \mathcal{F}[(v, A), B_i] + 2\pi h_{\mathrm{ex}} \sum d_i\xi_0(a_i) + \frac{1}{2}\int_{\Omega\setminus\cup B_i} |\nabla f|^2 + \frac{1}{2}\int_\Omega f^2 - o(1)$$

where

$$\mathcal{F}[(v, A), B_i] \geq \pi b^2|d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|). \tag{141}$$

This estimate is the starting point of the main argument of [15].

Proof of Proposition 71. Let $\tilde{\Omega} := \Omega \setminus \cup\overline{B_i}$. With (141) we get

$$\mathcal{F}[(v, A), \cup B_i] \geq \pi b^2 \sum_i |d_i|[|\ln\varepsilon| - C\ln|\ln\varepsilon|].$$

On the other hand, letting $f := h - h_{\mathrm{ex}}h_0$ and since $\alpha|\nabla v - \imath Av|^2 \geq |\nabla h|^2$, we get

$$\begin{aligned} &\frac{1}{2}\int_{\tilde{\Omega}} \alpha|\nabla v - \imath Av|^2 + |h - h_{\mathrm{ex}}|^2 \\ \geq\ & h_{\mathrm{ex}}^2\mathbf{J_0} + \frac{1}{2}\|f\|^2_{H^1(\tilde{\Omega})} + h_{\mathrm{ex}}\int_{\tilde{\Omega}} \nabla f\cdot\nabla(h_0 - 1) + f(h_0 - 1) + o(1). \end{aligned}$$

Before refining the above lower bound we make some preliminary claims. We first note that from (137) we have $\|h - h_{\mathrm{ex}}\|^2_{H^1(\Omega)} \leq C\|\nabla v - \imath Av\|^2_{L^2(\Omega)} = \mathcal{O}(h_{\mathrm{ex}}^2)$. Then $\|f\|^2_{H^1(\Omega)} = \mathcal{O}(h_{\mathrm{ex}}^2)$. Consequently for $g \in \{f, h\}$ we have

$$h_{\mathrm{ex}}\int_{\cup B_i\cap\Omega} |\nabla g\cdot\nabla(h_0 - 1)| + |g(h_0 - 1)| \leq C\|g\|_{H^1(\Omega)}h_{\mathrm{ex}}\sum r_i = o(1). \tag{142}$$

We also observe that

$$\int_\Omega -A^\perp\cdot\nabla(h_0 - 1) + h(h_0 - 1) = 0. \tag{143}$$

With (23) we get $\|A\|_{L^\infty(\Omega)} \leq Ch_{\mathrm{ex}}$ and then [with (137)]

$$\begin{aligned} \sum_{B_i\subset\Omega}\left|\int_{\partial B_i}\partial_\tau\varphi(h_0 - h_0(a_i))\right| &= \sum_{B_i\subset\Omega}\left|\int_{\partial B_i}(h_0 - h_0(a_i))(\alpha^{-1}\nabla^\perp h + A)\cdot\tau\right| \\ &\leq \sum_{B_i\subset\Omega}\left[\left|\int_{\partial B_i}\alpha^{-1}(h_0 - h_0(a_i))\partial_\nu h\right| + Ch_{\mathrm{ex}}r_i\right]. \end{aligned}$$

If $B_i \subset \Omega$ we have

$$\begin{aligned}
&\left|\int_{\partial B_i} \alpha^{-1}(h_0 - h_0(a_i))\partial_\nu h\right| \\
= &\left|\int_{B_i} \alpha^{-1}\nabla h_0 \cdot \nabla h + (h_0 - h_0(a_i))\mathrm{div}(\alpha^{-1}\nabla h)\right| \\
\leq &\left|\int_{B_i} (h_0 - h_0(a_i))\mathrm{div}[v \wedge (\nabla^\perp v - \imath A^\perp v)]\right| + \mathcal{O}(h_{\mathrm{ex}} r_i) \\
\leq &\int_{B_i} |h_0 - h_0(a_i)|[2|\partial_1 v \wedge \partial_2 v| + 4|\nabla(|v|)||A| + |v|^2|h|] + \mathcal{O}(h_{\mathrm{ex}} r_i) \\
\leq &\; C r_i h_{\mathrm{ex}}^2.
\end{aligned}$$

And then

$$\sum_{B_i \subset \Omega} \left|\int_{\partial B_i} \partial_\tau \varphi (h_0 - h_0(a_i))\right| \leq C \sum_{B_i \subset \Omega} r_i h_{\mathrm{ex}}^2. \tag{144}$$

If $B_i \not\subset \Omega$, then $\|h_0 - 1\|_{L^\infty(B_i \cap \Omega)} \leq C r_i$ and

$$\begin{aligned}
&\left|\int_{\partial(B_i \cap \Omega)} (h_0 - 1)\partial_\tau \varphi\right| \\
\leq &\int_{B_i \cap \Omega} |\nabla h_0 \cdot \nabla h| + |h_0 - 1|\left[2|\partial_1 v \wedge \partial_2 v| + 4|\nabla(|v|)||A| + |v|^2|h|\right] \\
\leq &\; C r_i h_{\mathrm{ex}}^2. \qquad (145)
\end{aligned}$$

By combining (144) with (145) we deduce:

$$\sum \int_{\partial B_i \cap \Omega} (h_0 - 1)\partial_\tau \varphi = 2\pi \sum d_i (h_0(a_i) - 1) + o(1). \tag{146}$$

We used that if $B_i \not\subset \Omega$ then $d_i = 0$.

We end the preliminary claims by noting that

$$\int_\Omega |\alpha^{-1} - 1||\nabla h \cdot \nabla(h_0 - 1)| \leq C h_{\mathrm{ex}} \|\alpha^{-1} - 1\|_{L^2(\Omega)} = o(h_{\mathrm{ex}}^{-1}). \tag{147}$$

On the one hand, since $-\Delta f + f = -\Delta h + h$, we have with (142), (143), (146), (147) and integrations by parts:

$$\begin{aligned}
\int_{\tilde{\Omega}} \nabla f \cdot \nabla(h_0 - 1) + f(h_0 - 1) &= \int_\Omega \alpha^{-1}\nabla h \cdot \nabla(h_0 - 1) + h(h_0 - 1) + o(h_{\mathrm{ex}}^{-1}) \\
&= o(h_{\mathrm{ex}}^{-1}) + \sum_i \int_{\partial B_i} \partial_\tau \varphi (h_0 - 1) \\
&= o(h_{\mathrm{ex}}^{-1}) + 2\pi \sum_{B_i \subset \Omega} d_i [h_0(a_i) - 1] \\
&= o(h_{\mathrm{ex}}^{-1}) + 2\pi \sum_{B_i \subset \Omega} d_i \xi_0(a_i).
\end{aligned}$$

On the other hand, since $\|f\|_{L^4(\Omega)} \leq C h_{\mathrm{ex}}$, we get $\displaystyle\int_{\cup B_i} f^2 = o(h_{\mathrm{ex}}^{-4})$, and this estimate ends the proof.

□

B.3. **Estimate related with the signs of the d_i's.** By Proposition 71 we have:

$$0 \geq \pi b^2 \sum_i |d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|) + 2\pi h_{\rm ex}\sum_i d_i\xi_0(a_i) + \\ + \frac{1}{2}\int_{\Omega\setminus\cup B_i}|\nabla f|^2 + \frac{1}{2}\int_\Omega f^2 - o(1). \tag{148}$$

Denote $I_+ := \{i\in\mathcal{J}\,|\,d_i>0\}$, $I_- := \{i\in\mathcal{J}\,|\,d_i<0\}$, $D := \sum_{\mathcal{J}}|d_i|$, $D_+ := \sum_{i\in I_+} d_i$ and $D_- := \sum_{i\in I_-}|d_i|$.

With (148) we obtain $2h_{\rm ex}D_+\|\xi_0\|_{L^\infty(\Omega)} \geq b^2 D|\ln\varepsilon|\left(1-\frac{C\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)+o(1)$ and then:

$$D_- \leq D_+ \times \frac{C\ln|\ln\varepsilon|}{|\ln\varepsilon|} + o(1). \tag{149}$$

B.4. **Estimate related with** $\mathrm{dist}(a_i,\Lambda)$**.** From Lemma 1, there exist $\eta>0$ and $M\geq 1$ s.t., for $a\in\Omega$, $\xi_0(a)\geq \min\xi_0 + \eta\mathrm{dist}(a,\Lambda)^M$.

We let $I_0 := \{i\in I\,|\,\mathrm{dist}(a_i,\Lambda) < |\ln\varepsilon|^{-\frac{1}{2M}}\}$ and $D_0 := \sum_{i\in I_0}|d_i|$.

If $i\notin I_0$, then $|\xi_0(a_i)| \leq \|\xi_0\|_{L^\infty(\Omega)} - \dfrac{\eta}{\sqrt{|\ln\varepsilon|}}$. We thus have

$$\begin{aligned}\left|\sum_i d_i\xi_0(a_i)\right| &\leq \left|\sum_{i\in I_0} d_i\xi_0(a_i)\right| + \left|\sum_{i\notin I_0} d_i\xi_0(a_i)\right| \\ &\leq D_0\|\xi_0\|_{L^\infty(\Omega)} + (D-D_0)\left(\|\xi_0\|_{L^\infty(\Omega)} - \frac{\eta}{\sqrt{|\ln\varepsilon|}}\right) \\ &\leq D\|\xi_0\|_{L^\infty(\Omega)} - (D-D_0)\frac{\eta}{\sqrt{|\ln\varepsilon|}}.\end{aligned}$$

From (148) we may deduce

$$2h_{\rm ex}\left(D\|\xi_0\|_{L^\infty(\Omega)} - (D-D_0)\frac{\eta}{\sqrt{|\ln\varepsilon|}}\right) \geq b^2 D(|\ln\varepsilon| - C\ln|\ln\varepsilon|) - o(1)$$

and consequently

$$D - D_0 \leq CD\frac{\ln|\ln\varepsilon|}{\sqrt{|\ln\varepsilon|}} + o(1). \tag{150}$$

B.5. **Estimate of the two last terms in** (148)**.** We let $t \geq |\ln\varepsilon|^{-\frac{1}{2M}} \geq |\ln\varepsilon|^{-1/2}$ and then $t\geq\delta$ since $\delta|\ln\varepsilon|^{1/2}\leq 1$.

On the one hand, from Lemma E.1 in [6], by denoting $\mathscr{C}_t$ a circle with radius t we get:

$$\int_{\mathscr{C}_t\cap\Omega}(1-\alpha^{-1}) = \int_{\mathscr{C}_t\cap\Omega}|1-\alpha^{-1}| \leq C_b\lambda t. \tag{151}$$

We assume now that the center of $\mathscr{C}_t$ is in Λ and t is s.t. $\mathscr{C}_t\subset\tilde{\Omega} = \Omega\setminus\cup\overline{B_i}$. We denote also B_t the disk bounded by $\mathscr{C}_t$. On $\mathscr{C}_t$ we have $|v|=1$ and then $v=\mathrm{e}^{i\varphi}$ with φ locally defined.

By direct calculations, we have [with $f = h - h_{\rm ex}h_0$, ν the outward normal unit vector to $\mathscr{C}_t$ and $\tau=\nu^\perp$]:

$$\int_{\mathscr{C}_t}\alpha^{-1}\partial_\nu h = -\int_{\mathscr{C}_t}[\partial_\tau\varphi - A\cdot\tau] = -2\pi d_t + \int_{B_t} h \text{ with } d_t := \deg_{\mathscr{C}_t}(v).$$

On the other hand $\int_{\mathscr{C}_t} \alpha^{-1}\partial_\nu h_0 = \int_{B_t} h_0 + \int_{\mathscr{C}_t} (\alpha^{-1}-1)\partial_\nu h_0$. Note that

$$\left|\int_{\mathscr{C}_t} (\alpha^{-1}-1)\partial_\nu h_0\right| \leq \|\nabla h_0\|_{L^\infty(\Omega)} \int_{\mathscr{C}_t} |1-\alpha^{-1}| \leq C_b\lambda t \|\nabla h_0\|_{L^\infty(\Omega)}.$$

Then for $\varepsilon > 0$ sufficiently small: $-\int_{\mathscr{C}_t} \alpha^{-1}\partial_\nu f + \int_{B_t} f \geq 2\pi d_t - C\lambda h_{\rm ex} t$. Consequently we obtain

$$2\int_{\mathscr{C}_t} \alpha^{-2} \int_{\mathscr{C}_t} |\partial_\nu f|^2 + 2\pi t^2 \int_{B_t} f^2 \geq 4\pi^2 d_t^2 - Ct\lambda h_{\rm ex}|d_t|$$

and thus, by denoting $m_t := \int_{\mathscr{C}_t} \alpha^{-2}$, we get

$$\frac{1}{2}\int_{\mathscr{C}_t} |\partial_\nu f|^2 + \frac{\pi t^2}{2m_t}\int_{B_t} f^2 \geq \frac{\pi^2 d_t^2}{m_t} - \frac{Ct\lambda h_{\rm ex}|d_t|}{m_t}.$$

Since $2\pi t \leq m_t \leq b^{-4}2\pi t$, for sufficiently $\varepsilon > 0$ small we obtain

$$\frac{1}{2}\int_{\mathscr{C}_t} |\partial_\nu f|^2 + \frac{t}{4}\int_{B_t} f^2 \geq b^4\frac{\pi d_t^2}{2t} - C\lambda h_{\rm ex}|d_t| \geq b^4\frac{\pi d_t^2}{4t}. \tag{152}$$

Following exactly the argument in [15] we get

$$\frac{1}{2}\int_{\Omega\setminus\cup B_i} |\nabla f|^2 + \frac{1}{2}\int_\Omega f^2 \geq C' D^2 \ln|\ln\varepsilon| + o(1).$$

With (148) and $\xi_0(a_i) \leq -\|\xi_0\|_{L^\infty(\Omega)}$ there are $C_1, C_2 > 0$ [independent of ε] s.t.

$$(C_1 D^2 - C_2 D)\ln|\ln\varepsilon| \leq g(\varepsilon) \text{ with } g(\varepsilon)\to 0 \text{ for } \varepsilon \to 0.$$

This estimate implies $D \leq \dfrac{C_1}{C_2}$. Therefore with (149) and (150) we get the three first assertion of the theorem.

It remains to get (32) whose proof follows the same lines as in [15] [Section 4].

Appendix C. Proof of Proposition 11

Let $C_0 > 1$, $(v_\varepsilon)_{0<\varepsilon<1} \subset H^1(\Omega,\mathbb{C})$, $(h_{\rm ex})_{0<\varepsilon<1} \subset (0,\infty)$ and $(\xi_\varepsilon)_{0<\varepsilon<1} \subset H^1_0 \cap H^2 \cap W^{1,\infty}(\Omega,\mathbb{R})$ be s.t. (34) and (35) hold. For simplicity of the presentation we omit the index ε.

Let $\{(B(a_i,r_i),d_i)\,|\,i\in\mathcal{J}\}$ be as in the proposition and write $B_i := B(a_i,r_i)$.

In this proof the letter "C" stands for a quantity bounded by a power of C_0 whose value may differ from one line to another.

We let $A = \nabla^\perp\xi$ and $\tilde\Omega := \begin{cases}\Omega\setminus\cup\overline{B_i} & \text{if } |v|\not> 1/2 \text{ in } \Omega\\ \Omega & \text{if } |v|>1/2 \text{ in } \Omega\end{cases}$. The heart of the proof consists in estimating the quantity $\int_\Omega (v\wedge\nabla v)\cdot A$ in (30).

We first get with the help of (34) and (35) that if $|v|\not> 1/2$ in Ω then $\int_{\cup B_i} v\wedge\nabla v\cdot A = o(1)$.

We also claim that, letting $w := v/|v|$ in $\tilde\Omega$: $\int_{\tilde\Omega}(v\wedge\nabla v - w\wedge\nabla w)\cdot A = o(1)$.

In particular, if $|v|>1/2$ in Ω then we have $\int_\Omega (v\wedge\nabla v)\cdot A = o(1)$. We thus assume that $|v|\not> 1/2$ in Ω.

Then, with an integration by part we get

$$\begin{aligned} & -\int_\Omega v\wedge\nabla v\cdot A \\ = & -\sum_{B_i\subset\Omega}\left\{\xi(a_i)\int_{\partial B_i}(w\wedge\nabla^\perp w)\cdot\nu+\int_{\partial B_i}(\xi-\xi(a_i))(w\wedge\nabla^\perp w)\cdot\nu\right\}+ \\ & +\sum_{B_i\not\subset\Omega}\int_{\partial(B_i\cap\Omega)}\xi(w\wedge\nabla^\perp w)\cdot\nu. \end{aligned} \tag{153}$$

For $B_i\subset\Omega$ we immediately have :

$$\int_{\partial B_i}(w\wedge\nabla^\perp w)\cdot\nu=-2\pi d_i. \tag{154}$$

We now define $u:=\begin{cases}v & \text{in } \tilde\Omega\\ u_i & \text{in } B_i\cap\Omega\end{cases}$ where u_i is the harmonic extension of $\mathrm{tr}_{\partial(B_i\cap\Omega)}(v)$ in $B_i\cap\Omega$. By the Dirichlet principle we have for all i:

$$\|\nabla u\|_{L^2(B_i\cap\Omega)}\leq\|\nabla v\|_{L^2(B_i\cap\Omega)}=\mathcal{O}(|\ln\varepsilon|). \tag{155}$$

It is easy to check that $(w\wedge\nabla^\perp w)\cdot\nu=|u|^{-2}(u\wedge\nabla^\perp u)\cdot\nu$ on $\cup_i\partial B_i$. For $i\in\mathcal{J}$, we let

$$f_i=\begin{cases}\xi-\xi(a_i) & \text{if } B_i\subset\Omega\\ \xi & \text{if } B_i\not\subset\Omega\end{cases}\in H^2\cap W^{1,\infty}(B_i\cap\Omega).$$

From (35) we get

$$\|\nabla f_i\|_{L^\infty(B_i\cap\Omega)}\leq C|\ln\varepsilon|. \tag{156}$$

Our goal is now to estimate $\int_{\partial(B_i\cap\Omega)}f_i(w\wedge\nabla^\perp w)\cdot\nu$. We first consider the case where $i\in\mathcal{J}$ is s.t. $|u|\geq 1/2$ in $B_i\cap\Omega$. In this case we may write in B_i: $u=|u|\mathrm{e}^{\imath\phi}$ with $\phi\in H^1(B_i,\mathbb{R})$, $\|\phi\|_{H^1(B_i)}\leq C|\ln\varepsilon|$. We then have with (156) and an integration by parts

$$\left|\int_{\partial(B_i\cap\Omega)}f_i(w\wedge\nabla^\perp w)\cdot\nu\right|\leq\|\nabla f_i\|_{L^2(B_i\cap\Omega)}\|\nabla\phi\|_{L^2(B_i\cap\Omega)}\leq C|\ln\varepsilon|^2r_i. \tag{157}$$

We now assume $i\in\mathcal{J}$ is s.t. $|u|\not\geq 1/2$ in $B_i\cap\Omega$. By smoothness of $|u_i|^2\in C^\infty(B_i\cap\Omega,\mathbb{R})$, there exists $t_i\in]1/5,1/4[$, a regular value of $|u_i|^2$, s.t. $\omega_i:=\{|u_i|^2<t_i\}\neq\emptyset$. We denote $D_i:=\Omega\cap[B_i\setminus\overline{\omega}_i]$. Since $|u|^2\geq 1/4$ on $\partial B_i\cap\Omega$ we have $\partial D_i=(\partial B_i\cap\Omega)\cup\partial\omega_i\cup(\partial\Omega\cap\overline{D_i})$.

Letting $W:=\dfrac{u}{|u|}\wedge\nabla^\perp\left(\dfrac{u}{|u|}\right)$ we then get

$$\int_{\partial D_i}f_iW\cdot\nu=\int_{D_i}f_i\mathrm{div}(W)+\nabla f_i\cdot W. \tag{158}$$

It is standard to check that $\mathrm{div}\,(W)=0$ in D_i. Moreover:

$$\left|\int_{D_i}\nabla f_i\cdot W\right|\leq 2\|\nabla\xi\|_{L^2(B_i\cap\Omega)}\|\nabla u\|_{L^2(B_i\cap\Omega)}\leq C|\ln\varepsilon|^2r_i.$$

Consequently using (158) we may deduce

$$\left|\int_{\partial D_i}f_i\,W\cdot\nu\right|\leq C|\ln\varepsilon|^2r_i. \tag{159}$$

On the other hand, from (156), $\xi = 0$ on $\partial\Omega$ and $\operatorname{div}\left(u \wedge \nabla^\perp u\right) = -2\partial_1 u \wedge \partial_2 u$ in $B_i \cap \Omega$, we get

$$\begin{aligned} & \left| \int_{\partial D_i} f_i \, W \cdot \nu - \int_{\partial B_i \cap \Omega} f_i (w \wedge \nabla^\perp w) \cdot \nu \right| \\ = & \left| \int_{\partial \omega_i} f_i \, W \cdot \nu \right| \\ = & \frac{1}{t_i} \left| \int_{\omega_i} -2 f_i \partial_1 u \wedge \partial_2 u + \nabla f_i \cdot \left(u \wedge \nabla^\perp u\right) \right| \leq C |\ln \varepsilon|^3 r_i. \end{aligned} \tag{160}$$

We may conclude by using (153), (154), (157), (159) and (160):

$$-\int_\Omega v \wedge \nabla v \cdot A = 2\pi \sum_{B_i \subset \Omega} d_i \xi(a_i) + o(1).$$

The rest of the proof is exactly the same than in [17].

Appendix D. Proof of some results of Section 6.1.1

D.1. **Proof of Proposition 30.** We use the same notation than in Proposition 30. In this proof, the letter C is a quantity which depends only on Ω, N and $\sum_i |d_i|$, its value may change from one line to another.

We argue as in [13]. We let $\Phi_\star^{(\mathbf{z},\mathbf{d})} \in \cap_{0<p<2} W^{1,p}(\Omega, \mathbb{R}) \cap H^1_{\mathrm{loc}}(\Omega \setminus \{z_1, ..., z_N\}, \mathbb{R})$ be the unique solution of

$$\begin{cases} \Delta \Phi_\star^{(\mathbf{z},\mathbf{d})} = 2\pi \sum_{i=1}^N d_i \delta_{z_i} & \text{in } \Omega \\ \Phi_\star^{(\mathbf{z},\mathbf{d})} = 0 & \text{on } \partial\Omega \end{cases}.$$

and let $\Phi_{\tilde{r}} \in H^1(\Omega_{\tilde{r}}, \mathbb{R})$ be the unique solution of

$$\begin{cases} \Delta \Phi_{\tilde{r}} = 0 & \text{in } \Omega_{\tilde{r}} \\ \Phi_{\tilde{r}} = 0 & \text{on } \partial\Omega \\ \Phi_{\tilde{r}} = \mathrm{Cte}_i & \text{on } \partial B(z_i, \tilde{r}) \\ \displaystyle\int_{\partial B(z_i,\tilde{r})} \partial_\nu \Phi_{\tilde{r}} = 2\pi d_i & \text{for all } i \in \{1, ..., N\} \end{cases}. \tag{161}$$

We then have $\nabla^\perp \Phi_\star^{(\mathbf{z},\mathbf{d})} = w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})}$ and $\nabla^\perp \Phi_{\tilde{r}}^{(\mathbf{z},\mathbf{d})} = w_{\tilde{r}}^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_{\tilde{r}}^{(\mathbf{z},\mathbf{d})}$. It is important to note that if $w \in H^1(\Omega_{\tilde{r}}, \mathbb{S}^1)$, then $|\nabla w| = |w \wedge \nabla w|$.

We may decompose $\Phi_\star^{(\mathbf{z},\mathbf{d})}$ as $\Phi_\star^{(\mathbf{z},\mathbf{d})} = \sum_i d_i \Phi_{z_i}$ where, for $z \in \Omega$, Φ_z is the unique solution of

$$\begin{cases} \Delta \Phi_z = 2\pi \delta_z & \text{in } \Omega \\ \Phi_z = 0 & \text{on } \partial\Omega \end{cases}.$$

With a standard pointwise bound for the gradient of an harmonic function [see (2.31) in [10]] we have $\|\nabla \Phi_{z_i}\|_{L^\infty(\Omega \setminus \overline{B(z_i, \tilde{r})})} \leq C \dfrac{\|\Phi_{z_i}\|_{L^\infty(\Omega \setminus \overline{B(z_i, \tilde{r}/4)})}}{\tilde{r}}$. Thus

$$\|\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde{r}})} \leq C \frac{\sum_i |d_i| \|\Phi_{z_i}\|_{L^\infty(\Omega_{\tilde{r}/4})}}{\tilde{r}}. \tag{162}$$

Moreover, it is easy to check that $\Phi_{z_i} = \ln|x - z_i| + R_{z_i}$ where R_{z_i} is the harmonic extension of $-\ln|x - z_i|_{|\partial\Omega}$. From (162) and by the maximum principle we get for $\tilde{r} < \min\left\{[\operatorname{diam}(\Omega)]^{-1}; 1/4\right\}$

$$|\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}| \leq \frac{C(1 + |\ln \tilde{r}|)}{\tilde{r}} \text{ in } \Omega_{\tilde{r}} \tag{163}$$

which proves (56).

If there is $\eta > 0$ s.t. $\hbar > \eta$, then $\|R_{z_i}\|_{C^1(\Omega)} \leq C_\eta$ where C_η which depends only on η and Ω. We thus get $\|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde{r}})} \leq \frac{\tilde{C}_\eta}{\tilde{r}}$ [where $\tilde{C}_\eta$ depends only on η, N, $\sum|d_i|$ and Ω] and this estimates implies (60).

We now define $R_{(\mathbf{z},\mathbf{d})} := \sum_i d_i R_{z_i}$ in order to have $\Phi_\star^{(\mathbf{z},\mathbf{d})} = \sum_i d_i \ln|x-z_i| + R_{(\mathbf{z},\mathbf{d})}$.

From Lemma I.4 in [4] we have

$$\|\Phi_{\tilde{r}} - \Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde{r}})} \leq \sum_i \left[\sup_{\partial B(z_i,\tilde{r})} \sum_j \ln|x-z_j| - \inf_{\partial B(z_i,\tilde{r})} \sum_j \ln|x-z_j|\right] + \\ + \sum_i \left[\sup_{\partial B(z_i,\tilde{r})} R_{(\mathbf{z},\mathbf{d})} - \inf_{\partial B(z_i,\tilde{r})} R_{(\mathbf{z},\mathbf{d})}\right]. \tag{164}$$

If $N = 1$, then the first term of the RHS in (164) is 0. Otherwise, as in [17] [Proposition 5.1], we have

$$\sum_i \left[\sup_{\partial B(z_i,\tilde{r})} \sum_j \ln|x-z_j| - \inf_{\partial B(z_i,\tilde{r})} \sum_j \ln|x-z_j|\right] \leq \frac{C\tilde{r}}{\min_{i\neq j}|z_i-z_j|}. \tag{165}$$

And for $i \in \{1,...,N\}$, by harmonicity of $R_{(\mathbf{z},\mathbf{d})}$, for $0 < \rho < \frac{\hbar}{2}$ we get

$$\|\nabla R_{(\mathbf{z},\mathbf{d})}\|_{L^\infty(B(z_i,\rho))} \leq \frac{C\|R_{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega)}}{\mathrm{dist}(z_i,\partial\Omega)-\rho} \leq C\frac{1+|\ln(\hbar)|}{\hbar}. \tag{166}$$

Then

$$\sum_i \left[\sup_{\partial B(z_i,\tilde{r})} R_{(\mathbf{z},\mathbf{d})} - \inf_{\partial B(z_i,\tilde{r})} R_{(\mathbf{z},\mathbf{d})}\right] \leq C\frac{\tilde{r}(1+|\ln(\hbar)|)}{\hbar}. \tag{167}$$

We let

$$Y := \begin{cases} \dfrac{\tilde{r}(1+|\ln(\hbar)|)}{\hbar} & \text{if } N = 1 \\ \dfrac{\tilde{r}}{\min_{i\neq j}|z_i-z_j|} + \dfrac{\tilde{r}(1+|\ln(\hbar)|)}{\hbar} & \text{if } N \geq 2 \end{cases}. \tag{168}$$

By combining (164), (165) and (167) we get

$$\|\Phi_{\tilde{r}} - \Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde{r}})} \leq CY. \tag{169}$$

From (163) and (169) we immediately get

$$\begin{aligned} 0 &\leq \int_{\Omega_{\tilde{r}}} |\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2 - |\nabla\Phi_{\tilde{r}}|^2 + |\nabla(\Phi_\star^{(\mathbf{z},\mathbf{d})} - \Phi_{\tilde{r}})|^2 \\ &\leq C\,Y\tilde{r}\max_i \|\partial_\nu\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\partial B(z_i,\tilde{r}))}. \end{aligned} \tag{170}$$

On the other hand, for $i \in \{1,...,N\}$, we have [with (166)]

$$\|\partial_\nu\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(B(z_i,\tilde{r}))} \leq C\left(\frac{1}{\tilde{r}} + \frac{1+|\ln(\hbar)|}{\hbar}\right). \tag{171}$$

Using X defined in (57), from (170) and (171), we get

$$0 \leq \int_{\Omega_{\tilde{r}}} |\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2 - |\nabla\Phi_{\tilde{r}}|^2 + |\nabla(\Phi_\star^{(\mathbf{z},\mathbf{d})} - \Phi_{\tilde{r}})|^2 \leq CX. \tag{172}$$

From (172) we deduce (59) and since $\int_{\partial\Omega}(\varphi_\star - \varphi_{\tilde r}) = 0$, with a Poincaré inequality we obtain (58).

D.2. **Proof of Proposition 31.** Let $(\mathbf{z},\mathbf{d}) = (\mathbf{z},\mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$ and denote $\hbar := \min_i \operatorname{dist}(z_i, \partial\Omega) > 0$. Assume that $d_1, ..., d_N$ are independent of n. Let $\tilde r = \tilde r_n \to 0$ be s.t (50) holds.

In this proof the letter C stands for a quantity which depends only on Ω, N, C_1 and $\sum_i |d_i|$, its value may change from one line to another.

By Remark 29 and an integration by parts we have

$$\frac12\int_{\Omega_{\tilde r}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 = \frac12\int_{\Omega_{\tilde r}} |\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}|^2 = -\frac12\sum_i \int_{\partial B(z_i,\tilde r)} \Phi_\star^{(\mathbf{z},\mathbf{d})}\partial_\nu \Phi_\star^{(\mathbf{z},\mathbf{d})}. \tag{173}$$

For $i_0 \in \{1,...,N\}$, we fix $x_{i_0} \in \partial B(z_{i_0},\tilde r)$. Then [with $\nabla^\perp \Phi_\star^{(\mathbf{z},\mathbf{d})} = w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})}$]

$$\begin{aligned}&\int_{\partial B(z_{i_0},\tilde r)} \Phi_\star^{(\mathbf{z},\mathbf{d})}\partial_\nu \Phi_\star^{(\mathbf{z},\mathbf{d})}\\ &= \int_{\partial B(z_{i_0},\tilde r)} \left[\Phi_\star^{(\mathbf{z},\mathbf{d})} - \Phi_\star^{(\mathbf{z},\mathbf{d})}(x_{i_0})\right]\partial_\nu \Phi_\star^{(\mathbf{z},\mathbf{d})} + 2\pi d_{i_0}\Phi_\star^{(\mathbf{z},\mathbf{d})}(x_{i_0}).\end{aligned} \tag{174}$$

On the one hand, arguing as in the proof of (169), we get for $z \in \partial B(z_{i_0},\tilde r)$:

$$|\Phi_\star^{(\mathbf{z},\mathbf{d})}(z) - \Phi_\star^{(\mathbf{z},\mathbf{d})}(x_{i_0})| \le \sup_{\partial B(z_{i_0},\tilde r)} \Phi_\star^{(\mathbf{z},\mathbf{d})} - \inf_{\partial B(z_{i_0},\tilde r)} \Phi_\star^{(\mathbf{z},\mathbf{d})} \le CY.$$

Then, using (171), we obtain

$$\sum_i \left|\int_{\partial B(z_i,\tilde r)} \left[\Phi_\star^{(\mathbf{z},\mathbf{d})} - \Phi_\star^{(\mathbf{z},\mathbf{d})}(x_i)\right]\partial_\nu \Phi_\star^{(\mathbf{z},\mathbf{d})}\right| \le CX. \tag{175}$$

On the other hand, for $i_0 \in \{1,...,N\}$

$$\Phi_\star^{(\mathbf{z},\mathbf{d})}(x_{i_0}) - R_{(\mathbf{z},\mathbf{d})}(z_{i_0}) = -d_{i_0}|\ln\tilde r| + \sum_{j\neq i_0} d_j \ln|x_{i_0} - z_j| + \left[R_{(\mathbf{z},\mathbf{d})}(x_{i_0}) - R_{(\mathbf{z},\mathbf{d})}(z_{i_0})\right],$$

and with (166) we get $\left|R_{(\mathbf{z},\mathbf{d})}(x_{i_0}) - R_{(\mathbf{z},\mathbf{d})}(z_{i_0})\right| \le \dfrac{C(1+|\ln\hbar|)\tilde r}{\hbar}$. We then immediately get:

$$\Phi_\star^{(\mathbf{z},\mathbf{d})}(x_{i_0}) = R_{(\mathbf{z},\mathbf{d})}(z_{i_0}) - d_{i_0}|\ln\tilde r| + \sum_{j\neq i_0} d_j \ln|z_{i_0} - z_j| + \mathcal{O}(X). \tag{176}$$

With (174), (175) and (176) we may prove that (173) may be rewritten into

$$\frac12\int_{\Omega_{\tilde r}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 = \pi\sum_i \left[d_i^2|\ln\tilde r| - d_i R_{(\mathbf{z},\mathbf{d})}(z_i)\right] - \pi\sum_{j\neq i} d_j d_j \ln|z_i - z_j| + \mathcal{O}(X)$$

where "$\mathcal{O}(X)$" is quantity bounded by CX with C depending only on N, Ω and $\sum|d_i|$.

D.3. **Proof of Proposition 33.** Let $(\mathbf{z},\mathbf{d}) = (\mathbf{z},\mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$, $\tilde r \downarrow 0$ and $\eta > 0$ be as in the proposition.

In this proof the letter C stands for a quantity which depends only on Ω, N and $\sum_i |d_i|$, its value may change from one line to another.

We first claim that, for $i \neq j$, $B(z_i,\eta) \cap B(z_j,\eta) \neq \emptyset$, $B(z_i,\eta) \subset \Omega$ and $\eta = \chi\tilde r$ with $\chi \to \infty$. In particular we assume n sufficiently large to have $\eta > \tilde r$.

Since $\nabla^\perp \Phi_\star^{(\mathbf{z},\mathbf{d})} = w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})}$, for $i_0 \in \{1,...,N\}$ and $z \in \Omega \setminus \{z_1,...,z_N\}$, we have

$$w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})}(z) = d_{i_0} \nabla^\perp (\ln|z - z_{i_0}|) + \nabla^\perp \left[R_{(\mathbf{z},\mathbf{d})}(z) + \sum_{j \neq i_0} d_j \ln|z - z_j| \right].$$

For $j \in \{1,...,N\}$, let θ_j be the main determination of the argument of $\dfrac{z - z_j}{|z - z_j|}$ and let $\mathcal{R}$ be an harmonic conjugate of $R_{(\mathbf{z},\mathbf{d})}$. In $\Omega \setminus \{z_1,...,z_N\}$ we have

$$w_\star^{(\mathbf{z},\mathbf{d})} \wedge \nabla w_\star^{(\mathbf{z},\mathbf{d})} - d_{i_0} \nabla \theta_{i_0} = \nabla \left[\sum_{j \neq i_0} d_j \theta_j + \mathcal{R} \right].$$

Then for $z \in B(z_{i_0},\eta) \setminus \{z_{i_0}\}$ we have $w_\star^{(\mathbf{z},\mathbf{d})}(z) = \left(\dfrac{z - z_{i_0}}{|z - z_{i_0}|} \right)^{d_{i_0}} \mathrm{e}^{\imath \varphi_{i_0}(z)}$ with $\varphi_{i_0} = \sum_{j \neq i_0} d_j \tilde{\theta}_j + \mathcal{R} + \mathrm{Cte}_{i_0}$ where, for $j \neq i_0$, $\tilde{\theta}_j$ is a determination of the argument of $\dfrac{z - z_i}{|z - z_i|}$ which is globally defined in $B(z_{i_0},\eta)$. Note that $\varphi_{i_0} \in H^1(B(z_{i_0},\eta),\mathbb{R})$.

On the other hand, by direct calculations, we have $\left\| \sum_{j \neq i_0} d_j \nabla \tilde{\theta}_j \right\|_{L^\infty(B(z_{i_0},\eta))} \leq \dfrac{C}{\eta}$ and, since $R_{(\mathbf{z},\mathbf{d})}$ is harmonic, we also have from the definition of $\mathcal{R}$

$$\|\nabla \mathcal{R}\|_{L^\infty(B(z_{i_0},\eta))} = \|\nabla R_{(\mathbf{z},\mathbf{d})}\|_{L^\infty(B(z_{i_0},\eta))} \leq C \frac{\|R_{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega)}}{\mathrm{dist}(B(z_{i_0},\eta),\partial\Omega)} \leq C \frac{|\ln(\hbar)| + 1}{\hbar}.$$

We thus deduce

$$\|\nabla \varphi_{i_0}\|_{L^\infty(B(z_{i_0},\eta))} \leq C \left(\frac{1 + |\ln(\hbar)|}{\hbar} + \frac{1}{\eta} \right). \tag{177}$$

We switch to polar coordinates by letting for $i \in \{1,...,N\}$ and $\rho \in]\tilde{r},\eta[$, $\tilde{\varphi}_i(\rho,\theta) := \varphi_i(z_i + \rho \mathrm{e}^{\imath\theta})$. We then get, by (177) and a mean value argument, the existence of $\rho_n \in]\sqrt{\chi}\tilde{r},\eta[$ s.t.

$$\sum_i \int_0^{2\pi} |\partial_\theta \tilde{\varphi}_i(\rho_n,\theta)|^2 \,\mathrm{d}\theta \leq \frac{C}{\ln \chi} \left[\frac{\eta(|\ln(\hbar)| + 1)}{\hbar} + 1 \right]^2.$$

We let $Z := \dfrac{1}{\ln \chi} \left[\dfrac{\eta(|\ln(\hbar)| + 1)}{\hbar} + 1 \right]^2$ and by assumption we have $Z \to 0$.

We denote, for $i \in \{1,...,N\}$, $m_i = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \tilde{\varphi}_i(\rho_n,\theta) \,\mathrm{d}\theta$ in order to have

$$\int_0^{2\pi} |\tilde{\varphi}_i(\rho_n,\theta) - m_i|^2 \,\mathrm{d}\theta \leq CZ.$$

We then define $\phi_i \in H^1(B(z_i,\rho_n) \setminus \overline{B(z_i,\tilde{r})},\mathbb{R})$ using polar coordinates:

$$\tilde{\phi}_i(s,\theta) = \frac{s - \rho_n}{\tilde{r} - \rho_n} m_i + \frac{s - \tilde{r}}{\rho_n - \tilde{r}} \tilde{\varphi}(\rho_n,\theta) \text{ with } s \in (\tilde{r},\rho_n).$$

For $z_i + s\mathrm{e}^{\imath\theta} \in B(z_i,\rho_n) \setminus \overline{B(z_i,\tilde{r})}$, we let $\phi_i(z_i + s\mathrm{e}^{\imath\theta}) := \tilde{\phi}_i(s,\theta)$. By standard calculations we get $\displaystyle\int_{B(z_i,\rho_n)\setminus\overline{B(z_i,\tilde{r})}} |\nabla \phi_i|^2 \leq CZ$.

We conclude by defining $v = \begin{cases} w_\star^{(\mathbf{z},\mathbf{d})} & \text{in } \Omega \setminus \cup \overline{B(z_i,\rho_n)} \\ u_i \mathrm{e}^{\imath\phi_i} & \text{in } B(z_i,\rho_n) \setminus \overline{B(z_i,\tilde{r})} \end{cases}$ with $u_i(z) = \left(\frac{z-z_i}{|z-z_i|}\right)^{d_i}$. It is clear that $v \in H^1(\Omega_{\tilde{r}}, \mathbb{S}^1)$ and that for $i \in \{1,...,N\}$ we have $v(z_i + \tilde{r}\mathrm{e}^{\imath\theta}) = \mathrm{Cte}_i u_i$ [with $\mathrm{Cte}_i = \mathrm{e}^{\imath m_i}$]. Note that since $\deg_{\partial B(z_i,\tilde{r})}(w_\star^{(\mathbf{z},\mathbf{d})}) = d_i$ we have

$$\frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla u_i|^2 \leq \frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2$$

and

$$\frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla(u_i \mathrm{e}^{\imath\phi_i})|^2 = \frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla u_i|^2 + \frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla \phi_i|^2.$$

Consequently using (177) and $\rho_n < \eta$ we obtain

$$\sum_i \frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla v|^2 \leq \sum_i \frac{1}{2}\int_{B(z_i,\rho_n)\setminus B(z_i,\tilde{r})} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 + CZ.$$

Thus $\frac{1}{2}\int_{\Omega_{\tilde{r}}} |\nabla v|^2 \leq \frac{1}{2}\int_{\Omega_{\tilde{r}}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2 + CZ$. The last estimate and (59) end the proof.

Appendix E. Proof of Proposition 39

Proof. **Step 1. Selection of "good" points**

Let $d \in \mathbb{N}^*$ and consider $\mathbf{D} \in \Lambda_d$ which minimizes (72).

For $k \in \{1,...,N_0\}$, if $D_k \geq 1$ we let $(\tilde{z}_1^{(k)},...,\tilde{z}_{D_k}^{(k)}) \in [B(p_k, h_{\mathrm{ex}}^{-1/4})^{D_k}]^*$ which minimizes the infimum in the left hand side of (64) with $R = h_{\mathrm{ex}}^{-1/4}$, $p = p_k$ and $D = D_k$.

We then have the existence of C [depending only on Ω and d] s.t. $|p_k - \tilde{z}_i^{(k)}| \leq Ch_{\mathrm{ex}}^{-1/2}$ and if $D_k \geq 2$ then $|\tilde{z}_i^{(k)} - \tilde{z}_j^{(k)}| \geq h_{\mathrm{ex}}^{-1/2}/C$ for $i \neq j$.

We may choose [in an arbitrary way] $z_i^{(k)} \in B(\tilde{z}_i^{(k)}, \delta) \cap [\delta(\mathbb{Z}\times\mathbb{Z})]$. Since $\delta\sqrt{h_{\mathrm{ex}}} \to 0$, we still have [up to change the value C] $|p_k - z_i^{(k)}| \leq Ch_{\mathrm{ex}}^{-1/2}$ and if $D_k \geq 2$ then $|z_i^{(k)} - \tilde{z}_j^{(k)}| \geq h_{\mathrm{ex}}^{-1/2}/C$ for $i \neq j$.

For $i \in \{1,...,D_k\}$ we let $x_i^{(k)} := z_i^{(k)} + \lambda\delta x_0$ where $x_0 \in \omega$ is an arbitrary point of minimum of W^{micro} [defined in (70)].

Step 2. Construction of the test function

We construct test functions in subdomains of Ω and then we glue the test functions.

- We let $w_{h_{\mathrm{ex}}}^{\mathrm{macro}} \in H^1(\Omega_{h_{\mathrm{ex}}^{-1}}(\mathbf{z}), \mathbb{S}^1)$ be a minimizer of $I_{h_{\mathrm{ex}}^{-1}}^{\mathrm{Dir}}(\mathbf{z},\mathbf{d})$ [defined in (52)] with $\mathbf{d} = (1,...,1) \in \mathbb{Z}^d$ and $\mathbf{z} \in (\Omega^d)^*$ is a d-tuple s.t. $\{z_1,...,z_d\} = \{z_i^{(k)} \,|\, k \in \{1,...,N_0\}$ s.t. $D_k \geq 1$ and $i \in \{1,...,D_k\}\}$.
- For $k \in \{1,...,N_0\}$ s.t. $D_k \geq 1$ and $i \in \{1,...,D_k\}$, we let $w_{k,i}^{\mathrm{micro}} \in H^1[B(z_i^{(k)}, h_{\mathrm{ex}}^{-1}) \setminus \overline{B(x_i^{(k)}, \lambda\delta^2)}, \mathbb{S}^1]$ be a minimizer of the right hand side of (67) with $z_\varepsilon = z_i^{(k)}$, $x_\varepsilon = x_i^{(k)}$, $R = h_{\mathrm{ex}}^{-1}$ and $r = \lambda\delta^2$ [from (73) we have $R/r \to \infty$].

  We let also $u_{k,i} \in H^1[B(x_i^{(k)}, \lambda\delta^2), \mathbb{C}]$ be a minimizer of

$$u \mapsto \frac{1}{2}\int_{B(x_i^{(k)},\lambda\delta^2)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2$$

  with the Dirichlet boundary condition $u(x_i^{(k)} + \lambda\delta^2 \mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\theta}$.

By considering well chosen constants $\mathrm{Cte}_{k,i}^{(1)}$, $\mathrm{Cte}_{k,i}^{(2)}$ and Cte_k, we may glue the above test functions and we define $v \in H^1(\Omega, \mathbb{C})$:

$$v = \begin{cases} w_{h_{\mathrm{ex}}}^{\mathrm{macro}} & \text{in } \Omega_{h_{\mathrm{ex}}^{-1}}(\mathbf{z}) \\ \mathrm{Cte}_k & \text{in } B(z_i^{(k)}, h_{\mathrm{ex}}^{-1}) \text{ if } D_k = 0 \\ \mathrm{Cte}_{k,i}^{(1)} w_{k,i}^{\mathrm{micro}} & \text{in } B(z_i^{(k)}, h_{\mathrm{ex}}^{-1}) \setminus \overline{B(x_i^{(k)}, \lambda\delta^2)} \quad \Big| \quad k \in \{1, ..., N_0\} \text{ s.t. } D_k \geq 1 \text{ and } i \in \{1, ..., D_k\} \\ \mathrm{Cte}_{k,i}^{(2)} u_{k,i} & \text{in } B(x_i^{(k)}, \lambda\delta^2) \quad \Big| \quad k \in \{1, ..., N_0\} \text{ s.t. } D_k \geq 1 \text{ and } i \in \{1, ..., D_k\} \end{cases}.$$

Step 3. The energy of the test function

We first note that the configuration $(\mathbf{z}, \mathbf{d})$ is s.t. $\hbar(\mathbf{z}) > \frac{1}{2}\mathrm{dist}(\Lambda, \partial\Omega)$ and for $i \neq j$ we have $\dfrac{h_{\mathrm{ex}}^{-1}}{|z_i - z_j|} \to 0$, then we may apply Propositions 30, 31 and 33. We may also use Proposition 35. From these propositions we get

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_{h_{\mathrm{ex}}^{-1}}(\mathbf{z})} |\nabla v|^2 \\ &= \pi d \ln h_{\mathrm{ex}} + W_{N_0}^{\mathrm{macro}}(\mathbf{p}, \mathbf{D}) - \pi \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 2}}^{N_0} \sum_{i \neq j} \ln |z_i^{(k)} - z_j^{(k)}| + o(1). \end{aligned} \tag{178}$$

For $k \in \{1, ..., N_0\}$ s.t. $D_k \geq 1$ and $i \in \{1, ..., D_k\}$ with (67), (68) and (69) we get:

$$\begin{aligned} &\frac{1}{2}\int_{B(z_i^{(k)}, h_{\mathrm{ex}}^{-1}) \setminus \overline{B(x_i^{(k)}, \lambda\delta^2)}} \alpha |\nabla v|^2 \\ &= \pi |\ln(\lambda\delta h_{\mathrm{ex}})| + b^2 \pi |\ln(\delta)| + W^{\mathrm{micro}}(x_0) + o(1). \end{aligned} \tag{179}$$

From Lemma IX.1 in [4] and (25) [with $|\nabla v| \leq C\varepsilon^{-1}$], for $k \in \{1, ..., N_0\}$ s.t. $D_k \geq 1$ we have

$$\frac{1}{2}\int_{B(x_i^{(k)}, \lambda\delta^2)} \alpha |\nabla v|^2 + \frac{\alpha^2}{2\varepsilon^2}(1 - |v|^2)^2 = b^2 \pi \ln(b\lambda\delta^2/\varepsilon) + b^2\gamma + o(1) \tag{180}$$

where $\gamma \in \mathbb{R}$ is a universal constant.

In conclusion, by combining (178), (179) and (180) [note $\lambda\delta h_{\mathrm{ex}} \to 0$]:

$$\begin{aligned} F(v) \leq\ & d\pi \left[b^2 |\ln \varepsilon| + (1 - b^2)|\ln(\lambda\delta)|\right] + d\left[W^{\mathrm{micro}}(x_0) + b^2\gamma + b^2 \pi \ln b\right] + \\ &+ W_{N_0}^{\mathrm{macro}}(\mathbf{p}, \mathbf{D}) - \pi \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 2}}^{N_0} \sum_{i \neq j} \ln |z_i^{(k)} - z_j^{(k)}| + o(1). \end{aligned} \tag{181}$$

Step 4. Definition of the magnetic potential and conclusion

Let $A_{(\mathbf{z},\mathbf{1})}$ be given by Definition 19 with $(\mathbf{a}, \mathbf{d}) = (\mathbf{z}, \mathbf{1})$. It is clear that we have

$$-\pi \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 2}}^{N_0} \sum_{i \neq j} \ln |z_i^{(k)} - z_j^{(k)}| \leq C |\ln \delta|$$

where C depends only on d and Ω.

Consequently, for $\varepsilon > 0$ sufficiently small and $C_0 > \pi d$ we have $F(v) \leq C_0 |\ln \varepsilon|$. Therefore, with Remark 20, the configuration $(v, A_{(\mathbf{z},\mathbf{1})}) \in \mathscr{H}$ is s.t. $\mathcal{F}(v, A_{(\mathbf{z},\mathbf{1})}) \leq \mathcal{F}(v, 0) + o(1) \leq C_0 |\ln \varepsilon|^2 + \mathcal{H}^2(\Omega) h_{\mathrm{ex}}^2$.

Using Proposition11 and Lemma 18 we get

$$\mathcal{F}(v, A_{(\mathbf{z},\mathbf{1})}) \quad = \quad h_{\rm ex}^2 \mathbf{J_0} + 2\pi h_{\rm ex} \sum_{i=1}^{d} \xi_0(z_i) + F(v) + \tilde{V}[\zeta_{(\mathbf{z},\mathbf{1})}] + o(1)$$

where $\zeta_{(\mathbf{z},\mathbf{1})}$ is the unique solution of (42) with $(\mathbf{a},\mathbf{d}) = (\mathbf{z},\mathbf{1})$.

We now use Assertion 3 of Proposition 23 in order to get $\tilde{V}[\zeta_{(\mathbf{z},\mathbf{1})}] = \tilde{V}[\zeta_{(\mathbf{p},\mathbf{D})}] + o(1)$ and then

$$\mathcal{F}(v, A_{(\mathbf{z},\mathbf{1})}) = h_{\rm ex}^2 \mathbf{J_0} + 2\pi h_{\rm ex} \sum_{i=1}^{d} \xi_0(z_i) + F(v) + \tilde{V}_{(\mathbf{z},\mathbf{1})}[\zeta_{(\mathbf{p},\mathbf{D})}] + o(1). \tag{182}$$

We claim that, from the choice of the points $z_i^{(k)}, \tilde{z}_i^{(k)}$ we have $\xi_0(z_i^{(k)}) - \xi_0(\tilde{z}_i^{(k)}) = \mathcal{O}(\delta/\sqrt{h_{\rm ex}})$. Thus with Proposition 36 we have

$$\begin{aligned}
&-\pi \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 2}}^{N_0} \sum_{i \neq j} \ln |z_i^{(k)} - z_j^{(k)}| + 2\pi h_{\rm ex} \sum_{k=1}^{N_0} \sum_i \xi_0(z_i^{(k)}) - 2\pi d h_{\rm ex} \min_\Omega \xi_0 \\
= &\sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 1}}^{N_0} \left[-\pi \sum_{\substack{i,j \in \{1,\dots,D_k\} \\ i \neq j}} \ln |\tilde{z}_i^{(k)} - \tilde{z}_j^{(k)}| + 2\pi h_{\rm ex} \sum_{i=1}^{D_k} \left[\xi_0(\tilde{z}_i^{(k)}) - \min_\Omega \xi_0 \right] \right] + o(1) \\
= &\sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 1}}^{N_0} \left[\frac{\pi}{2}(D_k^2 - D_k) \ln\left(\frac{h_{\rm ex}}{D_k}\right) + C_{p_k, D_k} \right] + o(1).
\end{aligned}$$

We may now conclude:

$$\begin{aligned}
\mathcal{F}(v, B) \quad = \quad & h_{\rm ex}^2 \mathbf{J_0} + dM_\Omega \left[-h_{\rm ex} + H_{c_1}^0 \right] + \frac{\pi}{2} \ln h_{\rm ex} \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 1}}^{N_0} (D_k^2 - D_k) + \\
& + \overline{\mathcal{W}}_d + \frac{\pi}{2} \sum_{\substack{k=1 \\ \text{s.t. } D_k \geq 1}}^{N_0} (D_k - D_k^2) \ln D_k + o(1).
\end{aligned}$$

This estimate ends the proof of the proposition. □

Appendix F. Proof of Proposition 40

Let $h_{\rm ex}$ and $(v_\varepsilon, A_\varepsilon)$ be as in Proposition 40. Note that we may assume that $A_\varepsilon = A_{v_\varepsilon}$ given by Lemma 12 and then $\|A_\varepsilon\|_{L^\infty(\Omega)} = \mathcal{O}(h_{\rm ex})$. We drop the subscript ε. We first note that, by smoothness of Ω, there is $t_0 > 0$, s.t. letting $\Omega_{t_0} := \{x \in \mathbb{R}^2 \,|\, \mathrm{dist}(x, \Omega) < t_0\}$, we may extend by reflexion $v \in H^1(\Omega, \mathbb{C})$ into $u \in H^1(\Omega_{t_0}, \mathbb{C})$ letting $u = v$ in Ω and $u = v \circ \mathcal{S}_\Omega$ in $\Omega_{t_0} \setminus \overline{\Omega}$ where

$$\begin{array}{rccc}
\mathcal{S}_\Omega : & \Omega_{t_0} \setminus \overline{\Omega} & \to & \Omega \\
& x & \mapsto & \Pi(x) - \mathrm{dist}(x, \partial\Omega) \nu_{\Pi(x)}
\end{array}.$$

Here $\Pi : \Omega_{t_0} \setminus \overline{\Omega} \to \partial\Omega$ is the orthogonal projection on $\partial\Omega$ and, for $\sigma \in \partial\Omega$, ν_σ is the normal outward at σ.

Lemma 72. *Let $C_0 \geq 1$ and let $\{(v_\varepsilon, A_\varepsilon) \,|\, 0 < \varepsilon < 1\}$ be a family in the Coulomb gauge of quasi-minimizers of $\mathcal{F}$ in $\mathscr{H}$ for an intensity of the applied field $h_{\rm ex} = h_{\rm ex}(\varepsilon) \geq 0$ s.t. $\|\nabla |v|\|_{L^\infty(\Omega)} \leq C_0 \varepsilon^{-1}$.*

Under these hypotheses, for $\eta \in (0,1)$ there exists $\varepsilon_\eta, C_\eta > 0$ [depending on C_0] s.t. for $0 < \varepsilon < \varepsilon_\eta$, if $z \in \Omega$ is s.t.

$$b^2 \int_{B(z,\sqrt{\varepsilon}/2)} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq \frac{C_\eta}{3} |\ln \varepsilon|$$

with $u = \begin{cases} v & \text{in } \Omega \\ v \circ \mathcal{S}_\Omega & \text{in } \Omega_{t_0} \setminus \overline{\Omega} \end{cases}$, *then* $|v(z)| > \eta$.

In order to prove Proposition 40 we need the following lemma.

Lemma 73. *There exists $\varepsilon_\Omega > 0$ depending only on Ω s.t. for $0 < \varepsilon < \varepsilon_\Omega$, $z \in \Omega$ and $v \in H^1(\Omega, \mathbb{C})$, by defining u as in Lemma 72, the following inequality holds:*

$$\int_{B(z,\sqrt{\varepsilon}/2)} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq 3 \int_{B(z,\sqrt{\varepsilon})\cap\Omega} |\nabla v|^2 + \frac{b^2}{\varepsilon^2}(1-|v|^2)^2.$$

Proof of Lemma 73. In order to prove the lemma it suffices to check that by smoothness of Ω we have $\|\nabla(\mathcal{S}_\Omega^{-1})\|_{L^\infty(\Omega)}, \|\mathrm{jac}\,(\mathcal{S}_\Omega^{-1})\|_{L^\infty(\Omega)} = 1 + o(1)$. We then immediately obtain

$$\int_{B(z,\sqrt{\varepsilon}/2)\setminus\Omega} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq [1+o(1)] \int_{\mathcal{S}_\Omega[B(z,\sqrt{\varepsilon}/2)\setminus\Omega]} |\nabla v|^2 + \frac{b^2}{\varepsilon^2}(1-|v|^2)^2.$$

On the other hand, if $x \in B(z,\sqrt{\varepsilon}/2) \setminus \Omega$ then $|\mathcal{S}_\Omega(x) - z| \leq [1+o(1)]\sqrt{\varepsilon}/2 \leq \sqrt{\varepsilon}$ for sufficiently small $\varepsilon > 0$ [depending only on Ω]. Then $\mathcal{S}_\Omega[B(z,\sqrt{\varepsilon}/2) \setminus \Omega] \subset B(z,\sqrt{\varepsilon}) \cap \Omega$. The lemma follows from the monotonicity of the integral. □

By combining both lemmas we get Proposition 40.

Proof of Lemma 72. We argue by contradiction and we assume the existence of $\eta \in (0,1)$, $\varepsilon = \varepsilon_n \downarrow 0$ s.t. for all $n \geq 1$ there are $(v, A) = (v_n, A_n) \in \mathscr{H}$, $z = z_n \in \Omega$ and $h_{\mathrm{ex}} = h_{\mathrm{ex}}^{(n)} \geq 0$ s.t. (v, A) is a quasi-minimizers of $\mathcal{F}$ in $\mathscr{H}$ satisfying:

$$\int_{B(z,\sqrt{\varepsilon}/2)} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq \frac{|\ln\varepsilon|}{n} \tag{183}$$

with $u = u_n = \begin{cases} v & \text{in } \Omega \\ v \circ \mathcal{S}_\Omega & \text{in } \Omega_{t_0} \setminus \overline{\Omega} \end{cases}$ and $|v(z)| \leq \eta$. Up to replace v by $\underline{v}$ we may assume $|v| \leq 1$ in Ω.

We are going to prove that (183) implies

$$\frac{1}{\varepsilon^2} \int_{B(z,\varepsilon^{3/4})\cap\Omega} (1-|v|^2)^2 = o(1). \tag{184}$$

On the other hand, $\|\nabla|v|\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1})$ and then, from an argument in [4] [Theorem III.3], we will get, for sufficiently large n, $|v(z)| > \eta$. Clearly this contradiction will end the proof.

Since for $n \geq 1$ we have $\displaystyle\int_{\varepsilon^{3/4}/2}^{\sqrt{\varepsilon}/2} \frac{\mathrm{d}\rho}{\rho}\, \rho \int_{\partial B(z,\rho)} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq \frac{|\ln\varepsilon|}{n}$, there exists $\rho_n \in (\varepsilon^{3/4}, \sqrt{\varepsilon}/2)$ s.t. $\displaystyle\rho_n \int_{\partial B(z,\rho_n)} |\nabla u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq \frac{4}{n}$. Then we get :

$$\rho_n \int_{\partial B(z,\rho_n)} |\partial_\tau u|^2 + \frac{b^2}{\varepsilon^2}(1-|u|^2)^2 \leq \frac{4}{n}. \tag{185}$$

We switch in polar coordinate and we denote $\tilde{u}(\theta) := u(z+\rho_n \mathrm{e}^{\imath\theta})$. Estimate (185) becomes

$$\int_0^{2\pi} |\partial_\theta \tilde{u}|^2 + \frac{b^2\rho_n^2}{\varepsilon^2}(1-|\tilde{u}|^2)^2 \leq \frac{4}{n}. \tag{186}$$

On the one hand, $|\partial_\theta |\tilde{u}||^2 \leq |\partial_\theta \tilde{u}|^2$ and then $\displaystyle\int_0^{2\pi} |\partial_\theta |\tilde{u}|| \leq \frac{2\sqrt{2\pi}}{\sqrt{n}}$. Consequently in $[0,2\pi]$ we get $(1-|\tilde{u}|^2)^2 \geq \max_{[0,2\pi]}(1-|\tilde{u}|^2)^2 - \dfrac{2\sqrt{2\pi}}{\sqrt{n}}$. From (186) we deduce

$$\frac{4\varepsilon^2}{nb^2\rho_n^2} \geq \int_0^{2\pi} (1-|\tilde{u}|^2)^2 \geq 2\pi \left[\max_{[0,2\pi]}(1-|\tilde{u}|^2)^2 - \frac{2\sqrt{2\pi}}{\sqrt{n}}\right]$$

and thus for sufficiently large n we get $0 \leq \max_{[0,2\pi]}(1-|\tilde{u}|^2)^2 \leq \dfrac{100}{\sqrt{n}}$.

For a further use we define

$$\begin{array}{rccc} \chi_n : & B(z,\rho_n) & \to & [0,1] \\ & z+\rho \mathrm{e}^{\imath\theta} & \mapsto & (|\tilde{u}(\theta)|-1)\dfrac{\rho}{\rho_n}+1 \end{array}.$$

By direct calculations we have

$$\int_{B(z,\rho_n)} |\nabla \chi_n|^2 + \frac{1}{2\varepsilon^2}(1-\chi_n^2)^2 = \mathcal{O}\left(\frac{1}{n}\right). \tag{187}$$

On the other hand, for n sufficiently large, $|u|^2 \geq \dfrac{1}{2}$ in $\partial B(z,\rho_n)$. We thus may compute the degree of u on $\partial B(z,\rho_n)$ and we find $\left|\deg_{\partial B(z,\rho_n)}(u)\right| = \mathcal{O}\left(\dfrac{1}{n}\right)$ which implies, for sufficiently large n, $\deg_{\partial B(z,\rho_n)}(u) = 0$. Consequently, we may write $u = |u|\mathrm{e}^{\imath\varphi}$ with $\varphi = \varphi_n \in H^1(\partial B(z,\rho_n),\mathbb{R})$. Moreover, up to multiply u by a constant in $\mathbb{S}^1$, we may assume $\int_{\partial B(z,\rho_n)} \varphi = 0$.

We then consider $\tilde{\varphi} : [0,2\pi] \to \mathbb{R}$ defined by $\tilde{\varphi}(\theta) = \varphi(z+\rho_n \mathrm{e}^{\imath\theta})$, and thus

$$\mathcal{O}\left(\frac{1}{n}\right) = \rho_n \int_{\partial B(z,\rho_n)} |\nabla\varphi|^2 \geq \int_0^{2\pi} |\partial_\theta \tilde{\varphi}|^2.$$

Since $\displaystyle\int_0^{2\pi} \tilde{\varphi} = 0$, this estimate implies $\displaystyle\int_0^{2\pi} \tilde{\varphi}^2 = \mathcal{O}\left(\frac{1}{n}\right)$.

Letting $\psi = \psi_n : B(z,\rho_n) \to \mathbb{R}$, $z+\rho\mathrm{e}^{\imath\theta} \mapsto \dfrac{\rho}{\rho_n}\tilde{\varphi}(\theta)$, it is direct to check $\displaystyle\int_{B(z,\rho_n)} |\nabla\psi|^2 = \mathcal{O}\left(\frac{1}{n}\right)$.

We are now in position to end the proof by considering $V = V_n = \chi_n \mathrm{e}^{\imath\psi} \in H^1(B(z,\rho_n),\mathbb{C})$ in order to have $V = v$ on $\partial B(z,\rho_n) \cap \Omega$,

$$\frac{1}{2}\int_{\Omega\cap B(z,\rho_n)} |\nabla V|^2 + \frac{1}{2\varepsilon^2}(1-|V|^2)^2 = \mathcal{O}\left(\frac{1}{n}\right).$$

and [with $\|A\|_{L^\infty(\Omega)} = \mathcal{O}(h_{\mathrm{ex}})$]

$$\left|\int_{\Omega\cap B(z,\rho_n)} \alpha(V\wedge\nabla V)\cdot A\right| \leq C\frac{h_{\mathrm{ex}}\rho_n}{\sqrt{n}} = o(1).$$

Since $V = v$ on $\partial B(z,\rho_n)\cap\Omega$ we have $w := \begin{cases} v & \text{in } \Omega\setminus B(z,\rho_n) \\ V & \text{in } B(z,\rho_n)\cap\Omega \end{cases} \in H^1(\Omega,\mathbb{C})$. Considering the comparison configuration $(\tilde{w},A)$, from the quasi-minimality of (v,A) and the above estimates we get

$$\int_{\Omega\cap B(z,\rho_n)} |\nabla v|^2+\frac{1}{2\varepsilon^2}(1-|v|^2)^2 \leq b^{-4}\int_{\Omega\cap B(z,\rho_n)} |\nabla V|^2+\frac{1}{2\varepsilon^2}(1-|V|^2)^2+o(1)=o(1).$$

Since $\rho_n > \varepsilon^{3/4}$ we get (184) and thus this estimate ends the proof. □

Appendix G. Proof of Proposition 42

The proof of the proposition is an adaptation of the arguments presented in [2] [Section V] and also used in [17] [Proposition 3.2]. It is also inspired of the bad disk construction in [4]. Let μ, λ, δ, (v,A) and $h_{\rm ex}$ be as in the proposition.

Step 1. A first covering of $\{|v|\leq 1/2\}$

For $0<\varepsilon<\varepsilon_{1/2}$ [$\varepsilon_{1/2}>0$ is given by Proposition 40 with $\eta=1/2$] we consider a covering of Ω by disks $\{B(x_1^\varepsilon,4\sqrt{\varepsilon}),...,B(x_{N_\varepsilon}^\varepsilon,4\sqrt{\varepsilon})\}$ s.t., for $i\neq j$, $B(x_i^\varepsilon,\sqrt{\varepsilon})\cap B(x_j^\varepsilon,\sqrt{\varepsilon})=\emptyset$ and $x_i^\varepsilon\in\Omega$.

For the simplicity of the presentation we omit the dependance in ε.

We say that $B(x_i,4\sqrt{\varepsilon})$ is a *bad disk* if $\tilde{E}_\varepsilon[v,B(x_i,8\sqrt{\varepsilon})\cap\Omega]>C_{1/2}|\ln\varepsilon|$ where for a disk B we denoted

$$\tilde{E}_\varepsilon(v,B\cap\Omega):=\int_{B\cap\Omega}|\nabla v|^2+\frac{1}{\varepsilon^2}(1-|v|^2)^2$$

and $C_{1/2}>0$ is given by Proposition 40 with $\eta=1/2$. Let

$$J'=J'_\varepsilon:=\{i\in\{1,...,N_\varepsilon\}\,|\,B(x_i,4\sqrt{\varepsilon}) \text{ is a bad disk}\}.$$

We make two fundamental claims:

(1) There exists $M_0\geq 1$ [independent of ε] s.t. $\mathrm{Card}(J')\leq M_0$.
(2) If $B(x_i,4\sqrt{\varepsilon})$ is not a bad disk then $|v|\geq 1/2$ in $B(x_i,4\sqrt{\varepsilon})$.

The first claim is a direct consequence of (32) and $B(x_i^\varepsilon,\sqrt{\varepsilon})\cap B(x_j^\varepsilon,\sqrt{\varepsilon})=\emptyset$ for $i\neq j$.

The second claim is given by Proposition 40. Then $\cup_{i\in J'}B(x_i,4\sqrt{\varepsilon})$ is covering of $\{|v|\leq 1/2\}$ and $\mathrm{Card}(J')\leq M_0$.

Up to drop some disks, we may always assume that for $i\in J'$ we have $B(x_i,4\sqrt{\varepsilon})\cap\{|v|\leq 1/2\}\neq\emptyset$. Consequently using Corollary 41, for $i\in J'$ and $0<\varepsilon<\min\{\varepsilon_0,\varepsilon_{1/2}\}$ [ε_0 given by Corollary 41] we have $\mathrm{dist}(x_i,\Lambda)=\mathcal{O}(|\ln\varepsilon|^{-s_0})$.

If $|v|>1/2$ in Ω then there is nothing to prove. We then assume $J'\neq\emptyset$.

Step 2. Separation process

We replace the above bad disks with disks having same centers and with a radius ε^μ. Let $\varepsilon_\mu^{(1)}>0$ be s.t. $\min\{\varepsilon_0,\varepsilon_{1/2}\}\geq\varepsilon_\mu^{(1)}$, for $0<\varepsilon<\varepsilon_\mu^{(1)}$ we have $4\sqrt{\varepsilon}<\varepsilon^\mu$ and

$$\max_{i\in J'}\mathrm{dist}(B(x_i,\varepsilon^\mu),\Lambda)\leq\frac{1}{\ln|\ln\varepsilon|}.$$

In particular $\cup_{i\in J'}B(x_i,\varepsilon^\mu)$ is a covering of $\{|v|\leq 1/2\}$.

The goal of this step is to get a covering of $\{|v| \leq 1/2\}$ with disks $B(x_i, \varepsilon^s)$ where $i \in \tilde{J} = \tilde{J}_\varepsilon \subset J'$, $s = s_\varepsilon = 2^{-K}\mu$, $K = K_\varepsilon \in \{0, ..., M_0 - 1\}$ and s.t. for $i, j \in \tilde{J}$, $i \neq j$, we have

$$|x_i - x_j| \geq \varepsilon^{s/2}. \tag{188}$$

If $\mathrm{Card}(J') = 1$ or (188) is satisfied with $s = \mu$ [i.e. $K = 0$] then we let $\tilde{J} = J'$ and we obtained the desired result of this step. Otherwise, there are $i_0, j_0 \in J'$ [with $i_0 < j_0$] s.t. $|x_{i_0} - x_{j_0}| < \varepsilon^{\mu/2}$. In this case we let $J^{(1)} := J' \setminus \{i_0\}$ and we claim that $\mathrm{Card}(J^{(1)}) = \mathrm{Card}(J') - 1$.

If $\mathrm{Card}(J^{(1)}) = 1$ or $\mathrm{Card}(J^{(1)}) > 1$ with (188) holds with $s = 2^{-1}\mu$ [i.e. $K = 1$] for all $i, j \in J^{(1)}$ [$i \neq j$] then the goal of this step is done with $\tilde{J} = J^{(1)}$ and $s = 2^{-1}\mu$.

Otherwise, there exits $i_0, j_0 \in J^{(1)}$ [with $i_0 < j_0$] s.t. $|x_{i_0} - x_{j_0}| < \varepsilon^{s/2}$. We then let $J^{(2)} := J^{(1)} \setminus \{i_0\}$ and thus $\mathrm{Card}(J^{(2)}) = \mathrm{Card}(J^{(1)}) - 1$.

By noting that $\mathrm{Card}(J') \leq M_0$, the above process stops after at most $M_0 - 1$ iteration. We thus get the existence of $K = K_\varepsilon \in \{0, ..., M_0 - 1\}$ and $\emptyset \neq J^{(K)} = J_\varepsilon^{(K)} \subset J'$ s.t. $\mathrm{Card}(J^{(K)}) = 1$ or (188) is satisfied with $s = s_\varepsilon = 2^{-K}\mu$ and $i, j \in J^{(K)}$ [$i \neq j$].

We then denote $\tilde{J} := J^{(K)}$, $s = 2^{-K}\mu$ and we fix $0 < \varepsilon_\mu^{(2)} \leq \varepsilon_\mu^{(1)}$ s.t. for $0 < \varepsilon < \varepsilon_\mu^{(2)}$ we have

$$\max_{i \in \tilde{J}} \mathrm{dist}(B(x_i, \varepsilon^{s/4}), \Lambda) \leq \frac{1}{\ln|\ln\varepsilon|} < 10^{-1}\mathrm{dist}(\Lambda, \partial\Omega).$$

In particular $B(x_i, \varepsilon^{s/4}) \subset \Omega$ for $i \in \tilde{J}$.

Step 3. Definition of r

With Corollary 5.2 in [5], for a.e. $t \in \mathrm{Image}(|v|)$ the set $V(t) := \{x \in \Omega \,|\, |v(x)| = t\}$ is a finite union of curve. Moreover if a such curve is included in Ω then it is a Jordan curve.

Following the same strategy as in [2] [Lemma V.1], we have the existence of $t_\varepsilon \in [1 - 2|\ln\varepsilon|^{-2}, 1 - |\ln\varepsilon|^{-2}]$ s.t. $V(t_\varepsilon)$ is a finite union of Jordan curves s.t.

$$\mathcal{H}^1[V(t_\varepsilon)] \leq C\varepsilon|\ln\varepsilon|^5 \text{ with } C \text{ is independent of } \varepsilon. \tag{189}$$

We fix $0 < \varepsilon_\mu^{(3)} \leq \varepsilon_\mu^{(2)}$ s.t. for $0 < \varepsilon < \varepsilon_\mu^{(3)}$ we have $C\varepsilon|\ln\varepsilon|^5 \leq 10^{-2}\varepsilon^s$.

We denote for $i \in \tilde{J}$

$$\mathcal{A}_i = \mathcal{A}_i^\varepsilon := \{\rho \in [\varepsilon^s, \varepsilon^{2s/3}] \,|\, |v| \geq t_\varepsilon \text{ on } \partial B(x_i, \rho)\}. \tag{190}$$

From the continuity of $|v|$, it is clear that $[\varepsilon^s, \varepsilon^{2s/3}] = \mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i$ where

$$\mathcal{B}_i = \mathcal{B}_i^\varepsilon := \{\rho \in [\varepsilon^s, \varepsilon^{2s/3}] \,|\, \exists\, x \in \partial B(x_i, \rho) \text{ s.t. } |v(x)| = t_\varepsilon\}$$

and

$$\mathcal{C}_i = \mathcal{C}_i^\varepsilon := \{\rho \in [\varepsilon^s, \varepsilon^{2s/3}] \,|\, |v| < t_\varepsilon \text{ on } \partial B(x_i, \rho)\}.$$

We first claim that, since the function $\rho \mapsto \rho$ is increasing, we have

$$\begin{aligned}
\mathcal{O}(\varepsilon^2|\ln\varepsilon|) &= \int_{\mathcal{C}_i} \mathrm{d}\rho \int_{\partial B(x_i,\rho)} (1 - |v|^2)^2 \\
&\geq 2\pi(1 - t_\varepsilon^2)^2 \int_{\mathcal{C}_i} \rho \mathrm{d}\rho \\
&\geq 2\pi(1 - t_\varepsilon^2)^2 \int_0^{\mathcal{H}^1(\mathcal{C}_i)} \rho \mathrm{d}\rho = \pi(1 - t_\varepsilon^2)^2 \mathcal{H}^1(\mathcal{C}_i)^2.
\end{aligned}$$

Then $\mathcal{H}^1(\mathcal{C}_i) = \mathcal{O}(\varepsilon|\ln\varepsilon|^{5/2})$.

On the other hand one may prove that if I is a connected components of $\mathcal{B}_i$, then there is ρ_1, ρ_2 s.t. $I = [\rho_1, \rho_2]$. Since straight lines are geodesics, we obviously get

$$\mathcal{H}^1(I) = \rho_2 - \rho_1 \leq \mathcal{H}^1[V(t_\varepsilon) \cap \overline{B(x_i, \rho_2)} \setminus B(x_i, \rho_1)].$$

Moreover one may prove that if $[\rho_1, \rho_2]$ and $[\rho_1', \rho_2']$ are distinct connected component of $\mathcal{B}_i$ and if Γ is a connected component of $V(t_\varepsilon)$ s.t. $\Gamma \cap \overline{B(x_i, \rho_2)} \setminus B(x_i, \rho_1) \neq \emptyset$ then $\Gamma \cap \overline{B(x_i, \rho_2')} \setminus B(x_i, \rho_1') = \emptyset$ [here we used (189)]. One may conclude: $\mathcal{H}^1(\mathcal{B}_i) \leq \mathcal{H}^1(V(t_\varepsilon)) \leq C\varepsilon |\ln \varepsilon|^5$.

Consequently

$$\mathcal{H}^1(\mathcal{A}_i) \geq \mathcal{H}^1([\varepsilon^s, \varepsilon^{2s/3}]) - \mathcal{H}^1(\mathcal{B}_i) - \mathcal{H}^1(\mathcal{C}_i) \geq \varepsilon^{2s/3} - \varepsilon^s - \mathcal{H}^1(V(t_\varepsilon)) - \mathcal{O}(\varepsilon |\ln \varepsilon|^{5/2}).$$

Fix $0 < \varepsilon_\mu^{(4)} \leq \varepsilon_\mu^{(3)}$ s.t. for $0 < \varepsilon < \varepsilon_\mu^{(4)}$ we have $\mathcal{H}^1(\mathcal{A}_i) \geq \varepsilon^{2s/3} - \varepsilon^s - \sqrt{\varepsilon}$.

Define

$$\mathcal{A} = \mathcal{A}_{\mu,\varepsilon} := \cap_{i \in \tilde{J}} \mathcal{A}_i. \tag{191}$$

It is clear that $\mathcal{H}^1(\mathcal{A}) \geq \varepsilon^{2s/3} - \varepsilon^s - M_0\sqrt{\varepsilon}$

Since $\rho \mapsto 1/\rho$ is decreasing we have

$$\begin{aligned} \mathcal{O}(|\ln \varepsilon|) &\geq \int_{\mathcal{A}} \frac{\mathrm{d}\rho}{\rho} \sum_{i \in \tilde{J}} \rho \int_{\partial B(x_i, \rho)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 \\ &\geq \int_{\varepsilon^{2s/3} - \mathcal{H}^1(\mathcal{A})}^{\varepsilon^{2s/3}} \frac{\mathrm{d}\rho}{\rho} \times \inf_{\rho \in \mathcal{A}} \sum_{i \in \tilde{J}} \rho \int_{\partial B(x_i, \rho)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2. \end{aligned}$$

Consequently, there exist $r = r_{\mu,\varepsilon} \in \mathcal{A}$, $C_\mu \geq 1$ [C_μ is independent of ε] and $0 < \varepsilon_\mu^{(5)} \leq \varepsilon_\mu^{(4)}$ s.t. for $0 < \varepsilon < \varepsilon_\mu^{(5)}$ we have

$$\sum_{i \in \tilde{J}} r \int_{\partial B(x_i, r)} |\nabla v|^2 + \frac{1}{\varepsilon^2}(1 - |v|^2)^2 \leq C_\mu. \tag{192}$$

We finally let $J_\mu := \tilde{J}$, with (188) and (192) the result is proved.

Appendix H. Proof of Proposition 49

The proof is an adaptation of the proof of (VI.21) in [2].

Let $\tilde{\alpha} = \tilde{\alpha}_n \in L^\infty(\Omega, [\beta^2; 1])$, $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}, \mathbf{d})^{(n)} \in (\Omega^N)^* \times \mathbb{Z}^N$ and $u = u_n \in H^1(\Omega, \mathbb{C})$ be as in the proposition.

We first claim that up to consider $\underline{u}$ instead of u we may assume $|u| \leq 1$ in Ω. Note also that if $\int_{\Omega_{\tilde{r}}} |\nabla u|^2 \geq \beta^{-2} \int_{\Omega_{\tilde{r}}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2$, then there is nothing to prove. We thus may assume

$$\int_{\Omega_{\tilde{r}}} |\nabla u|^2 < \beta^{-2} \int_{\Omega_{\tilde{r}}} |\nabla w_\star^{(\mathbf{z},\mathbf{d})}|^2.$$

Let $w := u/|u| \in H^1(\Omega_{\tilde{r}}, \mathbb{S}^1)$. From Lemma I.1 in [4] we have $w \wedge \nabla w = \nabla^\perp \Phi_\star^{(\mathbf{z},\mathbf{d})} + \nabla H$ with $H = H_\varepsilon \in H^1(\Omega_{\tilde{r}}, \mathbb{R})$ and

$$\int_{\Omega_{\tilde{r}}} |\nabla H|^2 \leq (\beta^{-1} + 1)^2 \int_{\Omega_{\tilde{r}}} |\nabla \Phi_\star^{(\mathbf{z},\mathbf{d})}|^2. \tag{193}$$

Let $\Phi_{\tilde r}$ be the unique solution of (161). We have $\displaystyle\int_{\Omega_{\tilde r}} \nabla H\cdot\nabla^\perp\Phi_{\tilde r}=0$. Then letting $\rho=|u|$:

$$\int_{\Omega_{\tilde r}}\tilde\alpha\rho^2\nabla H\cdot\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}=\int_{\Omega_{\tilde r}}(\tilde\alpha\rho^2-1)\nabla H\cdot\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}+\int_{\Omega_{\tilde r}}\nabla H\cdot(\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}-\nabla^\perp\Phi_{\tilde r}).$$

But, from (172), there exists $C\geq 1$ s.t. $\left|\displaystyle\int_{\Omega_{\tilde r}}\nabla H\cdot(\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}-\nabla^\perp\Phi_{\tilde r})\right|\leq C\|\nabla H\|_{L^2(\Omega_{\tilde r})}\sqrt{X}$ where X is defined in (57).

Consequently, letting $\tilde C:=4C^2/\beta^2$ we get

$$\begin{aligned}2\int_{\Omega_{\tilde r}}\nabla H\cdot\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}+\int_{\Omega_{\tilde r}}\tilde\alpha\rho^2|\nabla H|^2 &= 2\int_{\Omega_{\tilde r}}\nabla H\cdot(\nabla^\perp\Phi_\star^{(\mathbf{z},\mathbf{d})}-\nabla^\perp\Phi_{\tilde r})+\int_{\Omega_{\tilde r}}\tilde\alpha\rho^2|\nabla H|^2\\ &\geq \|\nabla H\|_{L^2(\Omega_{\tilde r})}\left(\frac{\beta^2}{4}\|\nabla H\|_{L^2(\Omega_{\tilde r})}-2C\sqrt{X}\right)\\ &\geq -\tilde C X.\end{aligned}$$

Therefore

$$\int_{\Omega_{\tilde r}}\tilde\alpha\rho^2|\nabla w|^2\geq\int_{\Omega_{\tilde r}}|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2-\int_{\Omega_{\tilde r}}(1-\tilde\alpha\rho^2)|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2-2\int_{\Omega_{\tilde r}}(1-\tilde\alpha\rho^2)|\nabla H||\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|-\mathcal{O}(X).$$

On the other hand, using (56) and Corollary 32, we get

$$\begin{aligned}\left|\int_{\Omega_{\tilde r}}(1-\tilde\alpha\rho^2)|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2\right| &\leq \left|\int_{\Omega_{\tilde r}}(1-\rho^2)|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2\right|+\left|\int_{\Omega_{\tilde r}}(1-\tilde\alpha)|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|^2\right|\\ &\leq \|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde r})}\|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^2(\Omega_{\tilde r})}\,(K+L)\end{aligned}$$

and with (193):

$$\begin{aligned}\left|\int_{\Omega_{\tilde r}}(1-\tilde\alpha\rho^2)|\nabla H||\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|\right| &\leq \left|\int_{\Omega_{\tilde r}}(1-\rho^2)|\nabla H||\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|\right|+\left|\int_{\Omega_{\tilde r}}(1-\tilde\alpha)|\nabla H||\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}|\right|\\ &\leq \|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^\infty(\Omega_{\tilde r})}\|\nabla\Phi_\star^{(\mathbf{z},\mathbf{d})}\|_{L^2(\Omega_{\tilde r})}\,(K+L)\,(2\beta^{-1}+1).\end{aligned}$$

The proposition is thus proved.

Appendix I. Proof of Proposition 55

We prove the first assertion and we assume $\mathrm{Card}(J_\mu)\geq 2$. We let $\chi_1:=2h_{\rm ex}^{-1}\ln h_{\rm ex}$, $\chi_2:=2h_{\rm ex}^{-1/2}\ln h_{\rm ex}$ and $\Omega_{\chi_2}=\Omega\setminus\cup_{p\in\Lambda}\overline{B(p,\chi_2)}$.

In order to get sufficiently sharp estimates to prove the proposition, we decompose Ω_r in several subdomains. To this aim, we distinguish two cases for $p\in\Lambda$: either $\mathrm{Card}(J_p^{(y)})\geq 2$ or $\mathrm{Card}(J_p^{(y)})\in\{0,1\}$ where $J_p^{(y)}:=\{k\in J^{(y)}\,|\,y_k\in B(p,\chi_2)\}$ [the y_k's are introduced in Definition 51].

If $p\in\Lambda$ is s.t. $\mathrm{Card}(J_p^{(y)})\geq 2$, then with Lemma 48 [with $P=17$ and $\eta=\chi_1/2$], there are $\kappa_p=\kappa_{p,\varepsilon}\in\{17^0,...,17^{N_0-1}\}$ and $\tilde J_p^{(y)}\subset J_p^{(y)}$ s.t.

$$\bigcup_{k\in J_p^{(y)}}B(y_k,\chi_1/2)\subset\bigcup_{k\in\tilde J_p^{(y)}}B(y_k,\kappa_p\chi_1/2)\text{ and }|y_k-y_l|\geq 8\kappa_p\chi_1\text{ for }k,l\in\tilde J_p^{(y)},\ k\neq l.$$

We then let $\mathcal{D}_p:=B(p,\chi_2)\setminus\cup_{k\in\tilde J_p^{(y)}}\overline{B(y_k,\kappa_p\chi_1)}$ and, for $k\in\tilde J_p^{(y)}$, we write $\underline{d}_k:=\deg_{\partial B(y_k,\kappa_p\chi_1)}(v)$. We denote also $D_p:=\sum_{k\in\tilde J_p^{(y)}}\underline{d}_k$

If $p \in \Lambda$ is s.t. $J_p^{(y)} = \{k\}$, then we let $\mathcal{D}_p = B(p, \chi_2) \setminus \overline{B(y_k, \kappa\delta)}$ with κ given by Definition 51. We let also $D_p := \underline{d}_k := \deg_{\partial B(y_k, \kappa\delta)}(v)$.

Recall that we denoted (see Definition 51), for $k \in J^{(y)}$, $\tilde{d}_k := \deg_{\partial B(y_k, \kappa\delta)}(v)$. Consequently, if $J_p^{(y)} = \{k\}$, then $D_p = \underline{d}_k = \tilde{d}_k$.

If $J_p^{(y)} = \emptyset$ then we denote $D_p = 0$ and $\mathcal{D}_p = B(p, \chi_2)$.

The heart of the proof consists in proving that $\underline{d}_k = 1$ for all k. Indeed, we know that if $i \in J_\mu$ then $\deg_{\partial B(z_i, r)}(v) = 1$. Consequently $\underline{d}_k$ is the number of points z_i contained in a disk of radius at least χ_1.

We let:

- $\mathcal{R} := \bigcup_{k \in J^{(y)}} B(y_k, \kappa\delta) \setminus \bigcup_{i \in J_\mu} \overline{B(z_i, r)}$, κ given in Definition 51.
- For $p \in \Lambda$ s.t. $\mathrm{Card}(J_p^{(y)}) \geq 2$ and for $k \in \tilde{J}_p^{(y)}$ we denote

$$\mathcal{Q}_{k,p} := B(y_k, \kappa_p \chi_1) \setminus \bigcup_{\substack{l \in J^{(y)} \\ y_l \in B(y_k, \kappa_p \chi_1)}} \overline{B(y_l, \kappa\delta)}.$$

Moreover, by construction, we have [for sufficiently small ε]

$$\bigcup_{\substack{l \in J^{(y)} \\ y_l \in B(y_k, \kappa_p \chi_1)}} B(y_l, \kappa\delta) \subset \bigcup_{\substack{l \in J^{(y)} \\ y_l \in B(y_k, \kappa_p \chi_1)}} B(y_l, \chi_1/2) \subset B(y_k, \kappa_p \chi_1/2). \tag{194}$$

Thus

$$\begin{aligned} \frac{1}{2}\int_{\Omega_r} \alpha |\nabla v|^2 \quad &\geq \quad \frac{1}{2}\int_{\mathcal{R}} \alpha |\nabla v|^2 + \sum_{p \in \Lambda} \frac{1}{2}\int_{\mathcal{D}_p} \alpha |\nabla v|^2 + \\ &+ \sum_{\substack{p \in \Lambda \\ \mathrm{Card}(J_p^{(y)}) \geq 2}} \sum_{k \in \tilde{J}_p^{(y)}} \frac{1}{2}\int_{\mathcal{Q}_{k,p}} \alpha |\nabla v|^2 + \frac{1}{2}\int_{\Omega_{\chi_2}} \alpha |\nabla v|^2. \end{aligned} \tag{195}$$

From (101) and (102) we have

$$\frac{1}{2}\int_{\mathcal{R}} \alpha |\nabla v|^2 \geq d\pi \left[b^2 |\ln r| + (1 - b^2) |\ln \lambda| - b^2 |\ln \delta| \right] + \mathcal{O}(1). \tag{196}$$

If $J_p^{(y)} = \{k\}$, then with Corollary 47.1 we get

$$\frac{1}{2}\int_{\mathcal{D}_p} \alpha |\nabla v|^2 \geq \pi \underline{d}_k^2 \ln\left(\frac{\chi_2}{\delta}\right) + \mathcal{O}(1). \tag{197}$$

And if $\mathrm{Card}(J_p^{(y)}) \geq 2$, still with Corollary 47.1:

$$\frac{1}{2}\int_{\mathcal{D}_p} \alpha |\nabla v|^2 \geq \pi \sum_{k \in \tilde{J}_p^{(y)}} \underline{d}_k^2 \ln\left(\frac{\chi_2}{\chi_1}\right) + \mathcal{O}(1). \tag{198}$$

We continue by dealing with the case $\mathrm{Card}(J_p^{(y)}) \geq 2$. From Corollary 47.1 applied in $\mathcal{Q}_{k,p}$ for $k \in \tilde{J}_p^{(y)}$ [with (194)] we get

$$\sum_{k \in \tilde{J}_p^{(y)}} \frac{1}{2}\int_{\mathcal{Q}_{k,p}} \alpha |\nabla v|^2 \geq \pi \sum_{k \in \tilde{J}_p^{(y)}} \sum_{\substack{l \in J^{(y)} \\ y_l \in B(y_k, \kappa_p \chi_1)}} \tilde{d}_l^2 \ln\left(\frac{\chi_1}{\delta}\right) + \mathcal{O}(1) \tag{199}$$

In order to end the proof, using Propositions 30 & 31 & 49, we get

$$\frac{1}{2}\int_{\Omega_{\chi_2}} \alpha |\nabla v|^2 \geq \pi \sum_{p \in \Lambda} D_p^2 |\ln \chi_2| + \mathcal{O}(1). \tag{200}$$

We let

$$\Delta := \sum_{\substack{p\in\Lambda \text{ s.t.}\\ \mathrm{Card}(J_p^{(y)})\geq 2}} \sum_{k\in\tilde{J}_p^{(y)}} \underline{d}_k^2 + \sum_{\substack{p\in\Lambda \text{ s.t.}\\ J_p^{(y)}=\{k\}}} \underline{d}_k^2 \text{ and } \tilde{\Delta} := \sum_{k\in J^{(y)}} \tilde{d}_k^2.$$

From (195), (196), (197), (198), (199) and (200) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_r} \alpha|\nabla v|^2 \\
&\geq \mathcal{O}(1) + d\pi\left[b^2|\ln r| + (1-b^2)|\ln\lambda| - b^2|\ln\delta|\right] + \pi \sum_{\substack{p\in\Lambda \text{ s.t.}\\ J_p^{(y)}=\{k\}}} \underline{d}_k^2 \ln\left(\frac{\chi_2}{\delta}\right) + \\
&+\pi \sum_{\substack{p\in\Lambda\\ \mathrm{Card}(J_p^{(y)})\geq 2}} \left[\sum_{k\in\tilde{J}_p^{(y)}} \underline{d}_k^2 \ln\left(\frac{\chi_2}{\chi_1}\right) + \sum_{\substack{l\in J^{(y)}\\ y_l\in B(p,\chi_2+\lambda\delta)}} \tilde{d}_l^2 \ln\left(\frac{\chi_1}{\delta}\right)\right] + \pi\sum_{p\in\Lambda} D_p^2|\ln\chi_2| \\
&\geq d\pi\left[b^2|\ln r| + (1-b^2)|\ln(\lambda\delta)|\right] + \pi|\ln\chi_2|\left(\sum_{p\in\Lambda} D_p^2 - \Delta\right) + \pi|\ln\delta|(\tilde{\Delta}-d) + \\
&+\pi|\ln\chi_1| \sum_{\substack{p\in\Lambda\\ \mathrm{Card}(J_p^{(y)})\geq 2}} \left[\sum_{k\in\tilde{J}_p^{(y)}} \underline{d}_k^2 - \sum_{\substack{l\in J^{(y)}\\ y_l\in B(p,\chi_2+\lambda\delta)}} \tilde{d}_l^2\right] + \mathcal{O}(1).
\end{aligned}$$

Since $\underline{d}_k, \tilde{d}_l \geq 1$ for all k,l, from Lemma 54.1 we have $\sum_{p\in\Lambda} D_p^2 \geq \Delta \geq \tilde{\Delta} \geq d$ and moreover

$$\Delta = d \Leftrightarrow (\underline{d}_k = 1 \text{ for all } k)$$

and

$$\tilde{\Delta} = d \Leftrightarrow (\tilde{d}_l = 1 \text{ for all } l).$$

On the other hand since for $p\in\Lambda$ s.t. $J_p^{(y)} = \{k\}$ we have $\underline{d}_k = \tilde{d}_k$, we get

$$\Delta - \tilde{\Delta} = \sum_{\substack{p\in\Lambda\\ \mathrm{Card}(J_p^{(y)})\geq 2}} \left[\sum_{k\in\tilde{J}_p^{(y)}} \underline{d}_k^2 - \sum_{\substack{l\in J^{(y)}\\ y_l\in B(p,\chi_2+\lambda\delta)}} \tilde{d}_l^2\right].$$

Then (96) gives

$$\frac{\mathscr{L}_1(d)}{\pi}\ln h_{\mathrm{ex}} \geq \left(\sum_{p\in\Lambda} D_p^2 - \Delta\right)|\ln\chi_2| + (\tilde{\Delta}-d)|\ln\delta| + (\Delta-\tilde{\Delta})|\ln\chi_1| + \mathcal{O}(1).$$

Since $|\ln\chi_1| = \ln(h_{\mathrm{ex}}) + \mathcal{O}[\ln(\ln h_{\mathrm{ex}})]$ and $|\ln\chi_2| = \ln\sqrt{h_{\mathrm{ex}}} + \mathcal{O}[\ln(\ln h_{\mathrm{ex}})]$ we obtain

$$\begin{aligned}
&\left(\frac{\mathscr{L}_1(d)}{\pi} + \frac{d - \sum_{p\in\Lambda} D_p^2}{2}\right)\ln h_{\mathrm{ex}} \\
&\geq (\Delta-\tilde{\Delta})\ln\sqrt{h_{\mathrm{ex}}} + (\tilde{\Delta}-d)|\ln(\delta\sqrt{h_{\mathrm{ex}}})| + \mathcal{O}[\ln(\ln h_{\mathrm{ex}})]. \qquad (201)
\end{aligned}$$

From Lemma 54.2 and the definition of $\mathscr{L}_1(d)$ [see Lemma 37], we have

$$\frac{\mathscr{L}_1(d)}{\pi} + \frac{d - \sum_{p\in\Lambda} D_p^2}{2} \leq 0. \qquad (202)$$

Using (202) in (201), (4) and $\tilde{\Delta} - d \geq 0 \& \Delta - \tilde{\Delta} \geq 0$ we get $\tilde{\Delta} - d = \Delta - \tilde{\Delta} = 0$ and then $\Delta = d$, *i.e.* $\underline{d}_k = 1$ for all k.

On the other hand, with the help of (201) we may write

$$0 \geq \left(\frac{\mathscr{L}_1(d)}{\pi} + \frac{d - \sum_{p\in\Lambda} D_p^2}{2} \right) \ln h_{\rm ex} \geq \mathcal{O}[\ln(\ln h_{\rm ex})].$$

We may thus deduce $\dfrac{\mathscr{L}_1(d)}{\pi} + \dfrac{d - \sum_{p\in\Lambda} D_p^2}{2} = 0$ and then, with Lemma 54.2, for $p \in \Lambda$ we have $D_p \in \{\lfloor d/N_0 \rfloor; \lceil d/N_0 \rceil\}$.

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M. Dos Santos, Université Paris Est-Créteil, 61 avenue du Général de Gaulle, 94010 Créteil Cedex

E-mail address: mickael.dos-santos@u-pec.fr