Asymptotic behavior for the Vlasov-Poisson equations with strong external curved magnetic field. Part I : well prepared initial conditions

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Abstract

The subject matter of this paper concerns the magnetic confinement. We focus on the asymptotic behavior of the three dimensional Vlasov-Poisson system with strong external magnetic field. We investigate second order approximations, when taking into account the curvature of the magnetic lines. The study relies on multi-scale analysis and allows us to determine a regular reformulation for the Vlasov-Poisson equations with well prepared initial conditions, when the magnetic field becomes large.

Keywords: Vlasov-Poisson system, averaging, homogenization.

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1 Introduction

We denote by $f = f(t, x, v)$ the density of a population of charged particles of mass m, charge q, depending on time t, position x and velocity v. We consider the Vlasov-Poisson equations, with a strong external non vanishing magnetic field

$$\mathbf{B}^\varepsilon(x) = B^\varepsilon(x)e(x), \quad B^\varepsilon(x) = \frac{B(x)}{\varepsilon}, \quad |e(x)| = 1, \quad x \in \mathbb{R}^3$$

where $\varepsilon > 0$ is a small parameter. In the three dimensional setting the Vlasov equation writes

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m} \{E[f^\varepsilon(t)](x) + v \wedge \mathbf{B}^\varepsilon(x)\} \cdot \nabla_v f^\varepsilon = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (1)$$

The electric field $E[f^\varepsilon(t)] = -\nabla_x \Phi[f^\varepsilon(t)]$ derives from the potential

$$\Phi[f^\varepsilon(t)](x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f^\varepsilon(t, x', v')}{|x - x'|} \, \mathrm{d}v'\mathrm{d}x' \quad (2)$$

which satisfies the Poisson equation

$$-\Delta_x \Phi[f^\varepsilon(t)] = \frac{q}{\epsilon_0} \int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \, \mathrm{d}v, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

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whose fundamental solution is $z \to \frac{1}{4\pi|z|}, z \in \mathbb{R}^3 \setminus \{0\}$. Here ϵ_0 represents the electric permittivity. For any particle density $f = f(x,v)$, the notation $E[f]$ stands for the Poisson electric field

$$E[f](x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x',v') \frac{x-x'}{|x-x'|^3} \, \mathrm{d}v'\mathrm{d}x' \tag{3}$$

and $\rho[f], j[f]$ are the charge and current densities respectively

$$\rho[f] = q \int_{\mathbb{R}^3} f(\cdot,v) \, \mathrm{d}v, \quad j[f] = q \int_{\mathbb{R}^3} f(\cdot,v) v \, \mathrm{d}v.$$

The above system is supplemented by the initial condition

$$f^\varepsilon(0,x,v) = f_{\mathrm{in}}(x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{4}$$

We are interested in the asymptotic behavior of the problem (1), (3), (4) when ε goes to 0, see [11, 12, 13, 14, 21, 22, 9, 10, 15, 16, 18, 3, 4, 5, 6] for previous results. In [8], a regular (when $\varepsilon \searrow 0$) reformulation of the Vlasov-Poisson system was emphasized, in the two dimensional setting, with straight magnetic field lines. It is shown that the solutions of the Vlasov-Poisson reformulation (whose numerical approximation is not anymore penalized by the smallness of the parameter ε) coincide, up to a second order term, with the solutions of the Vlasov-Poisson system with strong external magnetic field. In this study we go further into this direction, by considering the three dimensional setting, with curved magnetic field lines and well prepared initial conditions. The derivation of the regular formulation follows by averaging techniques, by taking advantage of the invariants of the cyclotronic motion. We establish the well posedness of it. For any $k \in \mathbb{N}$, the notation C_b^k stands for k times continuously differentiable functions, whose all partial derivatives, up to order k, are bounded.

Theorem 1.1
Consider a non negative, smooth, compactly supported initial particle density $\tilde{f}_{\mathrm{in}} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and a smooth magnetic field $\mathbf{B}^\varepsilon = \frac{\mathbf{B}}{\varepsilon} \in C_b^2(\mathbb{R}^3)$ such that $\inf_{x\in\mathbb{R}^3} |\mathbf{B}^\varepsilon(x)| = B_0^\varepsilon > 0$ (that is $B_0^\varepsilon = \frac{B_0}{\varepsilon}$, $\inf_{x\in\mathbb{R}^3} |\mathbf{B}(x)| = B_0 > 0$), $\mathrm{div}_x \mathbf{B}^\varepsilon = 0$. For any $T > 0$, there is $\varepsilon_T > 0$ such that for $0 < \varepsilon \leq \varepsilon_T$ there exists a unique particle density $\tilde{f}_\varepsilon \in C_c^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, whose Poisson electric field belongs to $C^1([0,T] \times \mathbb{R}^3)$

$$E[\tilde{f}_\varepsilon(t)](x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{f}_\varepsilon(t,x',\tilde{v}') \frac{x-x'}{|x-x'|^3} \, \mathrm{d}\tilde{v}'\mathrm{d}x', \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

satisfying

$$\begin{aligned} &\partial_t \tilde{f}_\varepsilon + (\tilde{v}\cdot e) e \cdot \nabla_x \tilde{f}_\varepsilon + \frac{q}{m}(E[m^\varepsilon \tilde{f}_\varepsilon]\cdot e) e \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon - [\tilde{v} \wedge \partial_x e (\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon \\ &+ c[\, \tilde{v}_D^\varepsilon[\tilde{f}_\varepsilon] \,] \cdot \nabla_{x,\tilde{v}} \tilde{f}_\varepsilon + \frac{(\tilde{v}\cdot e)}{\omega_c^\varepsilon} \left((\partial_x e e \wedge e) \cdot \frac{\nabla_x \omega_c^\varepsilon}{\omega_c^\varepsilon} \right) \frac{\tilde{v} \wedge (e \wedge \tilde{v})}{2} \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon \\ &+ (\tilde{v}\cdot e)(\tilde{v}_{\wedge D}^\varepsilon[\tilde{f}_\varepsilon] \cdot \partial_x e e) e \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon + \left(\tilde{v}_{\wedge D}^\varepsilon[\tilde{f}_\varepsilon] \cdot \frac{\nabla_x \omega_c^\varepsilon}{\omega_c^\varepsilon} \right) \frac{\tilde{v} - (\tilde{v}\cdot e)e}{2} \cdot \nabla_{\tilde{v}} \tilde{f}_\varepsilon = 0 \end{aligned} \tag{5}$$

and

$$\tilde{f}_\varepsilon(0,x,\tilde{v}) = \tilde{f}_{\mathrm{in}}(x,\tilde{v}), \quad (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$$

where

$$m^\varepsilon = 1 + (\tilde{v}\cdot e) \, \frac{e \cdot \mathrm{rot}_x e}{\omega_c^\varepsilon}, \quad \tilde{v}_D^\varepsilon = \tilde{v}_{\wedge D}^\varepsilon + \tilde{v}_{GD}^\varepsilon + \tilde{v}_{CD}^\varepsilon$$

$$\tilde{v}_{\wedge D}^\varepsilon[\tilde{f}_\varepsilon] = \frac{E[\tilde{f}_\varepsilon] \wedge e}{B^\varepsilon}, \quad \tilde{v}_{GD}^\varepsilon = -\frac{m|\tilde{v} \wedge e|^2}{2qB^\varepsilon} \, \frac{\nabla_x B^\varepsilon \wedge e}{B^\varepsilon}, \quad \tilde{v}_{CD}^\varepsilon = -\frac{m(\tilde{v}\cdot e)^2}{qB^\varepsilon} \partial_x e e \wedge e$$

and for any vector field $\xi \cdot \nabla_x$, the notation $c[\xi] \cdot \nabla_{x,\tilde{v}}$ stands for the vector field

$$c[\xi] \cdot \nabla_{x,\tilde{v}} = \xi \cdot \nabla_x + (\partial_x e \xi \otimes e - e \otimes \partial_x e \xi)\tilde{v} \cdot \nabla_{\tilde{v}}.$$

Moreover, if for some integer $k \geq 2$ we have $\tilde{f}_{\rm in} \in C^k_c(\mathbb{R}^3 \times \mathbb{R}^3), \mathbf{B}^\varepsilon \in C^{k+1}(\mathbb{R}^3)$, then $\tilde{f}_\varepsilon \in C^k([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $E[\tilde{f}_\varepsilon] \in C^k([0,T] \times \mathbb{R}^3)$.

We prove that the solution of the previous model approximates the solution of the Vlasov-Poisson system (1), (3) up to a second order term with respect to ε.

Theorem 1.2
Let $\mathbf{B} \in C^4_b(\mathbb{R}^3)$ be a smooth magnetic field, such that $\inf_{x \in \mathbb{R}^3} |\mathbf{B}(x)| = B_0 > 0, \mathrm{div}_x \mathbf{B} = 0$. We consider a family of non negative, smooth, uniformly compactly supported particle densities $(\tilde{g}_\varepsilon)_{0<\varepsilon\leq 1} \subset C^3_c(\mathbb{R}^3 \times \mathbb{R}^3)$

$$\exists\, R_{\tilde{x}}, R_{\tilde{v}} > 0 \ :\ \mathrm{supp}\ \tilde{g}_\varepsilon \subset \{(\tilde{x},\tilde{v}) \ :\ |\tilde{x}| \leq R_{\tilde{x}} \text{ and } |\tilde{v}| \leq R_{\tilde{v}}\}, \quad \sup_{0<\varepsilon\leq 1} \|\tilde{g}_\varepsilon\|_{C^3(\mathbb{R}^3\times\mathbb{R}^3)} < +\infty.$$

We assume that $(\tilde{g}_\varepsilon)_{0<\varepsilon\leq 1}$ are well prepared i.e.,

$$\sup_{0<\varepsilon\leq 1} \frac{\|(\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} \tilde{g}_\varepsilon\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}}{\varepsilon^2} < +\infty, \quad \sup_{0<\varepsilon\leq 1} \frac{\|c_0[\tilde{g}_\varepsilon] \cdot \nabla_{x,\tilde{v}} (\tilde{g}_\varepsilon - \langle \tilde{g}_\varepsilon \rangle)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}}{\varepsilon} < +\infty$$

where $c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \cdot e) e \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e) e \cdot \nabla_{\tilde{v}} - [\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}}$. We denote by $(f^\varepsilon)_{\varepsilon>0}$ the solutions of the Vlasov-Poisson equations with external magnetic field (1), (3) on $[0,T]$, corresponding to the initial conditions

$$f^\varepsilon(0,x,v) = (\tilde{g}_\varepsilon + \varepsilon \tilde{g}^1_\varepsilon)\left(x + \varepsilon \frac{v \wedge e(x)}{\omega_c(x)}, v - \varepsilon \frac{E[\tilde{g}_\varepsilon] \wedge e(x)}{B(x)}\right), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$$

where

$$c_0[\tilde{g}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\, \tilde{g}_\varepsilon - \langle c_0[\tilde{g}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\, \tilde{g}_\varepsilon \rangle + b \cdot \nabla_{x,\tilde{v}}\, \tilde{g}^1_\varepsilon = 0, \ \ \langle \tilde{g}^1_\varepsilon \rangle = 0$$

and $T < \inf_{0<\varepsilon\leq 1} T(f^\varepsilon(0))$ see Theorem 2.1. For ε small enough, we consider the solution $\tilde{f}_\varepsilon$ on $[0,T]$ of the problem (5) corresponding to the initial condition $\tilde{f}_\varepsilon(0) = \langle \tilde{g}_\varepsilon \rangle, 0 < \varepsilon \leq \varepsilon_T$ cf. Theorem 1.1 (see also Proposition (5.8)). Therefore there exists a constant $C_T > 0$ such that for any $0 < \varepsilon \leq \varepsilon_T$

$$\sup_{t\in[0,T]} \left\{ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[f^\varepsilon(t,x,v) - (\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon)\left(t, x + \varepsilon \frac{v \wedge e}{\omega_c}, v - \varepsilon \frac{E[\tilde{f}_\varepsilon(t)] \wedge e}{B}\right)\right]^2 \mathrm{d}v\mathrm{d}x \right\}^{1/2} \leq C_T \varepsilon^2$$

where

$$c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}_\varepsilon - \left\langle c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\, \tilde{f}_\varepsilon \right\rangle + b \cdot \nabla_{x,\tilde{v}}\, \tilde{f}^1_\varepsilon = 0, \ \ \left\langle \tilde{f}^1_\varepsilon \right\rangle = 0.$$

The notation $\langle \cdot \rangle$ stands for the average along the characteristic flow of the vector field $\omega_c(x)\, (\tilde{v} \wedge e(x)) \cdot \nabla_{\tilde{v}}$, see Proposition 3.1.

Our paper is organized as follows. In Section 2 we discuss the well posedness of the Vlasov-Poisson problem with external magnetic field. We indicate uniform estimates with respect to the magnetic field. The average operators, together with their main properties are introduced in Section 3. The regular reformulation of the Vlasov-Poisson problem is derived in Sections 4, 5. The error estimate, when the initial conditions are well prepared, relies on the construction of a corrector term. The well posedness of the limit model is discussed in Section 6.

2 Classical solutions for the Vlasov-Poisson problem with external magnetic field

The Vlasov-Poisson equations are now well understood. We refer to [1] for weak solutions, and to [23, 17, 19] for strong solutions. For studying the Vlasov-Poisson equations with external magnetic field we can adapt the arguments in [17, 20], see also Theorem 2.1 [8] for two dimensional results. Motivated by the asymptotic behavior when the magnetic field becomes strong, we are looking for classical solutions, satisfying uniform bounds with respect to the magnetic field. At least locally in time such solutions exist, see Appendix A for the main lines of the proof.

Theorem 2.1
Consider a non negative, smooth, compactly supported initial particle density $f_{\rm in} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ *such that*

$$\text{supp } f_{\rm in} \subset \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \, : \, |x| \leq R_x^{\rm in}, \ \ |v| \leq R_v^{\rm in}\}$$

and a smooth magnetic field $\mathbf{B} \in C^1_b(\mathbb{R}^3)$. *Let* $T < T(f_{\rm in}) := \frac{m\epsilon_0}{q^2 R_v^{\rm in}(12\pi)^{1/3}\|f_{\rm in}\|_{L^1}^{1/3}\|f_{\rm in}\|_{L^\infty}^{2/3}}$.
There is a unique particle density $f \in C^1_c([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, *whose Poisson electric field is smooth* $E[f] \in C^1([0,T] \times \mathbb{R}^3)$, *satisfying*

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m}\left(E[f(t)] + v \wedge \mathbf{B}\right) \cdot \nabla_v f = 0, \ \ (t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{6}$$

$$E[f(t)](x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x',v') \frac{x-x'}{|x-x'|^3} \, \mathrm{d}v'\mathrm{d}x', \ \ (t,x) \in [0,T] \times \mathbb{R}^3 \tag{7}$$

$$f(0,x,v) = f_{\rm in}(x,v), \ \ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{8}$$

Moreover, if for some integer $k \geq 2$ *we have* $f_{\rm in} \in C^k_c(\mathbb{R}^3 \times \mathbb{R}^3), \mathbf{B} \in C^k_b(\mathbb{R}^3)$, *then* $f \in C^k([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ *and* $E[f] \in C^k([0,T] \times \mathbb{R}^3)$.

Remark 2.1
1. The solution constructed in Theorem 2.1 preserves the particle number and the total energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) \, \mathrm{d}v\mathrm{d}x = 0, \ \ t \in [0,T]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{m|v|^2}{2} f(t,x,v) \, \mathrm{d}v\mathrm{d}x + \frac{1}{8\pi\epsilon_0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho[f(t)](x)\rho[f(t)](x')}{|x-x'|} \, \mathrm{d}x'\mathrm{d}x\right\} = 0.$$

2. We have the following balance for the total momentum

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) mv \, \mathrm{d}v\mathrm{d}x - q\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) v \wedge \mathbf{B} \, \mathrm{d}v\mathrm{d}x &= \int_{\mathbb{R}^3} \rho[f(t)] E[f(t)] \, \mathrm{d}x \\ &= \epsilon_0 \int_{\mathbb{R}^3} \mathbf{1}_{\text{supp } \rho[f(t)]} \, \mathrm{div}_x E[f(t)] \, E[f(t)] \, \mathrm{d}x \\ &= \epsilon_0 \int_{\mathbb{R}^3} \mathbf{1}_{\text{supp } \rho[f(t)]} \, \mathrm{div}_x \left(E[f(t)] \otimes E[f(t)] - \frac{|E[f(t)]|^2}{2} I_3\right) \, \mathrm{d}x = 0.\end{aligned}$$

When the magnetic field is uniform, we obtain the conservation of the parallel momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) m(v \cdot e) \, \mathrm{d}v\mathrm{d}x = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) m(v \wedge e) \, \mathrm{d}v\mathrm{d}x = \frac{qB}{m}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v) m(v \wedge e) \, \mathrm{d}v\mathrm{d}x\right) \wedge e$$

saying that the orthogonal momentum rotates at the cyclotronic frequency $\omega_c = \frac{qB}{m}$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v)m(v\wedge e)\,\mathrm{d}v\mathrm{d}x = \cos(\omega_c t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_{\mathrm{in}}(x,v)m(v\wedge e)\,\mathrm{d}v\mathrm{d}x + \sin(\omega_c t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_{\mathrm{in}}(x,v)m(v\wedge e)\wedge e\,\mathrm{d}v\mathrm{d}x,\ \ t\in[0,T].$$

3 Average operators and main properties

We intend to investigate the asymptotic behavior of the particle densities $(f^\varepsilon)_{\varepsilon>0}$ satisfying (1), (3), (4) when $\varepsilon > 0$ becomes small. We assume that the initial particle density and the external magnetic field $\mathbf{B}^\varepsilon = \frac{B}{\varepsilon}e$ are smooth

$$f_{\mathrm{in}} \geq 0,\ \ f_{\mathrm{in}} \in C^1_c(\mathbb{R}^3\times\mathbb{R}^3),\ \ \mathbf{B} = Be \in C^1_b(\mathbb{R}^3)$$

and let us consider $T < T(f_{\mathrm{in}})$. Under the above assumptions, we know by Theorem 2.1 that there exists $\varepsilon_T > 0$ such that for every $0 < \varepsilon \leq \varepsilon_T$, there is a unique strong solution $f^\varepsilon \in C^1_c([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$, $E^\varepsilon := E[f^\varepsilon] \in C^1([0,T]\times\mathbb{R}^3)$ for the Vlasov-Poisson problem with external magnetic field $\mathbf{B}^\varepsilon = \frac{B}{\varepsilon}e$. As noticed in the proof of Theorem 2.1, we have uniform estimates with respect to ε for the L^∞ norm of the electric field E^ε and the size of the support of the particle density f^ε. Let us denote by $(X^\varepsilon, V^\varepsilon)(t;t_0,x,v)$ the characteristics associated to (1)

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = V^\varepsilon(t;t_0,x,v),\ \ \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \frac{q}{m}\left[E^\varepsilon(t,X^\varepsilon(t;t_0,x,v)) + V^\varepsilon(t;t_0,x,v)\wedge\mathbf{B}^\varepsilon(X^\varepsilon(t;t_0,x,v))\right]$$
$$X^\varepsilon(t;t_0,x,v) = x,\ \ V^\varepsilon(t;t_0,x,v) = v. \tag{9}$$

The strong external magnetic field induces a large cyclotronic frequency $\omega_c^\varepsilon = qB^\varepsilon/m = \omega_c/\varepsilon, \omega_c = qB/m$, and thus a fast dynamics. We are looking for quantities which are left invariant with respect to this fast motion. By direct computations we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X^\varepsilon(t) + \varepsilon\frac{V^\varepsilon(t)\wedge e(X^\varepsilon(t))}{\omega_c(X^\varepsilon(t))}\right] = (V^\varepsilon(t)\cdot e(X^\varepsilon(t))\ e(X^\varepsilon(t)) + \varepsilon\frac{E^\varepsilon(t,X^\varepsilon(t))}{B(X^\varepsilon(t))}\wedge e(X^\varepsilon(t))$$
$$+\varepsilon\frac{V^\varepsilon(t)\wedge\partial_x e(X^\varepsilon(t))V^\varepsilon(t)}{\omega_c(X^\varepsilon(t))} - \varepsilon\left(\nabla_x\omega_c(X^\varepsilon(t))\cdot V^\varepsilon(t)\right)\frac{V^\varepsilon(t)\wedge e(X^\varepsilon(t))}{\omega_c^2(X^\varepsilon(t))}$$

saying that the variations of $x+\varepsilon\frac{v\wedge e(x)}{\omega_c(x)}$, along the characteristic flow (9), over one cyclotronic period, is very small. Notice that the electro-magnetic force writes

$$\frac{q}{m}E^\varepsilon(t,x) + \frac{\omega_c}{\varepsilon}v\wedge e(x) = \frac{q}{m}(E^\varepsilon(t,x)\cdot e(x))\ e(x) + \frac{\omega_c(x)}{\varepsilon}\left(v - \varepsilon\frac{E^\varepsilon(t,x)\wedge e(x)}{B(x)}\right)\wedge e(x)$$

and therefore we introduce the relative velocity with respect to the electric cross field drift

$$\tilde{v} = v - \varepsilon\frac{E^\varepsilon(t,x)\wedge e(x)}{B(x)}. \tag{10}$$

Accordingly, at any time $t\in\mathbb{R}_+$, we consider the new particle density

$$\tilde{f}^\varepsilon(t,x,\tilde{v}) = f^\varepsilon\left(t,x,\tilde{v} + \varepsilon\frac{E[f^\varepsilon(t)](x)\wedge e(x)}{B(x)}\right),\ \ (x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3. \tag{11}$$

It is easily seen that the particle densities $f^\varepsilon, \tilde{f}^\varepsilon$ have the same charge density

$$\rho[\tilde{f}^\varepsilon(t)] = q\int_{\mathbb{R}^3}\tilde{f}^\varepsilon(t,\cdot,\tilde{v})\,\mathrm{d}\tilde{v} = q\int_{\mathbb{R}^3} f^\varepsilon(t,\cdot,v)\,\mathrm{d}v = \rho[f^\varepsilon(t)],\ \ t\in\mathbb{R}_+$$

implying that the Poisson electric fields corresponding to $f^\varepsilon, \tilde{f}^\varepsilon$ coincide

$$E[f^\varepsilon(t)] = E[\tilde{f}^\varepsilon(t)], \quad t \in [0,T].$$

Therefore we can use the same notation $E^\varepsilon(t)$ for denoting them. We assume that the magnetic field satisfies

$$B_0 := \inf_{x\in\mathbb{R}^3} |B(x)| > 0 \ \text{ or equivalently } \omega_0 := \inf_{x\in\mathbb{R}^3} |\omega_c(x)| > 0 \tag{12}$$

and therefore (10), (11) are well defined. Notice that the particle densities $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ are smooth, $\tilde{f}^\varepsilon \in C_c^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ and uniformly compactly supported with respect to ε (use the uniform bound for the electric fields $(E^\varepsilon)_\varepsilon$ and the hypothesis (12)). Appealing to the chain rule leads to the following problem in the phase space $(x,\tilde{v})$

$$\begin{aligned}\partial_t \tilde{f}^\varepsilon + \left(\tilde{v} + \varepsilon\frac{E^\varepsilon\wedge e}{B}\right)\cdot\nabla_x\tilde{f}^\varepsilon - \varepsilon\left[\frac{\partial_t E^\varepsilon\wedge e}{B} + \partial_x\left(\frac{E^\varepsilon\wedge e}{B}\right)\left(\tilde{v}+\varepsilon\frac{E^\varepsilon\wedge e}{B}\right)\right]\cdot\nabla_{\tilde{v}}\tilde{f}^\varepsilon \\ + \left[\frac{\omega_c}{\varepsilon}\,\tilde{v}\wedge e + \frac{q}{m}(E^\varepsilon\cdot e)\,e\right]\cdot\nabla_{\tilde{v}}\tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v})\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3 \\ \tilde{f}^\varepsilon(0,x,\tilde{v}) = f_{\text{in}}\left(x,\tilde{v}+\varepsilon\frac{E[f_{\text{in}}](x)\wedge e(x)}{B(x)}\right), \quad (x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3.\end{aligned} \tag{13}$$

We are looking for a representation formula for the time derivative of the electric field E^ε, in terms of the particle density $\tilde{f}^\varepsilon$. Thanks to the continuity equation

$$\partial_t\rho[f^\varepsilon] + \text{div}_x j[f^\varepsilon] = 0$$

we write

$$\begin{aligned}\partial_t E[f^\varepsilon] &= \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3}\partial_t\rho[f^\varepsilon(t)](x-x')\frac{x'}{|x'|^3}\,\mathrm{d}x' \\ &= -\frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3}\text{div}_x j[f^\varepsilon](x-x')\frac{x'}{|x'|^3}\,\mathrm{d}x' \\ &= -\frac{1}{4\pi\epsilon_0}\text{div}_x\int_{\mathbb{R}^3}\frac{x'}{|x'|^3}\otimes j[f^\varepsilon(t)](x-x')\,\mathrm{d}x' \\ &= -\frac{1}{4\pi\epsilon_0}\text{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes\left(j[\tilde{f}^\varepsilon(t)](x') + \varepsilon\rho[\tilde{f}^\varepsilon(t)](x')\frac{E^\varepsilon(t,x')\wedge e(x')}{B(x')}\right)\,\mathrm{d}x'.\end{aligned}$$

We introduce as well the new Larmor center $\tilde{x} = x + \varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}$, which is a second order approximation of the Larmor center $x+\varepsilon\frac{v\wedge e(x)}{\omega_c(x)}$. The idea will be to decompose the transport field in the Vlasov equation in such a way that $\tilde{x}$ remains invariant with respect to the fast dynamics. We will distinguish between the orthogonal and parallel directions, taking as reference direction the magnetic line passing through the new Larmor center $\tilde{x}$, that is $e(\tilde{x})$ (which is left invariant with respect to the fast dynamics)

$$\tilde{v} = [\tilde{v} - (\tilde{v}\cdot e(\tilde{x}))e(\tilde{x})] + (\tilde{v}\cdot e(\tilde{x}))e(\tilde{x}).$$

Finally the Vlasov equation (13) writes

$$\partial_t\tilde{f}^\varepsilon + c^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v})\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3 \tag{14}$$

where the autonomous vector field $\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}$ is given by

$$\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}} = [\tilde{v} - (\tilde{v}\cdot e(\tilde{x}))\,e(\tilde{x}) + \varepsilon A_x^\varepsilon(x,\tilde{v})]\cdot\nabla_x + \frac{\omega_c(x)}{\varepsilon}\,(\tilde{v}\wedge e(\tilde{x}))\cdot\nabla_{\tilde{v}}$$

and for any particle density $\tilde{f}$, $a^\varepsilon[\tilde{f}]\cdot\nabla_{x,\tilde{v}}$, $c^\varepsilon[\tilde{f}]\cdot\nabla_{x,\tilde{v}}$ stand for the vector fields

$$\begin{aligned} a^\varepsilon[\tilde{f}]\cdot\nabla_{x,\tilde{v}} &= \left(\frac{E[\tilde{f}]\wedge e}{B} - A_x^\varepsilon\right)\cdot\nabla_x + \left[-\partial_x\left(\frac{E[\tilde{f}]\wedge e}{B}\right)\left(\tilde{v}+\varepsilon\frac{E[\tilde{f}]\wedge e}{B}\right)\right. \\ &\left. + \frac{1}{4\pi\epsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes\left(j[\tilde{f}]+\varepsilon\rho[\tilde{f}]\frac{E[\tilde{f}]\wedge e}{B}\right)(x')\,\mathrm{d}x'\wedge e(x)\right]\cdot\nabla_{\tilde{v}} \end{aligned} \tag{15}$$

$$\begin{aligned} c^\varepsilon[\tilde{f}]\cdot\nabla_{x,\tilde{v}} &= (\tilde{v}\cdot e(\tilde{x}))\,e(\tilde{x})\cdot\nabla_x + \left[\omega_c(x)\tilde{v}\wedge\frac{e(x)-e(\tilde{x})}{\varepsilon} + \frac{q}{m}(E[\tilde{f}]\cdot e(x))\,e(x)\right]\cdot\nabla_{\tilde{v}} \\ &= \left[\frac{q}{m}(E[\tilde{f}]\cdot e(x))\,e(x) - \omega_c\tilde{v}\wedge\int_0^1\partial_x e\left(x+\varepsilon s\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\right)\,\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\,\mathrm{d}s\right]\cdot\nabla_{\tilde{v}} \\ &+ (\tilde{v}\cdot e(\tilde{x}))\,e(\tilde{x})\cdot\nabla_x. \end{aligned} \tag{16}$$

The vector field $A_x^\varepsilon(x,\tilde{v})\cdot\nabla_x$ will be determined by imposing that the Larmor center $\tilde{x}$ is left invariant by the fast dynamics

$$b^\varepsilon\cdot\nabla_{x,\tilde{v}}\,\left(x+\varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\right)=0.$$

After some computations, the above condition writes

$$\left[I_3+\varepsilon\partial_x\left(\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]A_x^\varepsilon(x,\tilde{v}) = -\partial_x\left(\frac{\tilde{v}\wedge e}{\omega_c}\right)[\tilde{v}-(\tilde{v}\cdot e(\tilde{x}))\,e(\tilde{x})] - \frac{e(\tilde{x})-e(x)}{\varepsilon}\wedge(\tilde{v}\wedge e(\tilde{x}))$$

and therefore $A_x^\varepsilon(x,\tilde{v})$ is well defined for a.a. $(x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3$. Notice that for ε small enough, that is

$$\varepsilon\left\|\partial_x\left(\frac{\tilde{v}\wedge e}{\omega_c}\right)\right\|_{L^\infty}<1$$

the vector field A_x^ε is well defined on $\mathbb{R}^3\times\mathbb{R}^3$. In particular A_x^ε is well defined if

$$\varepsilon|\tilde{v}|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0}+\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)<1.$$

Remark 3.1
The vector field in the Vlasov equation (14) *is divergence free*

$$\mathrm{div}_{x,\tilde{v}}\left(c^\varepsilon[\tilde{f}]+\varepsilon a^\varepsilon[\tilde{f}]+\frac{b^\varepsilon}{\varepsilon}\right)=\varepsilon\mathrm{div}_x\left(\frac{E[\tilde{f}]\wedge e}{B}\right)-\varepsilon\mathrm{div}_{\tilde{v}}\left[\partial_x\left(\frac{E[\tilde{f}]\wedge e}{B}\right)\tilde{v}\right]=0.$$

We intend to study the asymptotic behavior of (14), when ε goes to 0 by averaging with respect to the flow of the fast dynamics generated by the advection field $\frac{b^\varepsilon(x,\tilde{v})}{\varepsilon}\cdot\nabla_{x,\tilde{v}}$ cf. [3, 4, 5, 6, 7]. In order to do that, we concentrate on the main properties of this flow. As in the two dimensional framework, we establish the periodicity of the fast dynamics.

Proposition 3.1
Let $B\in C_b^1(\mathbb{R}^3)$ verifying (12) *and $e\in C_b^2(\mathbb{R}^3)$. We denote by $(\mathcal{X}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$ the characteristic flow of the autonomous vector field $b^\varepsilon(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}$*

$$\frac{\mathrm{d}\mathcal{X}^\varepsilon}{\mathrm{d}s}=\varepsilon[I_3-e(\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}))\otimes e(\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}))]\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})+\varepsilon^2A_x^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$$

$$\frac{\mathrm{d}\tilde{\mathcal{V}}^\varepsilon}{\mathrm{d}s} = \omega_c(\mathcal{X}^\varepsilon(s;x,\tilde{v}))\ \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}) \wedge e(\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}))$$

$$\mathcal{X}^\varepsilon(0;x,\tilde{v}) = x, \ \ \tilde{\mathcal{V}}^\varepsilon(0;x,\tilde{v}) = \tilde{v}$$

and by $(\mathcal{X}(s;x,\tilde{v}),\tilde{\mathcal{V}}(s;x,\tilde{v}))$ *the characteristic flow of the autonomous vector field* $b(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} \ = \omega_c(x)\ (\tilde{v} \wedge e(x)) \cdot \nabla_{\tilde{v}}$

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}s} = 0, \ \ \frac{\mathrm{d}\tilde{\mathcal{V}}}{\mathrm{d}s} = \omega_c(\mathcal{X}(s;x,\tilde{v}))\ \tilde{\mathcal{V}}(s;x,\tilde{v}) \wedge e(\mathcal{X}(s;x,\tilde{v})), \ \ \mathcal{X}(0;x,\tilde{v}) = x, \ \ \tilde{\mathcal{V}}(0;x,\tilde{v}) = \tilde{v}.$$

1. *For any* $(x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ *and* $\varepsilon > 0$ *such that*

$$\varepsilon|\tilde{v}| \left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2} \right) < 1 \tag{17}$$

the characteristic $s \to (\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v})$ *is periodic, with smallest period* $S^\varepsilon(x,\tilde{v}) > 0$.

2. *For any* $(x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ *and* $\varepsilon > 0$ *such that*

$$\varepsilon|\tilde{v}| \left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2} \right) \leq \frac{1}{2}$$

we have

$$|\mathcal{X}^\varepsilon(s;x,\tilde{v}) - \mathcal{X}(s;x,\tilde{v})| = |\mathcal{X}^\varepsilon(s;x,\tilde{v}) - x| \leq \varepsilon \frac{2|\tilde{v}|}{\omega_0}, \ \ s \in \mathbb{R}$$

$$\frac{2\pi}{\|\omega_c\|_{L^\infty}} \leq S^\varepsilon(x,\tilde{v}) \leq \frac{2\pi}{\omega_0}, \ \ \frac{2\pi}{\|\omega_c\|_{L^\infty}} \leq S(x,\tilde{v}) := \frac{2\pi}{|\omega_c(x)|} \leq \frac{2\pi}{\omega_0}$$

$$|S^\varepsilon(x,\tilde{v}) - S(x,\tilde{v})| \leq \varepsilon \|\nabla \omega_c\|_{L^\infty} \frac{4\pi|\tilde{v}|}{\omega_0^3},$$

$$|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}) - \tilde{\mathcal{V}}(s;x,\tilde{v})| \leq \varepsilon|\tilde{v}|^2 \left(5\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + 4\pi \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2} \right), \ \ s \in \left[0, \frac{2\pi}{\omega_0}\right]$$

and

$$|A_x^\varepsilon(x,\tilde{v})| \leq 4|\tilde{v}|^2 \left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2} \right)$$

$$|A_x^\varepsilon(x,\tilde{v}) - A_x(x,\tilde{v})| \leq \varepsilon|\tilde{v}|^3 \left[7 \left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2} \right)^2 + \frac{1}{2} \frac{\|\partial_x^2 e\|_{L^\infty}}{\omega_0^2} \right]$$

where

$$A_x(x,\tilde{v}) = -\partial_x \left(\frac{\tilde{v} \wedge e(x)}{\omega_c(x)} \right) [\tilde{v} - (\tilde{v} \cdot e(x))\ e(x)] - \partial_x e \frac{\tilde{v} \wedge e(x)}{\omega_c(x)} \wedge (\tilde{v} \wedge e(x)).$$

In particular, when $\nabla_x \omega_c = 0$, *we have* $S^\varepsilon(x,\tilde{v}) = S(x,\tilde{v}) = 2\pi/|\omega_c|$.

3. *For any continuous function* $u \in C(\mathbb{R}^3 \times \mathbb{R}^3)$ *we define the averages along the flows of* $b \cdot \nabla_{x,\tilde{v}}$, $b^\varepsilon \cdot \nabla_{x,\tilde{v}}$

$$\langle u \rangle (x,\tilde{v}) = \frac{1}{S(x,\tilde{v})} \int_0^{S(x,\tilde{v})} u(\mathcal{X}(s;x,\tilde{v}), \tilde{\mathcal{V}}(s;x,\tilde{v}))\ \mathrm{d}s, \ \ (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$$

$$\langle u \rangle_\varepsilon (x,\tilde{v}) = \frac{1}{S^\varepsilon(x,\tilde{v})} \int_0^{S^\varepsilon(x,\tilde{v})} u(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))\ \mathrm{d}s, \ \ (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

For any $R_x, R_{\tilde{v}} \in \mathbb{R}_+$ we have

$$\| \langle u \rangle \|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))} \leq \|u\|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))}$$

$$\| \langle u \rangle_\varepsilon \|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))} \leq \|u\|_{L^\infty(B(R_x^\varepsilon)\times B(R_{\tilde{v}}))}, \quad R_x^\varepsilon = R_x + 2\varepsilon R_{\tilde{v}}/\omega_0$$

where $B(R)$ stands for the closed ball of radius R in $\mathbb{R}^3$.

4. *If u is Lipschitz continuous, then for any $(x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\varepsilon > 0$ such that $\varepsilon|\tilde{v}|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2}\right) \leq \frac{1}{2}$ we have*

$$\frac{|\langle u \rangle_\varepsilon (x,\tilde{v}) - \langle u \rangle (x,\tilde{v})|}{\varepsilon} \leq \mathrm{Lip}(u)\frac{|\tilde{v}|}{\omega_0}\left[2 + 5\|\partial_x e\|_{L^\infty}|\tilde{v}| + 4\pi\|\nabla_x\omega_c\|_{L^\infty}\frac{|\tilde{v}|}{\omega_0}\right] + \sup_{|\tilde{v}'|=|\tilde{v}|}|u(x,\tilde{v}')|\|\nabla_x\omega_c\|_{L^\infty}\frac{4|\tilde{v}|}{\omega_0^2}.$$

5. *For any function $u \in C_c^1(\mathbb{R}^3\times\mathbb{R}^3)$ we have the inequality*

$$\|u - \langle u \rangle\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq \frac{2\pi}{\omega_0}\|b\cdot\nabla_{x,\tilde{v}}\, u\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}.$$

6. *For any function $u \in C^1(\mathbb{R}^3\times\mathbb{R}^3)$, we have $\langle u\rangle \in C^1(\mathbb{R}^3\times\mathbb{R}^3)$ and*

$$\left\langle c^i\cdot\nabla_{x,\tilde{v}}\, u\right\rangle = c^i\cdot\nabla_{x,\tilde{v}}\, \langle u\rangle, \quad \left\langle \mathrm{div}_{x,\tilde{v}}(uc^i)\right\rangle = \mathrm{div}_{x,\tilde{v}}(\langle u\rangle\, c^i), \quad 1\leq i\leq 6$$

where

$$c^i\cdot\nabla_{x,\tilde{v}} = \partial_{x_i} + (\partial_{x_i}e\otimes e - e\otimes\partial_{x_i}e)\tilde{v}\cdot\nabla_{\tilde{v}}, \quad i\in\{1,2,3\}$$

$$c^4\cdot\nabla_{x,\tilde{v}} = [\tilde{v} - (\tilde{v}\cdot e)\, e]\cdot\nabla_{\tilde{v}}, \quad c^5\cdot\nabla_{x,\tilde{v}} = e\cdot\nabla_{\tilde{v}}, \quad c^6\cdot\nabla_{x,\tilde{v}} = (\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}.$$

The vector fields $\{c^i\cdot\nabla_{x,\tilde{v}}\ , i\neq 4\}$ are divergence free, and $\mathrm{div}_{x,\tilde{v}}c^4 = 2$.

Proof.
1. We use the notation $\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}) = \mathcal{X}^\varepsilon(s;x,\tilde{v}) + \varepsilon\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})\wedge e(\mathcal{X}^\varepsilon(s;x,\tilde{v}))/\omega_c(\mathcal{X}^\varepsilon(s;x,\tilde{v}))$. It is easily seen that $|\tilde{\mathcal{V}}^\varepsilon(s)| = |\tilde{v}|, s\in\mathbb{R}$ and therefore we have

$$\varepsilon|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right) < 1, \quad s\in\mathbb{R}$$

saying that $A_x^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$ is well defined for any $s\in\mathbb{R}$. By the definition of A_x^ε we know that $\tilde{\mathcal{X}}^\varepsilon(s)$ remains constant with respect to $s\in\mathbb{R}$

$$\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}) = x + \varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}, \quad s\in\mathbb{R}$$

implying that the parallel velocity is left invariant

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\tilde{\mathcal{V}}^\varepsilon(s)\cdot e(\tilde{\mathcal{X}}^\varepsilon(s))\right) = 0, \quad s\in\mathbb{R}$$

and that the orthogonal velocity rotates around $e(\tilde{x})$

$$\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}) = \mathcal{R}\left(-\int_0^s \omega_c(\mathcal{X}^\varepsilon(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma, e(\tilde{x})\right)\tilde{v}, \quad s\in\mathbb{R}.$$

Here the notation $\mathcal{R}(\theta, e)$ stands for the rotation of angle θ around the axis e

$$\mathcal{R}(\theta, e)\xi = \cos\theta(I_3 - e\otimes e)\xi - \sin\theta(\xi\wedge e) + (\xi\cdot e)\, e,\ \ \xi\in\mathbb{R}^3.$$

As ω_c has constant sign, there is a unique $S^\varepsilon(x,\tilde v) > 0$ such that

$$\operatorname{sgn}\omega_c \int_0^{S^\varepsilon(x,\tilde v)} \omega_c(\mathcal{X}^\varepsilon(\sigma;x,\tilde v))\,\mathrm{d}\sigma = \int_0^{S^\varepsilon(x,\tilde v)} |\omega_c(\mathcal{X}^\varepsilon(\sigma;x,\tilde v))|\,\mathrm{d}\sigma = 2\pi$$

and therefore $\tilde{\mathcal{V}}^\varepsilon(S^\varepsilon(x,\tilde v);x,\tilde v) = \tilde v$. We claim that $\mathcal{X}^\varepsilon(S^\varepsilon(x,\tilde v);x,\tilde v) = x$. It is enough to use the invariance of the Larmor center

$$\mathcal{X}^\varepsilon(S^\varepsilon) + \varepsilon\frac{\tilde{\mathcal{V}}^\varepsilon(S^\varepsilon)\wedge e(\mathcal{X}^\varepsilon(S^\varepsilon))}{\omega_c(\mathcal{X}^\varepsilon(S^\varepsilon))} = x + \varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}$$

and to observe that

$$\begin{aligned}
|\mathcal{X}^\varepsilon(S^\varepsilon) - x| &= \varepsilon\left|\frac{\tilde v\wedge e(x)}{\omega_c(x)} - \frac{\tilde v\wedge e(\mathcal{X}^\varepsilon(S^\varepsilon))}{\omega_c(\mathcal{X}^\varepsilon(S^\varepsilon))}\right| \\
&\le \varepsilon|\tilde v|\left|\frac{e(\mathcal{X}^\varepsilon(S^\epsilon))}{\omega_c(\mathcal{X}^\varepsilon(S^\varepsilon))} - \frac{e(x)}{\omega_c(x)}\right| \\
&\le \varepsilon|\tilde v|\left\|\partial_x\left(\frac{e}{\omega_c}\right)\right\|_{L^\infty}|\mathcal{X}^\varepsilon(S^\epsilon) - x| \\
&\le \varepsilon|\tilde v|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)|\mathcal{X}^\varepsilon(S^\varepsilon) - x|.
\end{aligned}$$

Our conclusion follows by (17).

2. By the definition of the vector field $A_x^\varepsilon(x,\tilde v)\cdot\nabla_x$, we deduce

$$\begin{aligned}
|A_x^\varepsilon(x,\tilde v)| &\le \left\|\partial_x\left(\frac{\tilde v\wedge e}{\omega_c}\right)\right\|_{L^\infty}|\tilde v| + \|\partial_x e\|_{L^\infty}\frac{|\tilde v|^2}{\omega_0} + \varepsilon\left\|\partial_x\left(\frac{\tilde v\wedge e}{\omega_c}\right)\right\|_{L^\infty}|A_x^\varepsilon(x,\tilde v)| \\
&\le 2|\tilde v|^2\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right) + \varepsilon|\tilde v|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)|A_x^\varepsilon(x,\tilde v)| \\
&\le 2|\tilde v|^2\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right) + \frac{|A_x^\varepsilon(x,\tilde v)|}{2}
\end{aligned}$$

implying that

$$|A_x^\varepsilon(x,\tilde v)| \le 4|\tilde v|^2\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right).$$

Notice that

$$\begin{aligned}
&\left|\left[I_3 + \varepsilon\partial_x\left(\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)\right]A_x^\varepsilon(x,\tilde v) - A_x(x,\tilde v)\right| \le 2\left\|\partial_x\left(\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)\right\|_{L^\infty}\varepsilon\|\partial_x e\|_{L^\infty}\frac{|\tilde v|^2}{\omega_0} \\
&+ \varepsilon\|\partial_x e\|_{L^\infty}^2\frac{|\tilde v|^3}{\omega_0^2} + |\tilde v|\left|\frac{e(\tilde x) - e(x)}{\varepsilon} - \partial_x e(x)\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right| \\
&\le \varepsilon\left(3\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + 2\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)\frac{\|\partial_x e\|_{L^\infty}}{\omega_0}|\tilde v|^3 + \frac{\varepsilon}{2}\frac{\|\partial_x e^2\|_{L^\infty}}{\omega_0^2}|\tilde v|^3
\end{aligned}$$

and therefore

$$\begin{aligned}
|A_x^\varepsilon(x,\tilde v) - A_x(x,\tilde v)| &\le \left|\left[I_3 + \varepsilon\partial_x\left(\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)\right]A_x^\varepsilon(x,\tilde v) - A_x(x,\tilde v)\right| \\
&\quad + \varepsilon|\tilde v|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)|A_x^\varepsilon(x,\tilde v)| \\
&\le 7\varepsilon|\tilde v|^3\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)^2 + \frac{\varepsilon\|\partial_x^2 e\|_{L^\infty}}{2\omega_0^2}|\tilde v|^3.
\end{aligned}$$

The invariances of $\tilde{x}$ and $|\tilde{v}|$ yield

$$|\mathcal{X}^\varepsilon(s)-x|=\left|\varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}-\varepsilon\frac{\tilde{\mathcal{V}}^\varepsilon(s)\wedge e(\mathcal{X}^\varepsilon(s))}{\omega_c(\mathcal{X}^\varepsilon(s))}\right|\leq\varepsilon\frac{2|\tilde{v}|}{\omega_0},\quad s\in\mathbb{R}.$$

It is easily seen that $2\pi/\|\omega_c\|_{L^\infty}\leq S^\varepsilon(x,\tilde{v})\leq 2\pi/\omega_0$. Notice that we have

$$|\omega_c(x)|-\|\nabla_x\omega_c\|_{L^\infty}\frac{2\varepsilon|\tilde{v}|}{\omega_0}\leq|\omega_c(\mathcal{X}^\varepsilon(\sigma))|\leq|\omega_c(x)|+\|\nabla_x\omega_c\|_{L^\infty}\frac{2\varepsilon|\tilde{v}|}{\omega_0}.$$

Integrating with respect to $\sigma\in[0,S^\varepsilon(x,\tilde{v})]$, we obtain

$$2\pi\left|\frac{1}{S^\varepsilon(x,\tilde{v})}-\frac{1}{S(x,\tilde{v})}\right|\leq\varepsilon\|\nabla_x\omega_c\|_{L^\infty}\frac{2|\tilde{v}|}{\omega_0}$$

and

$$|S^\varepsilon(x,\tilde{v})-S(x,\tilde{v})|=S^\varepsilon(x,\tilde{v})S(x,\tilde{v})\left|\frac{1}{S^\varepsilon(x,\tilde{v})}-\frac{1}{S(x,\tilde{v})}\right|\leq\varepsilon\|\nabla_x\omega_c\|_{L^\infty}\frac{4\pi|\tilde{v}|}{\omega_0^3}.$$

It remains to compare the velocities $\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}(s;x,\tilde{v})$. We will use the inequality

$$\|\mathcal{R}(\theta,e)-\mathcal{R}(\theta',e')\|\leq|\theta-\theta'|+5|e-e'|,\quad\theta,\theta'\in\mathbb{R},\quad|e|=|e'|=1.$$

For any $s\in[0,2\pi/\omega_0]$ we write

$$\begin{aligned}
|(\tilde{\mathcal{V}}^\varepsilon-\tilde{\mathcal{V}})(s;x,\tilde{v})|&=\left|\mathcal{R}\left(-\int_0^s\omega_c(\mathcal{X}^\varepsilon(\sigma))\,\mathrm{d}\sigma,e(\tilde{x})\right)\tilde{v}-\mathcal{R}\left(-\int_0^s\omega_c(\mathcal{X}(\sigma))\,\mathrm{d}\sigma,e(x)\right)\tilde{v}\right|\\
&\leq\left[\int_0^s|\omega_c(\mathcal{X}^\varepsilon(\sigma))-\omega_c(\mathcal{X}(\sigma))|\,\mathrm{d}\sigma+5|e(\tilde{x})-e(x)|\right]|\tilde{v}|\\
&\leq\left(\frac{2\pi}{\omega_0}\|\nabla_x\omega_c\|_{L^\infty}\frac{2\varepsilon|\tilde{v}|}{\omega_0}+5\|\partial_x e\|_{L^\infty}\frac{\varepsilon|\tilde{v}|}{\omega_0}\right)|\tilde{v}|\\
&=\varepsilon|\tilde{v}|^2\left(5\frac{\|\partial_x e\|_{L^\infty}}{\omega_0}+4\pi\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right).
\end{aligned}$$

3. It is a direct consequence of the invariances $\mathcal{X}(s;x,\tilde{v})=x$, $|\tilde{\mathcal{V}}(s;x,\tilde{v})|=|\tilde{v}|$

$$\mathcal{X}^\varepsilon(s;x,\tilde{v})+\varepsilon\frac{\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})\wedge e(\mathcal{X}^\varepsilon(s;x,\tilde{v}))}{\omega_c(\mathcal{X}^\varepsilon(s;x,\tilde{v}))}=x+\varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)},\quad|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})|=|\tilde{v}|.$$

4. It is a direct consequence of the previous statements. We have

$$\begin{aligned}
|\langle u\rangle_\varepsilon(x,\tilde{v})-\langle u\rangle(x,\tilde{v})|&\leq\frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})}|u(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))-u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s\\
&+\left|\frac{1}{S^\varepsilon(x,\tilde{v})}-\frac{1}{S(x,\tilde{v})}\right|\int_0^{S^\varepsilon(x,\tilde{v})}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s+\frac{1}{S(x,\tilde{v})}\left|\int_{S(x,\tilde{v})}^{S^\varepsilon(x,\tilde{v})}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s\right|
\end{aligned}$$

Our conclusion follows by noticing that

$$\begin{aligned}
\frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})}|u(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))-u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s&\leq\mathrm{Lip}(u)\\
\sup_{0\leq s\leq 2\pi/\omega_0}\left(|\mathcal{X}^\varepsilon(s)-\mathcal{X}(s)|+|\tilde{\mathcal{V}}^\varepsilon(s)-\tilde{\mathcal{V}}(s)|\right)&\\
\leq\varepsilon\mathrm{Lip}(u)\frac{|\tilde{v}|}{\omega_0}\left[2+5\|\partial_x e\|_{L^\infty}|\tilde{v}|+4\pi\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0}|\tilde{v}|\right]&
\end{aligned}$$

and

$$\left|\frac{1}{S^\varepsilon(x,\tilde v)}-\frac{1}{S(x,\tilde v)}\right|\int_0^{S^\varepsilon(x,\tilde v)}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s+\frac{1}{S(x,\tilde v)}\left|\int_{S(x,\tilde v)}^{S^\varepsilon(x,\tilde v)}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s\right|$$
$$\leq 4\varepsilon|\tilde v|\sup_{|\tilde v'|=|\tilde v|}|u(x,\tilde v')|\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}.$$

5. For any $(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3$

$$\begin{aligned}\langle u\rangle(x,\tilde v)-u(x,\tilde v)&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}[u(\mathcal{X}(s;x,\tilde v),\tilde{\mathcal{V}}(s;x,\tilde v))-u(x,\tilde v)]\,\mathrm{d}s\\&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}\int_0^s\frac{\mathrm{d}}{\mathrm{d}\sigma}u(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}s\\&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}\int_0^s(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}s\\&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\theta}{|\omega_c(x)|}}(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}\theta\end{aligned}$$

implying that

$$|\langle u\rangle(x,\tilde v)-u(x,\tilde v)|\leq\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega_0}}|(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))|\,\mathrm{d}\sigma\mathrm{d}\theta.$$

Taking into account that $(x,\tilde v)\to(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))$ is measure preserving, it is easily seen that

$$\|\langle u\rangle-u\|_{L^2}\leq\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\|(b\cdot\nabla u)(\mathcal{X}(\sigma;\cdot,\cdot),\tilde{\mathcal{V}}(\sigma;\cdot,\cdot))\|_{L^2}\,\mathrm{d}\sigma\mathrm{d}\theta=\frac{2\pi}{\omega_0}\|b\cdot\nabla u\|_{L^2}.$$

6. For any $(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3$ we have

$$\begin{aligned}\langle u\rangle(x,\tilde v)&=\frac{1}{2\pi}\int_0^{2\pi}u(x,\mathcal{R}(-\theta,e(x))\tilde v)\,\mathrm{d}\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}u(x,\cos\theta(\tilde v-(\tilde v\cdot e(x))e(x)+\sin\theta(\tilde v\wedge e(x))+(\tilde v\cdot e(x))e(x))\,\mathrm{d}\theta\end{aligned}$$

and therefore $\langle u\rangle\in C^1(\mathbb{R}^3\times\mathbb{R}^3)$, provided that $e\in C^1(\mathbb{R}^3)$. By direct computations we check that all the vector fields $(c^i\cdot\nabla_{x,\tilde v})_{1\leq i\leq 6}$ are in involution with respect to $(\tilde v\wedge e)\cdot\nabla_{\tilde v}$. Thanks to the commutation between the flows of $c^i\cdot\nabla_{x,\tilde v}$ and $(\tilde v\wedge e)\cdot\nabla_{\tilde v}$, we deduce easily that the average operator along the flow of $(\tilde v\wedge e)\cdot\nabla_{\tilde v}$ commutes with the flow of $c^i\cdot\nabla_{x,\tilde v}$, and thus with $c^i\cdot\nabla_{x,\tilde v}$, for $1\leq i\leq 6$. It remains to observe that the average along the flow of $(\tilde v\wedge e)\cdot\nabla_{\tilde v}$ coincides with the average along the flow of $\omega_c(\tilde v\wedge e)\cdot\nabla_{\tilde v}$. The divergences of the vector fields $(c^i\cdot\nabla_{x,\tilde v})_{1\leq i\leq 6}$ are constant along the flow of $b\cdot\nabla_{x,\tilde v}$, implying that

$$\begin{aligned}\mathrm{div}_{x,\tilde v}(\langle u\rangle c^i)&=c^i\cdot\nabla_{x,\tilde v}\ \langle u\rangle+\langle u\rangle\,\mathrm{div}_{x,\tilde v}c^i\\&=\left\langle c^i\cdot\nabla_{x,\tilde v}\ u\right\rangle+\left\langle u\,\mathrm{div}_{x,\tilde v}c^i\right\rangle\\&=\left\langle\mathrm{div}_{x,\tilde v}(uc^i)\right\rangle,\ \ 1\leq i\leq 6.\end{aligned}$$

□

Remark 3.2
For further developments, notice that for any vector field $\xi(x)\cdot\nabla_x$, the vector field

$$c[\xi]\cdot\nabla_{x,\tilde v}=\xi(x)\cdot\nabla_x+\tilde v\wedge(\partial_x e\xi\wedge e)\cdot\nabla_{\tilde v}=\xi(x)\cdot\nabla_x+(\partial_x e\xi\otimes e-e\otimes\partial_x e\xi)\tilde v\cdot\nabla_{\tilde v}$$

is in involution with respect to $(\tilde v\wedge e)\cdot\nabla_{\tilde v}$.

Notice that the periods S, S^ε are left invariant along the flows of $b\cdot\nabla_{x,\tilde v}$, $b^\varepsilon\cdot\nabla_{x,\tilde v}$ respectively, as well as the averages $\langle u\rangle$, $\langle u\rangle_\varepsilon$. If u is a C^1 function, we have

$$\langle b\cdot\nabla_{x,\tilde v}\,u\rangle(x,\tilde v)=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}\frac{\mathrm{d}}{\mathrm{d}s}u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))\,\mathrm{d}s=0,\quad(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3$$

and similarly $\langle b^\varepsilon\cdot\nabla_{x,\tilde v}\,u\rangle_\varepsilon=0$.
We introduce the application $T^\varepsilon:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3\times\mathbb{R}^3$, given by

$$T^\varepsilon(x,\tilde v)=\left(x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)},\tilde v\right),\quad(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3.$$

It is easily seen that for any application $\varphi(x,\tilde v)$ in the kernel of $b\cdot\nabla_{x,\tilde v}$ *i.e.,* $\varphi(x,\tilde v)=\psi(x,|\tilde v\wedge e(x)|,\tilde v\cdot e(x))$, the composition $\varphi\circ T^\varepsilon$ belongs to the kernel of $b^\varepsilon\cdot\nabla_{x,\tilde v}$. Indeed we have

$$(\varphi\circ T^\varepsilon)(x,\tilde v)=\varphi(\tilde x,\tilde v)=\psi(\tilde x,|\tilde v\wedge e(\tilde x)|,\tilde v\cdot e(\tilde x))$$

saying that $\varphi\circ T^\varepsilon$ is left invariant along the flow of $b^\varepsilon\cdot\nabla_{x,\tilde v}$. Another useful formula is

$$\partial_{x,\tilde v}T^\varepsilon b^\varepsilon=\lambda^\varepsilon b\circ T^\varepsilon,\quad\lambda^\varepsilon(x,\tilde v)=\frac{\omega_c(x)}{\omega_c(\tilde x)},\quad\tilde x=x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}\tag{18}$$

and can be obtained by direct computation. Notice that for any $\tilde v\in\mathbb{R}^3,\varepsilon>0$ such that (17) holds true, the application $x\to x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}$ is a C^1 diffeomorphism of $\mathbb{R}^3$. Indeed, the above application is injective on $\mathbb{R}^3$, because

$$x_1+\varepsilon\frac{\tilde v\wedge e(x_1)}{\omega_c(x_1)}=x_2+\varepsilon\frac{\tilde v\wedge e(x_2)}{\omega_c(x_2)}$$

implies

$$|x_1-x_2|=\varepsilon\left|\tilde v\wedge\left(\frac{e(x_2)}{\omega_c(x_2)}-\frac{e(x_1)}{\omega_c(x_1)}\right)\right|\le\varepsilon|\tilde v|\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0}+\frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right)|x_1-x_2|$$

and therefore $x_1=x_2$, thanks to (17). Observe that the Jacobian matrix $I_3+\varepsilon\partial_x\left(\frac{\tilde v\wedge e(x)}{\omega_c(x)}\right)$ is invertible, for any $x\in\mathbb{R}^3$ and therefore the image of $x\to x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}$ is an open non empty set in $\mathbb{R}^3$. It is easily seen that this image is also a closed set in $\mathbb{R}^3$. As $\mathbb{R}^3$ is connected, we deduce that $\{x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)},\ x\in\mathbb{R}^3\}=\mathbb{R}^3$ and that $x\to x+\varepsilon\frac{\tilde v\wedge e(x)}{\omega_c(x)}$ is a C^1 diffeomorphism of $\mathbb{R}^3$.
We establish now the following relation between the averages along the flows of $b\cdot\nabla_{x,\tilde v}$, $b^\varepsilon\cdot\nabla_{x,\tilde v}$.

Proposition 3.2
Consider $(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3$ and $\varepsilon>0$ satisfying (17).

1. *For any $s \in \mathbb{R}$ we have*

$$T^\varepsilon((\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v})) = (\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v}); T^\varepsilon(x,\tilde{v}))$$

and

$$\Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}) = \int_0^{S^\varepsilon(x,\tilde{v})} \lambda^\varepsilon((\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma = S(T^\varepsilon(x,\tilde{v}))$$

where the function Λ^ε is defined by

$$\Lambda^\varepsilon(s;x,\tilde{v}) = \int_0^s \lambda^\varepsilon((\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma.$$

2. *Let $u \in C^0(\mathbb{R}^3 \times \mathbb{R}^3)$ be a function such that $\operatorname{supp} u \subset \{(x,\tilde{v}) : |\tilde{v}| \le R\}$ for some $R > 0$. For any $\varepsilon > 0$ satisfying $\varepsilon R\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x \omega_c\|_{L^\infty}}{\omega_0^2}\right) < 1$ we have*

$$\langle \lambda^\varepsilon u \circ T^\varepsilon\rangle_\varepsilon = \langle \lambda^\varepsilon\rangle_\varepsilon \langle u\rangle \circ T^\varepsilon.$$

3. *Assume that $\nabla_x \omega_c = 0$. Let $u \in C^0(\mathbb{R}^3 \times \mathbb{R}^3)$ be a function such that $\operatorname{supp} u \subset \{(x,\tilde{v}) : |\tilde{v}| \le R\}$ for some $R > 0$. For any $\varepsilon > 0$ satisfying $\varepsilon R\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} < 1$ we have $\langle u \circ T^\varepsilon\rangle_\varepsilon = \langle u\rangle \circ T^\varepsilon$.*

Proof.
1. As $A_x^\varepsilon(x,\tilde{v})$ is well defined for any $(x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\varepsilon > 0$ satisfying (17), and since $|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})| = |\tilde{v}|$, we deduce that $(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$ exists for any $s \in \mathbb{R}$. Let us consider $\Lambda^\varepsilon(s;x,\tilde{v}) = \int_0^s \lambda^\varepsilon((\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma, s \in \mathbb{R}$ and $\gamma^\varepsilon(s;x,\tilde{v}) = T^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$. Observe, thanks to (18), that

$$\begin{aligned}\frac{\mathrm{d}\gamma^\varepsilon}{\mathrm{d}s} &= \partial T^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))b^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))\\ &= \lambda^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))b(T^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})))\\ &= \frac{\mathrm{d}\Lambda^\varepsilon}{\mathrm{d}s}(s;x,\tilde{v})b(\gamma^\varepsilon(s;x,\tilde{v})).\end{aligned}$$

Notice also that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v}); T^\varepsilon(x,\tilde{v})) = \frac{\mathrm{d}\Lambda^\varepsilon}{\mathrm{d}s}(s;x,\tilde{v})b((\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v}); T^\varepsilon(x,\tilde{v}))).$$

Since $\gamma^\varepsilon(s;x,\tilde{v})$ and $(\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v}); T^\varepsilon(x,\tilde{v}))$ coincide at $s = 0$

$$\gamma^\varepsilon(0;x,\tilde{v}) = T^\varepsilon(x,\tilde{v}) = (\mathcal{X}, \tilde{\mathcal{V}})(0; T^\varepsilon(x,\tilde{v}))$$

we deduce that

$$T^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})) = \gamma^\varepsilon(s;x,\tilde{v}) = (\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v}); T^\varepsilon(x,\tilde{v})), \quad s \in \mathbb{R}. \tag{19}$$

Taking $s = S^\varepsilon(x,\tilde{v})$ in (19) we obtain

$$T^\varepsilon(x,\tilde{v}) = T^\varepsilon(\mathcal{X}^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v})) = (\mathcal{X}, \tilde{\mathcal{V}})(\Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}); T^\varepsilon(x,\tilde{v}))$$

saying that $\Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v})$ is a positive period of $(\mathcal{X}, \tilde{\mathcal{V}})(\cdot; T^\varepsilon(x,\tilde{v}))$ and therefore

$$\Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}) \ge S(T^\varepsilon(s,\tilde{v})). \tag{20}$$

As $\lambda^\varepsilon \geq \frac{\omega_0}{\|\omega_c\|_{L^\infty}}$, we have $\Lambda^\varepsilon(s;x,\tilde{v}) \geq s\frac{\omega_0}{\|\omega_c\|_{L^\infty}}$, $s \in \mathbb{R}_+$, implying that $\lim_{s\to+\infty}\Lambda^\varepsilon(s;x,\tilde{v}) = +\infty$. Therefore $s \in \mathbb{R}_+ \to \Lambda^\varepsilon(s;x,\tilde{v}) \in \mathbb{R}_+$ is a strictly increasing bijection, and there is a unique $\overline{s} > 0$ such that $\Lambda^\varepsilon(\overline{s};x,\tilde{v}) = S(T^\varepsilon(x,\tilde{v})) > 0$. Taking $s = \overline{s}$ in (19) leads to

$$T^\varepsilon(\mathcal{X}^\varepsilon(\overline{s};x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(\overline{s};x,\tilde{v})) = (\mathcal{X},\tilde{\mathcal{V}})(S(T^\varepsilon(x,\tilde{v}));T^\varepsilon(x,\tilde{v})) = T^\varepsilon(x,\tilde{v}).$$

As $(x,\tilde{v}),\varepsilon$ satisfy (17), we deduce that $(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\overline{s};x,\tilde{v}) = (x,\tilde{v})$, saying that $\overline{s}$ is a positive period of $(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\cdot;x,\tilde{v})$. Therefore we have $\overline{s} \geq S^\varepsilon(x,\tilde{v})$, implying that

$$S(T^\varepsilon(x,\tilde{v})) = \Lambda^\varepsilon(\overline{s};x,\tilde{v}) \geq \Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}). \tag{21}$$

Combining (20), (21) we obtain

$$\int_0^{S^\varepsilon(x,\tilde{v})} \lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma = \Lambda^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}) = S(T^\varepsilon(x,\tilde{v})).$$

2. Consider first $(x,\tilde{v}) \in \mathbb{R}^3\times\mathbb{R}^3$ such that $|\tilde{v}| > R$. Obviously we have $|\tilde{\mathcal{V}}(s;x,\tilde{v})| = |\tilde{v}| > R$ and $\langle u\rangle(x,\tilde{v}) = \frac{1}{S(x,\tilde{v})}\int_0^{S(x,\tilde{v})} u(\mathcal{X}(s;x,\tilde{v}),\tilde{\mathcal{V}}(s;x,\tilde{v}))\,\mathrm{d}s = 0$. Similarly we have $|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})| = |\tilde{v}| > R$ and

$$\langle\lambda^\varepsilon u\circ T^\varepsilon\rangle_\varepsilon = \frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})} \lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v}))u(\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))\,\mathrm{d}s = 0$$

where $\tilde{\mathcal{X}}^\varepsilon = \mathcal{X}^\varepsilon + \varepsilon\frac{\tilde{\mathcal{V}}^\varepsilon\wedge e(\mathcal{X}^\varepsilon)}{\omega_c(\mathcal{X}^\varepsilon)}$. Therefore our conclusion holds true in this case. Consider now $|\tilde{v}| \leq R$. Thanks to the first statement, we can write

$$\begin{aligned}
\langle\lambda^\varepsilon u\circ T^\varepsilon\rangle_\varepsilon(x,\tilde{v}) &= \frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})} \lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v}))u(\tilde{\mathcal{X}}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))\,\mathrm{d}s\\
&= \frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})} \frac{\mathrm{d}}{\mathrm{d}s}\Lambda^\varepsilon(s;x,\tilde{v})u((\mathcal{X},\tilde{\mathcal{V}})(\Lambda^\varepsilon(s;x,\tilde{v});T^\varepsilon(x,\tilde{v})))\,\mathrm{d}s\\
&= \frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S(T^\varepsilon(x,\tilde{v}))} u((\mathcal{X},\tilde{\mathcal{V}})(\sigma;T^\varepsilon(x,\tilde{v})))\,\mathrm{d}\sigma\\
&= \frac{1}{S^\varepsilon(x,\tilde{v})}\int_0^{S^\varepsilon(x,\tilde{v})} \lambda^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma\,\langle u\rangle(T^\varepsilon(x,\tilde{v}))\\
&= (\langle\lambda^\varepsilon\rangle_\varepsilon\langle u\rangle\circ T^\varepsilon)(x,\tilde{v}).
\end{aligned}$$

3. If $|\tilde{v}| > R$ then as before we have $\langle u\circ T^\varepsilon\rangle_\varepsilon(x,\tilde{v}) = 0 = \langle u\rangle(T^\varepsilon(x,\tilde{v}))$. If $|\tilde{v}| \leq R$, then $\lambda^\varepsilon = 1, \Lambda^\varepsilon(s;x,\tilde{v}) = s$, $T^\varepsilon((\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v})) = (\mathcal{X},\tilde{\mathcal{V}})(s;T^\varepsilon(x,\tilde{v})), S^\varepsilon(x,\tilde{v}) = S(T^\varepsilon(x,\tilde{v}))$ and $\langle u\circ T^\varepsilon\rangle_\varepsilon(x,\tilde{v}) = \langle u\rangle(T^\varepsilon(x,\tilde{v}))$. □

Remark 3.3
The conclusions of the second and third statement in Proposition 3.2 remain valid for $|\tilde{v}| \leq R, \varepsilon R\left(\frac{\|\partial_x e\|_{L^\infty}}{\omega_0} + \frac{\|\nabla_x\omega_c\|_{L^\infty}}{\omega_0^2}\right) < 1$ *if* $u \in C^0(\mathbb{R}^3\times\mathbb{R}^3)$.

When establishing the convergence toward the limit model, we need to introduce a corrector term. More exactly, we need to invert the operator $(\tilde{v}\wedge e(x))\cdot\nabla_{\tilde{v}}$ on the set of zero average functions. We will use the following result.

Proposition 3.3
Let $z \in C^0(\mathbb{R}^3\times\mathbb{R}^3)$ *be a continuous function, of zero average*

$$\langle z\rangle(x,\tilde{v}) = \frac{1}{2\pi}\int_0^{2\pi} z\,(x,\mathcal{R}(-\theta,e(x))\tilde{v})\;\mathrm{d}\theta = 0,\quad (x,\tilde{v}) \in \mathbb{R}^3\times\mathbb{R}^3.$$

1. *There is a unique continuous function u of zero average whose derivative along the flow of $(\tilde{v}\wedge e(x))\cdot\nabla_{\tilde{v}}$ is z*

$$(\tilde{v}\wedge e(x))\cdot\nabla_{\tilde{v}}u = z(x,\tilde{v}),\ \ (x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3.$$

If z is bounded, so is u, and

$$\|u\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}\leq \pi\|z\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))},\ \textit{for any } R_x,R_{\tilde{v}}>0.$$

If $\mathrm{supp}\, z\subset B(R_x)\times B(R_{\tilde{v}})$, *then* $\mathrm{supp}\, u\subset B(R_x)\times B(R_{\tilde{v}})$.

2. *If z is of class C^1, then so is u and we have for any $R_x,R_{\tilde{v}}>0$*

$$\|\nabla_{\tilde{v}}u\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}\leq \pi\sqrt{3}\|\nabla_{\tilde{v}}z\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}$$

$$\|\nabla_x u\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}\leq C\left(\|\nabla_x z\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}+R_{\tilde{v}}\|\nabla_{\tilde{v}}z\|_{C^0(B(R_x)\times B(R_{\tilde{v}}))}\right)$$

for some constant C depending on $\|\partial_x e\|_{L^\infty}$.

Proof.
1. It is easily seen that

$$u(x,\tilde{v})=\frac{1}{2\pi}\int_0^{2\pi}(\theta-2\pi)z(x,\mathcal{R}(-\theta,e(x))\tilde{v})\,\mathrm{d}\theta,\ \ (x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3.$$

2. We appeal to the vector fields $(c^i\cdot\nabla_{x,\tilde{v}}\,)_{1\leq i\leq 6}$ which are in involution with $(\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}$, see the last statement in Proposition 3.1, Remark 3.2. We have

$$(\tilde{v}\wedge e(x))\cdot\nabla_{\tilde{v}}(c^i\cdot\nabla_{x,\tilde{v}}\,u)=c^i\cdot\nabla_{x,\tilde{v}}\,z$$

$$\left\langle c^i\cdot\nabla_{x,\tilde{v}}\,z\right\rangle=c^i\cdot\nabla_{x,\tilde{v}}\,\langle z\rangle=0,\ \ \left\langle c^i\cdot\nabla_{x,\tilde{v}}\,u\right\rangle=c^i\cdot\nabla_{x,\tilde{v}}\,\langle u\rangle=0.$$

By the previous statement we deduce that

$$\|c^i\cdot\nabla_{x,\tilde{v}}\,u\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)}\leq\pi\|c^i\cdot\nabla_{x,\tilde{v}}\,z\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)},\ \ 1\leq i\leq 6$$

and our conclusion follows immediately. □

4 The limit model and convergence result

We concentrate now on the formal derivation of the limit model in (14), as ε goes to 0. We expect that the solution of (14) writes

$$\tilde{f}^\varepsilon=\tilde{f}_\varepsilon\circ T^\varepsilon+\varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon+\varepsilon^2\tilde{f}^2_\varepsilon\circ T^\varepsilon+\dots \tag{22}$$

where $b\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon=0,\ \ \left\langle\tilde{f}^1_\varepsilon\right\rangle=0$. The idea is to split the contributions at any order into average and fluctuation. As $\tilde{f}_\varepsilon\in\ker b\cdot\nabla_{x,\tilde{v}}$, we know that $\tilde{f}_\varepsilon\circ T^\varepsilon\in\ker b^\varepsilon\cdot\nabla_{x,\tilde{v}}$ and thus $\left\langle\tilde{f}_\varepsilon\circ T^\varepsilon\right\rangle_\varepsilon=\tilde{f}_\varepsilon\circ T^\varepsilon$. By Proposition 3.2 we also have $\left\langle\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon\right\rangle_\varepsilon=\langle\lambda^\varepsilon\rangle_\varepsilon\left\langle\tilde{f}^1_\varepsilon\right\rangle\circ T^\varepsilon=0$ and therefore

$$\left\langle\tilde{f}^\varepsilon\right\rangle_\varepsilon=\tilde{f}_\varepsilon\circ T^\varepsilon+\mathcal{O}(\varepsilon^2),\ \ \tilde{f}^\varepsilon-\left\langle\tilde{f}^\varepsilon\right\rangle_\varepsilon=\varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon+\mathcal{O}(\varepsilon^2).$$

Accordingly, at the leading order, the particle density $\tilde{f}^\varepsilon$ has no fluctuation (provided that the initial condition will be well prepared) and the averages at the orders $1,\varepsilon$ combine together

in $\tilde{f}_\varepsilon \circ T^\varepsilon$. For any smooth compactly supported particle density $\tilde{f} = \tilde{f}(x,\tilde{v})$ we introduce the notations, motivated by (15), (16)

$$\begin{aligned} a[\tilde{f}]\cdot\nabla_{x,\tilde{v}} \ &= \left(\frac{E[\tilde{f}]\wedge e}{B} - A_x(x,\tilde{v})\right)\cdot\nabla_x - \partial_x\left(\frac{E[\tilde{f}]\wedge e}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}} \\ &\quad + \frac{1}{4\pi\epsilon_0 B}\left[\mathrm{div}_x \int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes j[\tilde{f}](x')\,\mathrm{d}x'\wedge e(x)\right]\cdot\nabla_{\tilde{v}} \\ c_0[\tilde{f}]\cdot\nabla_{x,\tilde{v}} \ &= (\tilde{v}\cdot e)\, e\cdot\nabla_x + \frac{q}{m}(E[\tilde{f}]\cdot e)\, e\cdot\nabla_{\tilde{v}} - [\tilde{v}\wedge\partial_x e(\tilde{v}\wedge e)]\cdot\nabla_{\tilde{v}} \qquad (23) \\ c_1[\tilde{f}]\cdot\nabla_{x,\tilde{v}} \ &= \lim_{\varepsilon\searrow 0}\frac{\partial T^\varepsilon c^\varepsilon[\tilde{f}\circ T^\varepsilon] - \lambda^\varepsilon(x,\tilde{v})(c_0[\tilde{f}]\circ T^\varepsilon)}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\,. \qquad (24) \end{aligned}$$

The last notation is justified by the expansion

$$c^\varepsilon[\tilde{f}\circ T^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,(\tilde{f}\circ T^\varepsilon) = \lambda^\varepsilon(x,\tilde{v})(c_0[\tilde{f}]\cdot\nabla_{x,\tilde{v}}\,\tilde{f})\circ T^\varepsilon + \varepsilon(c_1[\tilde{f}]\cdot\nabla_{x,\tilde{v}}\,\tilde{f})\circ T^\varepsilon + \mathcal{O}(\varepsilon^2) \quad (25)$$

for any smooth particle density $\tilde{f}$, which will be used in the sequel. The expression for the vector field $c_1[\tilde{f}]\cdot\nabla_{x,\tilde{v}}$ follows by straightforward computations, see Proposition 5.6, using the definition

$$c^\varepsilon[\tilde{f}]\cdot\nabla_{x,\tilde{v}} \ = (\tilde{v}\cdot e(\tilde{x}))\, e(\tilde{x})\cdot\nabla_x + \frac{q}{m}(E[\tilde{f}]\cdot e(x))\, e(x)\cdot\nabla_{\tilde{v}} - \omega_c(x)\tilde{v}\wedge\frac{e(\tilde{x})-e(x)}{\varepsilon}\cdot\nabla_{\tilde{v}}.$$

Taking the average of (14) along the flow $(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)$ yields

$$\partial_t\left\langle\tilde{f}^\varepsilon(t)\right\rangle_\varepsilon + \left\langle c^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon(t)\right\rangle_\varepsilon + \varepsilon\left\langle a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon(t)\right\rangle_\varepsilon = 0. \quad (26)$$

Motivated by (22), we have

$$\partial_t\left\langle\tilde{f}^\varepsilon\right\rangle_\varepsilon = (\partial_t\tilde{f}_\varepsilon)\circ T^\varepsilon + \mathcal{O}(\varepsilon^2).$$

For the contribution of the term $\varepsilon a^\varepsilon[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon$ observe that

$$\begin{aligned} a^\varepsilon[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon &= a[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon + \mathcal{O}(\varepsilon) \qquad (27)\\ &= a[\tilde{f}_\varepsilon\circ T^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,(\tilde{f}_\varepsilon\circ T^\varepsilon) + \mathcal{O}(\varepsilon) \\ &= \lambda^\varepsilon(a[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon)\circ T^\varepsilon + \mathcal{O}(\varepsilon). \end{aligned}$$

It remains to analyze the contribution of $c^\varepsilon[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon$. Since $\left\langle\tilde{f}^1_\varepsilon\right\rangle = 0$, we have $\rho[\tilde{f}^1_\varepsilon] = 0, E[\tilde{f}^1_\varepsilon] = 0, E[\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon] = \mathcal{O}(\varepsilon)$ and therefore

$$\begin{aligned} c^\varepsilon[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon &= c^\varepsilon[\tilde{f}_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,(\tilde{f}_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon) + \mathcal{O}(\varepsilon^2) \qquad (28)\\ &= c^\varepsilon[\tilde{f}_\varepsilon\circ T^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,(\tilde{f}_\varepsilon\circ T^\varepsilon) + \varepsilon c^\varepsilon[\tilde{f}_\varepsilon\circ T^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,(\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon) + \mathcal{O}(\varepsilon^2) \\ &= \lambda^\varepsilon(c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon)\circ T^\varepsilon + \varepsilon\lambda^\varepsilon(c_1[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon)\circ T^\varepsilon \\ &\quad + \varepsilon\lambda^\varepsilon(c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^1_\varepsilon)\circ T^\varepsilon + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Thanks to the second statement in Proposition 3.2, we have

$$\begin{aligned} \varepsilon\left\langle a^\varepsilon[\tilde{f}^\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}^\varepsilon\right\rangle_\varepsilon &= \varepsilon\left\langle\lambda^\varepsilon(a[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon)\circ T^\varepsilon + \mathcal{O}(\varepsilon^2)\right\rangle_\varepsilon \qquad (29)\\ &= \varepsilon\left\langle\lambda^\varepsilon\right\rangle_\varepsilon\left\langle a[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon\right\rangle\circ T^\varepsilon + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon\left\langle a[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon\right\rangle\circ T^\varepsilon + \mathcal{O}(\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned}\left\langle c^\varepsilon[\tilde f^\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^\varepsilon\right\rangle_\varepsilon &= \left\langle \lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon\right\rangle_\varepsilon + \varepsilon\left\langle \lambda^\varepsilon(c_1[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon\right\rangle_\varepsilon \\ &+ \varepsilon\left\langle \lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon)\circ T^\varepsilon\right\rangle_\varepsilon + \mathcal{O}(\varepsilon^2) \\ &= \langle\lambda^\varepsilon\rangle_\varepsilon\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon + \varepsilon\,\langle\lambda^\varepsilon\rangle_\varepsilon\left\langle c_1[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon \\ &+ \varepsilon\,\langle\lambda^\varepsilon\rangle_\varepsilon\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon\right\rangle\circ T^\varepsilon \\ &= \left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon + \varepsilon\left\langle c_1[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon \\ &+ \varepsilon\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon\right\rangle\circ T^\varepsilon + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{30}$$

where in the last equality we have used the relation $\langle\lambda^\varepsilon\rangle_\varepsilon = 1+\mathcal{O}(\varepsilon^2)$. By combining (26), (29), (30) and keeping all the terms up to the second order, we find the following model for the particle density $\tilde f_\varepsilon$

$$\partial_t\tilde f_\varepsilon + \left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle + \varepsilon\left\langle (a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle + \varepsilon\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon\right\rangle = 0,\ \ b\cdot\nabla\tilde f_\varepsilon = 0. \tag{31}$$

We need another equation for the fluctuation $\tilde f^1_\varepsilon$. Replacing in (14) the expressions in (27), (28) yields

$$\begin{aligned}&\partial_t\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon(\partial_t\tilde f^1_\varepsilon\circ T^\varepsilon) + \lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon + \varepsilon\lambda^\varepsilon((a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon \\ &+ \varepsilon\lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon)\circ T^\varepsilon + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde v}\,(\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde f^1_\varepsilon\circ T^\varepsilon + \varepsilon^2\tilde f^2_\varepsilon\circ T^\varepsilon) = \mathcal{O}(\varepsilon^2).\end{aligned} \tag{32}$$

Taking the difference between (32) and (31) (after composition with T^ε) leads to

$$\begin{aligned}&\varepsilon\lambda^\varepsilon(\partial_t\tilde f^1_\varepsilon\circ T^\varepsilon) + \lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon - \left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon \\ &\quad+\varepsilon\lambda^\varepsilon(a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon - \varepsilon\left\langle (a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon \\ &\quad+\varepsilon\lambda^\varepsilon(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon)\circ T^\varepsilon - \varepsilon\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon\right\rangle\circ T^\varepsilon \\ &\quad+ b^\varepsilon\cdot\nabla_{x,\tilde v}\,(\lambda^\varepsilon\tilde f^1_\varepsilon\circ T^\varepsilon + \varepsilon\tilde f^2_\varepsilon\circ T^\varepsilon) = \mathcal{O}(\varepsilon^2)\end{aligned}$$

because $\tilde f_\varepsilon\circ T^\varepsilon \in \ker b^\varepsilon\cdot\nabla_{x,\tilde v}$. As $\lambda^\varepsilon = 1+\mathcal{O}(\varepsilon)$, the previous equation also writes

$$\begin{aligned}&\varepsilon\partial_t\tilde f^1_\varepsilon\circ T^\varepsilon + \lambda^\varepsilon\left(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon - \left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\right)\circ T^\varepsilon + (\lambda^\varepsilon-1)\left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\circ T^\varepsilon \\ &+\varepsilon\left((a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon - \left\langle (a[\tilde f_\varepsilon]+c_1[\tilde f_\varepsilon])\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon\right\rangle\right)\circ T^\varepsilon \\ &+\varepsilon\left(c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon - \left\langle c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon\right\rangle\right)\circ T^\varepsilon \\ &+ b^\varepsilon\cdot\nabla_{x,\tilde v}\left[\tilde f^1_\varepsilon\circ T^\varepsilon - \varepsilon\frac{\tilde v\wedge e(\tilde x)}{\omega_c(\tilde x)}\cdot\frac{\nabla_x\omega_c(\tilde x)}{\omega_c(\tilde x)}\tilde f^1_\varepsilon\circ T^\varepsilon + \varepsilon\tilde f^2_\varepsilon\circ T^\varepsilon\right] = \mathcal{O}(\varepsilon^2).\end{aligned} \tag{33}$$

Notice that by (18) we have

$$b^\varepsilon\cdot\nabla_{x,\tilde v}\,(\tilde f^1_\varepsilon\circ T^\varepsilon) = b^\varepsilon\cdot\ {}^t\partial T^\varepsilon(\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon)\circ T^\varepsilon = \lambda^\varepsilon(b\cdot\nabla_{x,\tilde v}\,\tilde f^1_\varepsilon)\circ T^\varepsilon$$

and

$$\begin{aligned}\varepsilon b^\varepsilon \cdot \nabla_{x,\tilde v}\ &\left(\tilde f^2_\varepsilon \circ T^\varepsilon - \frac{\tilde v \wedge e(\tilde x)}{\omega_c(\tilde x)} \cdot \frac{\nabla_x \omega_c(\tilde x)}{\omega_c(\tilde x)} \tilde f^1_\varepsilon \circ T^\varepsilon\right)\\ &= \varepsilon b^\varepsilon \cdot \nabla_{x,\tilde v}\ \left[\left(\tilde f^2_\varepsilon - \frac{\tilde v \wedge e(x)}{\omega_c(x)} \cdot \frac{\nabla_x \omega_c(x)}{\omega_c(x)} \tilde f^1_\varepsilon\right) \circ T^\varepsilon\right]\\ &= \varepsilon \lambda^\varepsilon \left[b \cdot \nabla \left(\tilde f^2_\varepsilon - \frac{\tilde v \wedge e(x)}{\omega_c(x)} \cdot \frac{\nabla_x \omega_c(x)}{\omega_c(x)} \tilde f^1_\varepsilon\right)\right] \circ T^\varepsilon.\end{aligned}$$

The equality (33) suggests that the fluctuation $\tilde f^1_\varepsilon$ satisfies the problem

$$c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\ \tilde f_\varepsilon - \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\ \tilde f_\varepsilon \right\rangle + b \cdot \nabla_{x,\tilde v}\ \tilde f^1_\varepsilon = 0, \ \ \left\langle \tilde f^1_\varepsilon \right\rangle = 0. \tag{34}$$

Moreover, considering the contributions of order ε in (33) leads to the definition of the corrector $\tilde f^2_\varepsilon$

$$\begin{aligned}&\partial_t \tilde f^1_\varepsilon - \frac{\tilde v \wedge e(x)}{\omega_c(x)} \cdot \frac{\nabla_x \omega_c(x)}{\omega_c(x)} \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\ \tilde f_\varepsilon \right\rangle + (a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\ \tilde f_\varepsilon \\ &- \left\langle (a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\ \tilde f_\varepsilon \right\rangle + c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\ \tilde f^1_\varepsilon - \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\ \tilde f^1_\varepsilon \right\rangle \\ &+ b \cdot \nabla_{x,\tilde v}\ \left(\tilde f^2_\varepsilon - \frac{\tilde v \wedge e(x)}{\omega_c(x)} \cdot \frac{\nabla_x \omega_c(x)}{\omega_c(x)} \tilde f^1_\varepsilon\right) = 0.\end{aligned} \tag{35}$$

We have obtained the limit model (31), (34), to be supplemented by an initial condition. The well posedness of this model will be established in Section 6. We will discuss the existence/uniqueness of smooth solutions on any time interval $[0, T]$, if ε is small enough cf. Theorem 1.1. Notice that (31), (34) is a regular reformulation of the Vlasov-Poisson system with strong external magnetic field. Indeed, replacing ε by 0 in (31) leads to the zero order model

$$\partial_t \tilde f_0 + \left\langle (\tilde v \cdot e)\ e \cdot \nabla_x \tilde f_0 + \frac{q}{m} (E[\tilde f_0] \cdot e)\ e \cdot \nabla_{\tilde v} \tilde f_0 - [\tilde v \wedge (\partial_x e(\tilde v \wedge e)] \cdot \nabla_{\tilde v} \tilde f_0 \right\rangle = 0, \ \ b \cdot \nabla \tilde f_0 = 0.$$

We are ready now to establish rigorously the second order approximation $\tilde f^\varepsilon = \tilde f_\varepsilon \circ T^\varepsilon + \varepsilon \lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon + \mathcal{O}(\varepsilon^2)$.

Proof. (of Theorem 1.2)
Clearly we have

$$\sup_{\varepsilon > 0} \{\|\rho[\tilde g_\varepsilon]\|_{L^1(\mathbb{R}^3)} + \|\rho[\tilde g_\varepsilon]\|_{L^\infty(\mathbb{R}^3)}\} < +\infty.$$

By Proposition 3.3 we obtain supp $(\tilde g^1_\varepsilon) \subset \{(\tilde x, \tilde v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\tilde x| \le R_{\tilde x}, |\tilde v| \le R_{\tilde v}\}$ and it is easily seen that

$$\text{supp } f^\varepsilon(0) \subset \left\{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |v| \le R^\varepsilon_v := R_{\tilde v} + \varepsilon \frac{\|E[\tilde g_\varepsilon]\|_{L^\infty}}{B_0}, \ \ |x| \le R^\varepsilon_x := R_{\tilde x} + \varepsilon \frac{R^\varepsilon_v}{\omega_0}\right\}.$$

Therefore the particle densities $(f^\varepsilon(0))_{0<\varepsilon\le 1}$ are uniformly compactly supported and $\sup_{\varepsilon>0} \|f^\varepsilon(0)\|_{C^2(\mathbb{R}^3\times\mathbb{R}^3)} < +\infty$. Notice that $\inf_{\varepsilon>0} T(f^\varepsilon(0)) > 0$, see Theorem 2.1, and thus we can pick a time $0 < T < \inf_{\varepsilon>0} T(f^\varepsilon(0))$. By Theorem 2.1 we know that $(f^\varepsilon)_\varepsilon$ are uniformly compactly supported in $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and

$$\sup_{\varepsilon>0, t\in[0,T]} \left[\|f^\varepsilon(t)\|_{C^3(\mathbb{R}^3\times\mathbb{R}^3)} + \|\partial_t f^\varepsilon(t)\|_{C^2(\mathbb{R}^3\times\mathbb{R}^3)} + \|E[f^\varepsilon(t)]\|_{C^3(\mathbb{R}^3)}\right] < +\infty.$$

We deduce that $(\tilde{f}^\varepsilon)_\varepsilon$ are uniformly compactly supported in $[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$ and

$$\sup_{\varepsilon>0,t\in[0,T]}\left[\|\tilde{f}^\varepsilon(t)\|_{C^3(\mathbb{R}^3\times\mathbb{R}^3)}+\|\partial_t\tilde{f}^\varepsilon(t)\|_{C^2(\mathbb{R}^3\times\mathbb{R}^3)}\right]<+\infty.$$

Similarly, the particle densities $(\langle\tilde{g}_\varepsilon\rangle)_\varepsilon$ are uniformly compactly supported in $\mathbb{R}^3\times\mathbb{R}^3$, $\sup_{\varepsilon>0}\|\langle\tilde{g}_\varepsilon\rangle\|_{C^3(\mathbb{R}^3\times\mathbb{R}^3)}<+\infty$, and therefore we know by Theorem 1.1 that the particle densities $(\tilde{f}_\varepsilon)_\varepsilon$ are uniformly compactly supported in $[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$ and

$$\sup_{\varepsilon>0,t\in[0,T]}\left[\|\tilde{f}_\varepsilon(t)\|_{C^3(\mathbb{R}^3\times\mathbb{R}^3)}+\|\partial_t\tilde{f}_\varepsilon(t)\|_{C^2(\mathbb{R}^3\times\mathbb{R}^3)}+\|E[\tilde{f}_\varepsilon(t)]\|_{C^3(\mathbb{R}^3)}\right]<+\infty.$$

Clearly we have $c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon, \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle\in C^2_c(\mathbb{R}^3\times\mathbb{R}^3)$ and by Proposition 3.3 applied to

$$\frac{c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon}{\omega_c}-\left\langle\frac{c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon}{\omega_c}\right\rangle+(\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}\tilde{f}^1_\varepsilon=0,\ \ \left\langle\tilde{f}^1_\varepsilon\right\rangle=0$$

we deduce that $\tilde{f}^1_\varepsilon(t)\in C^2_c(\mathbb{R}^3\times\mathbb{R}^3)$. We also have $\partial_t(c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon), \partial_t\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle\in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)$ and by appealing one more time to Proposition 3.3 (noticing that $\left\langle\partial_t\tilde{f}^1_\varepsilon\right\rangle=\partial_t\left\langle\tilde{f}^1_\varepsilon\right\rangle=0$), we obtain $\partial_t\tilde{f}^1_\varepsilon(t)\in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)$ and

$$\sup_{\varepsilon>0,t\in[0,T]}\left[\|\tilde{f}^1_\varepsilon(t)\|_{C^2(\mathbb{R}^3\times\mathbb{R}^3)}+\|\partial_t\tilde{f}^1_\varepsilon(t)\|_{C^1(\mathbb{R}^3\times\mathbb{R}^3)}\right]<+\infty.$$

Finally we define the corrector $\tilde{f}^2_\varepsilon$ by solving (35). More exactly we define $\tilde{f}^2_\varepsilon=\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\cdot\frac{\nabla_x\omega_c}{\omega_c(x)}\tilde{f}^1_\varepsilon+u_\varepsilon$, where $\langle u_\varepsilon\rangle=0$ and

$$\begin{aligned}\partial_t\tilde{f}^1_\varepsilon-\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\cdot\frac{\nabla_x\omega_c}{\omega_c(x)}\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle+(a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\\-\left\langle(a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle+c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon-\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon\right\rangle+b\cdot\nabla_{x,\tilde{v}}\ u_\varepsilon=0.\end{aligned}$$

Thanks to Proposition 3.3, see also the expression of the field $c_1[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}$ cf. Proposition 5.6, it is easily seen that

$$\sup_{\varepsilon>0,t\in[0,T]}[\|u_\varepsilon(t)\|_{C^1(\mathbb{R}^3\times\mathbb{R}^3)}+\|\partial_t u_\varepsilon(t)\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)}]<+\infty$$

which implies

$$\sup_{\varepsilon>0,t\in[0,T]}[\|\tilde{f}^2_\varepsilon(t)\|_{C^1(\mathbb{R}^3\times\mathbb{R}^3)}+\|\partial_t\tilde{f}^2_\varepsilon(t)\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)}]<+\infty. \tag{36}$$

Multiplying (34) by λ^ε, one gets after composition with T^ε

$$\lambda^\varepsilon(c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon)\circ T^\varepsilon-\lambda^\varepsilon\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle\circ T^\varepsilon+b^\varepsilon\cdot\nabla_{x,\tilde{v}}\ (\tilde{f}^1_\varepsilon\circ T^\varepsilon)=0. \tag{37}$$

Similarly, multiplying (35) by $\varepsilon\lambda^\varepsilon$ yields, after composition with T^ε

$$\begin{aligned}\partial_t(\varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon)-\varepsilon\lambda^\varepsilon\frac{\tilde{v}\wedge e(\tilde{x})}{\omega_c(\tilde{x})}\cdot\frac{\nabla_x\omega_c(\tilde{x})}{\omega_c(\tilde{x})}\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle\circ T^\varepsilon \qquad (38)\\+\varepsilon\lambda^\varepsilon((a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon)\circ T^\varepsilon-\varepsilon\lambda^\varepsilon\left\langle(a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle\circ T^\varepsilon\\+\varepsilon\lambda^\varepsilon(c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon)\circ T^\varepsilon-\varepsilon\lambda^\varepsilon\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon\right\rangle\circ T^\varepsilon\\+\varepsilon b^\varepsilon\cdot\nabla_{x,\tilde{v}}\ (\tilde{f}^2_\varepsilon\circ T^\varepsilon)-\varepsilon b^\varepsilon\cdot\nabla_{x,\tilde{v}}\ \left[\left(\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\cdot\frac{\nabla_x\omega_c(x)}{\omega_c(x)}\tilde{f}^1_\varepsilon\right)\circ T^\varepsilon\right]=0.\end{aligned}$$

A straightforward computation shows that the functions

$$\delta_\varepsilon := b^\varepsilon \cdot \nabla_{x,\tilde v} \left[\left(\lambda^\varepsilon - 1 + \varepsilon \frac{\tilde v \wedge e(\tilde x)}{\omega_c(\tilde x)} \cdot \frac{\nabla_x \omega_c(\tilde x)}{\omega_c(\tilde x)} \right) (\tilde f^1_\varepsilon \circ T^\varepsilon) \right] + \left(\lambda^\varepsilon - 1 + \varepsilon \lambda^\varepsilon \frac{\tilde v \wedge e(\tilde x)}{\omega_c(\tilde x)} \cdot \frac{\nabla_x \omega_c(\tilde x)}{\omega_c(\tilde x)} \right) \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon \right\rangle \circ T^\varepsilon$$

satisfy

$$\sup_{\varepsilon>0, t\in[0,T]} \frac{\|\delta_\varepsilon(t)\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)}}{\varepsilon^2} < +\infty. \tag{39}$$

Combining (37), (38) we obtain

$$\begin{aligned} \partial_t(\varepsilon\lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon + \varepsilon^2 \tilde f^2_\varepsilon \circ T^\varepsilon) + \lambda^\varepsilon (c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon - \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon \right\rangle \circ T^\varepsilon \\ + \varepsilon((a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon - \varepsilon \left\langle (a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon \right\rangle \circ T^\varepsilon \\ + \varepsilon(c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^1_\varepsilon) \circ T^\varepsilon - \varepsilon \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^1_\varepsilon \right\rangle \circ T^\varepsilon \\ + \frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde v} \left(\varepsilon\lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon + \varepsilon^2 \tilde f^2_\varepsilon \circ T^\varepsilon \right) \\ = \varepsilon^2 \partial_t \tilde f^2_\varepsilon \circ T^\varepsilon + \delta_\varepsilon + \tilde\delta_\varepsilon \end{aligned} \tag{40}$$

where

$$\begin{aligned} \tilde\delta_\varepsilon := \varepsilon(1-\lambda_\varepsilon) \left[((a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon - \left\langle (a[\tilde f_\varepsilon] + c_1[\tilde f_\varepsilon]) \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon \right\rangle \circ T^\varepsilon \right] \\ + \varepsilon(1-\lambda_\varepsilon) \left[(c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^1_\varepsilon) \circ T^\varepsilon - \left\langle c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^1_\varepsilon \right\rangle \circ T^\varepsilon \right]. \end{aligned}$$

Clearly, the functions $(\tilde\delta_\varepsilon)_\varepsilon$ satisfy

$$\sup_{\varepsilon>0, t\in[0,T]} \frac{\|\tilde\delta_\varepsilon(t)\|_{C^0(\mathbb{R}^3\times\mathbb{R}^3)}}{\varepsilon^2} < +\infty. \tag{41}$$

Adding to (40) the equation (31) satisfied by $\tilde f_\varepsilon$ (after composition with T^ε), together with the constraint $\frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde v}\, (\tilde f_\varepsilon \circ T^\varepsilon) = 0$, we deduce

$$\begin{aligned} \partial_t(\tilde f_\varepsilon \circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon + \varepsilon^2 \tilde f^2_\varepsilon \circ T^\varepsilon) + \lambda^\varepsilon (c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon + \varepsilon(c_1[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon \\ + \varepsilon(c_0[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^1_\varepsilon) \circ T^\varepsilon + \varepsilon(a[\tilde f_\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f_\varepsilon) \circ T^\varepsilon \\ + \frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde v} \left(\tilde f_\varepsilon \circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon + \varepsilon^2 \tilde f^2_\varepsilon \circ T^\varepsilon \right) \\ = \varepsilon^2 \partial_t \tilde f^2_\varepsilon \circ T^\varepsilon + \delta_\varepsilon + \tilde\delta_\varepsilon. \end{aligned} \tag{42}$$

We compare (42) to the model of the particle density $\tilde f_\varepsilon$

$$\partial_t \tilde f^\varepsilon + c^\varepsilon[\tilde f^\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^\varepsilon + \varepsilon a^\varepsilon[\tilde f^\varepsilon] \cdot \nabla_{x,\tilde v}\, \tilde f^\varepsilon + \frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde v}\, \tilde f^\varepsilon = 0. \tag{43}$$

We are looking for an estimate of the L^2 norm of

$$r^\varepsilon := \tilde f^\varepsilon - \tilde f_\varepsilon \circ T^\varepsilon - \varepsilon\lambda^\varepsilon \tilde f^1_\varepsilon \circ T^\varepsilon - \varepsilon^2 \tilde f^2_\varepsilon \circ T^\varepsilon.$$

Taking the difference between (43) and (42) yields

$$\partial_t r^\varepsilon + \left(c^\varepsilon[\tilde f^\varepsilon] + \varepsilon a^\varepsilon[\tilde f^\varepsilon] + \frac{b^\varepsilon}{\varepsilon} \right) \cdot \nabla_{x,\tilde v}\, r^\varepsilon + \mathcal{T}^\varepsilon_c + \varepsilon \mathcal{T}^\varepsilon_a = -\varepsilon^2 \partial_t \tilde f^2_\varepsilon \circ T^\varepsilon - \delta_\varepsilon - \tilde\delta_\varepsilon \tag{44}$$

where the transport terms $\mathcal{T}_c^\varepsilon, \mathcal{T}_a^\varepsilon$ write

$$\mathcal{T}_c^\varepsilon(t,x,\tilde v) = c^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla_{x,\tilde v}\,(\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde f_\varepsilon^1\circ T^\varepsilon + \varepsilon^2\tilde f_\varepsilon^2\circ T^\varepsilon) - \lambda^\varepsilon(c_0[\tilde f_\varepsilon(t)]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon$$
$$-\varepsilon\left(c_1[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon + c_0[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon^1\right)\circ T^\varepsilon$$

$$\mathcal{T}_a^\varepsilon(t,x,\tilde v) := a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla_{x,\tilde v}\,(\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde f_\varepsilon^1\circ T^\varepsilon + \varepsilon^2\tilde f_\varepsilon^2\circ T^\varepsilon) - (a[\tilde f_\varepsilon]\cdot\nabla_{x,\tilde v}\,\tilde f_\varepsilon)\circ T^\varepsilon.$$

By Remark 3.1 we know that the vector field $\left(c^\varepsilon[\tilde f^\varepsilon] + \varepsilon a^\varepsilon[\tilde f^\varepsilon] + \frac{b^\varepsilon}{\varepsilon}\right)\cdot\nabla_{x,\tilde v}$ is divergence free and therefore, multiplying (44) by r^ε and integrating by parts yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(r^\varepsilon)^2\,\mathrm{d}\tilde v\mathrm{d}x + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{T}_c^\varepsilon(t,x,\tilde v)r^\varepsilon(t,x,\tilde v)\,\mathrm{d}\tilde v\mathrm{d}x + \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{T}_a^\varepsilon(t,x,\tilde v)r^\varepsilon(t,x,\tilde v)\,\mathrm{d}\tilde v\mathrm{d}x \tag{45}$$
$$= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\varepsilon^2\partial_t\tilde f_\varepsilon^2\circ T^\varepsilon + \delta_\varepsilon + \tilde\delta_\varepsilon\right)r^\varepsilon(t,x,\tilde v)\,\mathrm{d}\tilde v\mathrm{d}x.$$

We denote by C any constant depending on $m, \epsilon_0, q, T, \omega_c, e$ and the uniform bounds satisfied by the initial particle densities $(\tilde g_\varepsilon)_\varepsilon$, but not on ε. The bounds (36), (39), (41) and the uniform compactness of the supports of $\tilde f_\varepsilon, \tilde f_\varepsilon^1, \tilde f_\varepsilon^2$ imply immediately that

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\varepsilon^2\partial_t\tilde f_\varepsilon^2\circ T^\varepsilon + \delta_\varepsilon + \tilde\delta_\varepsilon\right)r^\varepsilon(t)\,\mathrm{d}\tilde v\mathrm{d}x\right| \le C\varepsilon^2\|r^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)},\ \ t\in[0,T],\ 0<\varepsilon\le\varepsilon_T. \tag{46}$$

We claim that the following inequalities hold true

$$\|\mathcal{T}_c^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \le C(\varepsilon^2 + \|r^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}),\ \ t\in[0,T],\ 0<\varepsilon\le\varepsilon_T \tag{47}$$

and

$$\|\mathcal{T}_a^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \le C(\varepsilon + \|r^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}),\ \ t\in[0,T],\ 0<\varepsilon\le\varepsilon_T. \tag{48}$$

Let us analyze first (48). By Proposition 3.1, we know that $|A_x^\varepsilon(x,\tilde v) - A_x(x,\tilde v)| \le C\varepsilon|\tilde v|^3$, and thanks to the uniform bounds of $(\tilde f^\varepsilon)_\varepsilon, (\tilde f_\varepsilon)_\varepsilon, (\tilde f_\varepsilon^1)_\varepsilon, (\tilde f_\varepsilon^2)_\varepsilon$ (together with the uniform compactness of their supports), we deduce that

$$\begin{aligned}
&\|a^\varepsilon[\tilde f^\varepsilon]\cdot\nabla(\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde f_\varepsilon^1\circ T^\varepsilon + \varepsilon^2\tilde f_\varepsilon^2\circ T^\varepsilon) - (a[\tilde f_\varepsilon]\cdot\nabla\tilde f_\varepsilon)\circ T^\varepsilon\|_{L^2}\\
&\quad\le C\varepsilon + \|a[\tilde f_\varepsilon]\cdot\nabla(\tilde f_\varepsilon\circ T^\varepsilon) - (a[\tilde f_\varepsilon]\cdot\nabla\tilde f_\varepsilon)\circ T^\varepsilon\|_{L^2}\\
&\quad\le C\varepsilon + \|(\partial T^\varepsilon a[\tilde f^\varepsilon] - a[\tilde f_\varepsilon]\circ T^\varepsilon)\cdot(\nabla\tilde f_\varepsilon)\circ T^\varepsilon\|_{L^2}\\
&\quad\le C\varepsilon + \|(a[\tilde f^\varepsilon] - a[\tilde f_\varepsilon\circ T^\varepsilon])\cdot(\nabla\tilde f_\varepsilon)\circ T^\varepsilon\|_{L^2}.
\end{aligned}$$

By elliptic regularity results, the quantity

$$\|(a[\tilde f^\varepsilon] - a[\tilde f_\varepsilon\circ T^\varepsilon])\cdot(\nabla\tilde f_\varepsilon)\circ T^\varepsilon\|_{L^2}$$

is bounded by the L^2 norms of charge and current densities

$$\|\rho[\tilde f^\varepsilon] - \rho[\tilde f_\varepsilon\circ T^\varepsilon]\|_{L^2} + \|j[\tilde f^\varepsilon] - j[\tilde f_\varepsilon\circ T^\varepsilon]\|_{L^2}$$

and thus by the L^2 norms of the particle densities $\|\tilde f^\varepsilon - \tilde f_\varepsilon\circ T^\varepsilon\|_{L^2} \le \|r^\varepsilon\|_{L^2} + C\varepsilon$, saying that (48) holds true. We concentrate now on (47). It is easily seen that

$$\begin{aligned}
&\|(c^\varepsilon[\tilde f^\varepsilon] - c^\varepsilon[\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde f_\varepsilon^1\circ T^\varepsilon])\cdot\nabla(\tilde f_\varepsilon\circ T^\varepsilon + \varepsilon\lambda^\varepsilon\tilde f_\varepsilon^1\circ T^\varepsilon + \varepsilon^2\tilde f_\varepsilon^2\circ T^\varepsilon)\|_{L^2} \qquad (49)\\
&\quad\le C\|\tilde f^\varepsilon - \tilde f_\varepsilon\circ T^\varepsilon - \varepsilon\lambda^\varepsilon\tilde f_\varepsilon^1\circ T^\varepsilon\|_{L^2}\\
&\quad\le C(\|r^\varepsilon\|_{L^2} + \varepsilon^2).
\end{aligned}$$

As the particle density $\tilde{f}_\varepsilon^1$ has zero average, we deduce that $\rho[\tilde{f}_\varepsilon^1] = 0$ and therefore

$$\|E[\varepsilon\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon]\|_{L^2} = \varepsilon\|E[\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon - \tilde{f}_\varepsilon^1]\|_{L^2} \leq C\varepsilon\|\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon - \tilde{f}_\varepsilon^1\|_{L^2} \leq C\varepsilon^2.$$

The above estimate allows us to write

$$\|(c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon] - c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon]) \cdot \nabla(\tilde{f}_\varepsilon \circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon + \varepsilon^2 \tilde{f}_\varepsilon^2 \circ T^\varepsilon)\|_{L^2} \leq C\varepsilon^2. \tag{50}$$

We are done if we establish the estimate

$$\begin{aligned}\|c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon] \cdot \nabla(\tilde{f}_\varepsilon \circ T^\varepsilon + \varepsilon\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon) - \lambda^\varepsilon(c_0[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon) \circ T^\varepsilon \\ - \varepsilon(c_1[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon + c_0[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon^1) \circ T^\varepsilon\|_{L^2} \leq C\varepsilon^2\end{aligned} \tag{51}$$

since, in that case (47) will be a consequence of (49), (50), (51). It is easily seen that

$$\|c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon] \cdot \nabla(\varepsilon\lambda^\varepsilon \tilde{f}_\varepsilon^1 \circ T^\varepsilon) - \varepsilon(c_0[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon^1) \circ T^\varepsilon\|_{L^2} \leq C\varepsilon^2.$$

It remains to prove that

$$\|c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon] \cdot \nabla(\tilde{f}_\varepsilon \circ T^\varepsilon) - \lambda^\varepsilon(c_0[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon) \circ T^\varepsilon - \varepsilon(c_1[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\|_{L^2} \leq C\varepsilon^2. \tag{52}$$

This comes by the definition of the vector fields c_0, c_1, c^ε and the regularity of the particle densities $(\tilde{f}_\varepsilon)_\varepsilon$, see (23), (24), (16), (25). Indeed, for any smooth, compactly supported particle density $\tilde{f}$ we have

$$\begin{aligned}c^\varepsilon[\tilde{f} \circ T^\varepsilon] \cdot \nabla(\tilde{f} \circ T^\varepsilon) - \lambda^\varepsilon(c_0[\tilde{f}] \cdot \nabla \tilde{f}) \circ T^\varepsilon - \varepsilon(c_1[\tilde{f}] \cdot \nabla \tilde{f}) \circ T^\varepsilon \\ = \varepsilon\left(\frac{F_{\tilde{f}}(\varepsilon, x, \tilde{v}) - F_{\tilde{f}}(0, x, \tilde{v})}{\varepsilon} - \partial_\varepsilon F_{\tilde{f}}(0, x, \tilde{v})\right) \cdot (\nabla \tilde{f}) \circ T^\varepsilon \\ + \varepsilon(c_1[\tilde{f}] - c_1[\tilde{f}] \circ T^\varepsilon) \cdot (\nabla \tilde{f}) \circ T^\varepsilon\end{aligned}$$

where $F_{\tilde{f}}(\varepsilon, x, \tilde{v}) = \partial T^\varepsilon c^\varepsilon[\tilde{f} \circ T^\varepsilon] - \lambda^\varepsilon(x, \tilde{v}) c_0[\tilde{f}] \circ T^\varepsilon$. Clearly, when $\tilde{f} \in C_c^2(\mathbb{R}^3 \times \mathbb{R}^3)$, the function $F_{\tilde{f}}$ is twice differentiable with respect to ε. Moreover, as

$$\sup_{t \in [0,T], 0 < \varepsilon \leq \varepsilon_T} \|\tilde{f}_\varepsilon(t)\|_{C^2(\mathbb{R}^3 \times \mathbb{R}^3)} < +\infty$$

and $\{\tilde{f}_\varepsilon(t) : t \in [0,T], 0 < \varepsilon \leq \varepsilon_T\}$ are uniformly compactly supported, we have

$$\begin{aligned}\|c^\varepsilon[\tilde{f}_\varepsilon \circ T^\varepsilon] \cdot \nabla(\tilde{f}_\varepsilon \circ T^\varepsilon) - \lambda^\varepsilon(c_0[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon) \circ T^\varepsilon - \varepsilon(c_1[\tilde{f}_\varepsilon] \cdot \nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\|_{L^2} \\ \leq \varepsilon\left\|\left(\frac{F_{\tilde{f}_\varepsilon}(\varepsilon) - F_{\tilde{f}_\varepsilon}(0)}{\varepsilon} - \partial_\varepsilon F_{\tilde{f}_\varepsilon}(0)\right) \cdot (\nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\right\|_{L^2} \\ + \left\|\varepsilon(c_1[\tilde{f}_\varepsilon] - c_1[\tilde{f}_\varepsilon] \circ T^\varepsilon) \cdot (\nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\right\|_{L^2} \\ \leq C\varepsilon^2 + \varepsilon\left\|(c_1[\tilde{f}_\varepsilon] - c_1[\tilde{f}_\varepsilon] \circ T^\varepsilon) \cdot (\nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\right\|_{L^2}\end{aligned}$$

By the expression of $c_1[\tilde{f}_\varepsilon]$, see Proposition 5.6, it will be clear that

$$\left\|(c_1[\tilde{f}_\varepsilon] - c_1[\tilde{f}_\varepsilon] \circ T^\varepsilon) \cdot (\nabla \tilde{f}_\varepsilon) \circ T^\varepsilon\right\|_{L^2} \leq C\varepsilon$$

and (52) follows. Coming back to (45), we obtain thanks to (46), (47), (48)

$$\|r^\varepsilon(t)\|_{L^2} \leq [\|r^\varepsilon(0)\|_{L^2} + Ct\varepsilon^2]\exp(Ct), \;\; 0 \leq t \leq T, \;\; 0 < \varepsilon \leq \varepsilon_T.$$

The well preparation of the initial particle densities $(f^\varepsilon(0))_\varepsilon$ guarantees that $\sup_{\varepsilon>0} \frac{\|r^\varepsilon(0)\|_{L^2}}{\varepsilon^2} < +\infty$. Indeed, for justifying this, it is enough to check

$$\sup_{\varepsilon>0} \frac{\|\tilde{f}^\varepsilon(0) - \tilde{f}_\varepsilon(0)\circ T^\varepsilon - \varepsilon \tilde{f}^1_\varepsilon(0)\circ T^\varepsilon\|_{L^2}}{\varepsilon^2} < +\infty. \tag{53}$$

As the family of electric fields $(E[\tilde{g}_\varepsilon])_\varepsilon$ is bounded in $L^\infty([0,T]\times\mathbb{R}^3)$, we have

$$\sup_{\varepsilon>0} \frac{\|\rho[f^\varepsilon(0)] - \rho[\tilde{g}_\varepsilon]\|_{L^1} + \|\rho[f^\varepsilon(0)] - \rho[\tilde{g}_\varepsilon]\|_{L^\infty}}{\varepsilon} < +\infty$$

implying that

$$\sup_{\varepsilon>0} \frac{\|E[f^\varepsilon(0)] - E[\tilde{g}_\varepsilon]\|_{L^\infty}}{\varepsilon} < +\infty.$$

By direct estimates we obtain

$$\sup_{\varepsilon>0} \frac{\|\tilde{f}^\varepsilon(0) - \tilde{g}_\varepsilon\circ T^\varepsilon - \varepsilon \tilde{g}^1_\varepsilon\circ T^\varepsilon\|_{L^2}}{\varepsilon^2} < +\infty \tag{54}$$

By the fifth statement in Proposition 3.1 we have

$$\sup_{\varepsilon>0} \frac{\|\tilde{g}_\varepsilon\circ T^\varepsilon - \tilde{f}_\varepsilon(0)\circ T^\varepsilon\|_{L^2}}{\varepsilon^2} \le C\sup_{\varepsilon>0} \frac{\|\tilde{g}_\varepsilon - \langle\tilde{g}_\varepsilon\rangle\|_{L^2}}{\varepsilon^2} \le 2\pi C\sup_{\varepsilon>0} \frac{\|(\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}\tilde{g}_\varepsilon\|_{L^2}}{\varepsilon^2} < +\infty \tag{55}$$

and

$$\begin{aligned}\sup_{\varepsilon>0} \frac{\|\tilde{g}^1_\varepsilon\circ T^\varepsilon - \tilde{f}^1_\varepsilon(0)\circ T^\varepsilon\|_{L^2}}{\varepsilon} &\le C\sup_{\varepsilon>0} \frac{\|\tilde{g}^1_\varepsilon - \tilde{f}^1_\varepsilon(0)\|_{L^2}}{\varepsilon} \\ &\le \frac{4\pi}{\omega_0} C\sup_{\varepsilon>0} \frac{\|c_0[\tilde{g}_\varepsilon]\cdot\nabla\tilde{g}_\varepsilon - c_0[\langle\tilde{g}_\varepsilon\rangle]\cdot\nabla\langle\tilde{g}_\varepsilon\rangle\|_{L^2}}{\varepsilon} \\ &= \frac{4\pi}{\omega_0} C\sup_{\varepsilon>0} \frac{\|c_0[\tilde{g}_\varepsilon]\cdot\nabla(\tilde{g}_\varepsilon - \langle\tilde{g}_\varepsilon\rangle)\|_{L^2}}{\varepsilon} < +\infty.\end{aligned} \tag{56}$$

Notice that in the last equality we have used $\rho[\tilde{g}_\varepsilon] = \rho[\langle\tilde{g}_\varepsilon\rangle]$, implying that $E[\tilde{g}_\varepsilon] = E[\langle\tilde{g}_\varepsilon\rangle]$ and therefore $c_0[\tilde{g}_\varepsilon] = c_0[\langle\tilde{g}_\varepsilon\rangle]$. Combining (54), (55), (56) yields (53) and therefore

$$\sup_{t\in[0,T],0<\varepsilon\le\varepsilon_T} \frac{\|r^\varepsilon(t)\|_{L^2}}{\varepsilon^2} < +\infty.$$

We deduce that

$$\sup_{t\in[0,T],0<\varepsilon\le\varepsilon_T} \frac{\|\tilde{f}^\varepsilon(t) - \tilde{f}_\varepsilon(t)\circ T^\varepsilon - \varepsilon\tilde{f}^1_\varepsilon(t)\circ T^\varepsilon\|_{L^2}}{\varepsilon^2} < +\infty$$

and in particular

$$\sup_{t\in[0,T],0<\varepsilon\le\varepsilon_T} \frac{\|E[\tilde{f}^\varepsilon(t)] - E[\tilde{f}_\varepsilon(t)]\|_{L^2}}{\varepsilon} < +\infty.$$

Finally one gets for any $t \in [0,T], 0 < \varepsilon \leq \varepsilon_T$

$$\left\{ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[f^\varepsilon(t,x,v) - (\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon) \left(t, x + \varepsilon \frac{v \wedge e(x)}{\omega_c(x)}, v - \varepsilon \frac{E[\tilde{f}_\varepsilon(t)] \wedge e(x)}{B(x)} \right) \right]^2 \mathrm{d}v\mathrm{d}x \right\}^{1/2}$$

$$= \left\{ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big[\tilde{f}^\varepsilon(t,x,\tilde{v}) - (\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon) \right.$$

$$\left. \left(t, x + \frac{\varepsilon}{\omega_c} \left(\tilde{v} + \varepsilon \frac{E[\tilde{f}^\varepsilon(t)] \wedge e}{B(x)} \right) \wedge e, \tilde{v} + \varepsilon \frac{(E[\tilde{f}^\varepsilon(t)] - E[\tilde{f}_\varepsilon(t)]) \wedge e}{B(x)} \right) \Big]^2 \mathrm{d}\tilde{v}\mathrm{d}x \right\}^{1/2}$$

$$\leq \|\tilde{f}^\varepsilon(t) - (\tilde{f}_\varepsilon(t) + \varepsilon \tilde{f}^1_\varepsilon(t)) \circ T^\varepsilon\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} + \left\{ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon) \left(t, x + \varepsilon \frac{\tilde{v} \wedge e(x)}{\omega_c(x)}, \tilde{v} \right) \right.\right.$$

$$\left.\left. -(\tilde{f}_\varepsilon + \varepsilon \tilde{f}^1_\varepsilon) \left(t, x + \varepsilon \frac{\tilde{v} \wedge e}{\omega_c} + \frac{\varepsilon^2}{\omega_c} \frac{(E[\tilde{f}^\varepsilon(t)] \wedge e) \wedge e}{B(x)}, \tilde{v} + \varepsilon \frac{E[\tilde{f}^\varepsilon - \tilde{f}_\varepsilon] \wedge e}{B(x)} \right) \right]^2 \mathrm{d}\tilde{v}\mathrm{d}x \right\}^{1/2}$$

$$\leq C\varepsilon^2 + C\varepsilon \|E[\tilde{f}^\varepsilon(t)] - E[\tilde{f}_\varepsilon(t)]\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^2.$$

□

5 Equivalent formulation of the limit model

In the previous section we proved a second order error estimate for the solution of the Vlasov-Poisson system with strong external magnetic fields (corresponding to smooth, well prepared initial conditions), with respect to the solution of

$$\partial_t \tilde{f}_\varepsilon + \left\langle c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon \right\rangle + \varepsilon \left\langle (a[\tilde{f}_\varepsilon] + c_1[\tilde{f}_\varepsilon]) \cdot \nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon \right\rangle + \varepsilon \left\langle c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon \right\rangle = 0, \quad b \cdot \nabla \tilde{f}_\varepsilon = 0 \tag{57}$$

$$c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon - \left\langle c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon \right\rangle + b \cdot \nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon = 0, \quad \left\langle \tilde{f}^1_\varepsilon \right\rangle = 0. \tag{58}$$

In order to establish the existence/uniqueness of smooth solution for the above system, we are looking for an equivalent formulation. More exactly, we will compute the averages of the vector fields $a[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ , c_0[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}\ , c_1[\tilde{f}_\varepsilon] \cdot \nabla_{x,\tilde{v}}$ cf. [7]. Let us consider the vector field $\overline{b}(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \wedge e(x)) \cdot \nabla_{\tilde{v}}$. By Proposition 3.1 (see proof of statement 6), Remark 3.2, we know that the vector fields $(c^i \cdot \nabla_{x,\tilde{v}})_{1\leq i\leq 6}$ are in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$ and that the average operators along the characteristic flows of $b \cdot \nabla_{x,\tilde{v}}\ , \overline{b} \cdot \nabla_{x,\tilde{v}}$ coincide. We introduce also the vector field

$$\nu(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} = -\frac{(\tilde{v} \cdot e)}{|\tilde{v} \wedge e|^2}\ {}^t\partial_x e(\tilde{v} \wedge e) \cdot \nabla_x + \frac{\tilde{v} \wedge e}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}}, \quad |\tilde{v} \wedge e| > 0$$

which will be used, together with the invariants of $b \cdot \nabla_{x,\tilde{v}}$, for computing the average vector fields. A straightforward computation leads to the following result.

Lemma 5.1
Assume that $e \in C^2(\mathbb{R}^3)$. Let us denote by $Y = (\mathcal{X}, \tilde{\mathcal{V}})$ the characteristic flow of the vector field $\overline{b} \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \wedge e(x)) \cdot \nabla_{\tilde{v}}$

$$Y(\theta; y) = (x, \mathcal{R}(-\theta, e(x))\tilde{v}) = (x, \cos\theta(\tilde{v} - (\tilde{v} \cdot e)e) + \sin\theta(\tilde{v} \wedge e) + (\tilde{v} \cdot e)e)$$

for any $\theta \in \mathbb{R}, y = (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$. The vector field $\nu(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$ verifies

$${}^t\partial Y(\theta; y)\nu(Y(\theta; y)) = \nu(y), \quad \theta \in \mathbb{R}, \quad y = (x,\tilde{v}), \quad |\tilde{v} \wedge e(x)| > 0.$$

The average of a vector field is defined as follows.

Proposition 5.1
Assume that $e \in C^1(\mathbb{R}^3)$. Let $\chi \cdot \nabla_{x,\tilde{v}}$ be a continuous vector field on $\mathbb{R}^3 \times \mathbb{R}^3$. There is a continuous vector field in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$, denoted $\langle \chi \rangle \cdot \nabla_{x,\tilde{v}}$ (the average of $\chi \cdot \nabla_{x,\tilde{v}}$ with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$), such that

$$\langle \chi \cdot \nabla_{x,\tilde{v}}\, u \rangle = \langle \chi \rangle \cdot \nabla_{x,\tilde{v}}\, u$$

for any function $u \in C^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\overline{b} \cdot \nabla_{x,\tilde{v}}\,) = C^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(b \cdot \nabla_{x,\tilde{v}}\,)$ and

$$\langle \chi \cdot \nu \rangle = \langle \chi \rangle \cdot \nu.$$

Proof.
Let us introduce the group $(\varphi(\theta))_{\theta \in \mathbb{R}}$ cf. [7]

$$\varphi(\theta)\chi = \partial Y(-\theta; Y(\theta;\cdot))\chi(Y(\theta;\cdot)), \quad \theta \in \mathbb{R}$$

and consider the vector field

$$\langle \chi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta)\chi \, \mathrm{d}\theta.$$

It is easily seen that if χ is continuous, so is $\langle \chi \rangle$, and that $\langle \chi \rangle$ is left invariant by the group $(\varphi(\theta))_{\theta \in \mathbb{R}}$, saying that $\langle \chi \rangle$ is in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$. For any function $u \in C^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\overline{b} \cdot \nabla_{x,\tilde{v}}\,)$ we have $u \circ Y(\theta;\cdot) = u$ and thus

$$\varphi(\theta)\chi \cdot \nabla u = \partial Y(-\theta; Y(\theta;\cdot))\chi(Y(\theta;\cdot)) \cdot {}^t\partial Y(\theta;\cdot)(\nabla u)(Y(\theta;\cdot)) = (\chi \cdot \nabla u)(Y(\theta;\cdot)).$$

We deduce that

$$\langle \chi \rangle \cdot \nabla u = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta)\chi \cdot \nabla u \, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} (\chi \cdot \nabla u)(Y(\theta;\cdot)) \, \mathrm{d}\theta = \langle \chi \cdot \nabla u \rangle .$$

Similarly, thanks to Lemma 5.1, we have for any $(x, \tilde{v})$ such that $|\tilde{v} \wedge e(x)| > 0$

$$\varphi(\theta)\chi \cdot \nu = \partial Y(-\theta; Y(\theta;\cdot))\chi(Y(\theta;\cdot)) \cdot {}^t\partial Y(\theta;\cdot)\nu(Y(\theta;\cdot)) = (\chi \cdot \nu)(Y(\theta;\cdot))$$

implying that

$$\langle \chi \rangle \cdot \nu = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta)\chi \cdot \nu \, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} (\chi \cdot \nu)(Y(\theta;\cdot)) \, \mathrm{d}\theta = \langle \chi \cdot \nu \rangle , \quad |\tilde{v} \wedge e(x)| > 0.$$

□

Remark 5.1
Recall that the vector fields

$$c^i \cdot \nabla_{x,\tilde{v}} = \partial_{x_i} + (\partial_{x_i} e \otimes e - e \otimes \partial_{x_i} e)\tilde{v} \cdot \nabla_{\tilde{v}}, \quad i \in \{1,2,3\}$$

$$c^4 \cdot \nabla_{x,\tilde{v}} = (\tilde{v} - (\tilde{v} \cdot e)\, e) \cdot \nabla_{\tilde{v}}, \quad c^5 \cdot \nabla_{x,\tilde{v}} = e \cdot \nabla_{\tilde{v}}, \quad c^6 \cdot \nabla_{x,\tilde{v}} = \overline{b} \cdot \nabla_{x,\tilde{v}} = (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}}$$

are in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$. For any continuous vector field $\chi(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$ we have

$$\chi(x,\tilde{v}) = \sum_{i=1}^{6} \alpha_i(x,\tilde{v}) c^i(x,\tilde{v}), \quad (x,\tilde{v}) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \mathcal{E}, \quad \mathcal{E} = \{(x,\tilde{v}) \ : \ b(x,\tilde{v}) = 0\}$$

where

$$(\alpha_1,\alpha_2,\alpha_3)=\chi_x,\ \ \alpha_4=\frac{\chi_{\tilde v}-(\partial_x e\chi_x\otimes e-e\otimes\partial_x e\chi_x)\tilde v}{|\tilde v\wedge e|^2}\cdot(\tilde v-(\tilde v\cdot e)\,e)$$

$$\alpha_5=[\chi_{\tilde v}-(\partial_x e\chi_x\otimes e-e\otimes\partial_x e\chi_x)\tilde v]\cdot e,\ \ \alpha_6=\frac{\chi_{\tilde v}-(\partial_x e\chi_x\otimes e-e\otimes\partial_x e\chi_x)\tilde v}{|\tilde v\wedge e|^2}\cdot(\tilde v\wedge e).$$

It is easily seen that for any vector field $d\cdot\nabla_{x,\tilde v}$ in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde v}$ and any function α, we have $\varphi(\theta)(\alpha d)=\alpha(Y(\theta;\cdot))\varphi(\theta)d=\alpha(Y(\theta;\cdot))d$, implying that $\langle\alpha d\rangle=\langle\alpha\rangle\,d$ and thus for any $(x,\tilde v)\in(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathcal{E}$ (see Remark 3.2 for the definition of $c[\langle\chi_x\rangle]\cdot\nabla_{x,\tilde v}$)

$$\begin{aligned}\langle\chi\rangle(x,\tilde v)\cdot\nabla_{x,\tilde v}\ &=\sum_{i=1}^{6}\langle\alpha_i\rangle(x,\tilde v)c^i(x,\tilde v)\cdot\nabla_{x,\tilde v}\\
&=c[\langle\chi_x\rangle]\cdot\nabla_{x,\tilde v}\ +\langle(\chi_{\tilde v}-(\tilde v\cdot e)\partial_x e\chi_x)\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\,\frac{\tilde v-(\tilde v\cdot e)\,e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}\\
&+\langle(\chi_{\tilde v}\cdot e)+(\partial_x e\chi_x\cdot\tilde v)\rangle\,e\cdot\nabla_{\tilde v}+\langle(\chi_{\tilde v}-(\tilde v\cdot e)\partial_x e\chi_x)\cdot(\tilde v\wedge e)\rangle\,\frac{\tilde v\wedge e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}\\
&=c[\langle\chi_x\rangle]\cdot\nabla_{x,\tilde v}\ +\partial_x e:\langle(\tilde v-(\tilde v\cdot e)\,e)\otimes\chi_x\rangle\,\frac{\tilde v\wedge(e\wedge\tilde v)}{|\tilde v\wedge e|^2}\\
&+\langle\chi_{\tilde v}\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\,\frac{\tilde v-(\tilde v\cdot e)\,e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}+\langle\chi_{\tilde v}\cdot e\rangle\ e\cdot\nabla_{\tilde v}\\
&+\langle(\chi_{\tilde v}-(\tilde v\cdot e)\partial_x e\chi_x)\cdot(\tilde v\wedge e)\rangle\,\frac{\tilde v\wedge e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}.\end{aligned}$$

In particular for any $(x,\tilde v)\in(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathcal{E}$ we have

$$\begin{aligned}\langle(\chi_x,0)\rangle\cdot\nabla_{x,\tilde v}\ &=c[\langle\chi_x\rangle]\cdot\nabla_{x,\tilde v}\ +\langle\partial_x e\chi_x\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\left(e-(\tilde v\cdot e)\frac{\tilde v-(\tilde v\cdot e)\,e}{|\tilde v\wedge e|^2}\right)\cdot\nabla_{\tilde v}\\
&-\frac{(\tilde v\cdot e)}{|\tilde v\wedge e|^2}\,\langle\partial_x e\chi_x\cdot(\tilde v\wedge e)\rangle\,(\tilde v\wedge e)\cdot\nabla_{\tilde v}\\
&=\langle\chi_x\rangle\cdot\nabla_x+\tilde v\wedge(\partial_x e\,\langle\chi_x\rangle\wedge e)\cdot\nabla_{\tilde v}+\langle\partial_x e\chi_x\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\,\frac{\tilde v\wedge(e\wedge\tilde v)}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}\\
&-\langle\partial_x e\chi_x\cdot(\tilde v\wedge e)\rangle\,\frac{(\tilde v\cdot e)}{|\tilde v\wedge e|^2}(\tilde v\wedge e)\cdot\nabla_{\tilde v}\end{aligned}$$

and

$$\begin{aligned}\langle(0,\chi_{\tilde v})\rangle\cdot\nabla_{x,\tilde v}\ &=\langle\chi_{\tilde v}\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\,\frac{\tilde v-(\tilde v\cdot e)\,e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}+\langle\chi_{\tilde v}\cdot e\rangle\,e\cdot\nabla_{\tilde v}\\
&+\langle\chi_{\tilde v}\cdot(\tilde v\wedge e)\rangle\,\frac{\tilde v\wedge e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}.\end{aligned}$$

The notation $\langle(\chi_x,0)\rangle\cdot\nabla_{x,\tilde v}$ stands for the average of the vector field $\chi_x\cdot\nabla_x$ and the notation $\langle\chi_x\rangle\cdot\nabla_x$ stands for $\langle\chi_{x_1}\rangle\,\partial_{x_1}+\langle\chi_{x_2}\rangle\,\partial_{x_2}+\langle\chi_{x_3}\rangle\,\partial_{x_3}$.

We need to eliminate $\tilde f^1_\varepsilon$ in (57), by solving (58). We will use the following result.

Proposition 5.2

Assume that $e\in C^2(\mathbb{R}^3)$. Let $\chi\cdot\nabla_{x,\tilde v}$ be a C^1 vector field on $\mathbb{R}^3\times\mathbb{R}^3$. There is a continuous vector field $\xi\cdot\nabla_{x,\tilde v}$ in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde v}$ such that for any function $u\in C^2(\mathbb{R}^3\times\mathbb{R}^3)\cap\ker(\overline{b}\cdot\nabla_{x,\tilde v}\,)$

$$\left\langle\chi\cdot\nabla_{x,\tilde v}\,u^1\right\rangle=\xi\cdot\nabla_{x,\tilde v}\,u$$

where $\chi\cdot\nabla_{x,\tilde v}\,u-\langle\chi\cdot\nabla_{x,\tilde v}\,u\rangle+\overline{b}\cdot\nabla_{x,\tilde v}\,u^1=0,\ \ \left\langle u^1\right\rangle=0$.

Proof.
We introduce the C^1 vector field

$$\zeta = \frac{1}{2\pi}\int_0^{2\pi}(\theta - 2\pi)\varphi(\theta)(\chi - \langle\chi\rangle)\,\mathrm{d}\theta.$$

Notice that for any $\theta \in \mathbb{R}$ we have $\nabla u = \nabla(u(Y(\theta;\cdot))) = {}^t\partial Y(\theta;\cdot)(\nabla u)(Y(\theta;\cdot))$, implying that

$$\begin{aligned}\varphi(\theta)(\chi - \langle\chi\rangle)\cdot\nabla u &= \partial Y(-\theta; Y(\theta;\cdot))(\chi - \langle\chi\rangle)(Y(\theta;\cdot))\cdot\ {}^t\partial Y(\theta;\cdot)(\nabla u)(Y(\theta;\cdot))\\ &= ((\chi - \langle\chi\rangle)\cdot\nabla u)\circ Y(\theta;\cdot)\\ &= -(\overline{b}\cdot\nabla_{x,\tilde{v}}\ u^1)\circ Y(\theta;\cdot).\end{aligned}$$

Therefore one gets

$$\begin{aligned}\zeta\cdot\nabla u &= -\frac{1}{2\pi}\int_0^{2\pi}(\theta - 2\pi)\frac{\mathrm{d}}{\mathrm{d}\theta}\{u^1\circ Y(\theta,\cdot)\}\,\mathrm{d}\theta\\ &= -\frac{1}{2\pi}[(\theta - 2\pi)u^1\circ Y(\theta;\cdot)]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi}u^1\circ Y(\theta;\cdot)\,\mathrm{d}\theta = -u^1.\end{aligned}$$

As the vector field $\langle\chi\rangle\cdot\nabla_{x,\tilde{v}}$ is in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde{v}}$ and u^1 has zero average, we have

$$\left\langle\langle\chi\rangle\cdot\nabla u^1\right\rangle = \langle\chi\rangle\cdot\nabla\left\langle u^1\right\rangle = 0$$

implying that

$$\left\langle\chi\cdot\nabla u^1\right\rangle = \left\langle(\chi - \langle\chi\rangle)\cdot\nabla u^1\right\rangle = -\left\langle(\chi - \langle\chi\rangle)\cdot\nabla(\zeta\cdot\nabla u)\right\rangle. \tag{59}$$

A straightforward computation shows that $\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\zeta = \varphi(\theta)(\chi - \langle\chi\rangle)$ and as before we have

$$\left((\chi - \langle\chi\rangle)\cdot\nabla u^1\right)(Y(\theta;\cdot)) = \varphi(\theta)(\chi - \langle\chi\rangle)\cdot\nabla(u^1\circ Y(\theta;\cdot)) = \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\zeta\cdot\nabla(u^1\circ Y(\theta;\cdot)).$$

After integration by parts one gets

$$\begin{aligned}\left\langle\chi\cdot\nabla u^1\right\rangle = \left\langle(\chi - \langle\chi\rangle)\cdot\nabla u^1\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\zeta\cdot\nabla(u^1\circ Y(\theta;\cdot))\,\mathrm{d}\theta\\ &= -\frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)\zeta\cdot\nabla[(\overline{b}\cdot\nabla u^1)\circ Y(\theta;\cdot)]\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)\zeta\cdot\nabla\{[(\chi - \langle\chi\rangle)\cdot\nabla u]\circ Y(\theta;\cdot)\}\,\mathrm{d}\theta\\ &= \left\langle\zeta\cdot\nabla((\chi - \langle\chi\rangle)\cdot\nabla u)\right\rangle.\end{aligned} \tag{60}$$

Combining (59), (60) yields

$$\left\langle\chi\cdot\nabla u^1\right\rangle = \frac{1}{2}\left\langle[\zeta,\chi - \langle\chi\rangle]\cdot\nabla u\right\rangle = \frac{1}{2}\left\langle[\zeta,\chi - \langle\chi\rangle]\right\rangle\cdot\nabla u$$

and our conclusion follows by taking $\xi = \frac{1}{2}\left\langle[\zeta,\chi - \langle\chi\rangle]\right\rangle$. □

We indicate now some formulae which will be used in the sequel (see Appendix B for the computation details). For any $\xi \in \mathbb{R}^3$, the notation $M[\xi]$ stands for the matrix of the linear application $v \in \mathbb{R}^3 \to \xi\wedge v \in \mathbb{R}^3$.

Proposition 5.3
We have the equalities

1.
$$\langle \tilde{v} \rangle = (\tilde{v} \cdot e)\, e, \;\; \langle M[\tilde{v}] \rangle = (\tilde{v} \cdot e) M[e]$$

2.
$$\langle \tilde{v} \otimes \tilde{v} \rangle = \frac{|\tilde{v} \wedge e|^2}{2}(I_3 - e \otimes e) + (\tilde{v} \cdot e)^2 e \otimes e$$

3.
$$\langle \tilde{v} \otimes (\tilde{v} - (\tilde{v} \cdot e)\, e) \rangle = \langle (\tilde{v} - (\tilde{v} \cdot e)\, e) \otimes (\tilde{v} - (\tilde{v} \cdot e)\, e) \rangle = \langle (\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) \rangle$$
$$= \frac{|\tilde{v} \wedge e|^2}{2}(I_3 - e \otimes e).$$

4.
$$\langle \tilde{v} \otimes (\tilde{v} \wedge e) \rangle = \frac{|\tilde{v} \wedge e|^2}{2} M[e], \;\; \langle \tilde{v} \cdot \partial_x e (\tilde{v} \wedge e) \rangle = \frac{|\tilde{v} \wedge e|^2}{2} \mathrm{rot}_x e \cdot e$$

5.
$$\left\langle \partial_x \left(\frac{\tilde{v} \wedge e(x)}{\omega_c(x)} \right) (\tilde{v} - (\tilde{v} \cdot e)\, e) \right\rangle = \left\langle M[\tilde{v}] \partial_x \left(\frac{e}{\omega_c} \right) (\tilde{v} - (\tilde{v} \cdot e)\, e) \right\rangle$$
$$= \frac{|\tilde{v} \wedge e|^2}{2} \left(\frac{\mathrm{rot}_x e \cdot e}{\omega_c} e - \frac{\nabla_x \omega_c \wedge e}{\omega_c^2} \right)$$

6.
$$\langle A_x \rangle = \frac{|\tilde{v} \wedge e|^2}{2} \, \frac{\nabla_x \omega_c \wedge e}{\omega_c^2}$$

7.
$$\langle (\tilde{v} - (\tilde{v} \cdot e)\, e) \otimes (\tilde{v} - (\tilde{v} \cdot e)\, e) \otimes (\tilde{v} - (\tilde{v} \cdot e)\, e) \rangle = 0$$

8.
$$\langle \partial_x e A_x \cdot (\tilde{v} - (\tilde{v} \cdot e)\, e) \rangle = \frac{(\tilde{v} \cdot e)|\tilde{v} \wedge e|^2}{2\omega_c} \mathrm{div}_x e \; (\mathrm{rot}_x e \cdot e)$$
$$\langle \partial_x e A_x \cdot (\tilde{v} \wedge e) \rangle = \frac{(\tilde{v} \cdot e)|\tilde{v} \wedge e|^2}{2\omega_c} \partial_x e M[e] \partial_x e : M[e]$$

9.
$$\left\langle \partial_x e \; \partial_x \left(\frac{\tilde{v} \wedge e(x)}{\omega_c(x)} \right) e \cdot (\tilde{v} - (\tilde{v} \cdot e)\, e) \right\rangle = -\frac{|\tilde{v} \wedge e|^2}{2} (\mathrm{rot}_x e \cdot e) \frac{\nabla_x \omega_c \cdot e}{\omega_c^2}$$

10.
$$\langle \tilde{v} \wedge \partial_x e (\tilde{v} \wedge e) \rangle = -\frac{|\tilde{v} \wedge e|^2}{2} \mathrm{div}_x e \; e$$

11.
$$\mathrm{trace}(\partial_x e M[e] \partial_x e) = -(\mathrm{rot}_x e \cdot e) \mathrm{div}_x e$$

12.
$$\mathrm{trace}(M[e] \partial_x e M[e] \partial_x e M[e]) = (\mathrm{rot}_x e \cdot e) \mathrm{div}_x e.$$

We compute now the average of the vector fields $a[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$, $c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$, $c_1[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$.

Proposition 5.4
For any particle density $\tilde{f} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$, the average of the vector field

$$\begin{aligned} a[\tilde{f}] \cdot \nabla_{x,\tilde{v}} \ &= (\tilde{v}_{\wedge D}[\tilde{f}] - A_x) \cdot \nabla_x - \partial_x(\tilde{v}_{\wedge D}[\tilde{f}])\tilde{v} \cdot \nabla_{\tilde{v}} \\ &+ \frac{1}{4\pi\epsilon_0 B}\left[\mathrm{div}_x \int_{\mathbb{R}^3} \frac{x-x'}{|x-x'|^3} \otimes j[\tilde{f}](x')\, \mathrm{d}x' \wedge e(x)\right] \cdot \nabla_{\tilde{v}} \end{aligned}$$

writes

$$\begin{aligned} &\left\langle a[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}} \ = c[\tilde{v}_{\wedge D}[\tilde{f}] + \tilde{v}_{GD}] \cdot \nabla_{x,\tilde{v}} \ - (\tilde{v} \cdot e)\left[\partial_x(\tilde{v}_{\wedge D}[\tilde{f}]) : e \otimes e\right] \ e \cdot \nabla_{\tilde{v}} \\ &- \left[\partial_x(\tilde{v}_{\wedge D}[\tilde{f}]) : (I_3 - e \otimes e)\right] \ \frac{\tilde{v} - (\tilde{v} \cdot e)\, e}{2} \cdot \nabla_{\tilde{v}} + \left[\partial_x(\tilde{v}_{\wedge D}[\tilde{f}]) : M[e]\right] \ \frac{\tilde{v} \wedge e}{2} \cdot \nabla_{\tilde{v}} \\ &- \frac{(\tilde{v} \cdot e)}{2\omega_c} \mathrm{div}_x e \ (\mathrm{rot}_x e \cdot e) \ (\tilde{v} \wedge (e \wedge \tilde{v})) \cdot \nabla_{\tilde{v}} + \frac{(\tilde{v} \cdot e)^2}{2\omega_c} \left[\partial_x e M[e] : M[e]\ {}^t\partial_x e\right] (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} \end{aligned}$$

where

$$\tilde{v}_{\wedge D}[\tilde{f}] = \frac{E[\tilde{f}] \wedge e}{B}, \ \ \tilde{v}_{GD} = -\frac{|\tilde{v} \wedge e|^2}{2} \frac{\nabla_x \omega_c \wedge e}{\omega_c^2}$$

and the notation $c[\xi] \cdot \nabla_{x,\tilde{v}}$ stands for the vector field

$$c[\xi] \cdot \nabla_{x,\tilde{v}} \ = \xi \cdot \nabla_x + (\partial_x e \xi \otimes e - e \otimes \partial_x e \xi)\tilde{v} \cdot \nabla_{\tilde{v}}.$$

Remark 5.2
We recognize here the electric cross field drift

$$\tilde{v}^\varepsilon_{\wedge D}[\tilde{f}] = \varepsilon \tilde{v}_{\wedge D}[\tilde{f}] = \varepsilon \frac{E[\tilde{f}] \wedge e}{B} = \frac{E[\tilde{f}] \wedge \mathbf{B}^\varepsilon}{|\mathbf{B}^\varepsilon|^2}$$

and the magnetic gradient drift

$$\tilde{v}^\varepsilon_{GD} = \varepsilon \tilde{v}_{GD} = -\varepsilon \frac{|\tilde{v} \wedge e|^2}{2} \frac{\nabla_x \omega_c \wedge e}{\omega_c^2} = -\frac{m|\tilde{v} \wedge e|^2}{2qB^\varepsilon} \frac{\nabla_x B^\varepsilon \wedge \mathbf{B}^\varepsilon}{(B^\varepsilon)^2} = -\mu^\varepsilon \frac{\nabla_x B^\varepsilon \wedge \mathbf{B}^\varepsilon}{q(B^\varepsilon)^2}$$

where $\mu^\varepsilon = m|\tilde{v} \wedge e|^2/(2B^\varepsilon)$ is the magnetic moment. We recall also the expression of the magnetic curvature drift

$$\tilde{v}^\varepsilon_{CD} = \varepsilon \tilde{v}_{CD} = -\varepsilon \frac{(\tilde{v} \cdot e)^2}{\omega_c} \ \partial_x e e \wedge e = -\frac{m(\tilde{v} \cdot e)^2}{qB^\varepsilon} \ \frac{\partial_x e \mathbf{B}^\varepsilon \wedge \mathbf{B}^\varepsilon}{(B^\varepsilon)^2}.$$

Proof. (of Proposition 5.4)
We compute all the averages on $(\mathbb{R}^3 \times \mathbb{R}^3) \setminus \mathcal{E}$ and extend them by continuity on $\mathbb{R}^3 \times \mathbb{R}^3$. By Remark 5.1 we have

$$\begin{aligned} \left\langle a[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}} \ &= c\left[\left\langle a_x[\tilde{f}] \right\rangle\right] \cdot \nabla_x \\ &+ \left\langle (a_{\tilde{v}}[\tilde{f}] - (\tilde{v} \cdot e)\partial_x e \ a_x[\tilde{f}]) \cdot (\tilde{v} - (\tilde{v} \cdot e) \ e) \right\rangle \frac{\tilde{v} - (\tilde{v} \cdot e) \ e}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}} \\ &+ \left\langle a_{\tilde{v}}[\tilde{f}] \cdot e + \partial_x e \ a_x[\tilde{f}] \cdot \tilde{v} \right\rangle \ e \cdot \nabla_{\tilde{v}} \\ &+ \left\langle (a_{\tilde{v}}[\tilde{f}] - (\tilde{v} \cdot e)\partial_x e \ a_x[\tilde{f}]) \cdot (\tilde{v} \wedge e) \right\rangle \frac{\tilde{v} \wedge e}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}}. \end{aligned}$$

By the sixth statement in Proposition 5.3 we obtain

$$\left\langle a_x[\tilde{f}] \right\rangle = \tilde{v}_{\wedge D}[\tilde{f}] - \langle A_x \rangle = \tilde{v}_{\wedge D}[\tilde{f}] + \tilde{v}_{GD}.$$

We write $a_{\tilde v}[\tilde f] = a^I_{\tilde v}[\tilde f] + a^{II}_{\tilde v}[\tilde f]$, where

$$a^I_{\tilde v}[\tilde f] = -\partial_x(\tilde v_{\wedge D}[\tilde f])\tilde v,\ \ a^{II}_{\tilde v}[\tilde f] = \frac{1}{4\pi\epsilon_0 B}\mathrm{div}_x\left(\int_{\mathbb{R}^3}\frac{x-x'}{|x-x'|^3}\otimes j[\tilde f](x')\,\mathrm{d}x'\right)\wedge e(x).$$

It is easily seen that the second part $a^{II}_{\tilde v}[\tilde f]$ gives no contribution

$$\left\langle a^{II}_{\tilde v}[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\,e)\right\rangle = \left\langle a^{II}_{\tilde v}[\tilde f]\cdot e\right\rangle = \left\langle a^{II}_{\tilde v}[\tilde f]\cdot(\tilde v\wedge e)\right\rangle = 0.$$

For $a^I_{\tilde v}[\tilde f]$ we obtain cf. Proposition 5.3

$$\begin{aligned}\left\langle a^I_{\tilde v}[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\,e)\right\rangle &= -\partial_x(\tilde v_{\wedge D}[\tilde f]) : \langle(\tilde v-(\tilde v\cdot e)\,e)\otimes\tilde v\rangle\\ &= -\frac{|\tilde v\wedge e|^2}{2}\partial_x(\tilde v_{\wedge D}[\tilde f]) : (I_3-e\otimes e)\end{aligned}$$

$$\left\langle a^I_{\tilde v}[\tilde f]\cdot e\right\rangle = -(\tilde v\cdot e)\partial_x(\tilde v_{\wedge D}[\tilde f]) : e\otimes e$$

and

$$\left\langle a^I_{\tilde v}[\tilde f]\cdot(\tilde v\wedge e)\right\rangle = -\partial_x(\tilde v_{\wedge D}[\tilde f]) : \langle(\tilde v\wedge e)\otimes\tilde v\rangle = \frac{|\tilde v\wedge e|^2}{2}\partial_x(\tilde v_{\wedge D}[\tilde f]) : M[e].$$

It remains to compute the contributions of $a_x[\tilde f]$. Thanks to Proposition 5.3 statement 8, we obtain

$$\begin{aligned}-\left\langle(\tilde v\cdot e)\partial_x e\ a_x[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\,e)\right\rangle &= (\tilde v\cdot e)\,\langle\partial_x e\ A_x\cdot(\tilde v-(\tilde v\cdot e)\,e)\rangle\\ &= \frac{(\tilde v\cdot e)^2\ |\tilde v\wedge e|^2}{2\omega_c}\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e)\end{aligned}$$

$$\left\langle\partial_x e\ a_x[\tilde f]\cdot\tilde v\right\rangle = \left\langle\partial_x e\ a_x[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\,e)\right\rangle = -\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2\omega_c}\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e)$$

and

$$\begin{aligned}-(\tilde v\cdot e)\left\langle\partial_x e\ a_x[\tilde f]\cdot(\tilde v\wedge e)\right\rangle &= (\tilde v\cdot e)\,\langle\partial_x e\ A_x\cdot(\tilde v\wedge e)\rangle\\ &= \frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2\omega_c}\partial_x e M[e] : M[e]\ {}^t\partial_x e.\end{aligned}$$

Combining the previous results, we deduce

$$\begin{aligned}\left\langle a[\tilde f]\right\rangle\cdot\nabla_{x,\tilde v}\ &= c[\tilde v_{\wedge D}[\tilde f]+\tilde v_{GD}]\cdot\nabla_{x,\tilde v}\ -(\tilde v\cdot e)\left[\partial_x(\tilde v_{\wedge D}[\tilde f]) : e\otimes e\right]\ e\cdot\nabla_{\tilde v}\\ &-\left[\partial_x(\tilde v_{\wedge D}[\tilde f]) : (I_3-e\otimes e)\right]\ \frac{\tilde v-(\tilde v\cdot e)\,e}{2}\cdot\nabla_{\tilde v}+\left[\partial_x(\tilde v_{\wedge D}[\tilde f]) : M[e]\right]\ \frac{\tilde v\wedge e}{2}\cdot\nabla_{\tilde v}\\ &-\frac{(\tilde v\cdot e)}{2\omega_c}\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e)\ (\tilde v\wedge(e\wedge\tilde v))\cdot\nabla_{\tilde v}+\frac{(\tilde v\cdot e)^2}{2\omega_c}\left[\partial_x e M[e] : M[e]\ {}^t\partial_x e\right](\tilde v\wedge e)\cdot\nabla_{\tilde v}.\end{aligned}$$

□

Proposition 5.5
For any particle density $\tilde{f} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$, the average of the vector field

$$c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}} \ = (\tilde{v} \cdot e(x))\ e(x) \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} - [\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e(x))] \cdot \nabla_{\tilde{v}}$$

writes

$$\begin{aligned}\left\langle c_0[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}} \ &= c[(\tilde{v} \cdot e)\ e] \cdot \nabla_{x,\tilde{v}} \ + \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} \\ &\quad + \frac{\mathrm{div}_x e}{2}[\tilde{v} \wedge (e \wedge \tilde{v})] \cdot \nabla_{\tilde{v}} + (\tilde{v} \cdot e)\frac{(\mathrm{rot}_x e \cdot e)}{2}\ (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}}\end{aligned}$$

where $c[\xi] \cdot \nabla_{x,\tilde{v}} \ = \xi \cdot \nabla_x + (\partial_x e\ \xi \otimes e - e \otimes \partial_x e\ \xi)\tilde{v} \cdot \nabla_{\tilde{v}}$.

Proof.
Thanks to Remark 5.1 we have

$$\langle (\tilde{v} \cdot e)\ e \cdot \nabla_x \rangle = (\tilde{v} \cdot e) c[e] \cdot \nabla_{x,\tilde{v}} \ = c[(\tilde{v} \cdot e)\ e] \cdot \nabla_{x,\tilde{v}} \ .$$

As the field $\frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}}$ is in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$, we obtain

$$\left\langle \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} \right\rangle = \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}}.$$

For the last contribution in $c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ one gets

$$\begin{aligned}\langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot \nabla_{\tilde{v}} \rangle &= \langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} - (\tilde{v} \cdot e)\ e) \rangle \ \frac{\tilde{v} - (\tilde{v} \cdot e)\ e}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}} \\ &+ \langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot e \rangle \ e \cdot \nabla_{\tilde{v}} + \langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} \wedge e) \rangle \ \frac{\tilde{v} \wedge e}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}}.\end{aligned}$$

Thanks to the seventh and second statements in Proposition 5.3 we have

$$\begin{aligned}\langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} - (\tilde{v} \cdot e)\ e) \rangle &= (\tilde{v} \cdot e)\ \langle [e \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} - (\tilde{v} \cdot e)\ e) \rangle \\ &= (\tilde{v} \cdot e)\ \langle [e \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot \tilde{v} \rangle \\ &= -(\tilde{v} \cdot e) M[e] \partial_x e M[e] : \langle \tilde{v} \otimes \tilde{v} \rangle \\ &= -\frac{(\tilde{v} \cdot e)|\tilde{v} \wedge e|^2}{2} \mathrm{trace}(M[e] \partial_x e M[e]) \\ &= (\tilde{v} \cdot e)\frac{|\tilde{v} \wedge e|^2}{2} \mathrm{trace}((I_3 - e \otimes e)\partial_x e) \\ &= \frac{(\tilde{v} \cdot e)|\tilde{v} \wedge e|^2}{2} \mathrm{div}_x e.\end{aligned}$$

Similarly one gets

$$\langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot e \rangle = \langle [e \wedge \partial_x e\ (e \wedge \tilde{v})] \cdot \tilde{v} \rangle = M[e] \partial_x e M[e] : \langle \tilde{v} \otimes \tilde{v} \rangle = -\frac{|\tilde{v} \wedge e|^2}{2} \mathrm{div}_x e$$

and

$$\begin{aligned}\langle [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} \wedge e) \rangle &= (\tilde{v} \cdot e)\ \langle [e \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot (\tilde{v} \wedge e) \rangle \\ &= (\tilde{v} \cdot e) M[e] \partial_x e : \langle (\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) \rangle \\ &= -\frac{(\tilde{v} \cdot e)|\tilde{v} \wedge e|^2}{2} (\mathrm{rot}_x e \cdot e)\end{aligned}$$

implying that

$$\begin{aligned}\left\langle c_0[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}} \ &= c[(\tilde{v} \cdot e)\ e] \cdot \nabla_{x,\tilde{v}} \ + \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} \\ &\quad + \frac{\mathrm{div}_x e}{2}[\tilde{v} \wedge (e \wedge \tilde{v})] \cdot \nabla_{\tilde{v}} + (\tilde{v} \cdot e)\frac{(\mathrm{rot}_x e \cdot e)}{2}\ (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}}.\end{aligned}$$

□

The computation of the vector field $c_1 \cdot \nabla_{x,\tilde{v}}$, together with its average is much more elaborated. The details can be found in Appendix B.

Proposition 5.6
For any particle density $\tilde{f} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$, the vector field

$$c_1[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = \lim_{\varepsilon \searrow 0} \frac{\partial T^\varepsilon c^\varepsilon[\tilde{f} \circ T^\varepsilon] - \lambda^\varepsilon(x,\tilde{v})(c_0[\tilde{f}] \circ T^\varepsilon)}{\varepsilon} \cdot \nabla_{x,\tilde{v}}$$

writes

$$\begin{aligned}
c_1[\tilde{f}] \cdot \nabla_{x,\tilde{v}} &= (\tilde{v}\cdot e)\left[\tilde{v} \wedge \partial_x\left(\frac{e}{\omega_c}\right) e - \partial_x e\ \frac{\tilde{v}\wedge e}{\omega_c} + \left(\nabla_x \omega_c \cdot \frac{\tilde{v}\wedge e}{\omega_c^2}\right)\ e\right] \cdot \nabla_x \\
&- \left[\left(\nabla_x \omega_c \cdot \frac{\tilde{v}\wedge e}{\omega_c^2}\right)\ \tilde{v}\wedge \partial_x e(\tilde{v}\wedge e) - \tilde{v}\wedge[\partial_x e(\tilde{v}\wedge \partial_x e\ \frac{\tilde{v}\wedge e}{\omega_c})]\right]\cdot \nabla_{\tilde{v}} \\
&+ \left[\frac{\tilde{v}}{2\omega_c} \wedge ((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e) : \nabla_x \otimes \nabla_x) e\right]\cdot \nabla_{\tilde{v}} + \frac{q}{m}\left(E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot \nabla_x \tilde{f}\right]\cdot e\right) e \cdot \nabla_{\tilde{v}} \\
&- \frac{q}{m}\left[\left(\partial_x E[\tilde{f}]\frac{\tilde{v}\wedge e}{\omega_c}\cdot e\right)\ e + \left(E[\tilde{f}]\cdot \partial_x e\ \frac{\tilde{v}\wedge e}{\omega_c}\right)\ e + (E[\tilde{f}]\cdot e)\partial_x e\ \frac{\tilde{v}\wedge e}{\omega_c}\right]\cdot \nabla_{\tilde{v}} \\
&+ \frac{q}{m}(E[\tilde{f}]\cdot e)\left(\nabla_x \omega_c \cdot \frac{\tilde{v}\wedge e}{\omega_c^2}\right)\ e \cdot \nabla_{\tilde{v}}
\end{aligned}$$

and its average is

$$\begin{aligned}
\left\langle c_1[\tilde{f}]\right\rangle \cdot \nabla_{x,\tilde{v}} &= c[\tilde{v}_{CD}]\cdot \nabla_{x,\tilde{v}} - \left[\frac{(\tilde{v}\cdot e)^2}{\omega_c}\mathrm{div}_x e + \frac{(E[\tilde{f}]\cdot e)}{B}\right]\frac{(\mathrm{rot}_x e\cdot e)}{2}(\tilde{v} - (\tilde{v}\cdot e)\ e)\cdot \nabla_{\tilde{v}} \\
&+ \left[\frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e + \frac{q}{m}\left(E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot \nabla_x \tilde{f}\right]\cdot e\right)\right]\ e\cdot \nabla_{\tilde{v}} \\
&+ \left[\frac{(\tilde{v}\cdot e)^2}{2\omega_c}\mathrm{trace}(\partial_x e M[e]\partial_x e M[e]) - \frac{|\tilde{v}\wedge e|^2}{4\omega_c}(\partial_x e : \partial_x e - |\partial_x e e|^2) - \frac{(E[\tilde{f}]\cdot e)}{2B}\mathrm{div}_x e\right](\tilde{v}\wedge e)\cdot \nabla_{\tilde{v}} \\
&+ \frac{(\tilde{v}\cdot e)}{2}(\partial_x e e \wedge e)\cdot \frac{\nabla_x \omega_c}{\omega_c^2}\ (\tilde{v}\wedge(e\wedge\tilde{v}))\cdot \nabla_{\tilde{v}} \\
&+ \frac{(\tilde{v}\cdot e)^2}{2\omega_c}\left[|\partial_x e e|^2 - \partial_x e e\cdot \frac{\nabla_x \omega_c}{\omega_c} + \mathrm{div}_x(\partial_x e e - \mathrm{div}_x e\ e)\right]\ (\tilde{v}\wedge e)\cdot \nabla_{\tilde{v}}
\end{aligned}$$

where $\tilde{v}_{CD} = -\frac{(\tilde{v}\cdot e(x))^2}{\omega_c(x)}\partial_x e e \wedge e$.

Remark 5.3
When the particle density satisfies $\tilde{f} \in C_c^1(\mathbb{R}^3\times\mathbb{R}^3) \cap \ker(b\cdot\nabla_{x,\tilde{v}}\ )$, then

$$\begin{aligned}
\int_{\mathbb{R}^3}(\tilde{v}\wedge e)\cdot\nabla_x \tilde{f}\ \mathrm{d}\tilde{v} &= \int_{\mathbb{R}^3}\mathrm{div}_x(\tilde{f}\tilde{v}\wedge e)\ \mathrm{d}\tilde{v} - \int_{\mathbb{R}^3}\tilde{f}\mathrm{div}_x(\tilde{v}\wedge e)\ \mathrm{d}\tilde{v} \\
&= \mathrm{div}_x \underbrace{\int_{\mathbb{R}^3}\tilde{f}\tilde{v}\wedge e\ \mathrm{d}\tilde{v}}_{=0} + \int_{\mathbb{R}^3}\tilde{f}\tilde{v}\cdot\mathrm{rot}_x e\ \mathrm{d}\tilde{v} \\
&= \int_{\mathbb{R}^3}(\tilde{v}\cdot e)\tilde{f}\ \mathrm{d}\tilde{v}\ (e\cdot\mathrm{rot}_x e).
\end{aligned}$$

We deduce that

$$E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot\nabla_x\tilde{f}\right] = E\left[(e\cdot\mathrm{rot}_x e)\frac{(\tilde{v}\cdot e)}{\omega_c}\tilde{f}\right]$$

and in that case

$$\left\langle c_1[\tilde f]\right\rangle\cdot\nabla_{x,\tilde v}\ \tilde f = c[\tilde v_{CD}]\cdot\nabla_{x,\tilde v}\ \tilde f + \frac{(\tilde v\cdot e)}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e\ (\tilde v\wedge(e\wedge\tilde v))\cdot\nabla_{\tilde v}\tilde f$$
$$-\frac{(E[\tilde f]\cdot e)}{B}\frac{(e\cdot\mathrm{rot}_x e)}{2}\ (\tilde v-(\tilde v\cdot e)\ e)\cdot\nabla_{\tilde v}\tilde f+\frac{q}{m}\left(E\left[(e\cdot\mathrm{rot}_x e)\frac{(\tilde v\cdot e)}{\omega_c}\tilde f\right]\cdot e\right)\ e\cdot\nabla_{\tilde v}\tilde f$$
$$+\frac{(\tilde v\cdot e)}{2}(\partial_x e e\wedge e)\cdot\frac{\nabla_x\omega_c}{\omega_c^2}\ (\tilde v\wedge(e\wedge\tilde v))\cdot\nabla_{\tilde v}\tilde f.$$

We claim that the following simplifications between terms in $\left\langle a[\tilde f]\right\rangle\cdot\nabla_{x,\tilde v}$ *and* $\left\langle c_1[\tilde f]\right\rangle\cdot\nabla_{x,\tilde v}$ *occur*

$$-\partial_x(\tilde v_{\wedge D}[\tilde f]):(I_3-e\otimes e)-\frac{(E[\tilde f]\cdot e)}{B}(e\cdot\mathrm{rot}_x e)=\tilde v_{\wedge D}[\tilde f]\cdot\frac{\nabla_x\omega_c}{\omega_c}$$
$$-\partial_x(\tilde v_{\wedge D}[\tilde f]):e\otimes e=\tilde v_{\wedge D}[\tilde f]\cdot\partial_x e e.$$

The last identity follows by taking the directional derivative $e\cdot\nabla_x$ *in the equality* $\tilde v_{\wedge D}[\tilde f]\cdot e=0$. *For the other one it is enough to check that*

$$-\mathrm{div}_x\tilde v_{\wedge D}[\tilde f]-\frac{(E[\tilde f]\cdot e)}{B}(e\cdot\mathrm{rot}_x e)=\tilde v_{\wedge D}[\tilde f]\cdot\frac{\nabla_x\omega_c}{\omega_c}+\tilde v_{\wedge D}[\tilde f]\cdot\partial_x e e.$$

Combining the conclusions of Propositions 5.4, 5.6 we obtain for any $\tilde f\in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)\cap\ker(b\cdot\nabla_{x,\tilde v}\ )$

$$\left(\left\langle a[\tilde f]\right\rangle+\left\langle c_1[\tilde f]\right\rangle\right)\cdot\nabla_{x,\tilde v}\ \tilde f = c[\tilde v_D[\tilde f]]\cdot\nabla_{x,\tilde v}\ \tilde f+\frac{(\tilde v\cdot e)}{\omega_c}(\partial_x e\ e\wedge e)\cdot\frac{\nabla\omega_c}{\omega_c}\frac{\tilde v\wedge(e\wedge\tilde v)}{2}\cdot\nabla_{\tilde v}\tilde f$$
$$+(\tilde v\cdot e)(\tilde v_{\wedge D}[\tilde f]\cdot\partial_x e e)\ e\cdot\nabla_{\tilde v}\tilde f+\left(\tilde v_{\wedge D}[\tilde f]\cdot\frac{\nabla_x\omega_c}{\omega_c}\right)\frac{\tilde v-(\tilde v\cdot e)\ e}{2}\cdot\nabla_{\tilde v}\tilde f$$
$$+\frac{q}{m}\left(E\left[(e\cdot\mathrm{rot}_x e)\frac{(\tilde v\cdot e)}{\omega_c}\tilde f\right]\cdot e\right)\ e\cdot\nabla_{\tilde v}\tilde f \qquad (61)$$

where $\tilde v_D[\tilde f]=\tilde v_{\wedge D}[\tilde f]+\tilde v_{GD}+\tilde v_{CD}$.

Notice that the above (reduced) vector field differs from the vector field $\left(\left\langle a[\tilde f]\right\rangle+\left\langle c_1[\tilde f]\right\rangle\right)\cdot\nabla_{x,\tilde v}$ by a term $\alpha\overline{b}\cdot\nabla_{x,\tilde v}$, with $\overline{b}\cdot\nabla_{x,\tilde v}\ \alpha=0$. Since, by construction, the vector field $\left(\left\langle a[\tilde f]\right\rangle+\left\langle c_1[\tilde f]\right\rangle\right)\cdot\nabla_{x,\tilde v}$ is in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde v}$, so is the above reduced vector field. Observe also that this reduction does not change the divergence, because $\mathrm{div}_{x,\tilde v}(\alpha\overline{b})=0$.

In order to obtain our limit model, we need to eliminate $\tilde f^1_\varepsilon$ from (57), thanks to (58). Appealing to Proposition 5.2 we prove (see Appendix B for details)

Proposition 5.7
Assume that $e\in C^2(\mathbb{R}^3),\omega_c\in C^1(\mathbb{R}^3)$. *Let* $\tilde f\in C^2_c(\mathbb{R}^3\times\mathbb{R}^3)\cap\ker(b\cdot\nabla_{x,\tilde v}\ )$ *and* $\tilde f^1$ *verifying*

$$c_0[\tilde f]\cdot\nabla_{x,\tilde v}\ \tilde f-\left\langle c_0[\tilde f]\cdot\nabla_{x,\tilde v}\ \tilde f\right\rangle+b\cdot\nabla_{x,\tilde v}\ \tilde f^1=0,\ \ \left\langle\tilde f^1\right\rangle=0.$$

Then we have $\left\langle c_0[\tilde f]\cdot\nabla_{x,\tilde v}\ \tilde f^1\right\rangle=0$.

Thanks to Propositions 5.4, 5.5, 5.6, 5.7 we obtain the following equivalent formulation for (57), (58).

Proposition 5.8
Assume that $e \in C^2(\mathbb{R}^3), \omega_c \in C^1(\mathbb{R}^3)$ *such that* $\inf_{x\in\mathbb{R}^3} |\omega_c(x)| = \omega_0 > 0, \mathrm{div}_x(\omega_c e) = 0$ *and let us consider* $\tilde{f}_\varepsilon \in C^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$. *Then* $\tilde{f}_\varepsilon$ *solves*

$$\partial_t \tilde{f}_\varepsilon + \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle + \varepsilon\left\langle (a[\tilde{f}_\varepsilon]+c_1[\tilde{f}_\varepsilon])\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle + \varepsilon\left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon\right\rangle = 0$$

$$b\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon = 0,\ \ c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon - \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon\right\rangle + b\cdot\nabla_{x,\tilde{v}}\ \tilde{f}^1_\varepsilon = 0,\ \ \left\langle \tilde{f}^1_\varepsilon\right\rangle = 0 \qquad (62)$$

iff $\tilde{f}_\varepsilon$ *solves*

$$\partial_t \tilde{f}_\varepsilon + \left\langle c_0[\tilde{f}_\varepsilon]\right\rangle\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon + \varepsilon\left(\left\langle a[\tilde{f}_\varepsilon]\right\rangle + \left\langle c_1[\tilde{f}_\varepsilon]\right\rangle\right)\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon = 0,\ \ b\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon = 0 \qquad (63)$$

$$\tilde{f}^1_\varepsilon = \left[\frac{(\tilde{v}\cdot e)}{\omega_c}\partial_x e e\cdot\frac{\tilde{v}\wedge e}{|\tilde{v}\wedge e|} - \partial_x e : \frac{(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\ e)+(\tilde{v}-(\tilde{v}\cdot e)\ e)\otimes(\tilde{v}\wedge e)}{4\omega_c|\tilde{v}\wedge e|}\right]$$
$$\frac{(\tilde{v}\wedge e)\wedge\tilde{v}}{|\tilde{v}\wedge e|}\cdot\nabla_{\tilde{v}}\tilde{f}_\varepsilon.$$

Proof.
We only need to solve (62) with respect to $\tilde{f}^1_\varepsilon$. We already know, see the proof of Proposition 5.2, that there is a vector field $\zeta[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}$ such that for any $u \in C^2(\mathbb{R}^3\times\mathbb{R}^3)\cap\ker(b\cdot\nabla_{x,\tilde{v}}\ )$, the function $u^1 \in C^1(\mathbb{R}^3\times\mathbb{R}^3)$ solving

$$c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ u - \left\langle c_0[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ u\right\rangle + b\cdot\nabla_{x,\tilde{v}}\ u^1 = 0,\ \ \left\langle u^1\right\rangle = 0$$

writes $u^1 = \zeta[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ u$. For any $(x,\tilde{v})\in\mathbb{R}^3\times\mathbb{R}^3, |\tilde{v}\wedge e(x)| > 0$ we have

$$\begin{aligned}\zeta[\tilde{f}_\varepsilon]\cdot\nabla_{x,\tilde{v}}\ &= c[\zeta_x]\cdot\nabla_{x,\tilde{v}}\ + \frac{\zeta_{\tilde{v}}\cdot(\tilde{v}-(\tilde{v}\cdot e)\ e)-(e\cdot\tilde{v})\partial_x e\ \zeta_x\cdot\tilde{v}}{|\tilde{v}\wedge e|^2}\ (\tilde{v}-(\tilde{v}\cdot e)\ e)\cdot\nabla_{\tilde{v}}\\ &+(\zeta_{\tilde{v}}\cdot e+\partial_x e\ \zeta_x\cdot\tilde{v})\ e\cdot\nabla_{\tilde{v}} + \frac{(\zeta_{\tilde{v}}-(e\cdot\tilde{v})\partial_x e\ \zeta_x)\cdot(\tilde{v}\wedge e)}{|\tilde{v}\wedge e|^2}\ (\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}.\end{aligned}$$

For $u = x_i, i\in\{1,2,3\}$, we obtain $u^1 = 0$ and thus $\zeta_x = 0$. If $u = |\tilde{v}|^2/2$, we deduce that $u^1 = 0$ and $\zeta_{\tilde{v}}\cdot\tilde{v} = 0$. When considering $u = \tilde{v}\cdot e$, we obtain (see the proof of Proposition 5.7)

$$-\omega_c u^1 = -(\tilde{v}\cdot e)\partial_x e e\cdot(\tilde{v}\wedge e) + \partial_x e : \frac{(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\ e)+(\tilde{v}-(\tilde{v}\cdot e)\ e)\otimes(\tilde{v}\wedge e)}{4}$$

implying that

$$\zeta_{\tilde{v}}\cdot e = \frac{(\tilde{v}\cdot e)}{\omega_c}\ \partial_x e e\cdot(\tilde{v}\wedge e) - \partial_x e : \frac{(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\ e)+(\tilde{v}-(\tilde{v}\cdot e)\ e)\otimes(\tilde{v}\wedge e)}{4\omega_c}.$$

Finally we deduce

$$\tilde{f}^1_\varepsilon = \frac{(\zeta_{\tilde{v}}\cdot e)}{|\tilde{v}\wedge e|^2}((\tilde{v}\wedge e)\wedge\tilde{v})\cdot\nabla_{\tilde{v}}\tilde{f}_\varepsilon.$$

□

6 Well posedness and main properties of the limit model

We concentrate on the existence/uniqueness and other properties of the model

$$\partial_t \tilde{f}_\varepsilon + \left\langle c_0[\tilde{f}_\varepsilon] \right\rangle \cdot \nabla_{x,\tilde{v}}\, \tilde{f}_\varepsilon + \varepsilon \left(\left\langle a[\tilde{f}_\varepsilon] \right\rangle + \left\langle c_1[\tilde{f}_\varepsilon] \right\rangle \right) \cdot \nabla_{x,\tilde{v}}\, \tilde{f}_\varepsilon = 0, \quad (t,x,\tilde{v}) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{64}$$

together with the initial condition

$$\tilde{f}_\varepsilon(0,x,\tilde{v}) = \tilde{f}_{\rm in}(x,\tilde{v}), \quad (x,\tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{65}$$

The initial density is assumed smooth, compactly supported, and $\mathbf{B}$ is a smooth external magnetic field satisfying

$$\mathbf{B} = Be \in C^1_b(\mathbb{R}^3), \quad B_0 = \inf_{x \in \mathbb{R}^3} |B(x)| > 0, \quad {\rm div}_x \mathbf{B} = 0. \tag{66}$$

We notice that the magnetic moment $\mu^\varepsilon(x,\tilde{v}) = \frac{m|\tilde{v}\wedge e(x)|^2}{2B^\varepsilon(x)} = \frac{q|\tilde{v}\wedge e(x)|^2}{2\omega_c^\varepsilon(x)}$ is left invariant by the vector fields $\left\langle c_0[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}}$, $\left(\left\langle a[\tilde{f}] \right\rangle + \left\langle c_1[\tilde{f}] \right\rangle \right) \cdot \nabla_{x,\tilde{v}}$.

Proposition 6.1
Let $\mathbf{B}$ *be a magnetic field satisfying* (66), $\mu^\varepsilon = \varepsilon\mu$, $\mu(x,\tilde{v}) = \frac{m|\tilde{v}\wedge e(x)|^2}{2B(x)} = \frac{q|\tilde{v}\wedge e(x)|^2}{2\omega_c(x)}$. *For any particle density* $\tilde{f} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ *we have*

$$\left\langle c_0[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}}\, \mu = 0, \quad \left(\left\langle a[\tilde{f}] \right\rangle + \left\langle c_1[\tilde{f}] \right\rangle \right) \cdot \nabla_{x,\tilde{v}}\, \mu = 0.$$

Proof.
It is easily seen that

$$\nabla_x \frac{\mu}{q} = -\frac{(\tilde{v}\cdot e)\, {}^t\partial_x e \tilde{v}}{\omega_c(x)} - \frac{\mu(x,\tilde{v})}{q}\frac{\nabla_x \omega_c}{\omega_c}, \quad \nabla_{\tilde{v}} \frac{\mu}{q} = \frac{\tilde{v} - (\tilde{v}\cdot e)\, e}{\omega_c}$$

and therefore

$$\begin{aligned}
\left\langle c_0[\tilde{f}] \right\rangle \cdot \nabla_{x,\tilde{v}}\, \frac{\mu}{q} &= -\frac{(\tilde{v}\cdot e)^2}{\omega_c}(\partial_x e e \cdot \tilde{v}) - (\tilde{v}\cdot e)\frac{\mu(x,\tilde{v})}{q}\left(e \cdot \frac{\nabla_x \omega_c}{\omega_c} \right) \\
&\quad + \frac{(\tilde{v}\cdot e)^2}{\omega_c}(\partial_x e e \cdot \tilde{v}) - (\tilde{v}\cdot e)\frac{\mu(x,\tilde{v})}{q}{\rm div}_x e \\
&= -(\tilde{v}\cdot e)\frac{\mu}{qB}{\rm div}_x \mathbf{B} = 0.
\end{aligned}$$

As the magnetic moment belongs to the kernel of $b \cdot \nabla_{x,\tilde{v}}$, we have by (61)

$$\begin{aligned}
\left(\left\langle a[\tilde{f}] \right\rangle + \left\langle c_1[\tilde{f}] \right\rangle \right) \cdot \nabla_{x,\tilde{v}}\, \mu &= c[\tilde{v}_D[\tilde{f}]] \cdot \nabla_{x,\tilde{v}}\, \mu \\
&\quad + \frac{(\tilde{v}\cdot e)}{\omega_c}\left((\partial_x e e \wedge e) \cdot \frac{\nabla_x \omega_c}{\omega_c} \right) \frac{\tilde{v} \wedge (e \wedge \tilde{v})}{2} \cdot \nabla_{\tilde{v}} \mu \\
&\quad + \left(\tilde{v}_{\wedge D}[\tilde{f}] \cdot \frac{\nabla_x \omega_c}{\omega_c} \right) \frac{\tilde{v} - (\tilde{v}\cdot e) e}{2} \cdot \nabla_{\tilde{v}} \mu.
\end{aligned}$$

It is easily seen that

$$\frac{(\tilde{v}\cdot e)}{\omega_c}\left((\partial_x e e \wedge e) \cdot \frac{\nabla_x \omega_c}{\omega_c} \right) \frac{\tilde{v} \wedge (e \wedge \tilde{v})}{2} \cdot \nabla_{\tilde{v}} \mu = -\frac{(\tilde{v}\cdot e)^2}{\omega_c}\left((\partial_x e e \wedge e) \cdot \frac{\nabla_x \omega_c}{\omega_c} \right) \mu$$

and

$$\left(\tilde{v}_{\wedge D}[\tilde{f}]\cdot\frac{\nabla_x\omega_c}{\omega_c}\right)\frac{\tilde{v}-(\tilde{v}\cdot e)e}{2}\cdot\nabla_{\tilde{v}}\mu=\left(\tilde{v}_{\wedge D}[\tilde{f}]\cdot\frac{\nabla_x\omega_c}{\omega_c}\right)\mu.$$

It remains to compute the derivative along $c[\tilde{v}_D[\tilde{f}]]\cdot\nabla_{x,\tilde{v}}$

$$\begin{aligned}
c[\tilde{v}_D[\tilde{f}]]\cdot\nabla_{x,\tilde{v}}\,\frac{\mu}{q} &= -\left[\frac{E[\tilde{f}]\wedge e}{B}-\frac{\mu}{q}\frac{\nabla_x\omega_c\wedge e}{\omega_c}-\frac{(\tilde{v}\cdot e)^2}{\omega_c}\partial_x e e\wedge e\right]\\
&\quad\cdot\left[\frac{(\tilde{v}\cdot e)\,{}^t\partial_x e\tilde{v}}{\omega_c}+\frac{\mu}{q}\frac{\nabla_x\omega_c}{\omega_c}\right]+\tilde{v}\wedge\left(\partial_x e\tilde{v}_D[\tilde{f}]\wedge e\right)\cdot\frac{\tilde{v}-(\tilde{v}\cdot e)e}{\omega_c}\\
&= -\frac{(\tilde{v}\cdot e)}{\omega_c}\left(\partial_x e\;\tilde{v}_D[\tilde{f}]\cdot\tilde{v}\right)-\frac{\mu}{q}\frac{\nabla_x\omega_c}{\omega_c}\cdot\left(\tilde{v}_{\wedge D}[\tilde{f}]-\frac{(\tilde{v}\cdot e)^2}{\omega_c}\partial_x e e\wedge e\right)\\
&\quad+\frac{(\tilde{v}\cdot e)}{\omega_c}\left(\partial_x e\;\tilde{v}_D[\tilde{f}]\cdot\tilde{v}\right)\\
&= -\frac{\mu}{q}\frac{\nabla_x\omega_c}{\omega_c}\cdot\left(\tilde{v}_{\wedge D}[\tilde{f}]-\frac{(\tilde{v}\cdot e)^2}{\omega_c}\partial_x e e\wedge e\right)
\end{aligned}$$

and therefore $\left(\left\langle a[\tilde{f}]\right\rangle+\left\langle c_1[\tilde{f}]\right\rangle\right)\cdot\nabla_{x,\tilde{v}}\,\mu=0.$ □

Remark 6.1
The previous computations show that the invariance of the magnetic moment holds true when replacing $E[\tilde{f}], E[(\tilde{v}\cdot e)(e\cdot\mathrm{rot}_x e)\tilde{f}/\omega_c]$ by any continuous vector fields.

Recall that

$$\tilde{f}^\varepsilon=\tilde{f}_\varepsilon\circ T^\varepsilon+\varepsilon\lambda^\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon+\mathcal{O}(\varepsilon^2)=\tilde{f}_\varepsilon\circ T^\varepsilon+\varepsilon\tilde{f}^1_\varepsilon\circ T^\varepsilon+\mathcal{O}(\varepsilon^2),\;\; b\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon=0,\;\;\left\langle\tilde{f}^1_\varepsilon\right\rangle=0.$$

As the function $\tilde{f}^1_\varepsilon$ has zero average, it is easily seen that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}^1_\varepsilon(t,T^\varepsilon(x,\tilde{v}))\,\mathrm{d}\tilde{v}\mathrm{d}x=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}^1_\varepsilon(t,\tilde{x},\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}+\mathcal{O}(\varepsilon)=\mathcal{O}(\varepsilon)$$

and therefore we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}^\varepsilon(t,x,\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}x &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,\tilde{x},\tilde{v})\left|\det\left(\frac{\partial x}{\partial\tilde{x}}\right)\right|\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}+\mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,\tilde{x},\tilde{v})\left|\det\left(I_3-\varepsilon M[\tilde{v}]\partial\left(\frac{e}{\omega_c}\right)(x(\tilde{x}))\right)\right|\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}+\mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,\tilde{x},\tilde{v})\left[1-\varepsilon\mathrm{trace}\left(M[\tilde{v}]\partial\left(\frac{e}{\omega_c}\right)(\tilde{x})\right)\right]\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x}+\mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\left[1+\varepsilon\tilde{v}\cdot\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)\right]\,\mathrm{d}\tilde{v}\mathrm{d}x+\mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\left\{1+\varepsilon\tilde{v}\cdot\left[\frac{\mathrm{rot}_x e}{\omega_c}-\frac{\nabla_x\omega_c}{\omega_c^2}\wedge e\right]\right\}\,\mathrm{d}\tilde{v}\mathrm{d}x+\mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})m^\varepsilon(x,\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}x+\mathcal{O}(\varepsilon^2)
\end{aligned}$$

where $m^\varepsilon(x,\tilde{v})=1+\varepsilon(\tilde{v}\cdot e)(e\cdot\mathrm{rot}_x e)/\omega_c(x)$. We have used the constraint $b\cdot\nabla_{x,\tilde{v}}\,\tilde{f}_\varepsilon=0$, which implies $\int_{\mathbb{R}^3}\tilde{f}_\varepsilon\tilde{v}\;\mathrm{d}\tilde{v}=\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(\tilde{v}\cdot e)\;\mathrm{d}\tilde{v}\;e$. As the total particle number $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}^\varepsilon(t,x,\tilde{v})\;\mathrm{d}\tilde{v}\mathrm{d}x$

is conserved, up to a second order, the total particle number $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})m^\varepsilon(x,\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}x$ should be preserved as well. Indeed, multiplying (64) by m^ε and integrating by parts yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})m^\varepsilon(x,\tilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\mathrm{div}_{x,\tilde{v}}\left[m^\varepsilon\left(\left\langle c_0[\tilde{f}_\varepsilon]\right\rangle+\varepsilon\left\langle a[\tilde{f}_\varepsilon]\right\rangle+\varepsilon\left\langle c_1[\tilde{f}_\varepsilon]\right\rangle\right)\right]\,\mathrm{d}\tilde{v}\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\mathrm{div}_{x,\tilde{v}}\left\langle c_0[\tilde{f}_\varepsilon]\right\rangle\,\mathrm{d}\tilde{v}\mathrm{d}x\\
&\quad+\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\mathrm{div}_{x,\tilde{v}}\left[(\tilde{v}\cdot e)\frac{e\cdot\mathrm{rot}_x e}{\omega_c}\left\langle c_0[\tilde{f}_\varepsilon]\right\rangle+\left\langle a[\tilde{f}_\varepsilon]\right\rangle+\left\langle c_1[\tilde{f}_\varepsilon]\right\rangle\right]\,\mathrm{d}\tilde{v}\mathrm{d}x\\
&\quad+\varepsilon^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{f}_\varepsilon(t,x,\tilde{v})\mathrm{div}_{x,\tilde{v}}\left[(\tilde{v}\cdot e)\frac{e\cdot\mathrm{rot}_x e}{\omega_c}\left(\left\langle a[\tilde{f}_\varepsilon]\right\rangle+\left\langle c_1[\tilde{f}_\varepsilon]\right\rangle\right)\right]\,\mathrm{d}\tilde{v}\mathrm{d}x.
\end{aligned}$$

Our conclusion follows thanks to the result below, whose proof is left to the reader.

Proposition 6.2
For any particle density $\tilde{f}\in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)$, the vector fields

$$\left\langle c_0[\tilde{f}]\right\rangle\cdot\nabla_{x,\tilde{v}}\ ,\quad\left[(\tilde{v}\cdot e)\frac{(e\cdot\mathrm{rot}_x e)}{\omega_c}\left\langle c_0[\tilde{f}]\right\rangle+\left\langle a[\tilde{f}]\right\rangle+\left\langle c_1[\tilde{f}]\right\rangle\right]\cdot\nabla_{x,\tilde{v}}$$

are divergence free. The above result holds true when replacing $E[\tilde{f}], E[(\tilde{v}\cdot e)(e\cdot\mathrm{rot}_x e)\tilde{f}/\omega_c]$ by any $E,\tilde{E}\in C^1(\mathbb{R}^3)$ such that $\mathrm{rot}_x E=0$.

The well posedness of (64), (65) follows by standard arguments, similar to that in the proof of Theorem 2.1. The details of the proof of Theorem 1.1 are left to the reader.

Remark 6.2
For any $T>0$ there is $\varepsilon_T>0$ such that the solutions $(\tilde{f}_\varepsilon)_\varepsilon$ of (64), (65) are uniformly compactly supported and uniformly bounded with respect to $\varepsilon\in]0,\varepsilon_T]$

$$\sup_{0<\varepsilon\leq\varepsilon_T,t\in[0,T]}\{\|\tilde{f}_\varepsilon(t)\|_{C^1(\mathbb{R}^3\times\mathbb{R}^3)}+\|E[\tilde{f}_\varepsilon(t)]\|_{C^1(\mathbb{R}^3)}\}<+\infty.$$

Remark 6.3
The model (64), (65) propagates the constraint $b\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon=0$. This is a consequence of the fact that the vector fields $\left\langle c_0[\tilde{f}_\varepsilon]\right\rangle,\left\langle c_1[\tilde{f}_\varepsilon]\right\rangle,\left\langle a[\tilde{f}_\varepsilon]\right\rangle$ are in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde{v}}$. If $\tilde{f}_\varepsilon$ solves (64), (65) with the initial condition $\tilde{f}_{\mathrm{in}}$, then $\tilde{f}^\theta_\varepsilon(t,x,\tilde{V})=\tilde{f}_\varepsilon(t,x,\mathcal{R}(\theta,e(x))\tilde{V})$ solves (64), (65) with the initial condition $\tilde{f}^\theta_{\mathrm{in}}(x,\tilde{V})=\tilde{f}_{\mathrm{in}}(x,\mathcal{R}(\theta,e(x))\tilde{V})$. In particular, if the initial particle density satisfies the constraint $\overline{b}\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_{\mathrm{in}}=0$, then $\tilde{f}^\theta_{\mathrm{in}}=\tilde{f}_{\mathrm{in}},\theta\in\mathbb{R}$ and by the uniqueness, we obtain $\tilde{f}^\theta_\varepsilon=\tilde{f}_\varepsilon,\theta\in\mathbb{R}$, saying that $\overline{b}\cdot\nabla_{x,\tilde{v}}\ \tilde{f}_\varepsilon=0$.

A Well posedness of the Vlasov-Poisson problem with external magnetic field

Proof. (of Theorem 2.1)
We are not indicating all the details, but only the a priori estimates, for smooth solutions of (6), (7), (8). Let f be a smooth solution corresponding to the non negative, initial particle

density $f_{\rm in} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$. We are looking for estimating $E[f], \partial_x E[f]$ in $C([0,T] \times \mathbb{R}^3)$, where $0 < T < T(f_{\rm in})$. For any $R > 0$ we have

$$\begin{aligned}|\epsilon_0 E[f(t)](x)| &\le \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbf{1}_{\{|x-x'|<R\}} \frac{\rho[f(t)](x')}{|x-x'|^2}\, \mathrm{d}x' + \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbf{1}_{\{|x-x'|\ge R\}} \frac{\rho[f(t)](x')}{|x-x'|^2}\, \mathrm{d}x' \\ &\le R \|\rho[f(t)]\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{4\pi R^2} \|\rho[f(t)]\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Using the total charge conservation, after minimization with respect to R, that is by taking $R = (\|\rho[f_{\rm in}]\|_{L^1(\mathbb{R}^3)}/4\pi\|\rho[f(t)]\|_{L^\infty(\mathbb{R}^3)})^{1/3}$, we obtain

$$\epsilon_0 \|E[f(t)]\|_{L^\infty(\mathbb{R}^3)} \le 3\| \rho[f(t)] \|^{2/3}_{L^\infty(\mathbb{R}^3)} \left(\frac{|q|}{4\pi} \|f_{\rm in}\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \right)^{1/3}. \tag{67}$$

By the characteristic equations of (6)

$$\frac{\mathrm{d}X}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V}{\mathrm{d}s} = \frac{q}{m} \left(E[f(s)](X(s)) + V(s) \wedge \mathbf{B}(X(s)) \right), \quad (s,t,x,v) \in [0,T]^2 \times \mathbb{R}^3 \times \mathbb{R}^3$$

with the conditions $X(s=t;t,x,v) = x, V(s=t;t,x,v) = v$, we have for any $0 \le s \le t \le T$

$$|X(s;t,x,v) - x| \le \int_s^t |V(\sigma;t,x,v)|\, \mathrm{d}\sigma \tag{68}$$

and

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} |V(s;t,x,v)|^2 = \frac{q}{m} E[f(s)](X(s;t,x,v)) \cdot V(s;t,x,v) \tag{69}$$

implying that

$$|\, |V(s;t,x,v)| - |v|\, | \le \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}\sigma.$$

Assuming that $\mathrm{supp}\, f_{\rm in} \subset \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ |x| \le R_x^{\rm in} \ \text{ and } \ |v| \le R_v^{\rm in}\}$ it is easily seen that for any $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|v| > R_v(t) := R_v^{\rm in} + \frac{|q|}{m} \int_0^t \|E[f(s)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s$ we have

$$|V(0;t,x,v)| \ge |v| - \frac{|q|}{m} \int_0^t \|E[f(s)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s > R_v^{\rm in}$$

and therefore

$$f(t,x,v) = f_{\rm in}(X(0;t,x,v), V(0;t,x,v)) = 0.$$

Consider now $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$|x| > R_x(t) := R_x^{\rm in} + tR_v(t) + \int_0^t \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}\sigma\, \mathrm{d}s.$$

If $|v| > R_v(t)$ we already know that $f(t,x,v) = 0$. If $|v| \le R_v(t)$, we have by (68), (69)

$$\begin{aligned}|X(0;t,x,v)| &\ge |x| - \int_0^t |V(s;t,x,v)|\, \mathrm{d}s \\ &\ge |x| - \int_0^t \left[|v| + \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}\sigma \right] \mathrm{d}s \\ &\ge |x| - tR_v(t) - \int_0^t \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}\sigma\, \mathrm{d}s > R_x^{\rm in}\end{aligned}$$

implying that $f(t,x,v) = f_{\rm in}(X(0;t,x,v),V(0;t,x,v)) = 0$. Therefore f is compactly supported

$$\text{supp } f(t) \subset \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ |x| \le R_x(t) \ \text{ and } |v| \le R_v(t)\}, \ \ t \in [0,T].$$

Notice that the above computations are not depending on the magnetic field $\mathbf{B}$ since the magnetic force does not change the kinetic energy cf. (69). The charge density is bounded by

$$|\rho[f(t)](x)| \ \ = |q| \int_{\mathbb{R}^3} f_{\rm in}(X(0;t,x,v),V(0;t,x,v)) \, {\rm d}v \le |q| \|f_{\rm in}\|_{L^\infty} \tfrac{4}{3}\pi R_v^3(t). \tag{70}$$

Combining (67), (70) yields

$$\begin{aligned}
\epsilon_0 \|E[f(t)]\|_{L^\infty(\mathbb{R}^3)} &\le 3 \left(\frac{|q|}{4\pi} \|f_{\rm in}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \right)^{1/3} \left(|q| \frac{4\pi}{3} \|f_{\rm in}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \right)^{2/3} R_v^2(t) \\
&= |q| (12\pi)^{1/3} \|f_{\rm in}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}^{1/3} \|f_{\rm in}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{2/3} \left(R_v^{\rm in} + \frac{|q|}{m} \int_0^t \|E[f(s)]\|_{L^\infty(\mathbb{R}^3)} \, {\rm d}s \right)^2
\end{aligned}$$

leading to the bound

$$\|E[f(t)]\|_{L^\infty(\mathbb{R}^3)} \le \frac{m^2 \epsilon_0}{|q|^3 (12\pi)^{1/3} \|f_{\rm in}\|_{L^1}^{1/3} \|f_{\rm in}\|_{L^\infty}^{2/3}} \frac{1}{(T(f_{\rm in}) - t)^2}, \ \ 0 \le t \le T < T(f_{\rm in}).$$

Observe that the above bound depends on m, ϵ_0, q, T and the initial particle density $f_{\rm in}$, but not on the magnetic field. Accordingly, we obtain a bound for the size of the support of f depending on m, ϵ_0, q, T, but not on the magnetic field

$$\sup_{0\le t\le T} [R_x(t) + R_v(t)] < +\infty, \ \ 0 < T < T(f_{\rm in}).$$

For the L^∞ bound of $\partial_x E$ we appeal to [2] where it was shown that there is a constant $\tilde{C}$ (depending only on m, ϵ_0) such that

$$\|\partial_x E[f(t)]\|_{L^\infty} \le \tilde{C} \left[(1 + \|\rho[f(t)]\|_{L^\infty})(1 + \ln^+ \|\nabla_x \rho[f(t)]\|_{L^\infty}) + \|\rho[f(t)]\|_{L^1} \right].$$

The notation $\ln^+$ stands for the positive part of ln. We already have a priori bounds for the L^∞ norm of $\rho[f(t)]$ (use the estimate for the size of the support of f) and for the L^1 norm of $\rho[f(t)]$ (use the conservation of the total charge), and therefore we have

$$\|\partial_x E[f(t)]\|_{L^\infty} \le C_1 (1 + \ln^+ \|\nabla_x \rho[f(t)]\|_{L^\infty}), \ \ t \in [0,T] \tag{71}$$

for some constant C_1 depending on m, ϵ_0, q, T. Using the characteristics of the Vlasov equation, we write

$$\begin{aligned}
\nabla_x \rho[f(t)] &= q \nabla_x \int_{\mathbb{R}^3} f_{\rm in}(X(0;t,x,v),V(0;t,x,v)) \, {\rm d}v \qquad (72) \\
&= q \int_{\mathbb{R}^3} \mathbf{1}_{\{|v| \le R_v(t)\}} \, {}^t\partial_x X(0;t,x,v) (\nabla_X f_{\rm in})(X(0;t,x,v),V(0;t,x,v)) \, {\rm d}v \\
&+ q \int_{\mathbb{R}^3} \mathbf{1}_{\{|v| \le R_v(t)\}} \, {}^t\partial_x V(0;t,x,v) (\nabla_V f_{\rm in})(X(0;t,x,v),V(0;t,x,v)) \, {\rm d}v.
\end{aligned}$$

It is easily seen that there is a constant $C_2(m,q,T,\|\mathbf{B}\|_{W^{1,\infty}})$ such that for any $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, |v| \le R_v(t)$, we have

$$|\partial_x X(0;t,x,v)| + |\partial_x V(0;t,x,v)| \le C_2 \exp\left(\int_0^t \|\partial_x E[f(s)]\|_{L^\infty} \, {\rm d}s \right), \ \ t \in [0,T].$$

Combining to (72) we obtain

$$\|\nabla_x \rho[f(t)]\|_{L^\infty} \leq C_3 \exp\left(\int_0^t \|\partial_x E[f(s)]\|_{L^\infty}\, \mathrm{d}s\right), \quad t \in [0,T]$$

for some constant $C_3(m, q, T, \|\mathbf{B}\|_{W^{1,\infty}}) \geq 1$, implying that

$$\ln^+ \|\nabla_x \rho[f(t)]\|_{L^\infty} \leq \ln C_3 + \int_0^t \|\partial_x E[f(s)]\|_{L^\infty}\, \mathrm{d}s, \quad t \in [0,T].$$

Coming back to (71), we deduce that

$$\|\partial_x E[f(t)]\|_{L^\infty} \leq C_4 \left(1 + \int_0^t \|\partial_x E[f(s)]\|_{L^\infty}\, \mathrm{d}s\right), \quad t \in [0,T]$$

and the a priori estimate for the L^∞ norm of $\partial_x E[f]$ follows by Gronwall lemma. □

B Proofs of Propositions 5.3, 5.6, 5.7

Proof. (of Proposition 5.3)
The above formulae come by direct computations. Let us indicate some details.
1. Clearly we have $\langle \tilde{v} \rangle = (\tilde{v} \cdot e)\, e$ and for any $\xi \in \mathbb{R}^3$

$$\langle M[\tilde{v}]\xi \rangle = -\langle M[\xi]\tilde{v} \rangle = -(\tilde{v} \cdot e) M[\xi] e = (\tilde{v} \cdot e) M[e]\xi$$

saying that $\langle M[\tilde{v}] \rangle = (\tilde{v} \cdot e) M[e]$.
2. It is easily seen that

$$\begin{aligned}
\langle \tilde{v} \otimes \tilde{v} \rangle &= \frac{1}{S(x,\tilde{v})} \int_0^{S(x,\tilde{v})} \tilde{\mathcal{V}}(s;x,\tilde{v}) \otimes \tilde{\mathcal{V}}(s;x,\tilde{v})\, \mathrm{d}s \\
&= \frac{1}{S(x,\tilde{v})} \int_0^{S(x,\tilde{v})} \{\cos(s\omega_c(x))[\tilde{v} - (\tilde{v} \cdot e)\, e] + \sin(s\omega_c(x))\tilde{v} \wedge e + (\tilde{v} \cdot e)\, e\} \\
&\otimes \{\cos(s\omega_c(x))[\tilde{v} - (\tilde{v} \cdot e)\, e] + \sin(s\omega_c(x))\tilde{v} \wedge e + (\tilde{v} \cdot e)\, e\}\, \mathrm{d}s \\
&= \frac{1}{2}(\tilde{v} - (\tilde{v} \cdot e)\, e) \otimes (\tilde{v} - (\tilde{v} \cdot e)\, e) + \frac{1}{2}(\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) + (\tilde{v} \cdot e)^2 e \otimes e \\
&= \frac{|\tilde{v} \wedge e|^2}{2}(I_3 - e \otimes e) + (\tilde{v} \cdot e)^2 e \otimes e.
\end{aligned}$$

3. It is a direct consequence of the second statement.
4. We write

$$\langle \tilde{v} \otimes (\tilde{v} \wedge e) \rangle = -\langle \tilde{v} \otimes M[e]\tilde{v} \rangle = \langle \tilde{v} \otimes \tilde{v} \rangle\, M[e] = \frac{|\tilde{v} \wedge e|^2}{2} M[e]$$

and therefore

$$\langle \tilde{v} \cdot \partial_x e(\tilde{v} \wedge e) \rangle = \partial_x e : \langle \tilde{v} \otimes (\tilde{v} \wedge e) \rangle = \frac{|\tilde{v} \wedge e|^2}{2} \partial_x e : M[e] = \frac{|\tilde{v} \wedge e|^2}{2} \mathrm{rot}_x e \cdot e.$$

5. For any vector $\xi \in \mathbb{R}^3$ we have

$$\begin{aligned}
&\left\langle M[\tilde{v}]\partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\cdot\xi\right\rangle = \left\langle \partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\cdot M[\xi]\tilde{v}\right\rangle\\
&= -\left\langle M[\xi]\partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\cdot\tilde{v}\right\rangle = -M[\xi]\partial_x\left(\frac{e}{\omega_c}\right):\langle\tilde{v}\otimes(\tilde{v}-(\tilde{v}\cdot e)\,e)\rangle\\
&= -\frac{|\tilde{v}\wedge e|^2}{2}M[\xi]\partial_x\left(\frac{e}{\omega_c}\right):(I_3-e\otimes e) = \frac{|\tilde{v}\wedge e|^2}{2}\partial_x\left(\frac{e}{\omega_c}\right):[M[\xi]-(M[\xi]e)\otimes e]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)\cdot\xi+\partial_x\left(\frac{e}{\omega_c}\right):M[e](\xi\otimes e)\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)\cdot\xi-M[e]\partial_x\left(\frac{e}{\omega_c}\right):\xi\otimes e\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)\cdot\xi-M[e]\partial_x\left(\frac{e}{\omega_c}\right)e\cdot\xi\right]
\end{aligned}$$

implying that

$$\begin{aligned}
\left\langle M[\tilde{v}]\partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle &= \frac{|\tilde{v}\wedge e|^2}{2}\left[\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)-e\wedge\partial_x\left(\frac{e}{\omega_c}\right)e\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\mathrm{rot}_x\left(\frac{e}{\omega_c}\right)-\frac{e\wedge\partial_x e\,e}{\omega_c}\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\frac{\mathrm{rot}_x e}{\omega_c}-\frac{\nabla_x\omega_c}{\omega_c^2}\wedge e-\frac{e\wedge M[\mathrm{rot}_x e]e}{\omega_c}\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left[\frac{\mathrm{rot}_x e}{\omega_c}-\frac{\nabla_x\omega_c}{\omega_c^2}\wedge e+\frac{(\mathrm{rot}_x e\wedge e)\wedge e}{\omega_c}\right]\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\left(\frac{\mathrm{rot}_x e\cdot e}{\omega_c}e-\frac{\nabla_x\omega_c\wedge e}{\omega_c^2}\right).
\end{aligned}$$

6. Thanks to the statements 4. and 5. we obtain

$$\begin{aligned}
\langle A_x\rangle &= -\left\langle\partial_x\left(\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle+\left\langle(\tilde{v}\wedge e)\wedge\partial_x e\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}\right\rangle\\
&= -\frac{|\tilde{v}\wedge e|^2}{2}\left(\frac{\mathrm{rot}_x e\cdot e}{\omega_c}e-\frac{\nabla_x\omega_c\wedge e}{\omega_c^2}\right)+\frac{|\tilde{v}\wedge e|^2}{2}\frac{\mathrm{rot}_x e\cdot e}{\omega_c}e\\
&= \frac{|\tilde{v}\wedge e|^2}{2}\frac{\nabla_x\omega_c\wedge e}{\omega_c^2}.
\end{aligned}$$

7. For any three vectors $\xi,\eta,\chi\in\mathbb{R}^3$, the notations $\xi\otimes\eta\otimes\chi$ stands for the components $\xi_i\eta_j\chi_k, i,j,k\in\{1,2,3\}$. We have for any $s\in\mathbb{R}$

$$\begin{aligned}
\tilde{\mathcal{V}}(s;x,\tilde{v})-(\tilde{\mathcal{V}}(s;x,\tilde{v})\cdot e(\mathcal{X}(s;x,\tilde{v})))e(\mathcal{X}(s;x,\tilde{v})) &= \cos(s\omega_c)(I_3-e(x)\otimes e(x))\tilde{v}\\
&\quad+\sin(s\omega_c(x))\tilde{v}\wedge e(x).
\end{aligned}$$

The conclusion follows observing that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^3\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta\sin^2\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^3\theta\,\mathrm{d}\theta = 0.$$

8. Most of the averages in the next computations will vanish, thanks to 7. We have

$$\begin{aligned}\partial_x e A_x \cdot (\tilde{v} - (\tilde{v}\cdot e)\, e) &= -M[\tilde{v}]\partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)\\ &+ (\tilde{v}\wedge e)\wedge \partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\cdot\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)\\ &= \partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[(\tilde{v} - (\tilde{v}\cdot e)\, e)\wedge\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)]\\ &+ (\tilde{v}\wedge e)\wedge \partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\cdot\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)\\ &+ (\tilde{v}\cdot e)\partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[e\wedge\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)].\end{aligned}$$

By the previous statement we know that

$$\left\langle \partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[(\tilde{v} - (\tilde{v}\cdot e)\, e)\wedge\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)]\right\rangle = 0$$

and

$$\left\langle (\tilde{v}\wedge e)\wedge \partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\cdot\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)\right\rangle = 0.$$

For the last term we have

$$\begin{aligned}(\tilde{v}\cdot e)&\left\langle \partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[e\wedge\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)]\right\rangle\\ &= (\tilde{v}\cdot e)\left\langle \partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot M[e]\ {}^t\partial_x e(\tilde{v} - (\tilde{v}\cdot e)\, e)\right\rangle\\ &= -(\tilde{v}\cdot e)\partial_x e M[e]\partial_x \left(\frac{e}{\omega_c}\right) : \langle(\tilde{v} - (\tilde{v}\cdot e)\, e)\otimes(\tilde{v} - (\tilde{v}\cdot e)\, e)\rangle\\ &= -(\tilde{v}\cdot e)\partial_x e M[e]\partial_x \left(\frac{e}{\omega_c}\right) : \frac{|\tilde{v}\wedge e|^2}{2}(I_3 - e\otimes e)\\ &= -(\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2}\partial_x e M[e]\partial_x \left(\frac{e}{\omega_c}\right) : I_3\\ &= -(\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2\omega_c}\partial_x e M[e]\partial_x e : I_3\\ &= (\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e)\end{aligned}$$

since $\mathrm{trace}(\partial_x e M[e]\partial_x e) = -\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e)$ cf. 11. We proved that

$$\langle \partial_x e A_x\cdot(\tilde{v} - (\tilde{v}\cdot e)\, e)\rangle = \frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{div}_x e\ (\mathrm{rot}_x e\cdot e).$$

Similarly, we write

$$\begin{aligned}\partial_x e A_x \cdot (\tilde{v} \wedge e) &= -M[\tilde{v}]\partial_x \left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot\ {}^t\partial_x e(\tilde{v}\wedge e)\\ &+ \left[(\tilde{v}\wedge e)\wedge \partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\right]\cdot\ {}^t\partial_x e(\tilde{v}\wedge e)\\ &= \partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[(\tilde{v} - (\tilde{v}\cdot e)\, e)\wedge\ {}^t\partial_x e(\tilde{v}\wedge e)]\\ &+ \left[(\tilde{v}\wedge e)\wedge \partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\right]\cdot\ {}^t\partial_x e(\tilde{v}\wedge e)\\ &+ (\tilde{v}\cdot e)\partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot[e\wedge\ {}^t\partial_x e(\tilde{v}\wedge e)].\end{aligned}$$

The averages of the first and second term vanish, cf. 7. and for the third term we obtain

$$\begin{aligned}&(\tilde{v}\cdot e)\left\langle \partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot M[e]\ {}^t\partial_x e(\tilde{v}\wedge e)\right\rangle\\ &= -(\tilde{v}\cdot e)\left\langle \partial_x e M[e]\partial_x\left(\frac{e}{\omega_c}\right)(\tilde{v} - (\tilde{v}\cdot e)\, e)\cdot(\tilde{v}\wedge e)\right\rangle\\ &= -(\tilde{v}\cdot e)\partial_x e M[e]\partial_x\left(\frac{e}{\omega_c}\right) : \langle(\tilde{v}\wedge e)\otimes(\tilde{v} - (\tilde{v}\cdot e)\, e)\rangle\\ &= (\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2}\partial_x e M[e]\partial_x\left(\frac{e}{\omega_c}\right) : M[e]\\ &= (\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2\omega_c}\partial_x e M[e]\partial_x e : M[e]\end{aligned}$$

and thus

$$\langle \partial_x e A_x\cdot(\tilde{v}\wedge e)\rangle = (\tilde{v}\cdot e)\frac{|\tilde{v}\wedge e|^2}{2\omega_c}\partial_x e M[e]\partial_x e : M[e].$$

9. We can write thanks to 3.

$$\begin{aligned}\left\langle \partial_x e\ \partial_x\left(\frac{\tilde{v}\wedge e}{\omega_c}\right) e\cdot(\tilde{v} - (\tilde{v}\cdot e)\, e)\right\rangle &= \left\langle \partial_x e M[\tilde{v}]\partial_x\left(\frac{e}{\omega_c}\right) e\cdot(\tilde{v} - (\tilde{v}\cdot e)\, e)\right\rangle\\ &= -\left\langle \partial_x e M\left[\partial_x\left(\frac{e}{\omega_c}\right) e\right]\tilde{v}\cdot(\tilde{v} - (\tilde{v}\cdot e)\, e)\right\rangle\\ &= -\partial_x e M\left[\partial_x\left(\frac{e}{\omega_c}\right) e\right] : \langle(\tilde{v} - (\tilde{v}\cdot e)\, e)\otimes\tilde{v}\rangle\\ &= -\frac{|\tilde{v}\wedge e|^2}{2}\partial_x e M\left[\partial_x\left(\frac{e}{\omega_c}\right) e\right] : (I_3 - e\otimes e)\\ &= \frac{|\tilde{v}\wedge e|^2}{2}\partial_x e : M\left[\partial_x\left(\frac{e}{\omega_c}\right) e\right]\\ &= \frac{|\tilde{v}\wedge e|^2}{2}\mathrm{rot}_x e\cdot\partial_x\left(\frac{e}{\omega_c}\right) e\\ &= \frac{|\tilde{v}\wedge e|^2}{2}\mathrm{rot}_x e\cdot\left(\frac{\partial_x e\ e}{\omega_c} - \frac{e\otimes\nabla_x\omega_c}{\omega_c^2}e\right)\\ &= \frac{|\tilde{v}\wedge e|^2}{2}\mathrm{rot}_x e\cdot\left(\frac{\mathrm{rot}_x e\wedge e}{\omega_c} - \frac{e\otimes\nabla_x\omega_c}{\omega_c^2}e\right)\\ &= -\frac{|\tilde{v}\wedge e|^2}{2}(\mathrm{rot}_x e\cdot e)\frac{\nabla_x\omega_c\cdot e}{\omega_c^2}.\end{aligned}$$

10. For any vector $\xi \in \mathbb{R}^3$ we have by 4.

$$\begin{aligned}\langle \tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)\rangle \cdot \xi &= \langle M[\tilde{v}]\partial_x e(\tilde{v} \wedge e) \cdot \xi\rangle \\ &= \langle \partial_x e(\tilde{v} \wedge e) \cdot M[\xi]\tilde{v}\rangle = -\langle M[\xi]\partial_x e(\tilde{v} \wedge e) \cdot \tilde{v}\rangle \\ &= -M[\xi]\partial_x e : \langle \tilde{v} \otimes (\tilde{v} \wedge e)\rangle = -M[\xi]\partial_x e : M[e]\frac{|\tilde{v} \wedge e|^2}{2} \\ &= \frac{|\tilde{v} \wedge e|^2}{2}\mathrm{trace}(M[e]M[\xi]\partial_x e) = \frac{|\tilde{v} \wedge e|^2}{2}\mathrm{trace}([\xi \otimes e - (\xi \cdot e)I_3]\partial_x e) \\ &= \frac{|\tilde{v} \wedge e|^2}{2}\mathrm{trace}(\xi \otimes {}^t\partial_x e e - (\xi \cdot e)\partial_x e) = -\frac{|\tilde{v} \wedge e|^2}{2}(\xi \cdot e)\mathrm{div}_x e.\end{aligned}$$

Therefore we deduce that

$$\langle \tilde{v} \wedge \partial_x e(\tilde{v} \wedge e)\rangle = -\frac{|\tilde{v} \wedge e|^2}{2}\mathrm{div}_x e\ e.$$

11. The matrix $\partial_x e M[e]\ {}^t\partial_x e$ is anti-symmetric and thus

$$\begin{aligned}\mathrm{trace}(\partial_x e M[e]\partial_x e) &= \mathrm{trace}(\partial_x e M[e](\partial_x e - {}^t\partial_x e)) \\ &= \mathrm{trace}(\partial_x e M[e]M[\mathrm{rot}_x e]) = \mathrm{trace}(\partial_x e(\mathrm{rot}_x e \otimes e - \mathrm{rot}_x e \cdot e\ I_3)) \\ &= \mathrm{trace}((\partial_x e\ \mathrm{rot}_x e) \otimes e - (\mathrm{rot}_x e \cdot e)\partial_x e) \\ &= -\mathrm{div}_x e\ (\mathrm{rot}_x e \cdot e).\end{aligned}$$

12. By the statement 11. we have

$$\begin{aligned}\mathrm{trace}(M[e]\partial_x e M[e]\partial_x e M[e]) &= \mathrm{trace}(M[e]M[e]\partial_x e M[e]\partial_x e) \\ &= \mathrm{trace}((e \otimes e - I_3)\partial_x e M[e]\partial_x e) \\ &= -\mathrm{trace}(\partial_x e M[e]\partial_x e) = \mathrm{div}_x e\ (\mathrm{rot}_x e \cdot e).\end{aligned}$$

□

Proof. (of Proposition 5.6)
Observe that the vector field $c^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ writes

$$\begin{aligned}c^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}} &= (\tilde{v} \cdot e(\tilde{x}))\ e(\tilde{x}) \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} \\ &\quad - \tilde{v} \wedge \int_0^1 \partial_x e(\tilde{x}(\tau))(\tilde{v} \wedge (x))\ \mathrm{d}\tau \cdot \nabla_{\tilde{v}}\end{aligned}$$

where $\tilde{x}(\tau) = x + \tau\varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)}, \tilde{x} = x + \varepsilon\frac{\tilde{v}\wedge e(x)}{\omega_c(x)} = \tilde{x}(1)$, and therefore

$$\begin{aligned}\lim_{\varepsilon\searrow 0} c^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}} &= (\tilde{v} \cdot e(x))\ e(x) \cdot \nabla_x + \frac{q}{m}(E[\tilde{f}] \cdot e(x))\ e(x) \cdot \nabla_{\tilde{v}} - (\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e(x))) \cdot \nabla_{\tilde{v}} \\ &= c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ .\end{aligned}$$

By direct computations one gets

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\frac{\partial T^\varepsilon c^\varepsilon[\tilde{f} \circ T^\varepsilon] - c^\varepsilon[\tilde{f} \circ T^\varepsilon]}{\varepsilon} &= \lim_{\varepsilon\searrow 0}\partial\left(\frac{\tilde{v} \wedge e}{\omega_c}, 0\right) c^\varepsilon[\tilde{f} \circ T^\varepsilon] \\ &= (\tilde{v} \cdot e(x))M[\tilde{v}]\partial_x\left(\frac{e}{\omega_c}\right)\ e(x) \cdot \nabla_x + \left[\frac{e}{\omega_c} \wedge (\tilde{v} \wedge \partial_x e(\tilde{v} \wedge e))\right] \cdot \nabla_x \\ &= (\tilde{v} \cdot e(x))M[\tilde{v}]\partial_x\left(\frac{e}{\omega_c}\right)\ e(x) \cdot \nabla_x - \frac{(\tilde{v} \cdot e(x))}{\omega_c(x)}\partial_x e(\tilde{v} \wedge e(x)) \cdot \nabla_x\end{aligned}$$

$$\lim_{\varepsilon\searrow 0}\frac{c^\varepsilon[\tilde f\circ T^\varepsilon]-c_0[\tilde f]\circ T^\varepsilon}{\varepsilon}=\frac{q}{m}\lim_{\varepsilon\searrow 0}\frac{(E[\tilde f\circ T^\varepsilon]\cdot e(x))\,e(x)-(E[\tilde f]\circ T^\varepsilon\cdot e(\tilde x))\,e(\tilde x)}{\varepsilon}\cdot\nabla_{\tilde v}$$
$$-\left(\tilde v\wedge\int_0^1\lim_{\varepsilon\searrow 0}\frac{\partial_x e(\tilde x(\tau))\;\tilde v\wedge e(x)-\partial_x e(\tilde x)\;\tilde v\wedge e(\tilde x)}{\varepsilon}\,\mathrm{d}\tau\right)\cdot\nabla_{\tilde v}$$
$$=\frac{q}{m}\left(E\left[\frac{\tilde v\wedge e}{\omega_c}\cdot\nabla_x\tilde f\right]\cdot e(x)\right)\;e(x)\cdot\nabla_{\tilde v}-\frac{q}{m}\left(\partial_x E[\tilde f]\;\frac{\tilde v\wedge e}{\omega_c}\cdot e(x)\right)\;e(x)\cdot\nabla_{\tilde v}$$
$$-\frac{q}{m}\left(E[\tilde f]\cdot\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}\right)\;e(x)\cdot\nabla_{\tilde v}-\frac{q}{m}(E[\tilde f]\cdot e(x))\;\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}\cdot\nabla_{\tilde v}$$
$$+\frac{1}{2\omega_c(x)}\left[\tilde v\wedge((\tilde v\wedge e)\otimes(\tilde v\wedge e):\nabla_x\otimes\nabla_x)e\right]\cdot\nabla_{\tilde v}+\tilde v\wedge\partial_x e(x)\left(\tilde v\wedge\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}\right)\cdot\nabla_{\tilde v}$$

and

$$\lim_{\varepsilon\searrow 0}\frac{c_0[\tilde f]\circ T^\varepsilon-\lambda^\varepsilon c_0[\tilde f]\circ T^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde v}\;=\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\;c_0[\tilde f]\cdot\nabla_{x,\tilde v}\;.$$

Therefore the coordinates of the vector field $c_1[\tilde f]\cdot\nabla_{x,\tilde v}$ are given by

$$c_{1x}[\tilde f]=(\tilde v\cdot e)\tilde v\wedge\partial_x\left(\frac{e}{\omega_c}\right)e-\frac{(\tilde v\cdot e)}{\omega_c}\partial_x e\;(\tilde v\wedge e)+(\tilde v\cdot e)\left(\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\right)e$$

$$c_{1\tilde v}[\tilde f]=\frac{q}{m}\left(E\left[\frac{\tilde v\wedge e}{\omega_c}\cdot\nabla_x\tilde f\right]\cdot e\right)\;e-\frac{q}{m}\left(\partial_x E[\tilde f]\;\frac{\tilde v\wedge e}{\omega_c}\cdot e\right)\;e$$
$$-\frac{q}{m}\left(E[\tilde f]\cdot\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}\right)\;e-\frac{q}{m}(E[\tilde f]\cdot e)\;\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}+\frac{q}{m}(E[\tilde f]\cdot e)\left(\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\right)e$$
$$+\tilde v\wedge\left[\partial_x e(x)\left(\tilde v\wedge\partial_x e\;\frac{\tilde v\wedge e}{\omega_c}\right)\right]-\left(\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\right)\tilde v\wedge\partial_x e\;(\tilde v\wedge e)$$
$$+\frac{1}{2\omega_c(x)}\tilde v\wedge((\tilde v\wedge e)\otimes(\tilde v\wedge e):\nabla_x\otimes\nabla_x)e.$$

The average of $c_1[\tilde f]\cdot\nabla_{x,\tilde v}$ follows thanks to Remark 5.1

$$\left\langle c_1[\tilde f]\right\rangle\cdot\nabla_{x,\tilde v}\;=c\left[\left\langle c_{1x}[\tilde f]\right\rangle\right]\cdot\nabla_{x,\tilde v}$$
$$+\left\langle(c_{1\tilde v}[\tilde f]-(\tilde v\cdot e)\partial_x e\;c_{1x}[\tilde f])\cdot(\tilde v-(\tilde v\cdot e)\;e)\right\rangle\frac{\tilde v-(\tilde v\cdot e)\;e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}$$
$$+\left\langle c_{1\tilde v}[\tilde f]\cdot e+\partial_x e\;c_{1x}[\tilde f]\cdot\tilde v\right\rangle e\cdot\nabla_{\tilde v}$$
$$+\left\langle(c_{1\tilde v}[\tilde f]-(\tilde v\cdot e)\partial_x e\;c_{1x}[\tilde f])\cdot(\tilde v\wedge e)\right\rangle\frac{\tilde v\wedge e}{|\tilde v\wedge e|^2}\cdot\nabla_{\tilde v}.$$

The average of $c_{1x}[\tilde f]$ is the magnetic curvature drift

$$\left\langle c_{1x}[\tilde f]\right\rangle=(\tilde v\cdot e)^2e\wedge\partial_x\left(\frac{e}{\omega_c}\right)e=\frac{(\tilde v\cdot e)^2}{\omega_c}e\wedge\partial_x ee=\tilde v_{CD}.$$

We analyze one by one the other averages. We have

$$\left\langle c_{1\tilde v}[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\;e)\right\rangle=-\frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e-\frac{(E[\tilde f]\cdot e)}{B}\frac{|\tilde v\wedge e|^2}{2}(e\cdot\mathrm{rot}_x e).$$

Indeed, the last two terms in $c_{1\tilde{v}}[\tilde{f}]$ give no contribution thanks to the seventh statement in Proposition 5.3

$$\begin{aligned}
&\langle[\tilde{v}\wedge((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)\,e]\cdot(\tilde{v}-(\tilde{v}\cdot e)\,e)\rangle\\
&=-(\tilde{v}\cdot e)\,\langle\tilde{v}\wedge((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)\,e\rangle\cdot e\\
&=-(\tilde{v}\cdot e)\,\langle(\tilde{v}-(\tilde{v}\cdot e)\,e)\wedge((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)\,e\rangle\cdot e=0
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)(\tilde{v}\wedge\partial_x e\,(\tilde{v}\wedge e))\cdot(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle\\
&=-(\tilde{v}\cdot e)\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)\tilde{v}\wedge\partial_x e\,(\tilde{v}\wedge e)\right\rangle\cdot e\\
&=-(\tilde{v}\cdot e)\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)(\tilde{v}-(\tilde{v}\cdot e)\,e)\wedge\partial_x e\,(\tilde{v}\wedge e)\right\rangle\cdot e=0.
\end{aligned}$$

Observe also that the average of the term $\tilde{v}\wedge\left[\partial_x e\left(\tilde{v}\wedge\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\cdot(\tilde{v}-(\tilde{v}\cdot e)\;e)$ can be computed as follows, cf. statement 12. in Proposition 5.3

$$\begin{aligned}
&\left\langle\tilde{v}\wedge\left[\partial_x e\left(\tilde{v}\wedge\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\cdot(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle\\
&=(\tilde{v}\cdot e)\left\langle\left\{e\wedge\left[\partial_x e\left(\tilde{v}\wedge\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\right\}\cdot(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle\\
&=(\tilde{v}\cdot e)^2\left\langle\left\{e\wedge\left[\partial_x e\left(e\wedge\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\right\}\cdot(\tilde{v}-(\tilde{v}\cdot e)\,e)\right\rangle\\
&=(\tilde{v}\cdot e)^2\left\langle M[e]\partial_x eM[e]\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\cdot\tilde{v}\right\rangle\\
&=-\frac{(\tilde{v}\cdot e)^2}{\omega_c}M[e]\partial_x eM[e]\partial_x eM[e]:\langle\tilde{v}\otimes\tilde{v}\rangle\\
&=-\frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{trace}(M[e]\partial_x eM[e]\partial_x eM[e])\\
&=-\frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e.
\end{aligned}$$

By the first and seventh statements in Proposition 5.3 we find as before

$$\begin{aligned}
\left\langle c_{1\tilde{v}}[\tilde{f}]\cdot e\right\rangle&=\frac{q}{m}E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot\nabla_x\tilde{f}\right]\cdot e+\underbrace{\left\langle\tilde{v}\wedge\left[\partial_x e\left(\tilde{v}\wedge\partial_x e\,\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\right\rangle\cdot e}_{T_1}\\
&=\frac{q}{m}E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot\nabla_x\tilde{f}\right]\cdot e+\frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e
\end{aligned}$$

and

$$\begin{aligned}\left\langle c_{1\tilde{v}}[\tilde{f}]\cdot(\tilde{v}\wedge e)\right\rangle &= -\frac{q}{m}\frac{(E[\tilde{f}]\cdot e)}{\omega_c}\partial_x e : \langle(\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e)\rangle \\ &+ \underbrace{\frac{1}{2\omega_c(x)}\left\langle[\tilde{v}\wedge((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)e]\cdot(\tilde{v}\wedge e)\right\rangle}_{T_2} \\ &- \underbrace{\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)(\tilde{v}\wedge\partial_x e\,(\tilde{v}\wedge e))\cdot(\tilde{v}\wedge e)\right\rangle}_{T_3} \\ &+ \underbrace{\left\langle\left\{\tilde{v}\wedge\left[\partial_x e\,\left(\tilde{v}\wedge\partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\right)\right]\right\}\cdot(\tilde{v}\wedge e)\right\rangle}_{T_4} \\ &= -\frac{(E[\tilde{f}]\cdot e)}{B}\frac{|\tilde{v}\wedge e|^2}{2}\mathrm{div}_x e - \frac{|\tilde{v}\wedge e|^4}{4\omega_c}(\partial_x e:\partial_x e - |\partial_x e e|^2) \\ &+ \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{trace}(\partial_x e M[e]\partial_x e M[e]).\end{aligned}$$

Indeed, we have

$$\begin{aligned}T_1 &= -\left\langle(\tilde{v}\wedge e)\cdot\partial_x e\left(\tilde{v}\wedge\partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\right)\right\rangle \\ &= -(\tilde{v}\cdot e)\left\langle(\tilde{v}\wedge e)\cdot\partial_x e\left(e\wedge\partial_x e\frac{\tilde{v}\wedge e}{\omega_c}\right)\right\rangle \\ &= \frac{(\tilde{v}\cdot e)}{\omega_c}M[e]\partial_x e M[e]\partial_x e M[e] : \langle\tilde{v}\otimes\tilde{v}\rangle \\ &= \frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{trace}(M[e]\partial_x e M[e]\partial_x e M[e]) \\ &= \frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e\end{aligned}$$

$$\begin{aligned}T_2 &= -\frac{1}{2\omega_c}\left\langle[(\tilde{v}\cdot e)\tilde{v}-|\tilde{v}|^2 e]\cdot((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)e\right\rangle \\ &= -\frac{1}{2\omega_c}\left\langle[(\tilde{v}\cdot e)^2 e-|\tilde{v}|^2 e]\cdot((\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e):\nabla_x\otimes\nabla_x)e\right\rangle \\ &= \frac{|\tilde{v}\wedge e|^2}{2\omega_c}e\cdot(\langle(\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e)\rangle:\nabla_x\otimes\nabla_x)\,e \\ &= \frac{|\tilde{v}\wedge e|^2}{2\omega_c}e\cdot\left[\frac{|\tilde{v}\wedge e|^2}{2}(I_3-e\otimes e):\nabla_x\otimes\nabla_x\right]e \\ &= \frac{|\tilde{v}\wedge e|^4}{4\omega_c}e\cdot[\Delta_x e-(e\otimes e:\nabla_x\otimes\nabla_x)e] \\ &= -\frac{|\tilde{v}\wedge e|^4}{4\omega_c}(\partial_x e:\partial_x e-|\partial_x e e|^2).\end{aligned}$$

$$\begin{aligned}-T_3 &= \left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)((\tilde{v}\cdot e)\tilde{v}-|\tilde{v}|^2 e)\cdot\partial_x e(\tilde{v}\wedge e)\right\rangle \\ &= -|\tilde{v}\wedge e|^2\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)e\cdot\partial_x e\,(\tilde{v}\wedge e)\right\rangle = 0\end{aligned}$$

and

$$\begin{aligned}
T_4 &= -\left\langle [(\tilde v\cdot e)\tilde v - |\tilde v|^2 e]\cdot \partial_x e\ \left(\tilde v\wedge \partial_x e\ \frac{\tilde v\wedge e}{\omega_c}\right)\right\rangle\\
&= -(\tilde v\cdot e)\left\langle \tilde v\cdot \partial_x e\ \left(\tilde v\wedge \partial_x e\ \frac{\tilde v\wedge e}{\omega_c}\right)\right\rangle\\
&= -(\tilde v\cdot e)^2\left\langle \tilde v\cdot \partial_x e\ \left(e\wedge \partial_x e\ \frac{\tilde v\wedge e}{\omega_c}\right)\right\rangle\\
&= -\frac{(\tilde v\cdot e)^2}{\omega_c}\partial_x eM[e]\partial_x e : \langle \tilde v\otimes(\tilde v\wedge e)\rangle\\
&= \frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2\omega_c}\mathrm{trace}(\partial_x eM[e]\partial_x eM[e]).
\end{aligned}$$

The contributions of the coordinates $c_{1x}[\tilde f]$ are given by (cf. the ninth and fourth statements in Proposition 5.3)

$$\begin{aligned}
&\left\langle \partial_x e\ c_{1x}[\tilde f]\cdot\tilde v\right\rangle = \left\langle \partial_x e\ c_{1x}[\tilde f]\cdot(\tilde v-(\tilde v\cdot e)\ e)\right\rangle\\
&= (\tilde v\cdot e)\left\langle \partial_x e\ \partial_x\left(\frac{\tilde v\wedge e}{\omega_c}\right)e\cdot(\tilde v-(\tilde v\cdot)\ e)\right\rangle\\
&-\frac{(\tilde v\cdot e)}{\omega_c}\langle \partial_x e\ \partial_x e(\tilde v\wedge e)\cdot\tilde v\rangle + (\tilde v\cdot e)\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\right)\partial_x e\ e\cdot(\tilde v-(\tilde v\cdot e)\ e)\right\rangle\\
&= -\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}(\mathrm{rot}_x e\cdot e)\frac{(\nabla_x\omega_c\cdot e)}{\omega_c^2} + \frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2\omega_c}\mathrm{trace}(M[e]\partial_x e\ \partial_x e)\\
&+\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}(\partial_x ee\wedge e)\cdot\frac{\nabla_x\omega_c}{\omega_c^2}\\
&= \frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}(\partial_x ee\wedge e)\cdot\frac{\nabla_x\omega_c}{\omega_c^2}
\end{aligned}$$

and

$$\begin{aligned}
-(\tilde v\cdot e)\left\langle \partial_x e\ c_{1x}[\tilde f]\cdot(\tilde v\wedge e)\right\rangle &= -(\tilde v\cdot e)^2\left\langle \partial_x e\left[\tilde v\wedge\partial_x\left(\frac{e}{\omega_c}\right)e\right]\cdot(\tilde v\wedge e)\right\rangle\\
&+\frac{(\tilde v\cdot e)^2}{\omega_c}\langle \partial_x e\ \partial_x e\ (\tilde v\wedge e)\cdot(\tilde v\wedge e)\rangle - (\tilde v\cdot e)^2\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde v\wedge e}{\omega_c^2}\right)\partial_x ee\cdot(\tilde v\wedge e)\right\rangle.
\end{aligned}$$

By straightforward computations, we obtain (using also $\mathrm{div}_x(\omega_c e)=0$)

$$\begin{aligned}
-(\tilde v\cdot e)^2\left\langle \partial_x e\left[\tilde v\wedge\partial_x\left(\frac{e}{\omega_c}\right)e\right]\cdot(\tilde v\wedge e)\right\rangle &= (\tilde v\cdot e)^2\left\langle \partial_x eM\left[\partial_x\left(\frac{e}{\omega_c}\right)e\right]\tilde v\cdot(\tilde v\wedge e)\right\rangle\\
&= (\tilde v\cdot e)^2\partial_x eM\left[\partial_x\left(\frac{e}{\omega_c}\right)e\right] : \langle(\tilde v\wedge e)\otimes\tilde v\rangle\\
&= -\frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2}\partial_x eM\left[\partial_x\left(\frac{e}{\omega_c}\right)e\right] : M[e]\\
&= \frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2}\partial_x e : M[e]M\left[\partial_x\left(\frac{e}{\omega_c}\right)e\right]\\
&= \frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2}\partial_x e : \left[\partial_x\left(\frac{e}{\omega_c}\right)e\otimes e - \left(\partial_x\left(\frac{e}{\omega_c}\right)e\cdot e\right)I_3\right]\\
&= \frac{(\tilde v\cdot e)^2|\tilde v\wedge e|^2}{2}\frac{|\partial_x ee|^2-(\mathrm{div}_x e)^2}{\omega_c}
\end{aligned}$$

$$\frac{(\tilde{v}\cdot e)^2}{\omega_c}\left\langle \partial_x e\partial_x e\ (\tilde{v}\wedge e)\cdot(\tilde{v}\wedge e)\right\rangle = \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}\partial_x e\partial_x e : (I_3 - e\otimes e)$$
$$= \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}\mathrm{trace}(\partial_x e\partial_x e)$$
$$= \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}[\mathrm{div}_x(\partial_x ee) - e\cdot\nabla_x \mathrm{div}_x e]$$

$$-(\tilde{v}\cdot e)^2\left\langle\left(\nabla_x\omega_c\cdot\frac{\tilde{v}\wedge e}{\omega_c^2}\right)\partial_x ee\cdot(\tilde{v}\wedge e)\right\rangle = -(\tilde{v}\cdot e)^2\partial_x ee\otimes\frac{\nabla_x\omega_c}{\omega_c^2} : \langle(\tilde{v}\wedge e)\otimes(\tilde{v}\wedge e)\rangle$$
$$= -\frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2}\partial_x ee\otimes\frac{\nabla_x\omega_c}{\omega_c^2} : (I_3 - e\otimes e)$$
$$= -\frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2}\partial_x ee\cdot\frac{\nabla_x\omega_c}{\omega_c^2}$$

$$-(\tilde{v}\cdot e)\left\langle\partial_x e\ c_{1x}[\tilde{f}]\cdot(\tilde{v}\wedge e)\right\rangle = \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2}\frac{|\partial_x ee|^2 - (\mathrm{div}_x e)^2}{\omega_c}$$
$$+\frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2}\frac{\mathrm{div}_x(\partial_x ee) - e\cdot\nabla_x\mathrm{div}_x e}{\omega_c} - \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2}\partial_x ee\cdot\frac{\nabla_x\omega_c}{\omega_c^2}$$
$$= \frac{(\tilde{v}\cdot e)^2|\tilde{v}\wedge e|^2}{2\omega_c}\left[|\partial_x ee|^2 - \partial_x ee\cdot\frac{\nabla_x\omega_c}{\omega_c} + \mathrm{div}_x(\partial_x ee - \mathrm{div}_x e\ e)\right].$$

Finally we deduce that

$$\left\langle c_1[\tilde{f}]\right\rangle\cdot\nabla_{x,\tilde{v}} = c[\tilde{v}_{CD}]\cdot\nabla_{x,\tilde{v}} - \frac{(\tilde{v}\cdot e)^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e\ (\tilde{v} - (\tilde{v}\cdot e)\ e)\cdot\nabla_{\tilde{v}}$$
$$-\frac{(E[\tilde{f}]\cdot e)}{B}\frac{(e\cdot\mathrm{rot}_x e)}{2}\ (\tilde{v} - (\tilde{v}\cdot e)\ e)\cdot\nabla_{\tilde{v}} + \frac{(\tilde{v}\cdot e)|\tilde{v}\wedge e|^2}{2\omega_c}(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e\ e\cdot\nabla_{\tilde{v}}$$
$$+\frac{q}{m}\left(E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot\nabla_x\tilde{f}\right]\cdot e\right)\ e\cdot\nabla_{\tilde{v}} + \frac{(\tilde{v}\cdot e)}{2}(\partial_x ee\wedge e)\cdot\frac{\nabla_x\omega_c}{\omega_c^2}\ (\tilde{v}\wedge(e\wedge\tilde{v}))\cdot\nabla_{\tilde{v}}$$
$$+\left[\frac{(\tilde{v}\cdot e)^2}{2\omega_c}\mathrm{trace}(\partial_x eM[e]\partial_x eM[e]) - \frac{|\tilde{v}\wedge e|^2}{4\omega_c}(\partial_x e : \partial_x e - |\partial_x ee|^2) - \frac{(E[\tilde{f}]\cdot e)}{B}\frac{\mathrm{div}_x e}{2}\right]$$
$$(\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}$$
$$+\frac{(\tilde{v}\cdot e)^2}{2\omega_c}\left[|\partial_x ee|^2 - \partial_x ee\cdot\frac{\nabla_x\omega_c}{\omega_c} + \mathrm{div}_x(\partial_x ee - \mathrm{div}_x e\ e)\right](\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}$$
$$= c[\tilde{v}_{CD}]\cdot\nabla_{x,\tilde{v}} - \frac{(E[\tilde{f}]\cdot e)}{B}\frac{(e\cdot\mathrm{rot}_x e)}{2}\ (\tilde{v} - (\tilde{v}\cdot e)\ e)\cdot\nabla_{\tilde{v}}$$
$$+\frac{q}{m}\left(E\left[\frac{\tilde{v}\wedge e}{\omega_c}\cdot\nabla_x\tilde{f}\right]\cdot e\right)\ e\cdot\nabla_{\tilde{v}}$$
$$+\left[\frac{(\mathrm{rot}_x e\cdot e)\mathrm{div}_x e}{2\omega_c}(\tilde{v}\cdot e) + \frac{(\tilde{v}\cdot e)}{2}(\partial_x ee\wedge e)\cdot\frac{\nabla_x\omega_c}{\omega_c^2}\right]\ (\tilde{v}\wedge(e\wedge\tilde{v}))\cdot\nabla_{\tilde{v}}$$
$$-\left[\frac{|\tilde{v}\wedge e|^2}{4\omega_c}(\partial_x e : \partial_x e - |\partial_x ee|^2) + \frac{(E[\tilde{f}]\cdot e)}{B}\frac{\mathrm{div}_x e}{2}\right]\ (\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}$$
$$+\frac{(\tilde{v}\cdot e)^2}{2\omega_c}\left[|\partial_x ee|^2 - \partial_x ee\cdot\frac{\nabla_x\omega_c}{\omega_c} + \mathrm{div}_x(\partial_x ee - \mathrm{div}_x e\ e) + \mathrm{trace}(\partial_x eM[e]\partial_x eM[e])\right]$$
$$(\tilde{v}\wedge e)\cdot\nabla_{\tilde{v}}.$$

□

Proof. (of Proposition 5.7)
By Proposition 5.2, applied to the vector field $\frac{c_0[\tilde{f}]}{\omega_c} \cdot \nabla_{x,\tilde{v}}$, there is a vector field $\xi \cdot \nabla_{x,\tilde{v}}$ in involution with respect to $\overline{b} \cdot \nabla_{x,\tilde{v}}$ such that for any function $u \in C^2(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker(\overline{b} \cdot \nabla_{x,\tilde{v}})$

$$\left\langle \frac{c_0[\tilde{f}]}{\omega_c} \cdot \nabla_{x,\tilde{v}}\ u^1 \right\rangle = \xi \cdot \nabla_{x,\tilde{v}}\ u$$

where

$$\frac{c_0[\tilde{f}]}{\omega_c} \cdot \nabla_{x,\tilde{v}}\ u - \left\langle \frac{c_0[\tilde{f}]}{\omega_c} \cdot \nabla_{x,\tilde{v}}\ u \right\rangle + \overline{b} \cdot \nabla_{x,\tilde{v}}\ u^1 = 0, \ \ \left\langle u^1 \right\rangle = 0.$$

For any $(x, \tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3, |\tilde{v} \wedge e(x)| > 0$ we have

$$\begin{aligned}
\xi(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} \ &= c[\xi_x] \cdot \nabla_{x,\tilde{v}} \ + \frac{\xi_{\tilde{v}} \cdot (\tilde{v} - (\tilde{v} \cdot e)\ e) - (e \cdot \tilde{v}) \partial_x e\ \xi_x \cdot \tilde{v}}{|\tilde{v} \wedge e|^2}\ (\tilde{v} - (\tilde{v} \cdot e)\ e) \cdot \nabla_{\tilde{v}} \\
&+ (\xi_{\tilde{v}} \cdot e + \partial_x e\ \xi_x \cdot \tilde{v})\ e \cdot \nabla_{\tilde{v}} + \frac{(\xi_{\tilde{v}} - (e \cdot \tilde{v}) \partial_x e\ \xi_x) \cdot (\tilde{v} \wedge e)}{|\tilde{v} \wedge e|^2}\ (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} \\
&= c[\xi_x] \cdot \nabla_{x,\tilde{v}} \ + (\partial_x e\ \xi_x \cdot \tilde{v})\ \frac{\tilde{v} \wedge (e \wedge \tilde{v})}{|\tilde{v} \wedge e|^2} \cdot \nabla_{\tilde{v}} \\
&+ \frac{\xi_{\tilde{v}} \cdot (\tilde{v} - (\tilde{v} \cdot e)\ e)}{|\tilde{v} \wedge e|^2}\ (\tilde{v} - (\tilde{v} \cdot e)\ e) \cdot \nabla_{\tilde{v}} + (\xi_{\tilde{v}} \cdot e)\ e \cdot \nabla_{\tilde{v}} \\
&+ \frac{(\xi_{\tilde{v}} - (e \cdot \tilde{v}) \partial_x e\ \xi_x) \cdot (\tilde{v} \wedge e)}{|\tilde{v} \wedge e|^2}\ (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}}.
\end{aligned}$$

For $u = x_i, i \in \{1, 2, 3\}$ we obtain

$$\omega_c \xi_x = \left\langle (c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ )\eta \right\rangle, \ \ c_0[\tilde{f}]_x - \left\langle c_0[\tilde{f}]_x \right\rangle + (b \cdot \nabla_{x,\tilde{v}}\ )\eta = 0, \ \ \langle \eta \rangle = 0.$$

Obviously we have $\left\langle c_0[\tilde{f}]_x \right\rangle = c_0[\tilde{f}]_x = (\tilde{v} \cdot e)e$, implying $\eta = 0$ and thus $\xi_x = 0$. For $u = \frac{|\tilde{v}|^2}{2}$ we obtain

$$\omega_c \xi_{\tilde{v}} \cdot \tilde{v} = \left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ u^1 \right\rangle, \ \ c_0[\tilde{f}]_{\tilde{v}} \cdot \tilde{v} - \left\langle c_0[\tilde{f}]_{\tilde{v}} \cdot \tilde{v} \right\rangle + b \cdot \nabla_{x,\tilde{v}}\ u^1 = 0, \ \ \left\langle u^1 \right\rangle = 0.$$

But $\left\langle c_0[\tilde{f}]_{\tilde{v}} \cdot \tilde{v} \right\rangle = \frac{q}{m}(E[\tilde{f}] \cdot e)(\tilde{v} \cdot e) = c_0[\tilde{f}]_{\tilde{v}} \cdot \tilde{v}$ and therefore $u^1 = 0, \xi_{\tilde{v}} \cdot \tilde{v} = 0$. We also take $u = (\tilde{v} \cdot e)$, implying that

$$\omega_c \xi_{\tilde{v}} \cdot e = \left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ u^1 \right\rangle$$

$$c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ (\tilde{v} \cdot e) - \left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ (\tilde{v} \cdot e) \right\rangle + b \cdot \nabla_{x,\tilde{v}}\ u^1 = 0, \ \ \left\langle u^1 \right\rangle = 0.$$

A straightforward computation leads to

$$\begin{aligned}
-\omega_c(\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} u^1 &= c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ (\tilde{v} \cdot e) - \left\langle c_0[\tilde{f}] \cdot \nabla_{x,\tilde{v}}\ (\tilde{v} \cdot e) \right\rangle \\
&= (\tilde{v} \cdot e) \partial_x e e \cdot \tilde{v} - [\tilde{v} \wedge \partial_x e\ (\tilde{v} \wedge e)] \cdot e - \mathrm{div}_x e\ \frac{|\tilde{v} \wedge e|^2}{2} \\
&= -(\tilde{v} \cdot e) \partial_x e e \cdot [(\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} (\tilde{v} \wedge e)] \\
&+ \partial_x e : [(\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) - \langle (\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) \rangle].
\end{aligned}$$

Observe that

$$\begin{aligned}
(\tilde{v} \wedge e) \otimes (\tilde{v} \wedge e) - \frac{|\tilde{v} \wedge e|^2}{2}(I_3 - e \otimes e) \\
= (\tilde{v} \wedge e) \cdot \nabla_{\tilde{v}} \frac{(\tilde{v} \wedge e) \otimes (\tilde{v} - (\tilde{v} \cdot e)\ e) + (\tilde{v} - (\tilde{v} \cdot e)\ e) \otimes (\tilde{v} \wedge e)}{4}
\end{aligned}$$

$$\langle(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\,e)+(\tilde{v}-(\tilde{v}\cdot e)\,e)\otimes(\tilde{v}\wedge e)\rangle=0$$

and therefore we obtain

$$-\omega_c u^1=-(\tilde{v}\cdot e)\partial_x ee\cdot(\tilde{v}\wedge e)+\partial_x e:\frac{(\tilde{v}\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\,e)+(\tilde{v}-(\tilde{v}\cdot e)\,e)\otimes(\tilde{v}\wedge e)}{4}.$$

We claim that $\left\langle c_0[\tilde{f}]\cdot\nabla_{x,\tilde{v}}\,u^1\right\rangle=0$ and therefore $\xi_{\tilde{v}}\cdot e$. For this we write the vector field $c_0[\tilde{f}]\cdot\nabla_{x,\tilde{v}}$ as follows

$$\begin{aligned}c_0[\tilde{f}]\cdot\nabla_{x,\tilde{v}}\;&=(\tilde{v}\cdot e)c[e]\cdot\nabla_{x,\tilde{v}}\;-(\tilde{v}\cdot e)(\partial_x ee\otimes e-e\otimes\partial_x ee)\tilde{v}\cdot\nabla_{\tilde{v}}\\&\quad+\frac{q}{m}(E[\tilde{f}]\cdot e)\;e\cdot\nabla_{\tilde{v}}-[\tilde{v}\wedge\partial_x e(\tilde{v}\wedge e)]\cdot\nabla_{\tilde{v}}.\end{aligned}$$

As the vector fields $(\tilde{v}\cdot e)c[e]\cdot\nabla_{x,\tilde{v}}$, $\frac{q}{m}(E[\tilde{f}]\cdot e)\;e\cdot\nabla_{\tilde{v}}$ are in involution with respect to $\overline{b}\cdot\nabla_{x,\tilde{v}}$, we have

$$\begin{aligned}\left\langle(\tilde{v}\cdot e)c[e]\cdot\nabla_{x,\tilde{v}}\;u^1\right\rangle&=(\tilde{v}\cdot e)c[e]\cdot\nabla_{x,\tilde{v}}\;\left\langle u^1\right\rangle=0\\\left\langle\frac{q}{m}(E[\tilde{f}]\cdot e)\;e\cdot\nabla_{\tilde{v}}u^1\right\rangle&=\frac{q}{m}(E[\tilde{f}]\cdot e)\;e\cdot\nabla_{\tilde{v}}\left\langle u^1\right\rangle=0.\end{aligned}$$

We are done if we prove that

$$(\tilde{v}\cdot e)\left\langle(\partial_x ee\otimes e-e\otimes\partial_x ee)\tilde{v}\cdot\nabla_{\tilde{v}}u^1\right\rangle=0,\;\;\left\langle[\tilde{v}\wedge(\partial_x e(\tilde{v}\wedge e))]\cdot\nabla_{\tilde{v}}u^1\right\rangle=0.$$

It is easily seen that

$$\begin{aligned}-\omega_c e\cdot\nabla_{\tilde{v}}u^1&=-\partial_x ee\cdot(\tilde{v}\wedge e)\\-\omega_c\partial_x ee\cdot\nabla_{\tilde{v}}u^1&=(\partial_x e+\;{}^t\partial_x e):\frac{(\partial_x ee\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\,e)+(\tilde{v}\wedge e)\otimes\partial_x ee}{4}\end{aligned}$$

and therefore

$$\begin{aligned}\left\langle(\partial_x ee\otimes e-e\otimes\partial_x ee)\tilde{v}\cdot\nabla_{\tilde{v}}u^1\right\rangle&=(\tilde{v}\cdot e)\left\langle\partial_x ee\cdot\nabla_{\tilde{v}}u^1\right\rangle-\left\langle(\partial_x ee\cdot\tilde{v})\;e\cdot\nabla_{\tilde{v}}u^1\right\rangle\\&=-\frac{1}{\omega_c}\left\langle(\partial_x ee\cdot\tilde{v})\partial_x ee\cdot(\tilde{v}\wedge e)\right\rangle\\&=-\frac{1}{\omega_c}\partial_x ee\otimes\partial_x ee:\langle\tilde{v}\otimes(\tilde{v}\wedge e)\rangle\\&=-\frac{|\tilde{v}\wedge e|^2}{2\omega_c}\partial_x ee\otimes\partial_x ee:M[e]=0.\end{aligned}$$

Similarly we write, by using the notation $\sigma(x,\tilde{v})=\tilde{v}\wedge\partial_x e\;(\tilde{v}\wedge e)$

$$\begin{aligned}-\omega_c(\tilde{v}\wedge\partial_x e\;(\tilde{v}\wedge e))\cdot\nabla_{\tilde{v}}u^1&=-(\sigma\cdot e)\partial_x ee\cdot(\tilde{v}\wedge e)-(\tilde{v}\cdot e)\partial_x ee\cdot(\sigma\wedge e)\\&\quad+(\partial_x e+\;{}^t\partial_x e):\frac{(\sigma\wedge e)\otimes(\tilde{v}-(\tilde{v}\cdot e)\,e)+(\tilde{v}\wedge e)\otimes(\sigma-(\sigma\cdot e)\,e)}{4}.\end{aligned}$$

Thanks to the seventh statement in Proposition 5.3 we have

$$\langle(\sigma\cdot e)\partial_x ee\cdot(\tilde{v}\wedge e)\rangle=\langle[(\tilde{v}-(\tilde{v}\cdot e)\,e)\wedge\partial_x e(\tilde{v}\wedge e)]\cdot e\;[\partial_x ee\cdot(\tilde{v}\wedge e)]\rangle=0.$$

Notice also that

$$\begin{aligned}\langle(\tilde{v}\cdot e)\partial_x ee\cdot(\sigma\wedge e)\rangle&=(\tilde{v}\cdot e)\partial_x ee\cdot\langle(\tilde{v}\wedge\partial_x e\;(\tilde{v}\wedge e))\wedge e\rangle\\&=(\tilde{v}\cdot e)^2\partial_x ee\cdot\langle\partial_x e\;(\tilde{v}\wedge e)\rangle=0\end{aligned}$$

$$\langle(\sigma\wedge e)\otimes(\tilde v-(\tilde v\cdot e)\,e)\rangle=(\tilde v\cdot e)\,\langle\partial_x e\,(\tilde v\wedge e)\otimes(\tilde v-(\tilde v\cdot e)\,e)\rangle$$
$$=(\tilde v\cdot e)\partial_x e\,\langle(\tilde v\wedge e)\otimes\tilde v\rangle=-\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}\partial_x eM[e]$$

and

$$\langle(\tilde v\wedge e)\otimes(I_3-e\otimes e)\sigma\rangle=\langle(\tilde v\wedge e)\otimes\sigma\rangle-\langle(\sigma\cdot e)(\tilde v\wedge e)\otimes e\rangle$$
$$=(\tilde v\cdot e)\,\langle(\tilde v\wedge e)\otimes(e\wedge\partial_x e(\tilde v\wedge e))\rangle-\langle[(\tilde v-(\tilde v\cdot e)\,e)\wedge\partial_x e(\tilde v\wedge e)]\cdot e\,(\tilde v\wedge e)\otimes e\rangle$$
$$=-(\tilde v\cdot e)\,\langle(\tilde v\wedge e)\otimes(\tilde v\wedge e)\rangle\ {}^t\partial_x eM[e]=-\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}(I_3-e\otimes e)\,{}^t\partial_x eM[e]$$
$$=-\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}\,{}^t\partial_x eM[e]+\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}e\otimes(\partial_x ee\wedge e).$$

For these computations we obtain

$$(\partial_x e+\ {}^t\partial_x e):\langle(\sigma\wedge e)\otimes(\tilde v-(\tilde v\cdot e)\,e)+(\tilde v\wedge e)\otimes(\sigma-(\sigma\cdot e)\,e)\rangle=-\frac{(\tilde v\cdot e)|\tilde v\wedge e|^2}{2}$$
$$(\partial_x e+\ {}^t\partial_x e):[(\partial_x e+\ {}^t\partial_x e)M[e]-e\otimes(\partial_x ee\wedge e)]=0.$$

Finally, as $\xi_x=0,\xi_{\tilde v}\cdot\tilde v=0,\xi_{\tilde v}\cdot e=0$, we have

$$\xi\cdot\nabla_{x,\tilde v}\ =\left(\xi_{\tilde v}\cdot\frac{\tilde v\wedge e}{|\tilde v\wedge e|^2}\right)\ (\tilde v\wedge e)\cdot\nabla_{\tilde v},\ \ |\tilde v\cdot e(x)|>0$$

and

$$\left\langle c_0[\tilde f]\cdot\nabla_{x,\tilde v}\ \tilde f^1\right\rangle=\omega_c\xi\cdot\nabla_{x,\tilde v}\ \tilde f=0,\ \ |\tilde v\wedge e(x)|>0.$$

Actually, by continuity, the above equality holds true for any $(x,\tilde v)\in\mathbb{R}^3\times\mathbb{R}^3$. □

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