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NECESSARY AND SUFFICIENT CONDITIONS FOR THE IDENTIFIABILITY OF OBSERVATION-DRIVEN MODELS

RANDAL DOUC, FRANÇOIS ROUEFF, AND TEPMONY SIM

ABSTRACT. In this contribution we are interested in proving that a given observation-driven model is identifiable. In the case of a GARCH(p, q) model, a simple sufficient condition has been established in [1] for showing the consistency of the quasi-maximum likelihood estimator. It turns out that this condition applies for a much larger class of observation-driven models, that we call the class of linearly observation-driven models. This class includes standard integer valued observation-driven time series, such as the log-linear Poisson GARCH or the NBIN-GARCH models.

1. INTRODUCTION

Observation-driven models were introduced in [6] and have received considerable attention since. They are commonly used for modeling various non-linear times series in applications ranging from economics (see [18]), environmental study (see [2]), epidemiology and public health study (see [24, 7, 9]), finance (see [16, 19, 10, 13]) and population dynamics (see [14]).

As often for non-linear time series the question of identifiability of the observation-driven models is a delicate one and is often appearing as an assumption used for proving the consistency (say) of the maximum likelihood estimator. A noticeable exception is the GARCH(p, q) model, for which an explicit sufficient condition appears in [1], their condition (2.27). We will in fact prove that this condition is necessary and sufficient for the identifiability and extends to a much larger class of observation-driven models than the GARCH(p, q) model. See [Theorem 8](#) below and the comments following this result.

Let us now formally introduce the class of observation-driven models. Throughout the paper we use the notation $u_{\ell:m} := (u_{\ell}, \dots, u_m)$ for $\ell \leq m$, with the convention that $u_{\ell:m}$ is the empty sequence if $\ell > m$, so that, for instance $(x_{0:(-1)}, y) = y$. The observation-driven time series model can formally be defined as follows.

Definition 1 (Observation-driven model (ODM)). Let (X, \mathcal{X}) and (Y, \mathcal{Y}) be two measurable spaces, called the *latent space* and the *observation space*, and let (Θ, Δ) be a compact metric space, called the *parameter space*. Let $\{(x_{1:p}, y_{1:q}) \mapsto \psi_{y_{1:q}}^{\theta}(x_{1:p}) : \theta \in \Theta\}$ be a family of measurable functions from $(X^p \times Y^q, \mathcal{X}^{\otimes p} \otimes \mathcal{Y}^{\otimes q})$ to (X, \mathcal{X}) , called the *link functions* and let $\{G^{\theta} : \theta \in \Theta\}$ be a family of probability kernels on $X \times Y$, called the *observation kernels*. A time series $\{Y_k : k \geq -q + 1\}$ valued in Y is said to be distributed according to an *observation-driven model of order (p, q)* (hereafter, ODM(p, q)) with link function

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ψ^θ and observation kernel G^θ if there exists a process $\{X_k : k \geq -p+1\}$ on $(\mathsf{X}, \mathcal{X})$ such that for all $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$,

$$(1.1) \quad \begin{aligned} Y_k | \mathcal{F}_k &\sim G^\theta(X_k; \cdot), \\ X_{k+1} &= \psi_{Y_{k-q+1:k}}^\theta(X_{(k-p+1):k}), \end{aligned}$$

where $\mathcal{F}_k = \sigma(X_{(-p+1):k}, Y_{(-q+1):(k-1)})$.

Unless differently stated, we always denote the link function by ψ^θ and the observation kernel by G^θ . Moreover we always assume that the model is dominated by a σ -finite measure ν on $(\mathsf{Y}, \mathcal{Y})$, that is, for all $\theta \in \Theta$, there exists a measurable function $g^\theta : \mathsf{X} \times \mathsf{Y} \rightarrow \mathbb{R}_+$ written as $(x, y) \mapsto g^\theta(x; y)$ such that for all $x \in \mathsf{X}$, $g^\theta(x; \cdot)$ is the density of $G^\theta(x; \cdot)$ with respect to ν . In addition, we always assume that for all $(x, y) \in \mathsf{X} \times \mathsf{Y}$ and all $\theta \in \Theta$,

$$(1.2) \quad g^\theta(x; y) > 0,$$

and also, to avoid a trivial degenerate case, that ν is non-degenerate, that is, its support contains at least two points.

One of the most popular examples in this class of models is the general GARCH(p, q) model introduced by [3], where $\mathsf{X} = \mathbb{R}_+$, $\mathsf{Y} = \mathbb{R}$, $G^\theta(x; \cdot)$ is (usually but not necessarily) the centered Gaussian distribution of variance x and the link function ψ^θ given by

$$\psi_{y_{1:q}}^\theta(x_{1:p}) = \omega + \sum_{i=1}^p a_i x_i + \sum_{i=1}^q b_i y_i^2,$$

with $\theta = (\omega, a_{1:p}, b_{1:q})$ with $\omega > 0$ and $a_{1:p}, b_{1:q} \geq 0$. This model was extensively studied, see, for example, [5, 11, 12, 17, 13] and the references therein. Many other examples have been derived from this class, see [4]. The fact that the function ψ^θ is linear with respect to the x_i 's and $\Upsilon(y_i)$'s for some mapping Υ (here $\Upsilon(y) = y^2$), as well as the fact that $G^\theta(x; \cdot)$ is a distribution parameterized that does not depend on θ and for which x governs a scale parameter are features that are often shared among these conditionally heteroscedastic models. If $G^\theta(x; \cdot)$ is a discrete distribution, this latter property is not satisfied, which seriously complicates the theoretical analysis of such models, as explained in [23]. Let us recall an example of such a model, namely, the log-linear Poisson GARCH(p, q), see [10].

Example 1. The Log-linear Poisson GARCH(p, q) Model is a LODM(p, q) parameterized by $\theta = (\omega, a_{1:p}, b_{1:q}) \in \Theta \subset \mathbb{R}^{p+q+1}$ with observations space $\mathsf{Y} = \mathbb{Z}_+$ and hidden variables space $\mathsf{X} = \mathbb{R}$, link function given by

$$\psi_y^\theta(x) = \omega + \sum_{i=1}^p a_i x_i + \sum_{i=1}^q b_i \log(1 + y_i),$$

and $G(x, \cdot)$ is the Poisson distribution with mean e^x , that is, ν is the counting measure on \mathbb{Z}_+ and $g^\theta(x; y) = e^{x y - e^x} / (y!)$.

Many other examples of count time series have been proposed. It is for instance possible to let the parameter θ contain a sub-parameter φ involved in the conditional distribution $G^\theta(x; \cdot)$. It is the case in our next example introduced in [25].

Example 2. The NBIN-GARCH(p, q) model is a LODM(p, q) parameterized by $\theta = (\omega, a_{1:p}, b_{1:q}, r) \in \Theta \subset \mathbb{R}_+^* \times \mathbb{R}_+^{p+q} \times \mathbb{R}_+^*$ with observations space $\mathcal{Y} = \mathbb{Z}_+$ and hidden variables space $\mathcal{X} = \mathbb{R}_+$, link function given by

$$\psi_y^\theta(x) = \omega + \sum_{i=1}^p a_i x_i + \sum_{i=1}^q b_i y_i,$$

and $G(x, \cdot)$ is the the negative binomial distribution with shape parameter $r > 0$ and mean $r x$, that is, ν is the counting measure on \mathbb{Z}_+ and

$$(1.3) \quad g^\theta(x; y) = \frac{\Gamma(r+y)}{y! \Gamma(r)} \left(\frac{1}{1+x} \right)^r \left(\frac{x}{1+x} \right)^y.$$

The rest of the paper is organized as follows. [Section 2](#) contains additional notation and definitions that will be used throughout the paper. Our main results can be found in [Section 3](#), some proofs of which are postponed to [Section 4](#).

2. PRELIMINARIES

2.1. Admissible mappings and linearly observation driven models. We introduce the following definition in order to simplify the form of link function.

Definition 2 (Admissible mapping). Consider an ODM(p, q) with link function ψ^θ . Let Υ be a measurable function from $(\mathcal{Y}, \mathcal{Y})$ to a measurable space $(\mathcal{U}, \mathcal{U})$. If there exists $\tilde{\psi}^\theta : \mathcal{X}^p \times \mathcal{U}^q \rightarrow \mathcal{X}$ measurable, called the *reduced link function* such that, for all $(x, y_{1:q}) \in \mathcal{X}^p \times \mathcal{Y}^q$,

$$(2.1) \quad \psi_{y_{1:q}}^\theta(x) = \tilde{\psi}_{u_{1:q}}^\theta(x) \quad \text{with} \quad u_k = \Upsilon(y_k) \quad \text{for} \quad 1 \leq k \leq q,$$

then we call Υ an *admissible mapping* and it follows that, setting $U_k = \Upsilon(Y_k)$ for all $k > -q$, $\{U_k : k > -q\}$ is an ODM(p, q) with link function $\tilde{\psi}^\theta$ and observation kernel $\tilde{G}(x, \cdot) = G(x, \Upsilon^{-1}(\cdot))$ on the observation space $(\mathcal{U}, \mathcal{U})$. Moreover, for all $k \geq 0$, the conditional distribution of (Y_k, X_{k+1}) given \mathcal{F}_k only depends on

$$(2.2) \quad Z_k = (X_{(k-p+1):k}, U_{(k-q+1):(k-1)}) \in \mathcal{Z},$$

where we defined

$$(2.3) \quad \mathcal{Z} = \mathcal{X}^p \times \mathcal{U}^{q-1} \quad \text{endowed with the } \sigma\text{-field } \mathcal{Z} = \mathcal{X}^{\otimes p} \otimes \mathcal{U}^{\otimes (q-1)}.$$

A subclass of ODM of order (p, q) consists in those having an admissible mapping Υ whose corresponding reduced mapping $\tilde{\psi}_u^\theta(x)$ is linear in (u, x) , that is, for all $x = x_{1:p} \in \mathcal{X}^p$, $u = u_{1:q} \in \mathcal{U}^q$,

$$(2.4) \quad \tilde{\psi}_u^\theta(x) = \omega + \sum_{i=1}^p a_i x_i + \sum_{i=1}^q b_i u_i,$$

In particular the parameters $(\omega, a_{1:p}, b_{1:q})$ must constitute a part of the parameter θ (possibly whole of it). We will denote this part by ϑ and the remaining part (if any) of θ by φ . We call such a model a *linearly observation-driven model* of order (p, q) . The formal definition is as follows.

Definition 3 (Linearly observation-driven model (LODM)). Consider an ODM of order (p, q) . It is called a *linearly observation-driven model of order (p, q)* (hereafter, LODM(p, q)) if

- (i) All $\theta \in \Theta$ can be written as $\theta = (\vartheta, \varphi)$ with $\vartheta \in \mathbb{R}^{1+p+q}$.

- (ii) The latent space X is a closed subset of \mathbb{R} , and, for all $x = x_{1:p} \in \mathsf{X}^p$, $y = y_{1:q} \in \mathsf{Y}^q$, and $\theta = (\vartheta, \varphi) \in \Theta$ with $\vartheta = (\omega, a_{1:p}, b_{1:q})$,

$$(2.5) \quad \psi_y^\theta(x) = \omega + \sum_{i=1}^p a_i x_i + \sum_{i=1}^q b_i \Upsilon(y_i) ,$$

where $\Upsilon : \mathsf{Y} \rightarrow \mathbb{R}$ is measurable with $\nu \circ \Upsilon^{-1}$ being non-degenerate.

Remark 1. The standard GARCH(p, q) model is a special case of LODM(p, q), in which case $\mathsf{X} = \mathbb{R}_+$, $\mathsf{Y} = \mathbb{R}$, $\Upsilon(y) = y^2$, and $G^\theta(x, \cdot)$ is a centered distribution with variance x , most commonly the normal distribution. Similarly the log-linear Poisson Garch model of [Example 1](#) is an LODM(p, q) by taking $\Upsilon(y) = \ln(1 + y)$. Obviously, the NBIN-GARCH model of [Example 2](#) is also an LODM(p, q). Among these three special cases, only the NBIN-GARCH model has an observation kernel G^θ depending on a subpart of the parameter θ through r , which appears in the right-hand side of (1.3). Hence this r corresponds to the φ appearing in the general setting of [Definition 3](#).

2.2. Iterations of the link function. We now introduce iterated versions of the link function ψ^θ . Let Υ be an admissible mapping as given by [Definition 2](#), with corresponding reduced link function $\tilde{\psi}^\theta$, and let Z be defined by (2.3). In this context, throughout the paper, for all $j \in \{1, \dots, p + q - 1\}$, we denote by $\Pi_j(z)$ the j -th entry of $z \in \mathsf{Z}$.

We define for any $y_{0:k} \in \mathsf{Y}^{k+1}$ the mapping $\psi^\theta \langle y_{0:k} \rangle : \mathsf{Z} \rightarrow \mathsf{X}$ through a set of recursive equations of order (p, q) . Namely, for all $n \in \mathbb{N}$, $y_{0:k} \in \mathsf{Y}^{k+1}$ and $z \in \mathsf{Z}$,

$$(2.6) \quad \psi^\theta \langle y_{0:k} \rangle (z) := x_{k+1} ,$$

where $x_{(-p+1):(k+1)}$ is defined by

$$(2.7) \quad \begin{cases} u_j = \Pi_{p+q+j}(z) , & -q < j \leq -1 , \\ u_j = \Upsilon(y_j) , & 0 \leq j \leq k , \\ x_j = \Pi_{p+j}(z) , & -p < j \leq 0 , \\ x_j = \tilde{\psi}_{u_{(j-q):(j-1)}}^\theta (x_{(j-p):(j-1)}) , & 1 \leq j \leq k + 1 , \end{cases}$$

In this set of equation the last line is applied recursively so that in fact, for all $j \geq 0$, x_{j+1} only depends on z and $y_{0:j}$.

3. MAIN RESULTS

To study the identifiability of such models we only consider the case where all processes in the model are ergodic. Namely, we use the following assumption.

- (A-1) For all $\theta \in \Theta$, there exists a unique stationary solution satisfying (1.1).

This ergodic property is the cornerstone for making statistical inference theory work and we provide simple general conditions in [8] for $p = q = 1$ and in [20, 21, Chapter 5] for the case of general order (p, q) .

We now introduce the notation that will allow us to refer to the stationary distribution of the model throughout the paper.

Definition 4 (Stationary distributions \mathbb{P}^θ and $\tilde{\mathbb{P}}^\theta$). Suppose that (A-1) holds. We define the distributions \mathbb{P}^θ and $\tilde{\mathbb{P}}^\theta$ as follows.

- a) \mathbb{P}^θ denotes the distribution on $((\mathsf{X} \times \mathsf{Y})^{\mathbb{Z}}, (\mathcal{X} \times \mathcal{Y})^{\otimes \mathbb{Z}})$ of the stationary solution of (1.1) extended on $k \in \mathbb{Z}$, with $\mathcal{F}_k = \sigma(X_{-\infty:k}, Y_{-\infty:(k-1)})$;

b) $\tilde{\mathbb{P}}^\theta$ denotes the projection of \mathbb{P}^θ on the component $\mathcal{Y}^{\mathbb{Z}}$.

We also use the symbols \mathbb{E}^θ and $\tilde{\mathbb{E}}^\theta$ to denote the expectations corresponding to \mathbb{P}^θ and $\tilde{\mathbb{P}}^\theta$, respectively. We further denote by π_X^θ and π_Y^θ the marginal distributions of X_0 and Y_0 under \mathbb{P}^θ , on (X, \mathcal{X}) and (Y, \mathcal{Y}) , respectively. Moreover, for all $\theta, \theta' \in \Theta$, we write $\theta \sim \theta'$ if and only if $\tilde{\mathbb{P}}^\theta = \tilde{\mathbb{P}}^{\theta'}$. This defines an equivalence relation on the parameter set Θ and the corresponding equivalence class of θ is denoted by $[\theta] := \{\theta' \in \Theta : \theta \sim \theta'\}$.

In this context, identifiable parameters are defined as all the parameters θ_* for which $[\theta_*]$ is reduced to the singleton $\{\theta_*\}$. Without identifiability, the consistency of any estimator of θ_* is not possible. Nevertheless a weaker notion of consistency, the *equivalence-class consistency* introduced by [15] can still be established and the consistency in the usual sense is then equivalent to θ_* being identifiable.

As a byproduct of the proof of (A-1), one usually obtains a function $V_X : X \rightarrow \mathbb{R}_+$ of interest, common to all $\theta \in \Theta$, such that, for all $\theta \in \Theta$, $\pi_X^\theta(V_X) < \infty$, see [20, Chapter 5]. It turns out that, since $\pi_Y^\theta = \pi_X^\theta G^\theta$, such a condition allows one to check moment conditions on π_Y^θ such as Conditions (3.8) and (3.11) below.

To investigate the identifiability of the model, we first introduce an assumption which says how much can be identified from a single observation of the conditional distribution $G^\theta(x, \cdot)$.

(A-2) We can write $\theta_* = (\vartheta_*, \varphi_*)$ and, for all $\theta = (\vartheta, \varphi)$ in Θ and $x, x' \in X$,

$$G^\theta(x; \cdot) = G^{\theta_*}(x'; \cdot) \quad \text{if and only if} \quad \varphi = \varphi_* \quad \text{and} \quad x = x'.$$

The “if” in Assumption (A-2) simply means that G^θ does not depend on the component ϑ of the parameter, so could be written as G^φ . The “only if” part then says that, adopting temporarily this notation, $(\varphi, x) \mapsto G^\varphi(x, \cdot)$ is one-to-one. In many examples G^θ does not depend on θ at all, in which case the component φ can be discarded. It is the case for the Log-linear Poisson GARCH model of Example 1 but not for the NBIN-GARCH model of Example 2, where $\varphi = r$.

Our approach to establish identifiability is given by the following general result.

Proposition 5. *Let p and q be two positive integers. Consider an ODM(p, q) with admissible mapping Υ and let $\theta_* \in \Theta$. Assume that (A-1) and (A-2) hold. Suppose moreover that, for all $\theta \in \Theta$, there exists a measurable function $\psi^\theta(\cdot) : \mathcal{Y}^{\mathbb{Z}^-} \rightarrow X$ such that*

$$(3.1) \quad X_1 = \psi^\theta \langle Y_{-\infty:0} \rangle \quad \mathbb{P}^\theta\text{-a.s.}$$

Then the equivalent class $[\theta_*]$ of Definition 4 coincides with the set $\langle \theta_* \rangle$ defined as the set of all $\theta = (\vartheta, \varphi_*) \in \Theta$ satisfying the two following equations

$$(3.2) \quad \psi^\theta \langle Y_{-\infty:0} \rangle = \psi^{\theta_*} \langle Y_{-\infty:0} \rangle \quad \tilde{\mathbb{P}}^{\theta_*}\text{-a.s.},$$

$$(3.3) \quad \psi^\theta \langle Y_{-\infty:0} \rangle = \psi_{Y_{(-q+1):0}}^\theta \left(\left(\psi^\theta \langle Y_{-\infty:j} \rangle \right)_{-p \leq j \leq -1} \right) \quad \tilde{\mathbb{P}}^{\theta_*}\text{-a.s.}$$

The proof is postponed to Section 4.1 for convenience. For the moment, let us provide important insights for $\psi^\theta \langle Y_{-\infty:0} \rangle$ appearing in Proposition 5. These insights are threefold. First we show in Lemma 6 that (3.1) can be verified using $\tilde{\mathbb{P}}^\theta$ only. Second, in Lemma 7, that (3.1) is implied by some Lipschitz condition on the iterates of the link function ψ^θ and a moment condition on π_Y . Finally, we examine the special case of LODM(p, q) for which the link function is linear, leading to Theorem 8 below, that will conclude this section.

Lemma 6. *Consider an ODM(p, q) for some $p, q \geq 1$ and assume that (A-1) holds true. Let $\theta \in \Theta$ and consider a measurable function $\psi^\theta \langle \cdot \rangle : \mathcal{Y}^{\mathbb{Z}^-} \rightarrow \mathcal{X}$. Then (3.1) is satisfied if and only if the two following equations hold.*

$$(3.4) \quad \psi^\theta \langle Y_{-\infty:0} \rangle = \psi_{Y_{-q+1:0}}^\theta \left((\psi^\theta \langle Y_{-\infty:j} \rangle)_{-p \leq j \leq -1} \right) \quad \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

$$(3.5) \quad \tilde{\mathbb{P}}^\theta [Y_1 \in \cdot | Y_{-\infty:0}] = G^\theta (\psi^\theta \langle Y_{-\infty:0} \rangle, \cdot) \quad \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

Proof. Suppose that (3.1) holds true. Since \mathbb{P}^θ is shift invariant, it can be extended to all time instants $k \in \mathbb{Z}$,

$$X_k = \psi^\theta \langle Y_{-\infty:(k-1)} \rangle \quad \mathbb{P}^\theta\text{-a.s.}$$

But then (3.4) and (3.5) follows from the model equations (1.1).

Suppose now that (3.4) and (3.5) hold true. Since \mathbb{P}^θ is shift invariant, they are extended to all time instants $k \in \mathbb{Z}$ in the form

$$\psi^\theta \langle Y_{-\infty:k-1} \rangle = \psi_{Y_{(k-q):(k-1)}}^\theta \left((\psi^\theta \langle Y_{-\infty:j} \rangle)_{k-p-1 \leq j \leq k-2} \right) \quad \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

$$\tilde{\mathbb{P}}^\theta [Y_k \in \cdot | Y_{-\infty:(k-1)}] = G^\theta (\psi^\theta \langle Y_{-\infty:(k-1)} \rangle, \cdot) \quad \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

Defining $X'_k = \psi^\theta \langle Y_{-\infty:(k-1)} \rangle$ for all $k \in \mathbb{Z}$, we see that $\{(X'_k, Y_k) : k \in \mathbb{Z}\}$ is a stationary sequence satisfying the model equations (1.1). By uniqueness of \mathbb{P}^θ assumed in (A-1), we get that $\{(X'_k, Y_k) : k \in \mathbb{Z}\}$ has distribution \mathbb{P}^θ and so (3.1) holds since it is true with X'_1 replacing X_1 , by the mere definition of X'_1 . \square

Whenever we need some metric on the space \mathcal{Z} , we assume the following.

(A-3) The σ -fields \mathcal{X} and \mathcal{U} are Borel ones, respectively associated to $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{U}, \delta_{\mathcal{U}})$, both assumed to be complete and separable metric spaces.

Recall that, for any finite \mathcal{Y} -valued sequence y , the mapping $\psi^\theta \langle y \rangle$ is defined by (2.6) following the recursion in (2.7). Define, for all $n \in \mathbb{Z}_+^*$, the Lipschitz constant for $\psi^\theta \langle y \rangle$, uniform over $y \in \mathcal{Y}^n$,

$$(3.6) \quad \text{Lip}_n^\theta = \sup \left\{ \frac{\delta_{\mathcal{X}}(\psi^\theta \langle y \rangle(z), \psi^\theta \langle y \rangle(z'))}{\delta_{\mathcal{Z}}(z, z')} : (z, z', y) \in \mathcal{Z}^2 \times \mathcal{Y}^n \right\},$$

where we set, for all $v \in \mathcal{Z}^2$,

$$(3.7) \quad \delta_{\mathcal{Z}}(v) = \left(\max_{1 \leq k \leq p} \delta_{\mathcal{X}} \circ \Pi_k^{\otimes 2}(v) \right) \vee \left(\max_{p < k < p+q} \delta_{\mathcal{U}} \circ \Pi_k^{\otimes 2}(v) \right).$$

Consider the following assumption on the link function.

(A-4) For all $\theta \in \Theta$, we have $\text{Lip}_1^\theta < \infty$ and $\text{Lip}_n^\theta \rightarrow 0$ as $n \rightarrow \infty$.

We have the following result, whose proof is postponed to Section 4.2 for convenience.

Lemma 7. *Let p and q be two positive integers. Consider an ODM(p, q) with admissible mapping Υ . Assume that (A-1), (A-3) and (A-4) hold, and take $\theta, \theta_* \in \Theta$. Suppose that there exists $x_1^{(i)} \in \mathcal{X}$ and $y_1^{(i)} \in \mathcal{Y}$ such that the constant vectors $x^{(i)} = (x_1^{(i)}, \dots, x_1^{(i)}) \in \mathcal{X}^p$ and $u^{(i)} = (\Upsilon(y_1^{(i)}), \dots, \Upsilon(y_1^{(i)})) \in \Upsilon(\mathcal{Y})^{q-1}$ satisfy*

$$(3.8) \quad \int \phi^{(i)} d\pi_{\mathcal{Y}}^\theta \quad \text{and} \quad \int \phi^{(i)} d\pi_{\mathcal{Y}}^{\theta_*} \quad \text{are finite,}$$

where, for all $y \in \mathcal{Y}$,

$$\phi^{(i)}(y) = \ln^+ \left(\delta_{\mathcal{X}} \left(x_1^{(i)}, \psi^\theta \langle y \rangle((x^{(i)}, u^{(i)})) \right) \vee \delta_{\mathcal{U}}(\Upsilon(y_1^{(i)}), \Upsilon(y)) \right)$$

with the convention $(x^{(i)}, u^{(i)}) = x^{(i)}$ and $\delta_U(\Upsilon(y_1^{(i)}), \Upsilon(y)) = 0$ if $q = 1$. Then

$$(3.9) \quad \psi^\theta \langle Y_{-\infty:0} \rangle = \lim_{n \rightarrow \infty} \psi^\theta \langle Y_{-n:0} \rangle((x^{(i)}, u^{(i)}))$$

is well defined $\tilde{\mathbb{P}}^{\theta^*}$ -a.s. and $\tilde{\mathbb{P}}^\theta$ -a.s. and satisfies (3.1). If moreover, $x \mapsto \psi_y^\theta(x)$ is continuous for all $y \in \mathcal{Y}^q$, then (3.3) holds.

We now examine the application of Proposition 5 for an LODM(p, q), as given by Definition 3. In this case, the recursion (2.7) is defined with $\tilde{\psi}^\theta$ given by (2.4). Start this recursion with two different z and z' and a common sequence $y_{0:k} \in \mathcal{Y}$, giving raise to sequences $u_{(-q+1):k+1}$ and $x_{(-p+1):k+1}$ on the one hand, and $u'_{(-q+1):k+1}$ and $x'_{(-p+1):k+1}$ on the other hand. The second line of (2.7) implies that $u'_j = u_j$ for all $j \geq 0$. Thus, using the fourth line and (2.4), we get that, for all $j \geq q$,

$$x_j - x'_j = \sum_{i=1}^p a_i(x_{j-i} - x'_{j-i})$$

Since $\psi^\theta \langle y \rangle(z) = x_{k+1}$ and $\psi^\theta \langle y \rangle(z') = x'_{k+1}$ by (2.6), we thus obtain that, for all $z, z' \in \mathcal{Z}$ and $y \in \mathcal{Y}^{n+1}$, $|\psi^\theta \langle y \rangle(z) - \psi^\theta \langle y \rangle(z')|$ does not depend on y and tends to zero if and only if $a_{1:p} \in \mathcal{S}_p$, where

$$(3.10) \quad \mathcal{S}_p = \left\{ c_{1:r} \in \mathbb{R}^p : \forall z \in \mathbb{C}, |z| \leq 1 \text{ implies } 1 - \sum_{k=1}^p c_k z^k \neq 0 \right\}.$$

We summarize this in the following remark for later reference.

Remark 2. For an LODM(p, q), \mathcal{X} is a closed subset of \mathbb{R} and $U = \mathbb{R}$, with $\delta_U(u, u') = |u - u'|$. Condition (A-3) follows. Moreover, the link function $\psi_y(x)$ is of the form (2.5), which yields that (A-4) is equivalent to the following condition, often referred to in the standard GARCH case as the *invertibility* condition, see [22] for a more general discussion on this topic.

(L-1) For all $\theta = (\vartheta, \varphi) \in \Theta$ with $\vartheta = (\omega, a_{1:p}, b_{1:q})$, we have $a_{1:p} \in \mathcal{S}_p$.

Remarkably, all LODMs share the same identifiability condition, which can be expressed as follows, using $a_{1:p}^*$ and $b_{1:q}^*$ to denote the true linear coefficients of the link function (2.5).

(L-2) The polynomials $P_p(\cdot; a_{1:p}^*)$ and $Q_q(\cdot; b_{1:q}^*)$ have no common complex roots, where we defined

$$P_p(z; a_{1:p}) = z^p - \sum_{k=1}^p a_k z^{p-k} \quad \text{and} \quad Q_q(z; b_{1:q}) = \sum_{k=0}^{q-1} b_{k+1} z^{q-1-k}.$$

We have the following result, whose proof can be found in Section 4.5.

Theorem 8. Consider an LODM(p, q) for some $p, q \geq 1$ and let $\theta_* \in \Theta$. Suppose that (A-1) holds with an invariant probability measure satisfying, for all $\theta \in \Theta$,

$$(3.11) \quad \int \ln^+(|\Upsilon(y)|) \pi_V^\theta(dy) < \infty.$$

Suppose moreover that (A-2) and (L-1) hold. Then the following assertions hold for any $\theta_* = (\omega^*, a_{1:p}^*, b_{1:q}^*, \varphi^*) \in \Theta$.

- (i) If (L-2) holds, then the equivalent class $[\theta_*]$ of Definition 4 is reduced to the singleton $\{\theta_*\}$.

(ii) If (L-2) does not hold and there exists a neighborhood of $(\omega^*, a_{1:p}^*, b_{2:q}^*)$ included in $\{(\omega, a_{1:p}, b_{2:q}) : \theta = (\omega, a_{1:p}, b_{1:q}, \varphi) \in \Theta\}$, then $[\theta_\star]$ at least contains a positive length curve going through θ_\star .

Remark 3. If $p = q = 1$, condition (L-2) is reduced to $b_1^* \neq 0$. Let us see what $b_1^* = 0$ would imply about the identifiability of the model in this simple case. Taking ψ^θ as in (2.5) with $p = q = 1$, if $b_1^* = 0$, then $\{X_k : k \in \mathbb{Z}_+\}$ is a deterministic sequence which, under the stationary distribution, has to be constantly equal to $x^* = \frac{\omega^*}{1-a_1^*}$. But since the distribution of $\{Y_n : n \in \mathbb{Z}\}$ is then uniquely defined by this constant, if one can find a parameter θ with corresponding coefficients ω, a_1, b_1 such that $b_1 = 0$, $(\omega, a_1) \neq (\omega^*, a_1^*)$ yielding the same constant $\omega/(1-a_1) = \omega^*/(1-a_1^*)$, and such that $G^\theta = G^{\theta^*}$ (because G^θ typically does not depend on (ω, a_1)), we see that the model is not identifiable.

Remark 4. Condition (L-2) holds for “many” parameters $a_{1:p}^*, b_{1:q}^*$, e.g. for Lebesgue almost all ones in \mathbb{R}^{p+q} .

Remark 5. The identifiability condition (L-2) is a well known sufficient condition in the standard GARCH(p, q) models, see [11, (A4)] or [1, Condition (2.27)]. Our result shows that it is much more general and does not apply only in the case where $G^\theta(x, \cdot)$ is a distribution parameterized by a scale parameter x . We moreover show that it is also a necessary condition.

4. POSTPONED PROOFS

4.1. **Proof of Proposition 5.** First observe that (3.1) implies for all $\theta \in \Theta$,

$$(4.1) \quad \tilde{\mathbb{P}}^\theta [Y_1 \in \cdot | Y_{-\infty:0}] = G^\theta(\psi^\theta \langle Y_{-\infty:0} \rangle; \cdot) \quad \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

Let us now show that any $\theta = (\vartheta, \varphi) \in [\theta_\star]$ belongs to $\langle \theta_\star \rangle$, that is, $\varphi = \varphi_\star$ and (3.2) and (3.3) hold true. Since $\tilde{\mathbb{P}}^\theta = \tilde{\mathbb{P}}^{\theta_\star}$, (4.1), which also holds with θ replaced by θ_\star , yields

$$G^\theta(\psi^\theta \langle Y_{-\infty:0} \rangle; \cdot) = G^{\theta_\star}(\psi^{\theta_\star} \langle Y_{-\infty:0} \rangle; \cdot) \quad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s.}$$

By (A-2), we obtain that $\theta = (\vartheta, \varphi_\star)$ and (3.2) holds. By Lemma 6, (3.1) implies (3.4), and using $\tilde{\mathbb{P}}^\theta = \tilde{\mathbb{P}}^{\theta_\star}$, we obtain (3.3). Thus $\theta \in \langle \theta_\star \rangle$.

It remains to show that $\langle \theta_\star \rangle \subseteq [\theta_\star]$. Let $\theta \in \langle \theta_\star \rangle$, that is, let $\theta = (\vartheta, \varphi_\star) \in \Theta$ such that (3.2) and (3.3) hold true. Since (3.1) holds with θ replaced by θ_\star , (3.2) gives that $X_1 = \psi^\theta \langle Y_{-\infty:0} \rangle$ $\mathbb{P}^{\theta_\star}$ -a.s. Since $\mathbb{P}^{\theta_\star}$ is shift invariant, we get, for all $k \in \mathbb{Z}$, $X_{k+1} = \psi^\theta \langle Y_{-\infty:k} \rangle$ $\mathbb{P}^{\theta_\star}$ -a.s. With (3.3), we obtain $X_1 = \psi_{Y_{(-q+1):0}}^\theta (X_{(-p+1):0})$ $\mathbb{P}^{\theta_\star}$ -a.s. Since $\mathbb{P}^{\theta_\star}$ is shift invariant, we thus have, for all $k \in \mathbb{Z}$,

$$(4.2) \quad X_{k+1} = \psi_{Y_{(k+1-q):k}}^\theta (X_{(k+1-p):k}) \quad \mathbb{P}^{\theta_\star}\text{-a.s.}$$

On the other hand, by definition of $\mathbb{P}^{\theta_\star}$ and using (A-2) with $\theta = (\vartheta, \varphi_\star)$, we have that

$$\mathbb{P}^{\theta_\star} [Y_1 \in \cdot | Y_{-\infty:0}, X_{-\infty:1}] = G^{\theta_\star}(X_1; \cdot) = G^\theta(X_1; \cdot) \quad \mathbb{P}^{\theta_\star}\text{-a.s.}$$

And using again that $\mathbb{P}^{\theta_\star}$ is shift-invariant, for all $k \in \mathbb{Z}$,

$$\mathbb{P}^{\theta_\star} [Y_k \in \cdot | Y_{-\infty:k-1}, X_{-\infty:k}] = G^\theta(X_k; \cdot) \quad \mathbb{P}^{\theta_\star}\text{-a.s.}$$

This, with (4.2), shows that $\mathbb{P}^{\theta_\star}$ is a shift-invariant solution of (1.1). By (A-1), we conclude that $\mathbb{P}^{\theta_\star} = \mathbb{P}^\theta$, and thus $\theta \in [\theta_\star]$.

4.2. **Proof of Lemma 7.** We start by introducing some useful notation related to the iterated link function described in Section 2.2. For all $y \in \mathsf{Y}$, we define $\Psi_y^\theta : \mathsf{Z} \rightarrow \mathsf{Z}$ by

$$(4.3) \quad \Psi_y^\theta : z = z_{1:(p+q-1)} \mapsto \left(z_{2:p}, \tilde{\psi}_{u_{1:p}}^\theta(z_{1:p}), z_{(p+2):(p+q-1)}, u_p \right),$$

where $u_{1:(p-1)} = z_{(p+1):(p+q-1)}$ and $u_p = \Upsilon(y)$, and $\tilde{\psi}^\theta$ is the reduced link function appearing in (2.1). The definition (4.3) is only valid for $q > 1$. If $q = 1$, then $\mathsf{Z} = \mathsf{X}^p$ and the definition of Ψ^θ boils down to

$$(4.4) \quad \Psi_y^\theta : z = z_{1:p} \mapsto \left(z_{2:p}, \tilde{\psi}_{u_{1:p}}^\theta(z_{1:p}) \right).$$

We further denote the successive composition of $\Psi_{y_0}^\theta, \Psi_{y_1}^\theta, \dots$, and $\Psi_{y_k}^\theta$ by

$$(4.5) \quad \Psi^\theta \langle y_{0:k} \rangle = \Psi_{y_k}^\theta \circ \Psi_{y_{k-1}}^\theta \circ \dots \circ \Psi_{y_0}^\theta.$$

Note that $\psi^\theta \langle y_{0:k} \rangle$ introduced Section 2.2 satisfies

$$(4.6) \quad \psi^\theta \langle y_{0:k} \rangle = \Pi_p \circ \Psi^\theta \langle y_{0:k} \rangle : \mathsf{Z} \rightarrow \mathsf{X}.$$

Conversely, we have, for all $k \geq 0$ and $y_{0:k} \in \mathsf{Y}^{k+1}$,

$$(4.7) \quad \Psi^\theta \langle y_{0:k} \rangle(z) = \left((\psi^\theta \langle y_{0:j} \rangle(z))_{k-p < j \leq k}, u_{(k-q+2):k} \right),$$

where we set $u_j = \Pi_{p+q+j}(z)$ for $-q < j \leq -1$ and $u_j = \Upsilon(y_j)$ for $0 \leq j \leq k$ and use the convention $\psi^\theta \langle y_{0:j} \rangle(z) = \Pi_{p-j}(z)$ for $-p < j \leq 0$.

We can now derive the following result.

Lemma 9. (A-4) implies that for all $\theta \in \Theta$, there exists $C > 0$ and $\rho \in (0, 1)$ such that $\text{Lip}_n^\theta \leq C \rho^n$ for all $n \in \mathbb{Z}_+^*$.

Proof. By (3.6), (3.7) and (4.7), we have, for all $n \in \mathbb{Z}_+^*$, using the convention $\text{Lip}_m^\theta = 1$ for $m \leq 0$,

$$(4.8) \quad \sup_{y \in \mathsf{Y}^n, v \in \mathsf{Z}^2} \frac{\delta_{\mathsf{Z}} \circ \Psi^\theta \langle y \rangle^{\otimes 2}(v)}{\delta_{\mathsf{Z}}(v)} \leq \mathbb{1}_{\{n < q\}} \vee \left(\max_{0 \leq j < p} \text{Lip}_{n-j}^\theta \right).$$

Hence (A-4) implies that there exists $m \geq 1$ and $L \in (0, 1)$ such that, for all $y \in \mathsf{Y}^{m+1}$, $\Psi^\theta \langle y \rangle$ is L -Lipschitz. Now observe that, by (4.6), for all $n = km + r$ with $k \geq 0$ and $0 \leq r < m$, for all $y = y_{-n:0} \in \mathsf{Y}^{n+1}$, we can write $\Psi^\theta \langle y \rangle$ as

$$\Psi^\theta \langle y_{1-m:0} \rangle \circ \Psi^\theta \langle y_{1-2m:(-m)} \rangle \circ \dots \circ \Psi^\theta \langle y_{(1-km):-(k-1)m} \rangle \circ \Psi^\theta \langle y_{-n:(-km)} \rangle,$$

and in this composition, the k first functions are L Lipschitz and the last one is $L' = 1 \vee \max \left\{ \text{Lip}_j^\theta : 0 < j \leq m \right\}$ -Lipschitz. Hence, for all $z, z' \in \mathsf{Z}$,

$$\delta_{\mathsf{X}}(\psi^\theta \langle y \rangle(z), \psi^\theta \langle y \rangle(z')) \leq \delta_{\mathsf{Z}}(\Psi^\theta \langle y \rangle(z), \Psi^\theta \langle y \rangle(z')) \leq L' L^k \delta_{\mathsf{Z}}(z, z').$$

Hence the result by setting $\rho = L^{1/m} \in (0, 1)$. \square

We can now prove Lemma 7. Denote, for all $n \in \mathbb{Z}_+$, $X^{(n)} = \psi^\theta \langle Y_{-n:0} \rangle(z^{(i)})$. Then, by (4.5) we have, for all $n \in \mathbb{Z}_+^*$,

$$X^{(n)} = \psi^\theta \langle Y_{-n+1:0} \rangle \circ \Psi_{Y_{-n}}^\theta(z^{(i)}),$$

and, by (3.6), we get

$$\delta_{\mathsf{X}} \left(X^{(n)}, X^{(n-1)} \right) \leq \text{Lip}_n^\theta \delta_{\mathsf{Z}} \left(z^{(i)}, \Psi_{Y_{-n}}^\theta(z^{(i)}) \right).$$

Using (3.7) with $z^{(i)} = (x^{(i)}, u^{(i)})$ and $x_1^{(i)} = \dots = x_p^{(i)}$ and $u_1^{(i)} = \dots = u_{q-1}^{(i)} = \Upsilon(y_1^{(i)})$ and setting $y^{(i)} = (y_1^{(i)}, \dots, y_1^{(i)}) \in \mathcal{Y}^{q-1}$ we get that

$$\delta_Z \left(z^{(i)}, \Psi_{Y_{-n}}^\theta(z^{(i)}) \right) = \delta_X \left(x_1^{(i)}, \psi_{(y^{(i)}, Y_{-n})}^\theta(x^{(i)}) \right) \vee \delta_U(\Upsilon(y_1^{(i)}), \Upsilon(Y_{-n}))$$

Hence, for all $\alpha > 0$, Condition (3.8) implies, for all $\theta, \theta_* \in \Theta$, as $n \rightarrow \infty$,

$$\delta_Z \left(z^{(i)}, \Psi_{Y_{-n}}^\theta(z^{(i)}) \right) = O(e^{\alpha n}) \quad \tilde{\mathbb{P}}^{\theta_*}, \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

We conclude with Lemma 9 that (A-4) implies that $\{X^{(n)} : n \in \mathbb{N}\}$ is a Cauchy sequence, hence converges in \mathbf{X} and $\psi^\theta \langle Y_{-\infty:0} \rangle$ is well defined in (3.9) $\tilde{\mathbb{P}}^{\theta_*}, \tilde{\mathbb{P}}^\theta$ -a.s. Moreover, since, as a solution to (1.1) we also have, under \mathbb{P}^θ , for all $n \in \mathbb{Z}_+$, $X_1 = \psi^\theta \langle Y_{-n:0} \rangle(Z_{-n})$, (3.6) also implies

$$\delta_X(X_1, X^{(n)}) \leq \text{Lip}_{n+1}^\theta \delta_Z \left(Z_{-n}, z^{(i)} \right) \quad \mathbb{P}^\theta\text{-a.s.}$$

Note that $\delta_Z(Z_{-n}, z^{(i)})$ is bounded in probability under \mathbb{P}^θ , hence $X^{(n)}$ converges to X_1 in probability if (A-4) holds. We thus obtain (3.1).

Finally, we check that (3.3) holds when ψ_y^θ is continuous for all $y \in \mathcal{Y}^q$. Since (3.9) holds $\tilde{\mathbb{P}}^{\theta_*}$ -a.s. and $\tilde{\mathbb{P}}^{\theta_*}$ is shift invariant, we have, for all $k \in \mathbb{Z}$,

$$\psi^\theta \langle Y_{-\infty:k} \rangle = \lim_{n \rightarrow \infty} \psi^\theta \langle Y_{-n:k} \rangle(z^{(i)}) \quad \tilde{\mathbb{P}}^{\theta_*}\text{-a.s.}$$

Observe that, for all $n \geq p \vee q$, we have, for all $y_{(-n):0} \in \mathcal{Y}^{n+1}$,

$$\psi^\theta \langle y_{(-n):0} \rangle(z^{(i)}) = \psi_{y_{(-q+1):0}}^\theta \left(\left(\psi^\theta \langle y_{(-n):(-j)} \rangle(z^{(i)}) \right)_{-p \leq j \leq -1} \right).$$

By continuity of ψ_y^θ and using the previous display, we can take the limit as $n \rightarrow \infty$ under $\tilde{\mathbb{P}}^{\theta_*}$ and obtain (3.3).

4.3. Some additional notation. We introduce some algebra notation that will be used hereafter. The transpose of a matrix \mathbf{M} is denoted by \mathbf{M}^T , the identity matrix of order n by \mathbf{I}_n (or simply \mathbf{I}), the max norm of $z \in \mathbb{R}^{p+q-1}$ by

$$|z|_\infty = \max \{ \Pi_k(z) : 1 \leq k \leq p+q-1 \}.$$

We further write ε_j for the j -th canonical vector in \mathbb{R}^{p+q-1} , $1 \leq j < p+q$, so that $\Pi_j(z) = \varepsilon_j^T z$. For given coefficients $\omega, a_1, \dots, a_p, b_1, \dots, b_q$, we define the $(p+q-1)$ -square matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_p & a_{p-1} & a_{p-2} & a_{p-3} & \cdots & a_1 & b_q & b_{q-1} & \cdots & b_3 & b_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the $(p + q - 1)$ -dimensional vectors \mathbf{b} and $\boldsymbol{\omega}$ by

$$\begin{aligned}\mathbf{b} &= b_1 \boldsymbol{\varepsilon}_p + \boldsymbol{\varepsilon}_{p+q-1}, \\ \boldsymbol{\omega} &= \omega \boldsymbol{\varepsilon}_p.\end{aligned}$$

In the case $q = 1$, we adopt the following convention: \mathbf{A} reduces to its top-left p -square bloc and \mathbf{b} reduces to $b_1 \boldsymbol{\varepsilon}_p$. In particular, in the case $p = q = 1$, \mathbf{A} , \mathbf{b} and $\boldsymbol{\omega}$ reduce to $\mathbf{A} = a_1$, $\mathbf{b} = b_1$ and $\boldsymbol{\omega} = \omega$, respectively.

We further denote by \mathbf{A}_* , \mathbf{b}_* and $\boldsymbol{\omega}_*$ the corresponding values of \mathbf{A} , \mathbf{b} and $\boldsymbol{\omega}$ at $(\omega, a_{1:p}, b_{1:q}) = (\omega^*, a_{1:p}^*, b_{1:q}^*)$, respectively. Recall the notation convention $\theta = (\vartheta, \varphi)$ with $\vartheta = (\omega, a_{1:p}, b_{1:q})$ in [Definition 3](#). We use the corresponding one for θ_* : $\theta_* = (\vartheta_*, \varphi_*)$ with $\vartheta_* = (\omega^*, a_{1:p}^*, b_{1:q}^*)$.

4.4. Useful lemmas. Throughout this section, we use the notation introduced in [Section 4.3](#), in particular for matrix and vectors \mathbf{A} , \mathbf{b} , $\boldsymbol{\varepsilon}_k \dots$. This section contains two lemmas used in the proof of [Theorem 8](#).

Lemma 10. *Let $a_{1:p}^*$ and $b_{1:q}^*$ be in \mathbb{R}^p and \mathbb{R}^q . Suppose that $a_{1:p}^* \in \mathcal{S}_p$, where \mathcal{S}_p is defined in [\(3.10\)](#). Then the following assertions hold.*

(i) *Condition [\(L-2\)](#) implies that, for all $a_{1:p} \in \mathcal{S}_p$ and $b_{2:q} \in \mathbb{R}^{q-1}$,*

$$(4.9) \quad \boldsymbol{\varepsilon}_p^T (\mathbf{A}^k - \mathbf{A}_*^k) \mathbf{b}_* = 0 \quad \text{for all } k \in \mathbb{Z}_+^* \implies \mathbf{A} = \mathbf{A}_*,$$

where $\boldsymbol{\varepsilon}_p$, \mathbf{A} , \mathbf{A}_* and \mathbf{b}_* are defined as in [Section 4.3](#).

(ii) *If, on the contrary, [\(L-2\)](#) does not hold, then there exists $c_{1:p+q-1}^* \in \mathbb{R}^{p+q-1} \setminus \{0\}$ such that for all $a_{1:p} \in \mathcal{S}_p \cap (a_{1:p}^* + \text{Span}(c_{1:p}^*))$ and $b_{2:q} \in b_{2:q}^* + \text{Span}(c_{(p+1):(p+q-1)}^*)$ we have*

$$\boldsymbol{\varepsilon}_p^T (\mathbf{A}^k - \mathbf{A}_*^k) \mathbf{b}_* = 0 \quad \text{for all } k \in \mathbb{Z}_+^*$$

Proof. Let $y = \{y_n : n \in \mathbb{Z}\}$ be a real valued sequence and consider the recursive equation

$$(4.10) \quad x_t = \sum_{k=1}^p a_k x_{t-k} + \sum_{k=1}^q b_k y_{t+1-k}, \quad t \in \mathbb{Z}.$$

If $a_{1:p} \in \mathcal{S}_p$ and y is ℓ^1 (absolutely summable), then there is a unique ℓ^1 solution $x = \{x_n : n \in \mathbb{Z}\}$, given by

$$\hat{x}(\lambda) \left(1 - \sum_{k=1}^p a_k e^{-i\lambda k} \right) = \left(\sum_{k=0}^{q-1} b_{k+1} e^{-i\lambda k} \right) \hat{y}(\lambda),$$

where $\hat{x}(\lambda) = \sum_{k \in \mathbb{Z}} x_k e^{-i\lambda k}$ is the Fourier series of x and \hat{y} is that of y . In particular, if y is the impulse sequence (that is $\hat{y}(\lambda) \equiv 1$), x has a Fourier series given by (using the same notation as in [\(L-2\)](#))

$$(4.11) \quad \hat{x}(\lambda) = \frac{\sum_{k=0}^{q-1} b_{k+1} e^{-i\lambda k}}{1 - \sum_{k=1}^p a_k e^{-i\lambda k}} = e^{i\lambda(p-q-1)} \frac{Q_p(e^{i\lambda}; b_{1:q})}{P_p(e^{i\lambda}; a_{1:p})},$$

Now, writing $\mathbf{z}_t = (x_{(t-p+1):t}, y_{(t-q+2):p})$, Equation [\(4.10\)](#) is equivalent to

$$\mathbf{z}_t = \mathbf{A} \mathbf{z}_{t-1} + y_t \mathbf{b};, \quad t \in \mathbb{Z}.$$

For all $a_{1:p} \in \mathcal{S}_p$, we have $|\lambda|_{\max}(\mathbf{A}) < 1$, and taking y to be the impulse sequence, we have that the unique ℓ^1 solution x must be the causal sequence defined by

$$x_t = \boldsymbol{\varepsilon}_p^T \mathbf{A}^t \mathbf{b}, \quad t \in \mathbb{Z}_+.$$

Hence the left-hand side of (4.9) means that the two equations defined as in (4.10) with coefficients $(a_{1:p}^*, b_{1:q}^*)$ and $(a_{1:p}, b_{1:q}^*)$ have the same ℓ^1 solution when y is impulse sequence, hence by (4.11), it is equivalent to the polynomial equation

$$Q^* \times P = P^* \times Q,$$

where we introduced the polynomials $P^*(z) = P_p(z; a_1^*, \dots, a_p^*)$, $Q^*(z) = Q_q(z; b_1^*, \dots, b_q^*)$, $P(z) = P_p(z; a_1, \dots, a_p)$ and $Q(z) = Q_q(z; b_1, b_2, \dots, b_q)$.

We deduce from this analysis and using these definitions that, to prove Lemma 10 (i), we only need to show that, whenever $a_{1:p}^*, a_{1:p} \in \mathcal{S}_p$, if Condition (L-2) holds, then

$$Q^*P = P^*Q \text{ implies } Q = Q^* \text{ and } P = P^*.$$

This is indeed true, since, under (L-2), by the Gauss Theorem, if $Q^*P = P^*Q$ implies that P^* divides P and, since they have the same degree p and are unitary, we get $P = P^*$, from which it follows that $Q = Q^*$.

Next, we prove Lemma 10 (ii). Using the analysis above, we suppose that P^* and Q^* have at least one common root and show that we can find two real polynomials C and D , with C non-zero (and $D = 0$ only in the special case $q = 1$), with degrees at most $p - 1$ and $q - 2$ such that, for all $d \in \mathbb{R}$,

$$(4.12) \quad Q^*(P^* + d C) = P^*(Q^* + d D).$$

The constraints on the degrees of C and D guaranty that $P^* + d C$ and $Q^* + d D$ can be expressed as P and Q above for some $a_{1:p} \in a_{1:p}^* + \text{Span}(c_{1:p}^*)$ and $b_{2:q} \in b_{2:q}^* + \text{Span}(c_{(p+1):(p+q-1)}^*)$, where $c_{1:p}^*$ and $c_{(p+1):(p+q-1)}^*$ are coefficients in the polynomials C and D . Let U denote the greater common divisor of P^* and Q^* , which has degree at least one by assumption. Now define C and D by $Q^* = D \times U$ and $P^* = C \times U$. Then the degrees of C and D are at most $p - 1$ and $q - 2$, as requested, and (4.12) can be readily checked. \square

Lemma 11. Define ε_p and \mathbf{A} as in Section 4.3 and suppose that $a_{1:p} \in \mathcal{S}_p$, where \mathcal{S}_p is defined in (3.10). Then $\sum_{k \geq 0} \varepsilon_p^T \mathbf{A}^k \varepsilon_p \neq 0$.

Proof. We have, for all $k \geq 0$, $\varepsilon_p^T \mathbf{A}^k \varepsilon_p = \tilde{\varepsilon}_p^T \tilde{\mathbf{A}}^k \tilde{\varepsilon}_p$, where $\tilde{\varepsilon}_p$ denotes the last canonical vector of \mathbb{R}^p and $\tilde{\mathbf{A}}$ is the companion matrix defined as the $p \times p$ upper left bloc of \mathbf{A} . Since $a_{1:p} \in \mathcal{S}_p$, we have $\sum_{k \geq 0} \tilde{\mathbf{A}}^k = (\mathbf{I}_p - \tilde{\mathbf{A}})^{-1}$. To obtain the result we observe that $\tilde{\varepsilon}_p^T (\mathbf{I}_p - \tilde{\mathbf{A}})^{-1} \tilde{\varepsilon}_p = 0$ implies that there exists $c_1, \dots, c_{p-1} \in \mathbb{R}$ such that $\tilde{\varepsilon}_p = (\mathbf{I}_p - \tilde{\mathbf{A}}) [c_1 \ \dots \ c_{p-1} \ 0]^T$. Using the companion matrix form of $\tilde{\mathbf{A}}$, and looking successively at the $p - 1, p - 2, \dots$ entries in the previous equation, we easily get iteratively that $c_{p-1} = 0, \dots, c_1 = 0$, and thus obtain a contradiction. \square

We can now prove the main theorem.

4.5. Proof of Theorem 8. By Remark 2, (A-4) holds as a consequence of (L-1). We apply Proposition 5. To this end we must first show that (3.1) holds. For this we apply Lemma 7. Take an arbitrary $x_1^{(i)} \in \mathcal{X}$, and if $q > 1$, $y^{(i)} \in \mathcal{Y}^{q-1}$. Then, since $\psi_y^\theta(x)$ is of the form (2.5), there exists constants $C_1, C_2 > 0$ only depending on $\theta, x_1^{(i)}$ and $y^{(i)}$ such that, for all $y \in \mathcal{Y}$,

$$\delta_{\mathcal{X}} \left(x_1^{(i)}, \psi_{(y^{(i)}, y)}^\theta(x^{(i)}) \right) \leq C_1 + C_2 |\Upsilon(y)|.$$

Thus (3.11) implies that (3.8) holds and we can apply Lemma 7 to get (3.1) and (3.9). Using the notation of Section 4.3, we have for all $k \geq 0$,

$$Z_k = \boldsymbol{\omega} + \mathbf{A}Z_{k-1} + \mathbf{b}U_{k-1}$$

where Z_k is defined in (2.2). Then, noting that $X_k = \boldsymbol{\varepsilon}_p^T Z_k$, (3.9) becomes

$$(4.13) \quad \psi^\theta \langle Y_{-\infty:0} \rangle = \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}^k (\boldsymbol{\omega} + \Upsilon(Y_{-k}) \mathbf{b}) \quad \tilde{\mathbb{P}}^{\theta_*}, \tilde{\mathbb{P}}^\theta\text{-a.s.}$$

Now we can prove Assertion (i) of Theorem 8. Suppose that (L-2) holds and take $\theta \in [\theta_*]$. By Proposition 5, this implies $\theta \in \langle \theta_* \rangle$ and thus $\theta = (\vartheta, \varphi_*)$ and satisfies (3.2). It remains to show that $\vartheta = \vartheta_*$. Since (4.13) holds both $\tilde{\mathbb{P}}^{\theta_*}$ -a.s. and $\tilde{\mathbb{P}}^\theta$ -a.s., and the same equation holds with θ , \mathbf{A} , $\boldsymbol{\omega}$ and \mathbf{b} replaced by θ_* , \mathbf{A}_* , $\boldsymbol{\omega}_*$ and \mathbf{b}_* , (3.2) implies

$$(4.14) \quad \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}^k (\boldsymbol{\omega} + \Upsilon(Y_{-k}) \mathbf{b}) = \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k (\boldsymbol{\omega}_* + \Upsilon(Y_{-k}) \mathbf{b}_*) \quad \tilde{\mathbb{P}}^{\theta_*}\text{-a.s.}$$

This implies that $\tilde{\mathbb{P}}^{\theta_*}$ -a.s.,

$$(b_1^* - b_1) \Upsilon(Y_0) = \boldsymbol{\omega} - \boldsymbol{\omega}_* + \sum_{k=1}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}^k (\boldsymbol{\omega} + \Upsilon(Y_{-k}) \mathbf{b}) - \sum_{k=1}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k (\boldsymbol{\omega}_* + \Upsilon(Y_{-k}) \mathbf{b}_*)$$

If $b_1 \neq b_1^*$, this implies that $\Upsilon(Y_0)$ is a deterministic function of $Y_{-\infty:1}$. Under \mathbb{P}^{θ_*} , by (3.1), the distribution of Y_0 given $Y_{-\infty:1}$ is $G^{\theta_*}(X_1, \cdot)$. By (1.2) $G^{\theta_*}(X_1, \cdot)$ has the same support as ν . Hence the distribution of Y_0 given $Y_{-\infty:1}$ has the same support as $\nu \circ \Upsilon^{-1}$ which is not degenerate by Definition 3-(ii), and we get a contradiction. So we must have $b_1 = b_1^*$, that is, $\mathbf{b} = \mathbf{b}_*$. Iterating, and using this line of reasoning, we get that, for all $k \geq 1$,

$$\boldsymbol{\varepsilon}_p^T \mathbf{A}^k \mathbf{b}_* = \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k \mathbf{b}_* .$$

Applying Lemma 10, we have that this implies $\mathbf{A} = \mathbf{A}_*$, that is, $a_{1:p} = a_{1:p}^*$ and $b_{2:q} = b_{2:q}^*$. Going back to (4.14), we also get that

$$\sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k \boldsymbol{\varepsilon}_p (\boldsymbol{\omega} - \boldsymbol{\omega}_*) = 0 ,$$

and we conclude with Lemma 11 that $\boldsymbol{\omega} = \boldsymbol{\omega}_*$. Collecting the obtained identities, we arrive at $\vartheta = \vartheta_*$ and this concludes the proof of Theorem 8 (i).

Next, we prove Theorem 8 (ii). Hence we suppose that (L-2) does not hold, and, since we want to show that $[\theta_*]$ contains a particular set, we use that $[\theta_*] = \langle \theta_* \rangle$ as given by Proposition 5. Hence, we only need to exhibit $\theta = (\vartheta, \varphi_*)$ such that (3.2) and (3.3) hold. By Lemma 7, (3.3) holds for all $\theta \in \Theta$ and we only need to impose (3.2) on ϑ . Using (4.13), which holds $\tilde{\mathbb{P}}^\theta, \tilde{\mathbb{P}}^{\theta_*}$ -a.s. for all θ , including $\theta = \theta_*$, it is thus sufficient to have

$$\text{for all } k \geq 0, \boldsymbol{\varepsilon}_p^T \mathbf{A}^k \mathbf{b} = \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k \mathbf{b}_* \quad \text{and} \quad \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}^k \boldsymbol{\omega} = \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_p^T \mathbf{A}_*^k \boldsymbol{\omega}_* .$$

The first relation with $k = 0$ corresponds to $b_1 = b_1^*$, which we assume in the following, so that $\mathbf{b} = \mathbf{b}_*$. By Lemma 10, we can find a line of \mathbb{R}^{p+q-1} , going through $(a_{1:p}^*, b_{2:q}^*)$ such that these relations hold for all $k \geq 0$, for all $(a_{1:p}, b_{2:q})$ on

this line. Now, by definition of ω_* and ω , and using [Lemma 11](#) with $a_{1:p} \in \mathcal{S}_p$, the last relation in the displayed sufficient condition can be written as

$$\omega = \frac{\sum_{k=0}^{\infty} \varepsilon_p^T \mathbf{A}_*^k \varepsilon_p}{\sum_{k=0}^{\infty} \varepsilon_p^T \mathbf{A}^k \varepsilon_p} \omega^*,$$

where the ratio tends to 1 as $(a_{1:p}, b_{2:q})$ approaches $(a_{1:p}^*, b_{2:q}^*)$. Hence we get that $[\theta_*]$ contains the part of the curve obtained by taking $(a_{1:p}, b_{2:q})$ on the previously exhibited line and sufficiently close to $(a_{1:p}^*, b_{2:q}^*)$ to have $a_{1:p} \in \mathcal{S}_p$, and setting ω as above.

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DÉPARTEMENT CITI, CNRS UMR 5157, TÉLÉCOM SUDPARIS, 91000 ÉVRY, FRANCE
E-mail address: randal.douc@telecom-sudparis.eu

LTCI, TÉLÉCOM PARISTECH, UNIVERSITÉ PARIS-SACLAY, 75013 PARIS, FRANCE
E-mail address: roueff@telecom-paristech.fr

DEPARTMENT OF FOUNDATION YEAR, INSTITUTE OF TECHNOLOGY OF CAMBODIA, 12156 PHNOM PENH, CAMBODIA
E-mail address: tepmony.sim@itc.edu.kh