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Liouville Quantum Gravity

Songyuan Li¹, Nicolaos Toumbas², Jan Troost¹

¹ *Laboratoire de Physique de l'Ecole Normale Supérieure
CNRS, ENS, Université PSL, Sorbonne Université
Paris, France*

² *Department of Physics, University of Cyprus, Nicosia 1678, Cyprus*

ABSTRACT: We define a three-dimensional quantum theory of gravity as the holographic dual of the Liouville conformal field theory. The theory is consistent and unitary by definition. The corresponding theory of gravity with negative cosmological constant has peculiar properties. The quantum theory has no normalizable AdS_3 vacuum. The model contains primary black holes with zero spin. All states can be interpreted as BTZ geometries dressed with boundary gravitons. There is a unique universal interaction between these states consistent with unitarity and the conformal symmetry of the model. This theory of gravity, though conceptually isolated from other models of quantum gravity, is worth scrutinizing.

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1 Introduction

Consistent theories of quantum gravity are hard to come by. Quantum gravity in four dimensions (or higher) may be asymptotically safe [1]. String theory provides us with a perturbative definition of quantum gravity in a number of backgrounds [2,3]. Moreover, string theory holography suggests a non-perturbative definition of quantum gravity in spaces with negative cosmological constant [4]. Another fruitful approach is to study quantum gravity in three dimensions or lower [5,6]. Still, the total present yield is a limited set of models of quantum gravity. We would like to extend the set.

In this paper, we study quantum gravity in three dimensions with negative cosmological constant. Firstly, we wish to remark that a large number of ultraviolet complete quantum theories of gravity in three-dimensional anti-de Sitter space are provided by compactifying string theory on a space-time with an AdS_3 factor. These theories have a large set of degrees of freedom and account microscopically for the Bekenstein-Hawking entropy of black holes [7,8] in AdS_3 [9]. Sufficient ingredients for this result are on the one hand that the effective central

charge coincides with the bare central charge that can be computed in gravity [10], and on the other hand the Cardy formula [11, 12]. These string theories contain a graviton and a multitude of scalars, fermions, and higher spin fields. They permit a holographic description in terms of a two-dimensional conformal field theory dual [4].

An alternative approach to quantum gravity in three dimensions with a negative cosmological constant is to quantize the gravitational action directly [6, 13].¹ This track often exploits the fact that the three-dimensional gravity action can be rewritten in terms of a Chern-Simons action (with gauge algebra $so(2, 2)$ or $so(3, 1)$ for the case of negative cosmological constant) [15, 16]. Recent attempts in this direction have concentrated on defining a conformal field theory dual with certain prescribed properties, like an $SL(2, \mathbb{C})$ invariant vacuum and a degree of holomorphic factorization [17–19]. We note also that a careful quantization around a global AdS_3 vacuum was performed in [20], giving rise to a (non-modular invariant) theory of reparameterizations on a coadjoint orbit of the Virasoro group.

We rather take a different approach, *defining* three-dimensional quantum gravity to be the gravitational dual of Liouville conformal field theory. Manifestly, this is inspired by the identification between the respective actions of these theories [21]. We propose their path integral measures to be identical as well. The main advantage of this proposal is that the resulting theory is patently consistent and unitary, because the Liouville theory is. Another advantage is that the spectrum and correlation functions are known. We also stress a number of distinct properties. Firstly, the Liouville conformal field theory does not have an $SL(2, \mathbb{C})$ invariant normalizable ground state. Thus, our quantum theory of gravity does not have a normalizable global AdS_3 ground state. The theory only has primary black hole states or BTZ geometries with zero angular momentum, which can be dressed with boundary gravitons. Relatedly, the effective central charge is one, independently of the value of the bare central charge. This has as a consequence that the Cardy formula will not reproduce the entropy of BTZ black holes.² We will argue that this is compatible with the topological nature of three-dimensional gravity.

Thus, the theory of gravity we define will be consistent but peculiar. As a consequence, the theory is conceptually isolated from many of the quantum theories of gravity that we admire. Our aim in this paper is to proceed with the interpretation of this theory of quantum gravity, and to investigate its properties. Hopefully, its original features will broaden our vision on the set of consistent theories of quantum gravity.

We remark that the idea that pure quantum gravity with negative cosmological constant could be equivalent to Liouville theory has a longer history. Firstly, in [22] the role of Liouville theory in two-dimensional quantum gravity was understood. The three-dimensional picture [21] was further developed in [23, 24], but different conclusions from ours were drawn. Point particle bulk states which are non-normalizable states in the boundary Liouville theory were added to the spectrum in the hope of restoring the Bekenstein-Hawking microscopic entropy counting, potentially at the cost of unitarity.³ The references [25–27] are very interesting. Especially, [27] quantizes pure gravity and identifies a projection that gives rise to the Hilbert

¹It has been argued that this may be a misguided approach since it is as if one quantizes fields that describe the thermodynamics (in contrast to the statistical mechanics) of the system [14].

²More precisely, the continuous spectrum renders it hard to make a definite statement, let alone positive.

³This has been a recurring theme: attempts were made to identify purported missing states, thus presumably destroying the consistency of the boundary conformal field theory.

space of Liouville theory, drawing conclusions that overlap with ours. The perspective is different, and room is left for adding spinning black holes [26, 27], a physical picture in terms of long strings instead of black holes [25, 27], et cetera. We briefly come back to the additional information that [27] provides in the conclusions. The papers [28, 29] argue that Liouville theory arises as a universal subsector of irrational conformal field theories dual to three-dimensional theories of gravity (like AdS_3 string theory). While our perspective is different, the identification of gravitational dynamics with properties of Liouville conformal field theory is relevant to our proposal.⁴

The plan of the paper is as follows. We review the map from conformal field theory energy-momentum tensors to bulk metrics in section 2. We map out energy-momentum tensor expectation values, exploiting meromorphy to a large degree. We then study the metrics we obtain in section 3. The analysis in these sections applies to all examples of Euclidean AdS_3 holography. In particular, it does not depend on the microscopic completion of the bulk gravitational theory provided by the boundary conformal field theory. It does provide a useful backdrop for the center piece that follows. We recall the properties of Liouville theory in section 4 and interpret them holographically. This provides us with a global picture of three-dimensional Liouville quantum gravity. We provide entries of the holographic dictionary and discuss a detailed map of the action as well as the state spaces of the theories. Euclidean quantum black hole mergers are described semi-classically as well as exactly. We summarize our findings and indicate open problems in section 5.

2 Conformal Field Theory and Metrics

In the Euclidean, the bulk solution to Einstein's equation with negative cosmological constant is determined by the boundary data [31]. We consider various boundary set-ups, with operator insertions in different topologies. In the case of the AdS_3/CFT_2 correspondence, the metric is determined by the expectation value of the energy-momentum tensor in the boundary conformal field theory (in the presence of operator insertions). The expectation value of the energy-momentum tensor of a two-dimensional conformal field theory is heavily constrained, and we analyze consequences of these generic properties for the bulk metric in this section. Most of these properties are implicitly or explicitly known – we present an elementary review.

2.1 The Energy-momentum Tensor and the Metric

The Fefferman-Graham asymptotic expansion of the solution to Einstein gravity with negative cosmological constant and given boundary data terminates quickly in three dimensions [32]. The technique used to determine it is the integration of the Chern-Simons equations of motion [32] with Brown-Henneaux boundary conditions [10]. The resulting metric is:

$$ds^2 = l^2 d\rho^2 + 4Gl(-Tdz^2 - \bar{T}d\bar{z}^2) + (l^2 e^{2\rho} + 16G^2 T\bar{T}e^{-2\rho})dzd\bar{z}, \quad (2.1)$$

for a planar or spherical boundary parameterized by the coordinates z and \bar{z} . The boundary data are captured by the expectation values $T = T_{zz}(z)$ and $\bar{T} = T_{\bar{z}\bar{z}}(\bar{z})$ of the energy-momentum tensor components of the boundary conformal field theory. Under conformal

⁴For other related work see [30].

transformations, the boundary coordinates transform in a standard way to leading order in the radial expansion. These standard transformations are accompanied by subleading corrections as well as a change in the radial coordinate, to preserve the form of the metric (2.1) [32]. Our goal in this section is to uncover generic properties of the metric (2.1) using the properties of the energy-momentum tensor components.

2.2 Energy-momentum Tensor Expectation Value

The energy-momentum tensor component $T(z)$ is meromorphic. When inserted in a two-dimensional Euclidean conformal field theory correlation function, it gives rise to meromorphic functions of the coordinate z of the point of insertion. We consider a planar boundary, which we compactify to a two-sphere, and assume the presence of n primary operator insertions. In order to determine the resulting bulk metric, we must compute the value of the energy-momentum tensor when inserted in the corresponding n -point function of the conformal field theory. The value of the energy-momentum tensor component T (and likewise of \bar{T}) is considerably constrained by the property of meromorphy. Its double and single poles are known, given the conformal weight of the primary operator insertions. It is also restricted to behave as z^{-4} at most for large z . This is for the point at infinity to be regular. Let us enumerate results for a low number of insertions n .

Firstly, the one-point function is zero because of conformal invariance. For the two-point function, with insertions at z_1 and z_2 , we expect to be able to expand the energy-momentum tensor expectation value in terms of a holomorphic function, and at most double pole singularities at z_1 and z_2 . Moreover, the residues of the double poles are prescribed. Imposing that the energy-momentum tensor behaves as z^{-4} for large z fixes its value uniquely to:

$$\frac{\langle T(z)O_1O_2 \rangle}{\langle O_1O_2 \rangle} \equiv T^{(2)}(z) = \frac{h}{(z-z_1)^2} + \frac{h}{(z-z_2)^2} - \frac{2h}{(z-z_1)(z-z_2)}. \quad (2.2)$$

We had to set $h_1 = h_2$ to obtain a non-zero two-point function. It is easy to check that also the subleading single pole behaviour is as expected for a conformal primary. There is an interesting special case, when one of the insertions is at infinity, and the other insertion is at zero.⁵

$$T^{0,\infty}(z) = \frac{h}{z^2}. \quad (2.3)$$

Next, we record the expectation value of the energy-momentum tensor component $T(z)$ in the presence of three primary conformal field theory operators with conformal dimensions h_i :

$$T^{(3)}(z) = h_1\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right)\left(\frac{1}{z-z_1} - \frac{1}{z-z_3}\right) + \text{cyclic}. \quad (2.4)$$

Higher point functions are also heavily constrained.

2.3 Energy-momentum Tensor on the Cylinder

In the following we also make use of a boundary manifold which is a cylinder. In two dimensions, it can be obtained by the conformal transformation $z = e^{-iw}$ from the plane. It is

⁵Regularity at infinity is now violated because of the insertion.

important that the conformal transformation maps the points $z = 0$ and $z = \infty$ to the open ends of the infinite cylinder. The expectation value of the energy-momentum tensor on the cylinder is determined from the value on the plane through the quasi-primary transformation property of the energy-momentum tensor:

$$(\partial_z w)^2 T'(w) = T(z) - \frac{c}{12} \{w, z\}, \quad (2.5)$$

where the Schwarzian $\{\cdot, \cdot\}$ for the map $z = e^{-iw}$ evaluates to:

$$\{w, z\} = \frac{1}{2z^2}. \quad (2.6)$$

Thus, the energy-momentum tensor component T_c on the cylinder equals

$$T_c(w) = -z^2 T(z) + \frac{c}{24}. \quad (2.7)$$

Using this map, we straightforwardly compute the expectation values of the energy-momentum tensor on the cylinder for the configurations discussed previously:

$$T_c^{0,\infty}(w) = -h + \frac{c}{24}, \quad (2.8)$$

$$T_c^{(2)}(w) = \frac{h}{4} \frac{\sin^2 \frac{w_1 - w_2}{2}}{\sin^2 \frac{w - w_1}{2} \sin^2 \frac{w - w_2}{2}} + \frac{c}{24}, \quad (2.9)$$

$$T_c^{(3)}(w) = h_1 \frac{\sin \frac{w_2 - w_1}{2}}{2 \sin \frac{w - w_1}{2} \sin \frac{w - w_2}{2}} \frac{\sin \frac{w_3 - w_1}{2}}{2 \sin \frac{w - w_1}{2} \sin \frac{w - w_3}{2}} + \text{cyclic} + \frac{c}{24}. \quad (2.10)$$

Thus, we have computed examples of energy-momentum tensor expectation values that universally arise in two-dimensional conformal field theories. We commence the study of the corresponding metrics in the next section, and continue the analysis in Appendix A.

3 Universal Metrics

In the previous section, we computed universal expectation values of the energy-momentum tensor. Through the elliptic boundary value problem of three-dimensional Euclidean Einstein gravity with negative cosmological constant [31], these translate into universal metrics [32] which we study in the present section.

3.1 The BTZ Black Hole Metric

We recorded the constant energy-momentum tensor expectation value $T_c^{0,\infty}$ (2.8) on the cylinder corresponding to a primary state of conformal dimension h , propagating from the infinite past to the infinite future. The resulting metric is straightforwardly determined. If we take equal left and right conformal dimensions, then the metric can be transformed to the Euclidean BTZ black hole metric with zero spin:

$$ds_{BTZ}^2 = l^2 \left(\frac{r^2}{l^2} - 8GM \right) dt^2 + \left(\frac{r^2}{l^2} - 8GM \right)^{-1} dr^2 + r^2 d\phi^2. \quad (3.1)$$

The equivalence is demonstrated through the identifications and coordinate transformations [32]:

$$\begin{aligned}
T_c^{0,\infty} = \bar{T}_c^{0,\infty} &= -\frac{Ml}{2} \\
w &= \phi + it \\
r_+^2 &= 8GMl^2 = 4l^2 e^{2\rho_0} \\
r^2 &= 8GMl^2 \cosh^2(\rho - \rho_0).
\end{aligned} \tag{3.2}$$

In the classical theory we need to suppose that the conformal dimension of the operator on the plane satisfies the bound $h \geq c/24$. The Euclidean metric (3.1) can be analytically continued to the Lorentzian BTZ black hole. This whole analysis is easily generalized to the case of a BTZ black hole with non-zero angular momentum [32].

The Horizon

We can also study the metric if we start out with a planar boundary and the energy-momentum tensor $T^{0,\infty}$ (2.3). A useful and generic tool of analysis is to determine the location of the event horizon, where the metric (2.1) degenerates. It satisfies the equation:

$$e^{2\rho_{hor}} = 4Gl^{-1}(T\bar{T})^{\frac{1}{2}}. \tag{3.3}$$

Thus, for instance, for the planar energy-momentum tensor $T^{0,\infty}$, the horizon radius is

$$e^{2\rho_{hor}} = 4Gl^{-1}h \frac{1}{|z|^2}, \tag{3.4}$$

as a function of the planar coordinate z . The first thing to note is that the horizon now reaches radial infinity (at the boundary insertion). If we put an infrared cut-off R on the coordinate radius e^ρ of the bulk AdS_3 space, then we cut out a little hole from the surface of the event horizon, parameterized by the angle of z , where $R^2 = 4Gl^{-1}h|z|^{-2}$. There is an analogous singularity at $z = \infty$. Of course, the excisions correspond to the incoming and outgoing state that is present in the cylindrical set-up. In other words, the horizon surface is a sphere with two marked points, where the horizon reaches the boundary, or a sphere with two small disks cut out, when we introduce an infrared cut-off in the bulk. We recuperate the standard cylinder topology of the BTZ horizon in this manner.

3.2 Properties of the Metrics

It should be clear that this reasoning is very generic. The singular behaviour of the operator product expansion of the energy-momentum tensor with a (conformal primary) operator insertion will guarantee that near a local insertion with sufficiently high conformal dimension, the metric can be locally transformed (through the exponential map) into a BTZ black hole metric with mass $Ml = 2h - c/12$ (where for simplicity only we assume zero spin). This is independent of the number and the precise nature of the insertions. Thus, for example, in the cases of the energy-momentum tensors $T^{(2)}$ and $T^{(3)}$, we can state that locally, near one of

the insertions, the metric looks like a BTZ black hole metric with (left) conformal dimension h_i . The horizon looks like a sphere with two or three peaks, reaching the AdS_3 boundary.

On the cylinder, we locally find the same picture. Care should be taken because of the fact that the unit operator gives rise to a constant energy-momentum tensor T_c at infinity. The non-zero horizon radius for the cylindrical horizon should not be confused with a black hole insertion. Otherwise, the cylindrical picture is the same as the planar picture – the reasoning can be applied locally. Note that the singularities in the metric only depend on the planar or cylindrical coordinates on a radial slice. Thus, they propagate radially inward towards the center on straight lines.

Let us make a few more generic remarks on the universal metrics. We note that on the horizon surface defined by equation (3.3), the determinant of the metric has a double zero. Thus, when we cross the surface, the nature of the metric remains the same, in the sense that we are still outside the horizon. One can see this in more detail in e.g. the BTZ metric, in which case the radial coordinate ρ touches the horizon at ρ_0 , and then returns to infinity.

In appendix A, we start the exploration of the metrics associated to the energy-momentum tensors $T^{(2)}$ and $T^{(3)}$ after analytic continuation to the Lorentzian cylinder. We believe these Euclidean and Lorentzian metrics are interesting, and since they are universal by meromorphy, they deserve further analysis. As background for the following section, it is more useful to us to turn to a different characterization of the same class of metrics, in terms of monodromies.

3.3 On Monodromies

It is known how the above picture is reflected in the Chern-Simons formulation of three-dimensional gravity. The Banados metric (2.1) is associated to the (left) Chern-Simons connection [32]:

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dw \\ -i\frac{4GT}{l}e^{-\rho}dw & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.5)$$

This connection is the left combination of dreibein and spin connection (and there is an analogous right combination). The presence of BTZ sources in the metric is tracked by the monodromies of the connection around the singularity [8]. The simplest example is again the standard BTZ black hole, where the left connection reads

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dw \\ 2iGM e^{-\rho}dw & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.6)$$

We integrate it over the angular part of w , at the horizon radius $e^{\rho_0} = \sqrt{2GM}$. The matrix is then constant and gives rise to a monodromy trace

$$\text{tr } P \exp \int A = 2 \cosh 2\pi\sqrt{2GM}. \quad (3.7)$$

Our next example is slightly more involved. We compute the monodromy around a Euclidean insertion on the plane. The left-connection equals

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dz \\ -i\frac{4G}{l}\frac{\hbar}{z^2}e^{-\rho}dz & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.8)$$

At fixed $e^{2\rho_{hor}} = \frac{4G}{l} h/|z|^2$, the connection reduces to

$$A = \begin{pmatrix} 0 & \sqrt{\frac{4G}{l}} h e^{i\theta} d\theta \\ \sqrt{\frac{4G}{l}} h e^{-i\theta} d\theta & 0 \end{pmatrix}. \quad (3.9)$$

We can gauge transform the connection to a constant matrix, and compute the resulting trace:

$$\text{tr } P \exp \int A = 2 \cosh 2\pi \sqrt{\frac{4G}{l} (h - c/24)}. \quad (3.10)$$

Thus we recognize Euclidean vertex operator insertions as BTZ geometries, locally. More generally, near a Euclidean insertion that gives rise to a double pole in the energy-momentum tensor component $T(z)$, the monodromy calculation and result reduces to the hyperbolic monodromy (3.10).

4 Liouville Quantum Gravity

In the previous sections, we discussed a few universal properties of the AdS_3/CFT_2 holographic duality. In this section, we commit to a very particular dual conformal field theory, namely Liouville theory. The boundary conformal field theory is then well studied. We take the holographic dictionary to *define* the bulk three-dimensional theory of quantum gravity, and call the resulting theory Liouville quantum gravity. In this section, we first recall properties of the unitary Liouville conformal field theory. We then provide a holographic gravitational interpretation of these basic properties, and critically discuss the ensuing theory of quantum gravity.

4.1 Properties of Liouville Theory

Liouville theory is a consistent unitary conformal field theory in two dimensions. See e.g. [33, 34] for an older and a more recent review. The spectrum and the three-point functions entirely determine the theory. The consistency of the theory hinges on crossing symmetry of the four-point function on the sphere and the covariant modularity of the torus one-point function. Both have been understood [35, 36]. Therefore, if we define three-dimensional quantum gravity as the dual of Liouville theory, it will be manifestly consistent as well as unitary. The question arises which properties the resulting theory of quantum gravity exhibits when expressed as a three-dimensional bulk theory through holographic duality. To answer the question, we firstly review various properties of Liouville theory and then translate them using a holographic dictionary.

The Liouville action S_L is

$$S_L = \frac{1}{4\pi} \int d^2z (\partial\phi\bar{\partial}\phi + \mu e^{2b\phi} + QR^{(2)}\phi), \quad (4.1)$$

where the parameter b is related to the bare central charge c of the theory by

$$c = 1 + 6Q^2 = 1 + (b + b^{-1})^2, \quad (4.2)$$

which ensures that the exponential interaction term in the action is marginal. The conformal dimensions of the Liouville vertex operator $e^{2\alpha\phi}$ is given by the formula

$$h = \alpha(Q - \alpha), \quad (4.3)$$

and the spectrum of momenta of the unitary Liouville theory is determined by the momenta $\alpha \in Q/2 + i\mathbb{R}^+$. Thus, the continuous spectrum has a lower limit conformal dimension $h_{min} = Q^2/4 = (c-1)/24$. The effective central charge relevant to the Cardy formula is then

$$c_{eff} = c - 24h_{min} = 1. \quad (4.4)$$

The effective central charge reflects the fact that we have a single boson, and this is visible in the ultraviolet growth of the number of states as a function of their conformal dimension [11].

The Liouville action (4.1) in the presence of operator insertions needs to be regularized. In the semi-classical theory, the semi-classical Liouville field $\varphi = 2b\phi$ satisfies the boundary conditions

$$\begin{aligned} \varphi &= -2\eta_i \log |z - z_i|^2 + O(1) \quad \text{at } z \rightarrow z_i \\ \varphi &= -2 \log |z|^2 + O(1) \quad \text{at } z \rightarrow \infty, \end{aligned} \quad (4.5)$$

in the presence of operator insertions with $\alpha_i \approx \eta_i/b$ and a linear dilaton charge at infinity. Following e.g. [38], we cut out small disks D_i of radius ϵ_i around each insertion, as well as around infinity, and define the regularized Liouville action:

$$S_L^{reg} = S_L + \varphi_\infty + 2 \log r_\infty - \sum_i (\eta_i \varphi_i + 2\eta_i^2 \log \epsilon_i), \quad (4.6)$$

where we defined the boundary contributions

$$\varphi_\infty = \frac{1}{2\pi r_\infty} \int_{\partial D_\infty} \varphi dl, \quad (4.7)$$

and

$$\varphi_i = \frac{1}{2\pi\epsilon_i} \int_{\partial D_i} \varphi dl, \quad (4.8)$$

which cancel the divergences and insure that the boundary conditions are consistent.

The Liouville three-point functions can be guessed by computing their pole structure and their semi-classical behaviour [37, 38]. The three-point functions can also be derived by requiring that null vectors in degenerate representations (reached by analytic continuation from the unitary spectrum) decouple [39]. Furthermore, the three-point functions can be numerically bootstrapped [40]. Explicitly, they are given by:

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{Y_0 Y(2\alpha_1) Y(2\alpha_2) Y(2\alpha_3)}{Y(\alpha_1 + \alpha_2 + \alpha_3 - Q) Y(\alpha_1 + \alpha_2 - \alpha_3) Y(\alpha_2 + \alpha_3 - \alpha_1) Y(\alpha_3 + \alpha_1 - \alpha_2)} \quad (4.9)$$

where the definition of the special function Y is

$$\log Y(x) = \int_0^\infty \frac{dt}{t} \left(\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right)\frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right), \quad (4.10)$$

and we need:

$$Y_0 = \partial_x Y(x)|_{x=0}. \quad (4.11)$$

The reflection amplitude codes the fact that an operator in Liouville theory at momentum $\alpha = Q/2 + ip$, where p is a positive real number, creates the same state as the operator at momentum $Q/2 - ip$. In the mini-superspace approximation to Liouville theory, a normalizable state consists of a wave bouncing off an exponential potential, and returning with equal and opposite momentum. We have an equation of the type

$$| - p \rangle = R(-p) | p \rangle, \quad (4.12)$$

where the reflection amplitude R is related to the three-point function by taking a limit in one of the momenta. See e.g. [34] for the detailed formulas. These data fix the Liouville conformal field theory uniquely, and it can be proven to satisfy crossing symmetry as well as consistency of the torus one-point function. Thus, we have a two-dimensional conformal field theory that is consistent and unitary, and that to a large extent is unique, as seen both from the null vector decoupling argument [39], and from the numerical bootstrap approach [40].

4.2 The Holographic Interpretation

In this subsection, we holographically interpret properties of Liouville theory. We first discuss the big picture of three-dimensional Liouville gravity, and subsequent subsections fill in more details.

4.2.1 The Big Picture

The Liouville conformal field theory has peculiar properties that translate into unexpected properties of its proposed three-dimensional quantum gravity dual. Firstly, there is no normalizable $SL(2, \mathbb{C})$ invariant vacuum state in the Euclidean conformal field theory. This translates holographically into the absence of an $so(3, 1)$ invariant quantum gravity ground state in the theory. In other words, the path integral of our quantum theory of gravity is such that the Euclidean AdS_3 space-time is not normalizable. The spectrum of the theory consists of primaries with conformal dimensions which form a continuous spectrum starting at a gap $(c - 1)/24$. These primaries create hyperbolic monodromies at their insertion, and translate holographically to BTZ space-times. Moreover, the descendants, generated by the action of the Fourier modes of the boundary energy-momentum tensor, can be interpreted as boundary gravitons localized at the boundary of the three-dimensional space-time.

Thus the spectrum of the boundary conformal field theory has the following bulk interpretation. The only excitations that we have in our three-dimensional theory of quantum gravity are primary BTZ black holes, dressed with boundary gravitons. The theory has no propagating gravitons or other particles. A naked black hole (i.e. with no boundary gravitons) corresponds to a single quantum state. It has zero entropy. The boundary gravitons give entropy to a state at given high mass $Ml = L_0 + \bar{L}_0$ as dictated by the Cardy formula, at effective central charge $c_{eff} = 1$.⁶⁷ The theory we define only has primary states of spin zero. Thus, there are no primary spinning black holes.

⁶In no sense do these degrees of freedom account fully for the traditional semi-classical Bekenstein-Hawking entropy of the three-dimensional black holes.

⁷We have divided by a space-time volume factor due to the continuous spectrum in making these statements.

In our quantum theory of gravity, it is hard to detect the horizon of a black hole. That is because there are no propagating degrees of freedom. The only degrees of freedom are the primary states corresponding to the black holes themselves, as well as the boundary degrees of freedom that live at spatial infinity. Thus, it is impossible to derive the traditional macroscopic picture of the thermodynamic properties of the BTZ black holes [42, 43] within our limited theory. This may be just as well, since we have just established that the microscopic entropy of the black holes we consider is zero. On the other hand, coupling to other degrees of freedom would re-establish a traditional thermodynamic picture, but it is challenging to do this consistently with the postulates of holographic duality and quantum gravity.⁸⁹¹⁰

The Liouville theory has one parameter which is the bare central charge c . A perturbation expansion exists in one over the bare central charge c , the $1/c$ expansion. In the dual gravitational theory, the expansion is in G_N/l where G_N is the three-dimensional Newton constant and l is the radius of curvature of the locally AdS_3 space-time.

There is a detail that we skipped over in our explanation. The AdS_3 space-time corresponds to a state with (planar) conformal dimension h equal to zero both classically and quantum mechanically. It therefore has weight $-c/24$ with respect to the cylindrical Hamiltonian. The minimal conformal dimension in the spectrum of Liouville theory is $Q^2/4 = (c-1)/24$. In the cylinder frame, this becomes a minimal mass of $-1/24$. This is negative. Starting at (planar) conformal dimension $(c-1)/24$, we already identify the resulting states with (quantum) BTZ geometries because they have hyperbolic monodromy quantum mechanically. Note that this quantum correction to the classical picture is a one-loop effect. We will comment further on one-loop effects in subsection 4.4.

4.2.2 The Action

Since the holographic relation between Liouville theory and the bulk gravitational theory is crucial to us, we carefully review the map between the boundary and the bulk action [21] in appendix B. We do modify the discussion of the zero modes of the bulk theory. In particular, we gauge a symmetry in order to guarantee that the left and right zero modes of the reduced, chiral Wess-Zumino-Witten models on the boundary are glued together, such that we reproduce the diagonal spectrum of the boundary Liouville theory. The details of the procedure are in appendix B.

4.2.3 The Reflection Amplitude

Each Liouville state has a gravitational dual state. The Liouville state satisfies a reflection property (4.12). We can ask whether there is a bulk analogue of the reflection property. Firstly,

⁸Relatedly, in our theory, since there is no way we can detect the horizon of a black hole, there is little reason to call the black hole geometry black. We could therefore always refer to the BTZ black hole as the BTZ geometry. We will refrain from this heavy change of nomenclature, and keep referring to the BTZ geometry as being a black hole. The reader should keep the essential conceptual caveat in mind.

⁹At least one way to consistently couple matter is provided by AdS_3 string theory. The resulting quantum theory of gravity has a multitude of extra degrees of freedom – enough to measure as well as account for the black hole entropy.

¹⁰Note that in string theory too, it is impossible to implement the standard low energy operation of adding a single scalar field. Our theory of quantum gravity shares this feature of being rigid.

we should note that the bulk geometry depends only on the boundary energy-momentum tensor, which is invariant under reflection. Thus, the bulk geometry is invariant under reflection. Thus, there is a pressing question as to how evaluating the bulk action can lead to a different result for on the one hand momentum p and on the other hand momentum $-p$.

As demonstrated in appendix B, the bulk action equals the boundary Liouville action. A subtlety arises when we allow primary operator insertions. This guarantees that the bulk and boundary action have additional divergences. Thus, we must regularize the bulk (and boundary) action in the presence of operator insertions. It would be interesting to analyze bulk regularization in the presence of operator insertions in generality. We concentrate on the following phenomenon.

When we introduce the microscopic picture for the boundary energy-momentum tensor provided by Liouville theory, there are natural candidate primary operator insertions, defined in terms of the Liouville field. In semi-classical Liouville theory, in order to evaluate the action, we regularize it. The regularization depends on the Liouville field, which in turn does depend on the choice of the sign of the momentum. (See e.g. equation (4.6).) Thus, the semi-classical (regularized Liouville, bulk, and boundary) action changes under a change of momentum. From the bulk perspective, the regularization is in terms of a field which is a non-local functional of the energy-momentum tensor. The Liouville regulator does have neat properties. It guarantees that independently of the choice of momentum (which is, roughly, a square root of the conformal dimension), the semi-classical actions will be related by a phase factor. Thus, they can be associated to the same (albeit reflected) state (as in equation (4.12)).¹¹

The semi-classical actions for the three-point correlator (see also subsection 4.3), for instance, are related by a factor which is the semi-classical limit $R_{s.c.}$ of the reflection amplitude [38]:

$$e^{-\frac{1}{b^2}S_{cl}(p_1,p_2,p_3)} \approx R_{s.c.}(-p_1)e^{-\frac{1}{b^2}S_{cl}(-p_1,p_2,p_3)}. \quad (4.13)$$

Finally, quantum Liouville theory and the quantum reflection amplitude R extend the semi-classical discussion to a theory that is exact in G_N/l .

4.2.4 The States

The Liouville partition function is

$$Z = V \int \frac{dp}{2\pi} \frac{e^{-\pi\tau_2 p^2}}{|\eta(q)|^2} = \frac{V}{2\pi\sqrt{\tau_2}|\eta(q)|^2}, \quad (4.14)$$

where $q = e^{2\pi i\tau}$, τ is the modulus of the rectangular torus and V is an infinite volume factor due to the Liouville zero mode. Conformal invariance implies that the partition function will depend on the product LT only where L is the size of the boundary circle and T is the temperature. We can express this product in terms of the parameter τ_2 . The partition function coincides with the partition function of one free boson. This matches the fact that the effective number of boundary degrees of freedom is of finite central charge, equal to

¹¹See also [23] for a detailed defense of the statement that natural regulators for the bulk gravity action lead to equality with the boundary Liouville action, albeit in the differing context of non-normalizable point particle excitations.

one. The partition function is the simplest holomorphically factorized partition function which satisfies the requirement of reproducing the spectrum of boundary gravitons as well as matching the bare Brown-Henneaux central charge. The continuous spectrum of momenta matches geometries in the bulk space-time which have singularities cloaked by horizons.

With a finite number of effective degrees of freedom, we do not expect a phase transition. The partition function is a continuous function of the temperature (times the volume) τ_2 . In this model there is no Hawking-Page transition. In fact, the partition function is independent of the bare central charge, and there is no semi-classical limit, nor a summed approximation over saddles.

4.3 Black Hole Correlators

We described how Liouville quantum gravity only has black hole states dressed with boundary gravitons. Moreover, the theory enjoys the full conformal invariance of a two-dimensional conformal field theory. Beyond the known spectrum of conformal dimensions, a natural observable to compute is therefore the correlation function of three operators that create (primary) black holes states. The correlators of descendent black holes, namely black holes dressed with boundary gravitons, are generated from these by the symmetries of the theory. If we consider the correlation function of more than three black holes, those can be reconstructed by factorization.

The three-point function respects global conformal invariance and is therefore proportional to

$$\langle O_1 O_2 O_3 \rangle = \frac{C_{123}}{|z_1 - z_2|^{2h_1+2h_2-2h_3} |z_2 - z_3|^{2h_2+2h_3-2h_1} |z_3 - z_1|^{2h_3+2h_1-2h_2}}. \quad (4.15)$$

In Liouville quantum gravity, the proportionality constant C_{123} is given by the Liouville three-point function. The semi-classical three-point function is again determined by the regularized Liouville action, which we interpreted as a regularization of the bulk action in the presence of operator insertions. The $1/c = 2G_N/(3l)$ corrections arise from integrating over the Liouville field configurations, which we consider to be the microscopic degrees of freedom of the theory. We study these statements in more detail below.

4.3.1 The Semi-Classical Liouville Three-point Function

In this subsection, we remind the reader of the calculation of the semi-classical approximation to the Liouville three-point function [38]. The boundary conditions on the semi-classical Liouville field $\varphi = 2b\phi$ are

$$\begin{aligned} \varphi &= -2\eta_i \log |z - z_i|^2 + O(1) \quad \text{at } z \rightarrow z_i \\ \varphi &= -2 \log |z|^2 + O(1) \quad \text{at } z \rightarrow \infty, \end{aligned} \quad (4.16)$$

while the action is regularized as before. In the semiclassical limit where $b \rightarrow 0$, the three point function in the Liouville theory with operator insertions $\exp(\eta_i \varphi(z_i))$ is given by $\exp(-\frac{1}{b^2} S_{cl})$ where S_{cl} is the classical Liouville action evaluated on the classical solution to the Liouville equation. This classical solution is given by [38]

$$\varphi_{\eta_1, \eta_2, \eta_3}(z|z_1, z_2, z_3) = -2 \log [a_1 \psi_1(z) \psi_1(\bar{z}) + a_2 \psi_2(z) \psi_2(\bar{z})], \quad (4.17)$$

where

$$\psi_1 = (z - z_1)^{\eta_1} (z - z_2)^{1 - \eta_1 - \eta_3} (z - z_3)^{\eta_3} \times {}_2F_1(\eta_1 + \eta_3 - \eta_2, \eta_1 + \eta_2 + \eta_3 - 1, 2\eta_1, x) \quad (4.18)$$

$$\psi_2 = (z - z_1)^{1 - \eta_1} (z - z_2)^{\eta_1 + \eta_3 - 1} (z - z_3)^{1 - \eta_3} \times {}_2F_1(1 + \eta_2 - \eta_1 - \eta_3, 2 - \eta_1 - \eta_2 - \eta_3, 2 - 2\eta_1, x), \quad (4.19)$$

with the variable x defined as

$$x = \frac{(z - z_1)z_{32}}{(z - z_2)z_{31}}. \quad (4.20)$$

The constants a_1 and a_2 are

$$a_1^2 = \frac{\pi \mu b^2}{|z_{13}|^{4\eta_3 + 4\eta_1 - 2} |z_{12}|^{2 - 4\eta_2} |z_{23}|^{2 - 4\eta_1}} \times \frac{\gamma(\eta_1 + \eta_2 + \eta_3 - 1) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 - \eta_3)}{\gamma^2(2\eta_1) \gamma(\eta_2 + \eta_3 - \eta_1)} \quad (4.21)$$

$$a_2 = -\frac{\pi \mu b^2}{|z_{13}|^2 (1 - 2\eta_1)^2 a_1}, \quad (4.22)$$

where the special function γ is a ratio of Γ functions,

$$\gamma(x) = \frac{\Gamma(1 - x)}{\Gamma(x)}. \quad (4.23)$$

The conformal dimensions of the operators in the semiclassical limit are given by ($\alpha_i = \eta_i/b$)

$$h_i = \alpha_i(Q - \alpha_i) \approx \frac{\eta_i(1 - \eta_i)}{b^2}, \quad (4.24)$$

and the energy-momentum tensor can indeed be computed to be

$$T(z) = \sum_{i=1}^3 \frac{h_i}{(z - z_i)^2} - \left[\frac{h_1 + h_2 - h_3}{(z - z_1)(z - z_2)} + \text{cyclic permutations} \right]. \quad (4.25)$$

We verify this prediction of holomorphy explicitly in appendix C using the semi-classical solution (4.17).

In summary, through the relation between the Einstein-Hilbert action in the bulk and the Liouville theory on the boundary [21], reviewed and complemented in appendix B, and the regularization of the Liouville action (4.6) which we take to be valid for Liouville quantum gravity as well, we equate the semi-classical approximation to the quantum Liouville three-point function [38] with the semi-classical approximation to the gravity action evaluated on the geometry (2.1) determined by the energy-momentum tensor (4.25). In this manner, Liouville theory propels itself as a proposal for a quantization of this classical theory of gravity.

We recall that this proposal satisfies a non-trivial consistency check, in that three-point functions of states with opposite momentum both associated to a given space-time energy-momentum tensor and metric are related through a phase factor (essentially given by the reflection amplitude).

4.4 One Loop Correction

The semi-classical expansion of Liouville theory can be continued to higher orders. In particular, at one loop one can compute the exact quantum conformal dimension of the Liouville vertex operators [44]. The details are as follows. By regularizing the one-loop determinant around the classical three-point configuration in an $SL(2, \mathbb{C})$ covariant manner, and by analyzing the dependence of the resulting one-loop determinant on the insertion points of the three Liouville operators, one finds the $O(b^2)$ correction to the semi-classical conformal dimension of the operator insertions and reproduces the exact conformal dimension [44]:

$$h_i = \alpha_i(Q - \alpha_i) = \frac{1}{b^2} \eta_i (1 - \eta_i) + \eta_i. \quad (4.26)$$

It is proven explicitly in [44] that there is no further correction at two loop order, and, from our knowledge of Liouville conformal field theory, we know the property holds to all orders. Thus, the conformal dimension is one loop exact.

The holographic interpretation of the result is that integrating over gravitons gives a one loop correction to the mass, which is one loop exact in Liouville quantum gravity. Note that here, we again identify the bulk gravitational measure with the Liouville measure.¹²

Finally, we note the following phenomenon. When we compute the minimal (real) conformal dimension using the leading order formula, we find $1/(4b^2)$ which equals $c/24$ in the semi-classical limit $b \rightarrow 0$. When we use the exact formula (4.26), with the correction linear in the momentum η_i , then the exact minimal conformal dimension $(c - 1)/24$ is again recuperated. Since the latter value for the minimal conformal dimension is the origin of the shift in the minimal mass of a black hole, we again recognize this as a one loop quantum gravity effect.¹³

5 Conclusions

Liouville quantum gravity is a consistent and unitary theory of quantum gravity. It is a gravitational theory because it is the boundary incarnation of an Einstein-Hilbert action in three-dimensional anti-de Sitter space. It is quantum mechanically consistent because Liouville conformal field theory is. The theory is a peculiar example of quantum gravity. The Euclidean three-dimensional anti-de Sitter space is a solution to the classical equations of motion, but it does not correspond to a normalizable quantum state. The spectrum of normalizable states consists of primary states whose associated metric has hyperbolic (black hole) monodromies and whose descendants are BTZ geometries dressed with boundary gravitons. The black hole mass formula receives one loop quantum corrections. The entropy of the primary BTZ geometries is zero, while an arbitrary state with given high mass has a degeneracy determined by the effective central charge equal to one, associated to the boundary gravitons. There is a unique non-trivial three-point function between the states in the theory which is determined

¹²From section 2 and appendix B it is clear that this entails a choice of how to integrate over metrics determined by energy-momentum tensor components.

¹³While formula (3.10) suggests a point particle spectrum of width $1/c$ for conformal dimensions between $(c - 1)/24$ to $c/24$, the one loop correction due to integration over the Liouville, radial mode suggests the interpretation we provided here. This remains to be rigorously substantiated.

by the Liouville conformal field theory. This translates into a universal bulk gravitational interaction which is non-perturbatively well-defined.

Classically, we can interpret Liouville theory as an effective field theory. Like gravity, it comes about as a universal and crucial low-energy subsector of certain theories of quantum gravity in three dimensions. However, it is hard to interpret the quantum Liouville theory as a universal effective field theory. The quantum Liouville theory corrections will differ from quantum corrections in other microscopic theories. Indeed, we defined our quantum gravitational measure as coinciding with the boundary Liouville measure. This fixes the definition of the microscopic theory.

Our quantum theory of gravity is isolated in the sense that it is hard to add matter to the theory consistently with unitarity, and for sure, the interpretation of the quantum theory of gravity, if it existed, would drastically change in the presence of non-topological degrees of freedom. The property that the theory is isolated in theory space is reminiscent of other theories of quantum gravity, like string theory, in which adding even a single degree of freedom in space-time is an equally challenging enterprise. Our model also deviates from the standard view on quantum gravity, which requires the theory to account for the Bekenstein-Hawking entropy. Our proposal evades this requirement since the three-dimensional theory of gravity is topological and therefore lacks local degrees of freedom.

In summary, we believe we defined an interesting theory of three-dimensional quantum gravity and encourage its further exploration. One direction of further research is to advance the Lorentzian interpretation of our Liouville quantum gravity. It would also be very interesting to understand even better the bulk counterpart to the definition of the theory in terms of the Liouville measure. A relevant analysis is in [27], where the Liouville Hilbert space is obtained after quantization of the bulk degrees of freedom, and imposing a powerful projection condition. Understanding the relation between the projection procedure and the Liouville measure would be useful. Independently, the analysis of the bulk measure in [20] may also be useful in the comparison of the bulk and boundary measures. It is clear that the central idea of our paper can be transplanted to a host of other three-dimensional theories of gravity, with supersymmetry, higher spins, W-symmetry, different boundary conditions, et cetera. We believe we opened up an avenue to enlarge our perspective on theories of quantum gravity by bringing a known speculation to its rigorous conclusion – we hope this turns out to be useful.

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A Lorentzian Conformal Field Theory and Metrics

In this appendix, we discuss Lorentzian metrics that solve Einstein’s equation with negative cosmological constant. We obtain the metrics through analytic continuation from the Eu-

clidean metrics described in section 3. In particular, we again exploit the Banados form of the metric (2.1), now in Lorentzian signature. We determine Lorentzian energy-momentum tensor components through analytic continuation. Plugging those into the Lorentzian Banados metric, we obtain interesting solutions to Einstein's equations. We discuss properties of these solutions, but leave their full investigation for future work.

A.1 The Lorentzian Metrics

The Lorentzian Banados solution reads:

$$ds^2 = l^2 d\rho^2 + 4Gl(-T(dw^+)^2 - \bar{T}(dw^-)^2) + (l^2 e^{2\rho} + 16G^2 T\bar{T}e^{-2\rho})dw^+ dw^- \quad (\text{A.1})$$

where T and \bar{T} indicate real and independent components of the Lorentzian energy-momentum tensor, and w^\pm are real coordinates. The first example metric is when T and \bar{T} are constant. We then reproduce the Lorentzian BTZ black hole metric. A second example is given by plugging in the Lorentzian version of the planar energy-momentum tensor:

$$T_L^{(0,\infty)} = \frac{h}{(z^+)^2} \quad \bar{T}_L^{(0,\infty)} = \frac{h}{(z^-)^2}. \quad (\text{A.2})$$

We obtain a horizon:

$$e^{4\rho_{hor}} = (4G/l)^2 h^2 (z^+ z^-)^{-2}. \quad (\text{A.3})$$

The horizon radius goes to infinity on the lines $z^+ = 0$ and $z^- = 0$. Where the lines intersect, the singularity is the product of the individual quadratic singularities, and becomes quartic. The horizon therefore forms Lorentzian mountain ranges with extreme peaks where the ranges cross. The singularities are located where the horizon intersects the boundary. When we introduce the cut-off on the radius, a ribbon in the form of a cross is cut out from the horizon surface.

Generically, it is subtle to analytically continue a complex valued energy-momentum tensor T_{zz} to a real-valued energy-momentum tensor component T_{++} . However, we have readied the expressions for the energy-momentum tensor components in section 3 in order to render this task eminently feasible. Thus, the energy-momentum tensors $T_c^{(2)}$ and $T_c^{(3)}$ become real when we replace the coordinates (w, \bar{w}) by real coordinates (w^+, w^-) , and choose the source points w_i^\pm to be real. This gives rise to energy-momentum tensors

$$\begin{aligned} T_{c,L}^{(2)}(w^+) &= \frac{h}{4} \frac{\sin^2 \frac{w_1^+ - w_2^+}{2}}{\sin^2 \frac{w^+ - w_1^+}{2} \sin^2 \frac{w^+ - w_2^+}{2}} + \frac{c}{24}, \\ T_{c,L}^{(3)} &= h_1 \frac{\sin \frac{w_2^+ - w_1^+}{2}}{2 \sin \frac{w^+ - w_1^+}{2} \sin \frac{w^+ - w_2^+}{2}} \frac{\sin \frac{w_3^+ - w_1^+}{2}}{2 \sin \frac{w^+ - w_1^+}{2} \sin \frac{w^+ - w_3^+}{2}} + \text{cyclic} + \frac{c}{24}. \end{aligned} \quad (\text{A.4})$$

We offer a few generic observations:

- When one light-cone coordinate coincides between two insertions, e.g. $w_1^+ = w_2^+$, then the light-cone energy-momentum tensor component $T(w^+)$ does not depend on the conformal dimensions h_1 nor h_2 (but only on h_3), and only on the difference $w_2^+ - w_3^+ = w_1^+ - w_3^+$, and in precisely the way of a two-point energy-momentum tensor expectation value.

- When three insertions have the same light-cone coordinate, the energy-momentum tensor component is constant. This follows from translational invariance.

These facts are a consequence of the operator product expansion.

Naively, the metric associated to the energy-momentum tensor $T_{c,L}^{(3)}$ should correspond to three continuously interacting black hole states, but in trying to substantiate this claim, we encountered difficulties. To identify the precise physical interpretation of these interesting Lorentzian metrics requires considerably more work.

B The Action

This appendix reviews the derivation of the Liouville action from the bulk Einstein-Hilbert action, via the Chern-Simons action in the bulk and chiral Wess-Zumino-Witten models on the boundary [21]. A crucial extra step that we discuss is the gauging of an anti-diagonal symmetry that enforces the gluing of the left and right moving zero modes.

B.1 From Einstein-Hilbert to Wess-Zumino-Witten

The Einstein-Hilbert action in three dimensions with a negative cosmological constant $\Lambda = -l^{-2}$ can be written as the difference of two $sl(2, \mathbb{R})$ Chern-Simons actions [15, 16],

$$S_{EH}[A, \tilde{A}] = S_{CS}[A] - S_{CS}[\tilde{A}], \quad (\text{B.1})$$

where the connection one-forms A and \tilde{A} are related to the dreibein e_μ^a and the (dualized) spin connection ω_μ^a through the formulas $A_\mu^a = \omega_\mu^a + \frac{1}{l}e_\mu^a$ and $\tilde{A}_\mu^a = \omega_\mu^a - \frac{1}{l}e_\mu^a$. The Chern-Simons action can be written in polar t, r, θ components as

$$S_{CS}[A] = \frac{l}{16\pi G_N} \int_M \text{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \quad (\text{B.2})$$

$$= \frac{l}{16\pi G_N} \int_M dt dr d\theta \text{tr}(A_\theta \dot{A}_r - A_r \dot{A}_\theta + 2A_\theta F_{r\theta}). \quad (\text{B.3})$$

The Brown-Henneaux boundary conditions at large radius r on the fields A and \tilde{A} are [10]

$$A \sim \begin{pmatrix} \frac{dr}{2r} & O(\frac{1}{r}) \\ \frac{r}{l} dx^+ & -\frac{dr}{r} \end{pmatrix}, \quad \tilde{A} \sim \begin{pmatrix} -\frac{dr}{2r} & \frac{r}{l} dx^- \\ O(\frac{1}{r}) & \frac{dr}{r} \end{pmatrix}, \quad (\text{B.4})$$

where $x^\pm = t \pm \theta$. This implies that $A_- = \tilde{A}_+ = 0$ on the boundary. If one computes the variation of the action δS_{EH} when the equation of motion and this boundary condition are satisfied one obtains a surface term $\frac{l}{16\pi G_N} \delta \int_{\Sigma_2} dt d\theta \text{tr}(A_\theta^2 + \tilde{A}_\theta^2)$, where Σ_2 is the asymptotic surface at $r \rightarrow \infty$. In order to make the action and the boundary condition compatible, we add a surface term to the Einstein-Hilbert action. The action becomes

$$S[A, \tilde{A}] = S_{CS}[A] - \frac{l}{16\pi G_N} \int_{\Sigma_2} dt d\theta \text{tr}(A_\theta^2) - S_{CS}[\tilde{A}] - \frac{l}{16\pi G_N} \int_{\Sigma_2} dt d\theta \text{tr}(\tilde{A}_\theta^2). \quad (\text{B.5})$$

The field components A_0 and \tilde{A}_0 are Lagrange multipliers that implement the constraints $F_{r\theta} = \tilde{F}_{r\theta} = 0$. Solving these constraints one gets

$$\begin{aligned} A_i &= G_1^{-1} \partial_i G_1, \\ \tilde{A} &= G_2^{-1} \partial_i G_2, \end{aligned} \tag{B.6}$$

for $i = r, \theta$, where $G_{1,2}$ are elements of $SL(2, \mathbb{R})$.¹⁴ On the boundary, the fields $G_{1,2}$ asymptotically behave as

$$G_1 \sim g_1(t, \theta) \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}, \quad G_2 \sim g_2(t, \theta) \begin{pmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{pmatrix}, \tag{B.7}$$

such that the Brown-Henneaux boundary condition (B.4) is satisfied. Exploiting this parameterization, the action (B.5) can be written as the difference of two chiral WZW actions [21]

$$S[A, \tilde{A}] = \frac{l}{16\pi G_N} (S_{WZW}^R[g_1] - S_{WZW}^L[g_2]). \tag{B.8}$$

Given the relation between the Einstein-Hilbert action and Chern-Simons theory on the one hand, and Chern-Simons theory and chiral Wess-Zumino-Witten models on the other hand, this is as expected. The two chiral WZW actions are given by

$$\begin{aligned} S_{WZW}^R &= \int_{\Sigma_2} dt d\theta \operatorname{tr} [g_1^{-1} \partial_t g_1 g_1^{-1} \partial_\theta g_1 - (g_1^{-1} \partial_\theta g_1)^2] + \Gamma[g_1] \\ S_{WZW}^L &= \int_{\Sigma_2} dt d\theta \operatorname{tr} [g_2^{-1} \partial_t g_2 g_2^{-1} \partial_\theta g_2 - (g_2^{-1} \partial_\theta g_2)^2] + \Gamma[g_2], \end{aligned} \tag{B.9}$$

where $\Gamma[g]$ is the three-dimensional part of the WZW action. Note that the chiral WZW actions are invariant under the transformations

$$\begin{aligned} g_1 &\rightarrow h_1(t) g_1 f_1(x^+) \\ g_2 &\rightarrow h_2(t) g_2 f_2(x^-), \end{aligned} \tag{B.10}$$

where $h_{1,2}, f_{1,2} \in SL(2, \mathbb{R})$.

Combining two chiral WZW to a non-chiral WZW by gauging a symmetry

We gauge a subgroup of the above symmetry group. This subgroup consists of transformations

$$\begin{aligned} g_1 &\rightarrow h(t) g_1 \\ g_2 &\rightarrow h(t) g_2, \end{aligned} \tag{B.11}$$

and will guarantee that the left and right zero modes of g_1 and g_2 are glued together. Since these transformations are already symmetries of the action, gauging the symmetry comes

¹⁴This is the solution only when one does not consider holonomies. For the case with non-zero holonomies, we refer to the Appendix of [45].

down to restricting the integral over configuration space – we only integrate over the zero modes a single time. This can be implemented by defining the fields

$$g = g_1^{-1}g_2, \quad u = -g_2^{-1}\partial_\theta g_1 g_1^{-1}g_2 - g_2^{-1}\partial_\theta g_2, \quad (\text{B.12})$$

which are invariant under the gauged subgroup. In terms of these fields, we find that the action can be written as

$$S[g, u] = \frac{l}{16\pi G_N} \left[\int_{\Sigma_2} dt d\theta \operatorname{tr} \left(u g^{-1} \partial_t g - \frac{1}{2} u^2 - \frac{1}{2} (g^{-1} \partial_\theta g)^2 \right) - \Gamma(g) \right]. \quad (\text{B.13})$$

The measures are related by

$$Dg Du = \frac{1}{V} Dg_1 Dg_2, \quad (\text{B.14})$$

where V is volume of the set $\{g(t) \in SL(2, \mathbb{R})\}$. Performing the functional integral over the field u , we get the non-chiral WZW action

$$S[g] = \frac{l}{16\pi G_N} \left[\int_{\Sigma_2} dt d\theta \operatorname{tr} (2g^{-1} \partial_+ g g^{-1} \partial_- g) - \Gamma(g) \right], \quad (\text{B.15})$$

where $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_\theta)$. Crucially, we have eliminated a zero mode.

B.2 From Wess-Zumino-Witten to Liouville

The next step amounts to Drinfeld-Sokolov or Hamiltonian reduction, induced by the boundary conditions. Firstly, we apply Gauss decomposition to write the remaining field g as

$$g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{1}{2}\varphi) & 0 \\ 0 & \exp(-\frac{1}{2}\varphi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}. \quad (\text{B.16})$$

The action will then become

$$S[g] = \frac{l}{8\pi G_N} \int dt d\theta \left[\frac{1}{2} \partial_+ \varphi \partial_- \varphi + 2 \partial_- X \partial_+ Y \exp(-\varphi) \right]. \quad (\text{B.17})$$

Consider now the action as defined on a cylinder $[t_1, t_2] \times S^1$ of finite height [21]. The boundary conditions (B.4) will impose the following constraints:

$$\partial_- X \exp(-\varphi) = 1, \quad \partial_+ Y \exp(-\varphi) = -1. \quad (\text{B.18})$$

In order to make the variation of the action zero when the equation of motion is satisfied, and in the presence of such constraint on the boundary, one improves the action by adding a term. The action then reads

$$S_{\text{improved}}[g] = S[g] - \frac{l}{8\pi G_N} \oint d\theta (X \partial_+ Y + Y \partial_- X) \exp(-\varphi) \Big|_{t_1}^{t_2}. \quad (\text{B.19})$$

Plugging in the constraints, the improved action becomes a Liouville action

$$S[\varphi] = \frac{l}{8\pi G_N} \int dt d\theta \left[\frac{1}{2} \partial_+ \varphi \partial_- \varphi + 2 \exp(\varphi) \right]. \quad (\text{B.20})$$

Alternatively, one can determine the final action as follows. One derives the equation of motion for the field φ from the action (B.17), and finds

$$\partial_+\partial_-\varphi = -2\partial_-X\partial_+Y \exp(-\varphi). \quad (\text{B.21})$$

Eliminating the fields X and Y using the constraint (B.18), one obtains

$$\partial_+\partial_-\varphi = 2 \exp(\varphi), \quad (\text{B.22})$$

which is the same as the equation of motion one gets starting from (B.20). Therefore the boundary action is the Liouville action (B.20).

C The Three-point Energy-momentum Tensor

In this appendix, we demonstrate explicitly that the semiclassical limit of the energy-momentum tensor in the presence of three insertions is given by formula (4.25), as it must be. The energy-momentum tensor of Liouville theory in the semiclassical limit is [38]

$$\begin{aligned} T(z) &= -(\partial\phi)^2 + Q\partial^2\phi \\ &= \frac{1}{2b^2}[-\frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi], \end{aligned} \quad (\text{C.1})$$

where we have used that $\varphi = 2b\phi$ and that $Q = \frac{1}{b} + b \rightarrow \frac{1}{b}$ in the semiclassical limit $b \rightarrow 0$. Plugging in the classical solution (4.17), the energy-momentum tensor can be written as

$$T(z) = -\frac{a_1\psi_1''(z)\psi_1(\bar{z}) + a_2\psi_2(z)''\psi_2(\bar{z})}{b^2(a_1\psi_1(z)\psi_1(\bar{z}) + a_2\psi_2(z)\psi_2(\bar{z}))}. \quad (\text{C.2})$$

We will show that

$$\frac{\psi_1''}{\psi_1} = \frac{\psi_2''}{\psi_2}, \quad (\text{C.3})$$

and therefore

$$T(z) = -\frac{\psi_1''}{b^2\psi_1}. \quad (\text{C.4})$$

Using the explicit form of the field ψ_1 given in (4.18), we find

$$\begin{aligned} \frac{\psi_1''}{\psi_1} &= \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{(1 - \eta_1 - \eta_3)(-\eta_1 - \eta_3)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\ &+ \frac{2\eta_1(1 - \eta_1 - \eta_3)}{(z - z_1)(z - z_2)} + \frac{2\eta_1\eta_3}{(z - z_1)(z - z_3)} + \frac{2(1 - \eta_1 - \eta_3)\eta_3}{(z - z_2)(z - z_3)} \\ &+ 2\left[\left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3}\right)\frac{\partial x}{\partial z} + \frac{\partial^2 x}{\partial z^2}\right] \frac{{}_2F_1'}{{}_2F_1} \\ &+ \left(\frac{\partial x}{\partial z}\right)^2 \frac{{}_2F_1''}{{}_2F_1}, \end{aligned} \quad (\text{C.5})$$

where the symbol ${}_2F_1$ stands for the hypergeometric function ${}_2F_1$ with arguments ${}_2F_1(\eta_1 + \eta_3 - \eta_2, \eta_1 + \eta_2 + \eta_3 - 1, 2\eta_1, x)$. Using the differential equation the hypergeometric function

satisfies, we can express the second order derivative of the hypergeometric function ${}_2F_1$ in terms of its first order derivative and itself, which yields

$$\begin{aligned}
\frac{\psi_1''}{\psi_1} &= \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{(1 - \eta_1 - \eta_3)(-\eta_1 - \eta_3)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\
&+ \frac{2\eta_1(1 - \eta_1 - \eta_3)}{(z - z_1)(z - z_2)} + \frac{2\eta_1\eta_3}{(z - z_1)(z - z_3)} + \frac{2(1 - \eta_1 - \eta_3)\eta_3}{(z - z_2)(z - z_3)} \\
&+ 2\left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3}\right) \frac{\partial x}{\partial z} \frac{{}_2F_1'}{{}_2F_1} \\
&+ \left[\frac{\partial^2 x}{\partial z^2} + \left(\frac{\partial x}{\partial z}\right)^2 \left(\frac{2(\eta_1 + \eta_3)}{1 - x} - \frac{2\eta_1}{x(1 - x)}\right)\right] \frac{{}_2F_1'}{{}_2F_1} \\
&+ \left(\frac{\partial x}{\partial z}\right)^2 \frac{(\eta_1 + \eta_3 - \eta_2)(\eta_1 + \eta_2 + \eta_3 - 1)}{x(1 - x)}. \tag{C.6}
\end{aligned}$$

Applying the definition of the quantity x given in (4.20), we obtain:

$$\frac{\partial x}{\partial z} = \frac{z_{12}z_{32}}{(z - z_2)^2 z_{31}}, \tag{C.7}$$

$$\frac{\partial^2 x}{\partial z^2} = -\frac{2z_{12}z_{32}}{(z - z_2)^3 z_{31}}, \tag{C.8}$$

$$1 - x = \frac{(z - z_3)z_{21}}{(z - z_2)z_{31}}, \tag{C.9}$$

$$x(1 - x) = \frac{(z - z_1)(z - z_3)z_{21}z_{32}}{(z - z_2)^2 z_{31}^2}. \tag{C.10}$$

We therefore find

$$\begin{aligned}
&\frac{\partial^2 x}{\partial z^2} + \left(\frac{\partial x}{\partial z}\right)^2 \left(\frac{2(\eta_1 + \eta_3)}{1 - x} - \frac{2\eta_1}{x(1 - x)}\right) \\
&= -2 \frac{\partial x}{\partial z} \left[\frac{1}{z - z_2} + \frac{(\eta_1 + \eta_3)z_{32}}{(z - z_2)(z - z_3)} - \frac{\eta_1 z_{31}}{(z - z_1)(z - z_3)} \right] \\
&= -2 \frac{\partial x}{\partial z} \left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3} \right), \tag{C.11}
\end{aligned}$$

and thus the total coefficient in front of $\frac{{}_2F_1'}{{}_2F_1}$ on the right hand side of (C.6) is zero. One can also simplify the remaining terms and get

$$\begin{aligned}
\frac{\psi_1''}{\psi_1} &= \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{\eta_2(\eta_2 - 1)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\
&- \left[\frac{\eta_1(\eta_1 - 1) + \eta_2(\eta_2 - 1) - \eta_3(\eta_3 - 1)}{(z - z_1)(z - z_2)} + \text{cyclic permutations} \right]. \tag{C.12}
\end{aligned}$$

Mutadis mutandis, equation (C.3) follows. The final result for the energy-momentum tensor then agrees with equation (4.25).

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