



**HAL**  
open science

# Hardy Fields, the Intermediate Value Property, and $\omega$ -freeness

Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven

► **To cite this version:**

Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. Hardy Fields, the Intermediate Value Property, and  $\omega$ -freeness. 2019. hal-02088484

**HAL Id: hal-02088484**

**<https://hal.science/hal-02088484v1>**

Preprint submitted on 2 Apr 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# HARDY FIELDS, THE INTERMEDIATE VALUE PROPERTY, AND $\omega$ -FREEDOM

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

ABSTRACT. We discuss the conjecture that every maximal Hardy field has the Intermediate Value Property for differential polynomials, and its equivalence to the statement that all maximal Hardy fields are elementarily equivalent to the differential field of transseries. As a modest but essential step towards establishing the conjecture we show that every maximal Hardy field is  $\omega$ -free.

## INTRODUCTION

Du Bois Reymond's "orders of infinity" were put on a firm basis by Hardy [8] and Hausdorff [9], leading to the notion of a Hardy field (Bourbaki [6]). A Hardy field is a field  $H$  of germs at  $+\infty$  of differentiable real-valued functions on intervals  $(a, +\infty)$  such that the germ of the derivative of any differentiable function whose germ is in  $H$  is also in  $H$ . (See Section 2 for more precision.) A Hardy field is naturally a differential field, and is an ordered field with the germ of  $f$  being  $> 0$  if and only if  $f(t) > 0$ , eventually.

If  $H$  is a Hardy field, then so is  $H(\mathbb{R})$  (obtained by adjoining the germs of the constant functions) and for any  $h \in H$ , the germ  $e^h$  generates a Hardy field  $H(e^h)$  over  $H$ , and so does any differentiable germ with derivative  $h$  [13]. Each Hardy field  $H$  has a unique Hardy field extension that is algebraic over  $H$  and real closed [12]. The ultimate extension result of this kind would be the following:

**Conjecture.** *Let  $H$  be a Hardy field,  $P(Y) \in H\{Y\}$  a differential polynomial and  $f < g$  in  $H$  such that  $P(f) < 0 < P(g)$ . Then there is an element  $\phi$  in a Hardy field extension of  $H$  such that  $f < \phi < g$  and  $P(\phi) = 0$ .*

In [7] this is proved for  $P$  of order 1. In [11] it is shown that there do exist Hardy fields with the intermediate value property for all differential polynomials. Every Hardy field extends to a maximal Hardy field, by Zorn, and so the Conjecture above is equivalent to maximal Hardy fields having the intermediate value property for differential polynomials. By the results mentioned earlier, maximal Hardy fields contain  $\mathbb{R}$  as a subfield and are Liouville closed in the sense of [2]. At the end of Section 1 we show that for Liouville closed Hardy fields containing  $\mathbb{R}$  the intermediate value property is equivalent to the conjunction of two other properties,  $\omega$ -freeness and newtonianity. These two notions are central in [2] in a more general setting. Roughly speaking,  $\omega$ -freeness controls the solvability of second-order homogeneous linear differential equations in suitable extensions, and newtonianity is a very strong version of differential-henselianity. (We did not consider the intermediate value property in [2] and mention it here mainly for expository reasons: it

---

*Date:* March 2019.

The first-named author was partially supported by NSF Grant DMS-1700439.

is easier to grasp than the more subtle and more fundamental notions of  $\omega$ -freeness and newtonianity.)

The main result in these seminar notes is that any Hardy field has an  $\omega$ -free Hardy field extension: Theorem 3.1. We do not present it here, but we also have a detailed outline for showing that any  $\omega$ -free Hardy field extends to a newtonian  $\omega$ -free Hardy field. At this stage (March 2019) that proof is not yet finished. If we finish the proof, it would follow that every maximal Hardy field is an  $\omega$ -free newtonian Liouville closed  $H$ -field with small derivation, in the terminology of [2]. Now the elementary theory  $T_{\text{small}}^{\text{nl}}$  of  $\omega$ -free newtonian Liouville closed  $H$ -fields with small derivation is *complete*, by [2, Corollary 16.6.3]. Thus finishing the proof alluded to would give that any two maximal Hardy fields are indistinguishable as to their elementary properties, that is, any two maximal Hardy fields would be elementarily equivalent as ordered differential fields.

The present seminar notes prove some results announced in our exposition [3]. There we also discuss another fundamental conjecture and partial results towards it, namely that the underlying ordered set of any maximal Hardy field is  $\eta_1$ . Our plan for proving it does depend on first establishing the conjecture that maximal Hardy fields are newtonian. The two conjectures together imply: all maximal Hardy fields are isomorphic as ordered differential fields, assuming the continuum hypothesis (CH); for more on this, see [3].

Let us add here a remark about maximal Hardy fields that is more set-theoretic in nature. Every Hardy field is contained in the ring  $\mathcal{C}$  of germs at  $+\infty$  of continuous real-valued functions on half-lines  $(a, +\infty)$ , and so there *at most*  $2^{\mathfrak{c}}$  many Hardy fields, where  $\mathfrak{c} = 2^{\aleph_0}$  is the cardinality of the continuum; note that  $\mathfrak{c}$  is also the cardinality of  $\mathcal{C}$ . It is worth mentioning that the two conjectures above imply that there are in fact  $2^{\mathfrak{c}}$  many different maximal Hardy fields: In an email to one of the authors, Ilijas Farah showed that there are  $2^{\mathfrak{c}}$  many maximal Hausdorff fields, Hausdorff fields being the subfields of the ring  $\mathcal{C}$  (without differentiability assumptions as in the case of Hardy fields). Farah's proof can easily be modified to give the same conclusion about the number of maximal Hardy fields assuming these conjectures.

Throughout we use the algebraic and valuation-theoretic tools from [2]. We need in addition analytic facts about real and complex solutions of linear differential equations; these facts and various generalities about Hardy fields are in Section 2.

The second-named author gave a talk about the above material in the *Séminaire de structures algébriques ordonnées* in honor of Paulo Ribenboim's 90th birthday. We dedicate this paper to Paulo in gratitude for his fundamental contributions to the theory of valuations, which is indispensable in our work.

**Notations and terminology.** Throughout,  $m, n$  range over  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Given an additively written abelian group  $A$  we let  $A^\neq := A \setminus \{0\}$ . Rings are commutative with identity 1, and for a ring  $R$  we let  $R^\times$  be the multiplicative group of units (consisting of the  $a \in R$  such that  $ab = 1$  for some  $b \in R$ ). A *differential ring* will be a ring  $R$  containing (an isomorphic copy of)  $\mathbb{Q}$  as a subring and equipped with a derivation  $\partial: R \rightarrow R$ ; note that then  $C_R := \{a \in R : \partial(a) = 0\}$  is a subring of  $R$ , called the ring of constants of  $R$ , and that  $\mathbb{Q} \subseteq C_R$ . If  $R$  is a

field, then so is  $C_R$ . An *ordered differential field* is an ordered field equipped with a derivation; such an ordered differential field is in particular a differential ring.

Let  $R$  be a differential ring and  $a \in R$ . When its derivation  $\partial$  is clear from the context we denote  $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$  by  $a', a'', \dots, a^{(n)}, \dots$ , and if  $a \in R^\times$ , then  $a^\dagger$  denotes  $a'/a$ , so  $(ab)^\dagger = a^\dagger + b^\dagger$  for  $a, b \in R^\times$ . In Sections 1 and 3 we need to consider the function  $\omega = \omega_R: R \rightarrow R$  given by  $\omega(z) = -2z' - z^2$ , and the function  $\sigma = \sigma_R: R^\times \rightarrow R$  given by  $\sigma(y) = \omega(z) + y^2$  for  $z := -y^\dagger$ .

We have the differential ring  $R\{Y\} = R[Y, Y', Y'', \dots]$  of differential polynomials in an indeterminate  $Y$ . We say that  $P = P(Y) \in R\{Y\}$  has order at most  $r \in \mathbb{N}$  if  $P \in R[Y, Y', \dots, Y^{(r)}]$ ; in this case  $P = \sum_{\mathbf{i}} P_{\mathbf{i}} Y^{\mathbf{i}}$ , as in [2, Section 4.2], with  $\mathbf{i}$  ranging over tuples  $(i_0, \dots, i_r) \in \mathbb{N}^{1+r}$ ,  $Y^{\mathbf{i}} := Y^{i_0} (Y')^{i_1} \dots (Y^{(r)})^{i_r}$ , coefficients  $P_{\mathbf{i}}$  in  $R$ , and  $P_{\mathbf{i}} \neq 0$  for only finitely many  $\mathbf{i}$ . For  $P \in R\{Y\}$  and  $a \in R$  we let  $P_{\times a}(Y) := P(aY)$ . For  $\phi \in R^\times$  we let  $R^\phi$  be the *compositional conjugate of  $R$  by  $\phi$* : the differential ring with the same underlying ring as  $R$  but with derivation  $\phi^{-1}\partial$  instead of  $\partial$ . We have an  $R$ -algebra isomorphism  $P \mapsto P^\phi: R\{Y\} \rightarrow R^\phi\{Y\}$  such that  $P^\phi(y) = P(y)$  for all  $y \in R$ ; see [2, Section 5.7].

For a field  $K$  we have  $K^\times = K^\neq$ , and a (Krull) valuation on  $K$  is a surjective map  $v: K^\times \rightarrow \Gamma$  onto an ordered abelian group  $\Gamma$  (additively written) satisfying the usual laws, and extended to  $v: K \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$  by  $v(0) = \infty$ , where the ordering on  $\Gamma$  is extended to a total ordering on  $\Gamma_\infty$  by  $\gamma < \infty$  for all  $\gamma \in \Gamma$ . A *valued field*  $K$  is a field (also denoted by  $K$ ) together with a valuation ring  $\mathcal{O}$  of that field, and the corresponding valuation  $v: K^\times \rightarrow \Gamma$  on the underlying field is such that  $\mathcal{O} = \{a \in K : va \geq 0\}$  as explained in [2, Section 3.1].

Let  $K$  be a valued field with valuation ring  $\mathcal{O}_K$  and valuation  $v: K^\times \rightarrow \Gamma_K$ . Then  $\mathcal{O}_K$  is a local ring with maximal ideal  $\mathfrak{o}_K = \{a \in K : va > 0\}$  and residue field  $\text{res}(K) = \mathcal{O}_K/\mathfrak{o}_K$ . If  $\text{res}(K)$  has characteristic zero, then  $K$  is said to be of equicharacteristic zero. When the ambient valued field  $K$  is clear from the context, then we denote  $\Gamma_K, \mathcal{O}_K, \mathfrak{o}_K$ , by  $\Gamma, \mathcal{O}, \mathfrak{o}$ , respectively, and for  $a, b \in K$  we set

$$\begin{aligned} a \asymp b &:\Leftrightarrow va = vb, & a \preceq b &:\Leftrightarrow va \geq vb, & a \prec b &:\Leftrightarrow va > vb, \\ a \succcurlyeq b &:\Leftrightarrow b \preceq a, & a \succ b &:\Leftrightarrow b \prec a, & a \sim b &:\Leftrightarrow a - b \prec a. \end{aligned}$$

It is easy to check that if  $a \sim b$ , then  $a, b \neq 0$ , and that  $\sim$  is an equivalence relation on  $K^\times$ . We use *pc-sequence* to abbreviate *pseudocauchy sequence*, and  $a_\rho \rightsquigarrow a$  indicates that the pc-sequence  $(a_\rho)$  pseudoconverges to  $a$ ; see [2, Sections 2.2, 3.2]. As in [2], a *valued differential field* is a valued field  $K$  of equicharacteristic zero that is also equipped with a derivation  $\partial: K \rightarrow K$ , and an *ordered valued differential field* is a valued differential field  $K$  equipped with an ordering on  $K$  making  $K$  an ordered field.

## 1. $H$ -FIELDS AND IVP

We recall from [2, Introduction] that an  *$H$ -field* is an ordered differential field  $K$  with constant field  $C$  such that:

- (H1)  $\partial(a) > 0$  for all  $a \in K$  with  $a > C$ ;
- (H2)  $\mathcal{O} = C + \mathfrak{o}$ , where  $\mathcal{O}$  is the convex hull of  $C$  in the ordered field  $K$ , and  $\mathfrak{o}$  is the maximal ideal of the valuation ring  $\mathcal{O}$ .

Let  $K$  be an  $H$ -field, and let  $\mathcal{O}$  and  $\mathfrak{o}$  be as in (H2). Thus  $K$  is a valued field with valuation ring  $\mathcal{O}$ . The residue morphism  $\mathcal{O} \rightarrow \text{res}(K) = \mathcal{O}/\mathfrak{o}$  restricts to an

isomorphism  $C \xrightarrow{\cong} \text{res}(K)$ . The valuation topology on  $K$  equals its order topology if  $C \neq K$ . We consider  $K$  as an  $\mathcal{L}$ -structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, <, \preceq\}$$

is the language of ordered valued differential fields. The symbols  $0, 1, +, -, \times, \partial, <$  are interpreted as usual in  $K$ , and  $\preceq$  encodes the valuation: for  $a, b \in K$ ,

$$a \preceq b \iff a \in \mathcal{O}b.$$

An  $H$ -field  $K$  is said to be *Liouville closed* if it is real closed and for all  $a \in K$  there exists  $b \in K$  with  $a = b'$  and also  $b \in K^\times$  with  $a = b^\dagger$ .

**Remarks on IVP.** Ordered valued differential subfields of  $H$ -fields are called pre- $H$ -fields, and are characterized in [2, Section 10.5]. Below we assume some familiarity with the  $H$ -asymptotic couple  $(\Gamma, \psi)$  of a pre- $H$ -field  $K$ , as explained in [2], and properties of  $K$  based on those of  $(\Gamma, \psi)$ , such as  $K$  having *asymptotic integration* and  $K$  having a *gap* [2, Sections 9.1, 9.2].

Let  $K$  be a pre- $H$ -field. We say that  $K$  has *IVP* (the *Intermediate Value Property*) if for all  $P(Y) \in K\{Y\}$  and  $f < g$  in  $K$  with  $P(f) < 0 < P(g)$  there is a  $\phi \in K$  such that  $f < \phi < g$  and  $P(\phi) = 0$ . Restricting this to  $P$  of order  $\leq r$ , where  $r \in \mathbb{N}$ , gives the notion of  $r$ -IVP. Thus  $K$  having 0-IVP is equivalent to  $K$  being real closed as an ordered field. In particular, if  $K$  has 0-IVP, then the  $H$ -asymptotic couple  $(\Gamma, \psi)$  of  $K$  is divisible. From [2, Section 2.4] recall our convention that  $K^> = \{a \in K : a > 0\}$ , and similarly with  $<$  replacing  $>$ .

**Lemma 1.1.** *Suppose  $\Gamma \neq \{0\}$  and  $K$  has 1-IVP. Then  $\partial K = K$ ,  $(K^>)^\dagger = (K^<)^\dagger$  is a convex subgroup of  $K$ ,  $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^\neq\}$  has no largest element, and  $\Psi$  is convex in  $\Gamma$ .*

*Proof.* We have  $y' = 0$  for  $y = 0$ , and  $y'$  takes arbitrarily large positive values in  $K$  as  $y$  ranges over  $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$ , since by [2, Lemma 9.2.6] the set  $(\Gamma^<)'$  is coinitial in  $\Gamma$ . Hence  $y'$  takes all positive values on  $K^>$ , and therefore also all negative values on  $K^<$ . Thus  $\partial K = K$ . Next, let  $a, b \in K^>$ , and suppose  $s \in K$  lies strictly between  $a^\dagger$  and  $b^\dagger$ . Then  $s = y^\dagger$  for some  $y \in K^>$  strictly between  $a$  and  $b$ ; this follows by noting that for  $y = a$  and  $y = b$  the signs of  $sy - y'$  are opposite.

Let  $\beta \in \Psi$  and take  $a \in K$  with  $v(a') = \beta$ . Then  $a \succ 1$ , since  $a \preceq 1$  would give  $v(a') > \Psi$ . Hence for  $\alpha = va < 0$  we have  $\alpha + \alpha^\dagger = \beta$ , so  $\alpha^\dagger > \beta$ . Thus  $\Psi$  has no largest element. Therefore the set  $\Psi$  is convex in  $\Gamma$ .  $\square$

Thus the ordered differential field  $\mathbb{T}_{\log}$  of logarithmic transseries, [2, Appendix A], does not have 1-IVP, although it is a newtonian  $\mathfrak{o}$ -free  $H$ -field.

Does IVP imply that  $K$  is an  $H$ -field? No: take an  $\aleph_0$ -saturated elementary extension of  $\mathbb{T}$  and let  $\Delta$  be as in [2, Example 10.1.7]. Then the  $\Delta$ -coarsening of  $K$  is a pre- $H$ -field with IVP and nontrivial value group, and has a gap, but it is not an  $H$ -field. On the other hand:

**Lemma 1.2.** *Suppose  $K$  has 1-IVP and has no gap. Then  $K$  is an  $H$ -field.*

*Proof.* In [2, Section 11.8] we defined

$$I(K) := \{y \in K : y \preceq f' \text{ for some } f \in \mathcal{O}\}.$$

Since  $K$  has no gap, we have

$$\partial \mathcal{O} \subseteq I(K) = \{y \in K : y \preceq f' \text{ for some } f \in \mathcal{O}\}.$$

Also  $\Gamma \neq \{0\}$ , and so  $(\Gamma, \psi)$  has asymptotic integration by Lemma 1.1. We show that  $K$  is an  $H$ -field by proving  $I(K) = \partial\mathcal{O}$ , so let  $g \in I(K)$ ,  $g < 0$ . Since  $(\Gamma^>)'$  has no least element we can take positive  $f \in \mathcal{O}$  such that  $f' > g$ . Since  $f' < 0$ , this gives  $f' < g$ . Since  $(\Gamma^>)'$  is cofinal in  $\Gamma$  we can also take positive  $h \in \mathcal{O}$  such that  $h' < g$ , which in view of  $h' < 0$  gives  $g < h'$ . Thus  $f' < g < h'$ , and so 1-IVP yields  $a \in \mathcal{O}$  with  $g = a'$ .  $\square$

We refer to Sections 11.6 and 14.2 of [2] for the definitions of  $\lambda$ -freeness and  $r$ -newtonianity ( $r \in \mathbb{N}$ ). From the introduction we recall that  $\omega(z) := -2z' - z^2$ .

**Lemma 1.3.** *Suppose  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ , and  $K$  has 1-IVP. Then  $K$  is  $\lambda$ -free and 1-newtonian, and the subset  $\omega(K)$  of  $K$  is downward closed.*

*Proof.* First we note that  $K$  has (asymptotic) integration, by Lemma 1.1. Assume towards a contradiction that  $K$  is not  $\lambda$ -free. We can arrange that  $K$  has small derivation, and thus  $K$  has an element  $x > 1$  with  $x' = 1$ , and so  $x > C$ . This leads to a pc-sequence  $(\lambda_\rho)$  and an element  $s \in K$  such that  $\lambda_\rho \rightsquigarrow -s$  with  $\lambda_\rho \sim x^{-1}$  for all  $\rho$ . Hence  $s \sim -x^{-1}$ , and  $s$  creates a gap over  $K$  by [2, Lemma 11.5.14]. Now note that for  $P := Y' + sY$  we have  $P(0) = 0$  and  $P(x^2) = 2x + sx^2 \sim x$ , so by 1-IVP we have  $P(y) = 1$  for some  $y \in K$ , contradicting [2, Lemma 11.5.12].

Let  $P \in K\{Y\}$  of order at most 1 have Newton degree 1; we have to show that  $P$  has a zero in  $\mathcal{O}$ . We know that  $K$  is  $\lambda$ -free, so by [2, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of  $K^\times$  to arrange that  $K$  has small derivation and  $P = D + R$  where  $D = cY + d$  or  $D = cY'$  with  $c, d \in C$ ,  $c \neq 0$ , and where  $R \prec^b 1$ . Then  $R(a) \prec^b 1$  for all  $a \in \mathcal{O}$ . If  $D = cY + d$ , then we can take  $a, b \in C$  with  $D(a) < 0$  and  $D(b) > 0$ , which in view of  $R(a) \prec D(a)$  and  $R(b) \prec D(b)$  gives  $P(a) < 0$  and  $P(b) > 0$ , and so  $P$  has a zero strictly between  $a$  and  $b$ , and thus a zero in  $\mathcal{O}$ . Next, suppose  $D = cY'$ . Then we take  $t \in \mathcal{O}^\neq$  with  $v(t^\dagger) = v(t)$ , that is,  $t' \asymp t^2$ , so

$$P(t) = ct' + R(t), \quad P(-t) = -ct' + R(-t), \quad R(t), R(-t) \prec t'.$$

Hence  $P(t)$  and  $P(-t)$  have opposite signs, so  $P$  has a zero strictly between  $t$  and  $-t$ , and thus  $P$  has a zero in  $\mathcal{O}$ .

From  $\omega(z) = -z^2 - 2z'$  we see that  $\omega(z) \rightarrow -\infty$  as  $z \rightarrow +\infty$  and as  $z \rightarrow -\infty$  in  $K$ , so  $\omega(K)$  is downward closed by 1-IVP.  $\square$

For results involving 2-IVP we need a minor variant of [2, Lemma 11.8.31]. Here  $\Gamma(K) = \{a^\dagger : a \in K \setminus \mathcal{O}\}$  as in [2, Section 11.8], and the superscripts  $\uparrow, \downarrow$  indicate upward, respectively downward, closure, as in [2, Section 2.1].

**Lemma 1.4.** *Let  $K$  be an  $H$ -field with asymptotic integration. Then*

$$K^{>} = I(K)^> \cup \Gamma(K)^\uparrow, \quad \sigma(K^{>} \setminus \Gamma(K)^\uparrow) \subseteq \omega(K)^\downarrow.$$

*Proof.* If  $a \in K$ ,  $a > I(K)$ , then  $a \geq b^\dagger$  for some  $b \in K^{>1}$ , and thus  $a \in \Gamma(K)^\uparrow$ . The inclusion involving  $\sigma$  now follows as in the proof of [2, Lemma 11.8.13].  $\square$

The concept of  $\omega$ -freeness is introduced in [2, Section 11.7].

**Lemma 1.5.** *Suppose  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ , and  $K$  has 2-IVP. Then  $K$  is 2-newtonian, the operator  $\partial^2 - a$  splits over  $K[i]$  for all  $a \in K$ , and  $K$  is  $\omega$ -free.*

*Proof.* Let  $P \in K\{Y\}$  of order at most 2 have Newton degree 1; we have to show that  $P$  has a zero in  $\mathcal{O}$ . Lemma 1.3 tells us that  $K$  is  $\lambda$ -free, and in view of [2, Corollary 13.3.7] and 2-IVP this allows us to repeat the argument in the proof of that lemma for differential polynomials of order at most 1 so that it applies to our  $P$  of order at most 2. Thus  $K$  is 2-newtonian.

By [2, Section 5.2] it remains to show that  $K = \omega(K) \cup \sigma(K^\times)$ . In view of Lemma 1.1 we can arrange by compositional conjugation that  $a^\dagger = -1$  for some  $a \prec 1$  in  $K^>$ . Below we fix such  $a$ . Let  $f \in K$ ; our job is to show that  $f \in \omega(K) \cup \sigma(K^\times)$ . Since  $\omega(0) = 0$ , we do have  $f \in \omega(K)$  if  $f \leq 0$ , by Lemma 1.3. So assume  $f > 0$ ; we show that then  $f \in \sigma(K^>)$ . Now for  $y \in K^>$ ,  $f = \sigma(y)$  is equivalent (by multiplying with  $y^2$ ) to  $P(y) = 0$ , where

$$P(Y) := 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [2, Section 13.7]. We have  $P(0) = 0$  and  $P(y) \rightarrow +\infty$  as  $y \rightarrow +\infty$  (because of the term  $y^4$ ). In view of 2-IVP it will suffice to show that for some  $y > 0$  in  $K$  we have  $P(y) < 0$ . Now with  $y \in K^>$  and  $z := -y^\dagger$  we have

$$P(y) = y^2(\sigma(y) - f) = y^2(\omega(z) + y^2 - f),$$

hence

$$P(a) = a^2(\omega(1) + a^2 - f) = a^2(-1 + a^2 - f) < 0.$$

As to  $\omega$ -freeness, this now follows from Lemma 1.4 and [2, Corollary 11.8.30].  $\square$

It follows that Liouville closed  $H$ -fields having 2-IVP are Schwarz closed as defined in [2, Section 11.8]. (There exist  $H$ -fields with a non-trivial derivation that have IVP but are not Liouville closed; see [1, Section 14].)

**Corollary 1.6.** *Suppose  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ , and  $K$  has IVP. Then  $K$  is  $\omega$ -free and newtonian.*

*Proof.* Showing that every  $P \in K\{Y\}$  of Newton degree 1 has a zero in  $\mathcal{O}$  is done just as in the proof of Lemma 1.3.  $\square$

**Corollary 1.7.** *Let  $K$  be a Liouville closed  $H$ -field. Then*

$$K \text{ has IVP} \iff K \text{ is } \omega\text{-free and newtonian.}$$

*Proof.* The forward direction is part of Corollary 1.6. For the backward direction we appeal to the main results from the book [10] to the effect that  $\mathbb{T}_{\mathfrak{g}}$ , the ordered differential field of grid-based transseries (cf. [2, Appendix A]), is a newtonian Liouville closed  $H$ -field with small derivation, and has IVP. In particular, it is a model of the theory  $T_{\text{small}}^{\text{nl}}$ , which we mentioned in the introduction. This theory is complete by [2, Corollary 16.6.3], so every model of it has IVP. If  $K$  is  $\omega$ -free and newtonian but its derivation is not small, then it nevertheless has IVP: some compositional conjugate  $K^\phi$  with  $\phi \in K^>$  has small derivation and is Liouville closed,  $\omega$ -free and newtonian.  $\square$

## 2. PRELIMINARIES ON HARDY FIELDS

We begin with some results from Boshernitzan [5] on ordered fields of germs of continuous functions. Next we prove some easy facts about extending ordered fields inside an ambient partially ordered ring, as needed later.

**Germ of continuous functions.** As in [2, Section 9.1] we let  $\mathcal{G}$  be the ring of germs at  $+\infty$  of real-valued functions whose domain is a subset of  $\mathbb{R}$  containing an interval  $(a, +\infty)$ ,  $a \in \mathbb{R}$ ; the domain may vary and the ring operations are defined as usual. If  $g \in \mathcal{G}$  is the germ of a real-valued function on a subset of  $\mathbb{R}$  containing an interval  $(a, +\infty)$ ,  $a \in \mathbb{R}$ , then we simplify notation by letting  $g$  also denote this function if the resulting ambiguity is harmless. With this convention, given a property  $P$  of real numbers and  $g \in \mathcal{G}$  we say that  $P(g(t))$  *holds eventually* if  $P(g(t))$  holds for all sufficiently large real  $t$ . We identify each real number  $r$  with the germ at  $+\infty$  of the function  $\mathbb{R} \rightarrow \mathbb{R}$  that takes the constant value  $r$ . This makes the field  $\mathbb{R}$  into a subring of  $\mathcal{G}$ . We call a germ  $g \in \mathcal{G}$  *continuous* if it is the germ of a continuous function  $(a, +\infty) \rightarrow \mathbb{R}$  for some  $a \in \mathbb{R}$ , and we let  $\mathcal{C} \supseteq \mathbb{R}$  be the subring of  $\mathcal{G}$  consisting of the continuous germs  $g \in \mathcal{C}$ . We let  $x$  denote the germ at  $+\infty$  of the identity function on  $\mathbb{R}$ .

**Asymptotic relations on  $\mathcal{C}$ .** Note that the multiplicative group  $\mathcal{C}^\times$  of  $\mathcal{C}$  consists of the  $f \in \mathcal{C}$  such that  $f(t) \neq 0$ , eventually. Thus for  $f \in \mathcal{C}^\times$ , either  $f(t) > 0$ , eventually, or  $f(t) < 0$ , eventually. Although  $\mathcal{C}$  is not a valued field, it will be convenient to equip  $\mathcal{C}$  with the asymptotic relations  $\preceq, \prec, \sim$  (which are defined on any valued field) as follows: for  $f, g \in \mathcal{C}$ ,

$$\begin{aligned} f \preceq g & : \iff \text{there exists } c \in \mathbb{R}^> \text{ such that eventually } |f(t)| \leq c|g(t)|, \\ f \prec g & : \iff g \in \mathcal{C}^\times \text{ and } \lim_{t \rightarrow \infty} f(t)/g(t) = 0, \\ f \sim g & : \iff g \in \mathcal{C}^\times \text{ and } \lim_{t \rightarrow \infty} f(t)/g(t) = 1 \\ & \iff f - g \prec g. \end{aligned}$$

Thus  $\preceq$  is a transitive and reflexive binary relation on  $\mathcal{C}$ , and  $\sim$  is an equivalence relation on  $\mathcal{C}^\times$ . Moreover, for  $f, g, h \in \mathcal{C}$  we have

$$f \prec g \Rightarrow f \preceq g, \quad f \preceq g \prec h \Rightarrow f \prec h, \quad f \prec g \preceq h \Rightarrow f \prec h.$$

Note that  $\prec$  is a transitive binary relation on  $\mathcal{C}$ . For  $f, g \in \mathcal{C}$  we also set

$$f \succ g : \iff f \preceq g \text{ and } g \preceq f, \quad f \succcurlyeq g : \iff g \preceq f, \quad f \succcurlyeq g : \iff g \prec f,$$

so  $\succ$  is an equivalence relation on  $\mathcal{C}$ .

**Subfields of  $\mathcal{C}$ .** Let  $K$  be a subfield of  $\mathcal{C}$ , that is, a subring of  $\mathcal{C}$  that happens to be a field. (In the introduction we called such  $K$  a *Hausdorff field*.) Then  $K$  itself has the subfield  $K \cap \mathbb{R}$ . Every nonzero  $f \in K$  has a multiplicative inverse in  $K$ , so eventually  $f(t) \neq 0$ , hence either eventually  $f(t) < 0$  or eventually  $f(t) > 0$  (by eventual continuity of  $f$ ). We make  $K$  an ordered field by declaring

$$f > 0 : \iff f(t) > 0, \text{ eventually.}$$

We now have [5, Propositions 3.4 and 3.6]:

**Lemma 2.1.** *Let  $K^{\text{rc}}$  consist of the  $y \in \mathcal{C}$  with  $P(y) = 0$  for some  $P(Y) \in K[Y]^\neq$ . Then  $K^{\text{rc}}$  is the unique real closed subfield of  $\mathcal{C}$  that extends  $K$  and is algebraic over  $K$ . In particular,  $K^{\text{rc}}$  is a real closure of the ordered field  $K$ .*

In [5] this lemma assumes  $K \supseteq \mathbb{R}$ , but this is not really needed in the proof. The ordered field  $K$  has a convex subring

$$\mathcal{O} = \{f \in K : |f| \leq n \text{ for some } n\},$$



which is a valuation ring of  $K$ , and we consider  $K$  accordingly as a valued ordered field. Restricting  $\preccurlyeq$ ,  $\prec$ ,  $\sim$  from the previous subsection to  $K$  gives exactly the asymptotic relations  $\preccurlyeq$ ,  $\prec$ ,  $\sim$  on  $K$  that it comes equipped with as a valued field.

**Composition and compositional inversion.** Let  $g \in \mathcal{C}$  be eventually strictly increasing with  $\lim_{t \rightarrow +\infty} g(t) = +\infty$ . Then its compositional inverse  $g^{\text{inv}} \in \mathcal{C}$  is given by  $g^{\text{inv}}(g(t)) = t$ , eventually, and the composition operation

$$f \mapsto f \circ g : \mathcal{C} \rightarrow \mathcal{C}, \quad (f \circ g)(t) := f(g(t)) \text{ eventually,}$$

is an automorphism of the ring  $\mathcal{C}$  that is the identity on the subring  $\mathbb{R}$ , with inverse  $f \mapsto f \circ g^{\text{inv}}$ . In particular,  $g \circ g^{\text{inv}} = g^{\text{inv}} \circ g = x$ , and  $f \mapsto f \circ g$  maps each subfield  $K$  of  $\mathcal{C}$  isomorphically (as an ordered field) onto the subfield  $K \circ g$  of  $\mathcal{C}$ . Note that if the subfield  $K$  of  $\mathcal{C}$  contains  $x$ , then  $K \circ g$  contains  $g$ .

**Extending ordered fields inside an ambient partially ordered ring.** Let  $R$  be a commutative ring with  $1 \neq 0$ , equipped with a translation-invariant partial ordering  $\leq$  such that  $r^2 \geq 0$  for all  $r \in R$ , and  $rs \geq 0$  for all  $r, s \in R$  with  $r, s \geq 0$ . It follows that for  $a, b, r \in R$  we have: if  $a \leq b$  and  $r \geq 0$ , then  $ar \leq br$ ; if  $a$  is a unit and  $a > 0$ , then  $a^{-1} = a \cdot (a^{-1})^2 > 0$ ; if  $a, b$  are units, and  $0 < a \leq b$ , then  $0 < b^{-1} \leq a^{-1}$ . Relevant cases:  $R = \mathcal{G}$  and  $R = \mathcal{C}$ , with partial ordering given by

$$f \leq g : \iff f(t) \leq g(t), \text{ eventually.}$$

Call a subset  $K$  of  $R$  *totally ordered* if the partial ordering of  $R$  induces a total ordering on  $K$ . An *ordered subfield of  $R$*  is a subfield  $K$  of  $R$  that is totally ordered as a subset of  $R$ ; note that then  $K$  equipped with the induced partial ordering is indeed an ordered field, in the usual sense of that term. (Thus any subfield of  $\mathcal{C}$  with the above partial ordering is an ordered subfield of  $\mathcal{C}$ .)

We identify  $\mathbb{Z}$  with its image in  $R$  via the unique ring embedding  $\mathbb{Z} \rightarrow R$ , and this makes  $\mathbb{Z}$  with its usual ordering into an ordered subring of  $R$ .

**Lemma 2.2.** *Assume  $D$  is a totally ordered subring of  $R$  and every nonzero element of  $D$  is a unit of  $R$ . Then  $D$  generates an ordered subfield  $\text{Frac } D$  of  $R$ .*

*Proof.* It is clear that  $D$  generates a subfield  $\text{Frac } D$  of  $R$ . For  $a \in D$ ,  $a > 0$ , we have  $a^{-1} > 0$ . It follows that  $\text{Frac } D$  is totally ordered.  $\square$

Thus if every  $n \geq 1$  is a unit of  $R$ , then we may identify  $\mathbb{Q}$  with its image in  $R$  via the unique ring embedding  $\mathbb{Q} \rightarrow R$ , making  $\mathbb{Q}$  into an ordered subfield of  $R$ .

**Lemma 2.3.** *Suppose  $K$  is an ordered subfield of  $R$ , all  $g \in R$  with  $g > K$  are units of  $R$ , and  $K < f \in R$ . Then we have an ordered subfield  $K(f)$  of  $R$ .*

*Proof.* For  $P(Y) \in K[Y] \setminus K$  of degree  $d \geq 1$  with leading coefficient  $a > 0$  we have  $P(f) = af^d(1 + \varepsilon)$  with  $-1/n < \varepsilon < 1/n$  for all  $n \geq 1$ , in particular,  $P(f) > K$  is a unit of  $R$ . It remains to appeal to Lemma 2.2.  $\square$

**Lemma 2.4.** *Assume  $K$  is a real closed ordered subfield of  $R$ . Let  $A$  be a nonempty downward closed subset of  $K$  such that  $A$  has no largest element and  $B := K \setminus A$  is nonempty and has no least element. Let  $f \in R$  be such that  $A < f < B$ . Then the subring  $K[f]$  has the following properties:*

- (i)  $K[f]$  is a domain;
- (ii)  $K[f]$  is totally ordered;
- (iii)  $K$  is cofinal in  $K[f]$ ;

- (iv) for all  $g \in K[f] \setminus K$  and  $a \in K$ , if  $a < g$ , then  $a < b < g$  for some  $b \in K$ , and if  $g < a$ , then  $g < b < a$  for some  $b \in K$ .

*Proof.* Let  $P \in K[Y] \setminus K$ ; to obtain (i) and (ii) it suffices to show that then  $P(f) < 0$  or  $P(f) > 0$ . We have

$$P(Y) = cQ(Y)(Y - a_1) \cdots (Y - a_n)$$

where  $c \in K^\times$ ,  $Q(Y)$  is a product of monic quadratic irreducibles in  $K[Y]$ , and  $a_1, \dots, a_n \in K$ . This gives  $\delta \in K^>$  such that  $Q(r) \geq \delta$  for all  $r \in R$ . Assume  $c > 0$ . (The case  $c < 0$  is handled similarly.) We can arrange that  $m \leq n$  is such that  $a_i \in A$  for  $1 \leq i \leq m$  and  $a_j \in B$  for  $m < j \leq n$ . Take  $\varepsilon > 0$  in  $K$  such that  $a_i + \varepsilon \leq f$  for  $1 \leq i \leq m$  and  $f \leq a_j - \varepsilon$  for  $m < j \leq n$ . Then

$$P(f) = cQ(f)(f - a_1) \cdots (f - a_m)(f - a_{m+1}) \cdots (f - a_n),$$

and  $(f - a_1) \cdots (f - a_m) \geq \varepsilon^m$ . If  $n - m$  is even, then  $(f - a_{m+1}) \cdots (f - a_n) \geq \varepsilon^{n-m}$ , so  $P(f) \geq a\delta\varepsilon^n > 0$ . If  $n - m$  is odd, then  $(f - a_{m+1}) \cdots (f - a_n) \leq -\varepsilon^{n-m}$ , so  $P(f) \leq -a\delta\varepsilon^n < 0$ . These estimates also yield (iii) and (iv).  $\square$

**Lemma 2.5.** *With  $K, A, f$  as in Lemma 2.4, suppose all  $g \in R$  with  $g \geq 1$  are units of  $R$ . Then we have an ordered subfield  $K(f)$  of  $R$  such that (iii) and (iv) of Lemma 2.4 go through for  $K(f)$  in place of  $K[f]$ .*

*Proof.* Note that if  $g \in R$  and  $g \geq \delta \in K^>$ , then  $g\delta^{-1} \geq 1$ , so  $g$  is a unit of  $R$  and  $0 < g^{-1} \leq \delta^{-1}$ . For  $Q \in K[Y]^\neq$  with  $Q(f) > 0$  we can take  $\delta \in K^>$  such that  $Q(f) \geq \delta$ , and thus  $Q(f)$  is a unit of  $R$  and  $0 < Q(f)^{-1} \leq \delta^{-1}$ . Thus we have an ordered subfield  $K(f)$  of  $R$  by Lemma 2.2, and the rest now follows easily.  $\square$

**Adjoining pseudolimits and increasing the value group.** Let  $K$  be a real closed subfield of  $\mathcal{C}$ , and view  $K$  as an ordered valued field as before. Let  $(a_\rho)$  be a strictly increasing divergent pc-sequence in  $K$ . Set

$$A := \{a \in K : a < a_\rho \text{ for some } \rho\}, \quad B := \{b \in K : b > a_\rho \text{ for all } \rho\},$$

so  $A$  is nonempty and downward closed without a largest element. Moreover,  $B = K \setminus A$  is nonempty and has no least element, since a least element of  $B$  would be a limit and thus a pseudolimit of  $(a_\rho)$ . Let  $f \in \mathcal{C}$  satisfy  $A < f < B$ . Then we have an ordered subfield  $K(f)$  of  $\mathcal{C}$ , and:

**Lemma 2.6.**  *$K(f)$  is an immediate valued field extension of  $K$  with  $a_\rho \rightsquigarrow f$ .*

*Proof.* We can assume that  $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$  for all indices  $\tau > \sigma > \rho$ . Set  $d_\rho := a_{s(\rho)} - a_\rho$  ( $s(\rho) :=$  successor of  $\rho$ ). Then  $a_\rho + 2d_\rho \in B$  for all indices  $\rho$ ; see the discussion preceding [2, Lemma 2.4.2]. It then follows from that lemma that  $a_\rho \rightsquigarrow f$ . Now  $(a_\rho)$  is a divergent pc-sequence in the henselian valued field  $K$ , so it is of transcendental type over  $K$ , and thus  $K(f)$  is an immediate extension of  $K$ .  $\square$

**Lemma 2.7.** *Suppose  $K$  is a subfield of  $\mathcal{C}$  with divisible value group  $\Gamma = v(K^\times)$ . Let  $P$  be a nonempty upward closed subset of  $\Gamma$ , and let  $f \in \mathcal{C}$  be such that  $a < f$  for all  $a \in K^>$  with  $va \in P$ , and  $f < b$  for all  $b \in K^>$  with  $vb < P$ . Then  $f$  generates a subfield  $K(f)$  of  $\mathcal{C}$ , with  $P > vf > Q$ ,  $Q := \Gamma \setminus P$ .*

*Proof.* For any positive  $a \in K^{\text{rc}}$  there is  $b \in K^>$  with  $a \asymp b$  and  $a < b$ , and also an element  $b \in K^>$  with  $a \asymp b$  and  $a > b$ . Thus we can replace  $K$  by  $K^{\text{rc}}$  and arrange in this way that  $K$  is real closed. Set

$$A := \{a \in K : a \leq 0 \text{ or } va \in P\}, \quad B := K \setminus A.$$

Then we are in the situation of Lemma 2.4 for  $R = \mathcal{C}$ , so by that lemma and Lemma 2.5 we have an ordered subfield  $K(f)$  of  $\mathcal{C}$ . Clearly then  $P > vf > Q$ .  $\square$

**Notational conventions on functions and germs.** Let  $r$  range over  $\mathbb{N} \cup \{\infty\}$ , and let  $U$  be a nonempty open subset of  $\mathbb{R}$ . Then  $\mathcal{C}^r(U)$  denotes the  $\mathbb{R}$ -algebra of  $r$ -times continuously differentiable functions  $U \rightarrow \mathbb{R}$ , with the usual pointwise defined algebra operations. (We use “ $\mathcal{C}$ ” instead of “ $C$ ” since  $C$  will often denote the constant field of a differential field.) For  $r = 0$  this is the  $\mathbb{R}$ -algebra  $\mathcal{C}(U)$  of continuous real-valued functions on  $U$ , so

$$\mathcal{C}(U) = \mathcal{C}^0(U) \supseteq \mathcal{C}^1(U) \supseteq \mathcal{C}^2(U) \supseteq \dots \supseteq \mathcal{C}^\infty(U).$$

For  $r \geq 1$  we have the derivation  $f \mapsto f' : \mathcal{C}^r(U) \rightarrow \mathcal{C}^{r-1}(U)$  (with  $\infty - 1 := \infty$ ). This makes  $\mathcal{C}^\infty(U)$  a differential ring, with its subalgebra  $\mathcal{C}^\omega(U)$  of real-analytic functions  $U \rightarrow \mathbb{R}$  as a differential subring. The algebra operations on the algebras below are also defined pointwise.

Let  $a$  range over  $\mathbb{R}$ . Then  $\mathcal{C}_a^r$  denotes the  $\mathbb{R}$ -algebra of functions  $[a, +\infty) \rightarrow \mathbb{R}$  that extend to a function in  $\mathcal{C}^r(U)$  for some open  $U \supseteq [a, +\infty)$ . Thus  $\mathcal{C}_a^0$  is the  $\mathbb{R}$ -algebra of real-valued continuous functions on  $[a, +\infty)$ , and

$$\mathcal{C}_a^0 \supseteq \mathcal{C}_a^1 \supseteq \mathcal{C}_a^2 \supseteq \dots \supseteq \mathcal{C}_a^\infty.$$

We also have the subalgebra  $\mathcal{C}_a^\omega$  of  $\mathcal{C}_a^\infty$ , consisting of the functions  $[a, +\infty) \rightarrow \mathbb{R}$  that extend to a real-analytic function  $U \rightarrow \mathbb{R}$  for some open  $U \supseteq [a, +\infty)$ . For  $r \geq 1$  we have the derivation  $f \mapsto f' : \mathcal{C}_a^r \rightarrow \mathcal{C}_a^{r-1}$ . This makes  $\mathcal{C}_a^\infty$  a differential ring with  $\mathcal{C}_a^\omega$  as a differential subring.

For each of the algebras  $A$  above we also consider its complexification  $A[i]$  which consists by definition of the  $\mathbb{C}$ -valued functions  $f = g + hi$  with  $g, h \in A$ , so  $g = \operatorname{Re} f$  and  $h = \operatorname{Im} f$  for such  $f$ . We consider  $A[i]$  as a  $\mathbb{C}$ -algebra with respect to the natural pointwise defined algebra operations. We identify each complex number with the corresponding constant function to make  $\mathbb{C}$  a subfield of  $A[i]$  and  $\mathbb{R}$  a subfield of  $A$ . (This justifies the notation  $A[i]$ .) For  $r \geq 1$  we extend  $g \mapsto g' : \mathcal{C}_a^r \rightarrow \mathcal{C}_a^{r-1}$  to the derivation

$$g + hi \mapsto g' + h'i : \mathcal{C}_a^r[i] \rightarrow \mathcal{C}_a^{r-1}[i] \quad (g, h \in \mathcal{C}_a^r[i]),$$

which for  $r = \infty$  makes  $\mathcal{C}_a^\infty$  a differential subring of  $\mathcal{C}_a^\infty[i]$ . We also use the map

$$f \mapsto f^\dagger := f'/f : \mathcal{C}_a^1[i]^\times = (\mathcal{C}_a^1[i])^\times \rightarrow \mathcal{C}_a^0[i],$$

with

$$(fg)^\dagger = f^\dagger + g^\dagger \quad \text{for } f, g \in \mathcal{C}_a^1[i]^\times,$$

in particular the fact that  $f \in \mathcal{C}_a^1[i]^\times$  and  $f^\dagger \in \mathcal{C}_a^0[i]$  are related by

$$f(t) = f(a) \exp \left[ \int_a^t f^\dagger(s) ds \right] \quad (t \geq a).$$

Let  $\mathcal{C}^r$  be the partially ordered subring of  $\mathcal{C}$  consisting of the germs at  $+\infty$  of the functions in  $\bigcup_a \mathcal{C}_a^r$ ; thus  $\mathcal{C}^0 = \mathcal{C}$  consists of the germs at  $+\infty$  of the continuous real valued functions on intervals  $[a, +\infty)$ ,  $a \in \mathbb{R}$ . Note that  $\mathcal{C}^r$  with its partial ordering satisfies the conditions on  $R$  from the previous subsection. Also, every  $g \geq 1$  in  $\mathcal{C}^r$  is a unit of  $\mathcal{C}^r$ , so Lemmas 2.3 and 2.5 apply to ordered subfields of  $\mathcal{C}^r$ . We have

$$\mathcal{C}^0 \supseteq \mathcal{C}^1 \supseteq \mathcal{C}^2 \supseteq \dots \supseteq \mathcal{C}^\infty,$$

and we set  $\mathcal{C}^{<\infty} := \bigcap_r \mathcal{C}^r$ . Thus  $\mathcal{C}^{<\infty}$  is naturally a differential ring with  $\mathbb{R}$  as its ring of constants. Note that  $\mathcal{C}^{<\infty}$  has  $\mathcal{C}^\infty$  as a differential subring. The differential

ring  $\mathcal{C}^\infty$  has in turn the differential subring  $\mathcal{C}^\omega$ , whose elements are the germs at  $+\infty$  of the functions in  $\bigcup_a \mathcal{C}_a^\omega$ .

**Second-order differential equations.** Let  $f \in \mathcal{C}_a^0$ , that is,  $f: [a, \infty) \rightarrow \mathbb{R}$  is continuous. We consider the differential equation

$$Y'' + fY = 0.$$

The solutions  $y \in \mathcal{C}_a^2$  form an  $\mathbb{R}$ -linear subspace  $\text{Sol}(f)$  of  $\mathcal{C}_a^2$ . The solutions  $y \in \mathcal{C}_a^2[i]$  are the  $y_1 + y_2i$  with  $y_1, y_2 \in \text{Sol}(f)$  and form a  $\mathbb{C}$ -linear subspace  $\text{Sol}_{\mathbb{C}}(f)$  of  $\mathcal{C}_a^2[i]$ . For any complex numbers  $c, d$  there is a unique solution  $y \in \mathcal{C}_a^2[i]$  with  $y(a) = c$  and  $y'(a) = d$ , and the map that assigns to  $(c, d) \in \mathbb{C}^2$  this unique solution is an isomorphism  $\mathbb{C}^2 \rightarrow \text{Sol}_{\mathbb{C}}(f)$  of  $\mathbb{C}$ -linear spaces; it restricts to an  $\mathbb{R}$ -linear bijection  $\mathbb{R}^2 \rightarrow \text{Sol}(f)$ . Induction on  $r \in \mathbb{N}$  shows:  $f \in \mathcal{C}_a^r \Rightarrow \text{Sol}(f) \subseteq \mathcal{C}_a^{r+2}$ . Thus  $f \in \mathcal{C}_a^\infty \Rightarrow \text{Sol}(f) \subseteq \mathcal{C}_a^\infty$ . It is also well-known that  $f \in \mathcal{C}_a^\omega \Rightarrow \text{Sol}(f) \subseteq \mathcal{C}_a^\omega$ . From [4, Chapter 2, Lemma 1] we recall:

**Lemma 2.8** (Gronwall's Lemma). *Let the constant  $C \in \mathbb{R}^{\geq}$  and the functions  $v, y \in \mathcal{C}_a^0$  be such that  $v(t), y(t) \geq 0$  for all  $t \geq a$  and*

$$y(t) \leq C + \int_a^t v(s)y(s) ds \quad \text{for all } t \geq a.$$

Then

$$y(t) \leq C \exp \left[ \int_a^t v(s) ds \right] \quad \text{for all } t \geq a.$$

In the rest of this subsection we assume that  $a \geq 1$  and that  $c \in \mathbb{R}^>$  is such that  $|f(t)| \leq c/t^2$  for all  $t \geq a$ . Under this hypothesis, the lemma above yields the following bound on the growth of the solutions  $y \in \text{Sol}(f)$ ; the proof we give is similar to that of [4, Chapter 6, Theorem 5].

**Proposition 2.9.** *Let  $y \in \text{Sol}(f)$ . Then there is  $C \in \mathbb{R}^{\geq}$  such that  $|y(t)| \leq Ct^{c+1}$  and  $|y'(t)| \leq Ct^c$  for all  $t \geq a$ .*

*Proof.* Let  $t$  range over  $[a, +\infty)$ . Integrating  $y'' = -fy$  twice between  $a$  and  $t$ , we obtain constants  $c_1, c_2$  such that for all  $t$ ,

$$y(t) = c_1 + c_2 t - \int_a^t \int_a^{t_1} f(t_2)y(t_2) dt_2 dt_1 = c_1 + c_2 t - \int_a^t (t-s)f(s)y(s) ds$$

and hence, with  $C := |c_1| + |c_2|$ ,

$$|y(t)| \leq Ct + t \int_a^t |f(s)| \cdot |y(s)| ds,$$

so

$$\frac{|y(t)|}{t} \leq C + \int_a^t s|f(s)| \cdot \frac{|y(s)|}{s} ds.$$

Hence by the lemma above,

$$\frac{|y(t)|}{t} \leq C \exp \left[ \int_a^t s|f(s)| ds \right] \leq C \exp \left[ \int_1^t c/s ds \right] = Ct^c$$

and thus  $|y(t)| \leq Ct^{c+1}$ . Now

$$y'(t) = c_2 - \int_a^t f(s)y(s) ds$$

and thus

$$|y'(t)| \leq |c_2| + \int_a^t |f(s)y(s)| ds \leq C + Cc \int_1^t s^{c-1} ds = C + Cc \left[ \frac{t^c}{c} - \frac{1}{c} \right] = Ct^c.$$

□

Let  $y_1, y_2 \in \text{Sol}(f)$  be  $\mathbb{R}$ -linearly independent. The Wronskian  $w := y_1 y_2' - y_1' y_2$  satisfies  $w' = 0$  (Abel's identity), so  $w \in \mathbb{R}^\times$ . It follows that  $y_1$  and  $y_2$  cannot be simultaneously very small:

**Lemma 2.10.** *There is a positive constant  $d$  such that*

$$\max(|y_1(t)|, |y_2(t)|) \geq dt^{-c} \quad \text{for all } t \geq a.$$

*Proof.* Proposition 2.9 yields  $C \in \mathbb{R}^>$  such that  $|y_i'(t)| \leq Ct^c$  for  $i = 1, 2$  and all  $t \geq a$ . Hence  $|w| \leq 2 \max(|y_1(t)|, |y_2(t)|)Ct^c$  for  $t \geq a$ , so

$$\max(|y_1(t)|, |y_2(t)|) \geq \frac{|w|}{2C} t^{-c} \quad (t \geq a). \quad \square$$

**Corollary 2.11.** *Set  $y := y_1 + y_2 i$  and  $z := y^\dagger$ . Then for some  $D \in \mathbb{R}^>$ ,*

$$|z(t)| \leq Dt^{2c} \quad \text{for all } t \geq a.$$

*Proof.* Take  $C$  as in the proof of Lemma 2.10, and  $d$  as in that lemma. Then

$$|z(t)| = \frac{|y_1'(t) + y_2'(t)i|}{|y_1(t) + y_2(t)i|} \leq \frac{|y_1'(t)| + |y_2'(t)|}{\max(|y_1(t)|, |y_2(t)|)} \leq \left( \frac{2C}{d} \right) t^{2c}$$

for  $t \geq a$ . □

**Changing variables.** Let now  $K$  be a differential field,  $f \in K$ , and consider the differential polynomial  $P(Y) := 4Y'' + fY$ . (The factor 4 is to simplify certain expressions, in conformity with [2, Section 9.2].) Which ‘‘changes of variable’’ preserve the general form of  $P$ ? Here is an answer:

**Lemma 2.12.** *For  $g \in K^\times$  and  $\phi := g^{-2}$  we have*

$$g^3 P_{\times g}^\phi(Y) = 4Y'' + g^3 P(g)Y.$$

*Proof.* Let  $g, \phi \in K^\times$ . Then

$$\begin{aligned} P_{\times g}(Y) &= 4gY'' + 8g'Y' + (4g'' + fg)Y = 4gY'' + 8g'Y' + P(g)Y, \quad \text{so} \\ P_{\times g}^\phi(Y) &= 4g(\phi^2 Y'' + \phi' Y') + 8g' \phi Y' + P(g)Y \\ &= 4g\phi^2 Y'' + (4g\phi' + 8g'\phi)Y' + P(g)Y. \end{aligned}$$

Now  $4g\phi' + 8g'\phi = 0$  is equivalent to  $\phi^\dagger = -2g^\dagger$ , which holds for  $\phi = g^{-2}$ . For this  $\phi$  we get  $P_{\times g}^\phi(Y) = g^{-3}(4Y'' + g^3 P(g)Y)$ , that is,  $g^3 P_{\times g}^\phi(Y) = 4Y'' + g^3 P(g)Y$ . □

**Hardy fields.** A *Hardy field* is a subfield of  $\mathcal{C}^{<\infty}$  that is closed under the derivation of  $\mathcal{C}^{<\infty}$ . A Hardy field  $H$  is considered as an ordered valued differential field in the obvious way, and has  $\mathbb{R} \cap H$  as its field of constants. Hardy fields are pre- $H$ -fields, and  $H$ -fields if they contain  $\mathbb{R}$ . Here are some well-known extension results:

**Proposition 2.13.** *Any Hardy field  $H$  has the following Hardy field extensions:*

- (i)  $H(\mathbb{R})$ , the subfield of  $\mathcal{C}^{<\infty}$  generated by  $H$  and  $\mathbb{R}$ ;
- (ii)  $H^{\text{rc}}$ , the real closure of  $H$  as defined in Lemma 2.1;
- (iii)  $H(e^f)$  for any  $f \in H$ ;

- (iv)  $H(f)$  for any  $f \in \mathcal{C}^1$  with  $f' \in H$ ;
- (v)  $H(\log f)$  for any  $f \in H^>$ .

If  $H$  is contained in  $\mathcal{C}^\infty$ , then so are the Hardy fields in (i), (ii), (iii), (iv), (v); likewise with  $\mathcal{C}^\omega$  instead of  $\mathcal{C}^\infty$ .

Note that (v) is a special case of (iv), since  $(\log f)' = f'/f \in H$  for  $f \in H^>$ . Another special case of (iv) is that  $H(x)$  is a Hardy field. A consequence of the Proposition is that any Hardy field  $H$  has a smallest real closed Hardy field extension  $H^*$  with  $\mathbb{R} \subseteq H^*$  such that for all  $f \in H^*$  we have  $e^f \in H^*$  and  $g' = f$  for some  $g \in H^*$ . Note that then  $H^*$  is Liouville closed as defined in [2, Section 10.6].

We also have the following more general extension result from Rosenlicht [13], attributed there to M. Singer:

**Proposition 2.14.** *Let  $H$  be a Hardy field and  $p(Y), q(Y) \in H[Y]$ . Suppose  $f \in \mathcal{C}^1$  is a solution of the differential equation  $y'q(y) = p(y)$  and  $q(f)$  is a unit of  $\mathcal{C}^1$ . Then  $f$  generates a Hardy field  $H(f)$  over  $H$ .*

### Compositional inversion and compositional conjugation in Hardy fields.

Let  $H$  be a Hardy field, and let  $g \in \mathcal{C}^1$  be such that  $g > \mathbb{R}$  and  $g' \in H$ . Then we have a Hardy field  $H(g)$ , and the compositional inverse  $g^{\text{inv}} \in \mathcal{C}^1$  of  $g$  satisfies

$$g^{\text{inv}} > \mathbb{R}, \quad (g^{\text{inv}})' = (1/g') \circ g^{\text{inv}} \in H \circ g^{\text{inv}}$$

and yields an ordered field isomorphism

$$h \mapsto h \circ g^{\text{inv}} : H \rightarrow H \circ g^{\text{inv}}$$

such that for all  $h \in H$ ,

$$(h \circ g^{\text{inv}})' = (h' \circ g^{\text{inv}}) \cdot (g^{\text{inv}})' = (h'/g') \circ g^{\text{inv}} \in H \circ g^{\text{inv}}.$$

Thus  $H \circ g^{\text{inv}}$  is again a Hardy field, and for  $\phi = g'$  this yields an isomorphism

$$h \mapsto h \circ g^{\text{inv}} : H^\phi \rightarrow H \circ g^{\text{inv}}$$

of pre- $H$ -fields. If  $H \subseteq \mathcal{C}^\infty$  and  $g \in \mathcal{C}^\infty$ , then  $H \circ g^{\text{inv}} \subseteq \mathcal{C}^\infty$ ; likewise with  $\mathcal{C}^\omega$  instead of  $\mathcal{C}^\infty$ . For later use, a  $\mathcal{C}^\infty$ -Hardy field is a Hardy field  $H \subseteq \mathcal{C}^\infty$ , and a  $\mathcal{C}^\omega$ -Hardy field (also called an *analytic Hardy field*) is a Hardy field  $H \subseteq \mathcal{C}^\omega$ .

### 3. EXTENDING HARDY FIELDS TO $\omega$ -FREE HARDY FIELDS

In this section we assume familiarity with [2, Sections 5.2, 11.5–11.8]. Here we summarize some of this material, and then use this to prove Theorem 3.1 below. In the *Notations and terminology* at the end of the introduction we defined for any differential ring  $R$  functions  $\omega: R \rightarrow R$  and  $\sigma: R^\times \rightarrow R$ . We define likewise

$$\omega : \mathcal{C}_a^1 \rightarrow \mathcal{C}_a^0, \quad \sigma : (\mathcal{C}_a^1)^\times \rightarrow \mathcal{C}_a^0$$

by

$$\omega(z) = -2z' - z^2 \quad \text{and} \quad \sigma(y) = \omega(z) + y^2 \text{ for } z := -y^\dagger.$$

To clarify this role of  $\omega$  and  $\sigma$  in connection with second-order linear differential equations, let  $f \in \mathcal{C}_a^0$  and consider the differential equation

$$4Y'' + fY = 0.$$

Suppose  $y \in \mathcal{C}_a^2$  is a *non-oscillating* solution, that is, a solution with  $y(t) \neq 0$  for all sufficiently large  $t$ , say for all  $t \geq b$ , where  $b \geq a$ . Then  $z \in \mathcal{C}_b^1$  given by  $z(t) = 2y'(t)/y(t)$  satisfies the first-order differential equation  $-2z' - z^2 = f$

on  $[b, \infty)$ . Thus if the germ of  $f$  at  $+\infty$  belongs to a Hardy field  $H$ , then by Proposition 2.14 the germ of  $z$  at  $+\infty$  (also denoted by  $z$ ) generates a Hardy field  $H(z)$  with  $\omega(z) = f$ , which in turn yields a Hardy field  $H(z, y)$  with  $y$  now denoting its germ at  $+\infty$  so that  $y \in (\mathcal{C}^{<\infty})^\times$  and  $2y^\dagger = z$  in  $\mathcal{C}^{<\infty}$ . Thus  $y_1 := y$  lies in a Hardy field extension of  $H$ . The germ  $y_2$  of the function  $[b, +\infty) \rightarrow \mathbb{R}$  given by  $t \mapsto y(t) \int_b^t \frac{1}{y(s)^2} ds$  also satisfies  $4y_2'' + fy_2 = 0$ , and  $y_1, y_2$  are  $\mathbb{R}$ -linearly independent [4, Chapter 6, Lemma 3]. By Proposition 2.13(iv),  $y_2$  lies in a Hardy field extension of  $H\langle y_1 \rangle = H(y, z)$ ; see also [13, Theorem 2, Corollary 2].

There might not exist a non-oscillating solution  $y$ , but we do have  $\mathbb{R}$ -linearly independent solutions  $y_1, y_2 \in \mathcal{C}_a^2$ . We saw before that  $w := y_1 y_2' - y_1' y_2 \in \mathbb{R}^\times$ . Set  $y := y_1 + y_2 i$ . Then  $4y'' + fy = 0$  and  $y(t) \neq 0$  for all  $t \geq a$ , and for  $z \in \mathcal{C}_a^1[i]$  given by  $z(t) = 2y'(t)/y(t)$  we have  $-2z' - z^2 = f$ . Now

$$z = \frac{2y_1' + 2iy_2'}{y_1 + iy_2} = \frac{2y_1'y_1 + 2y_2'y_2 - 2i(y_1'y_2 - y_1y_2')}{y_1^2 + y_2^2} = \frac{2(y_1'y_1 + y_2'y_2) + 2iw}{y_1^2 + y_2^2},$$

$$\text{so } \operatorname{Re}(z) = \frac{2(y_1'y_1 + y_2'y_2)}{y_1^2 + y_2^2} \in \mathcal{C}_a^1, \quad \operatorname{Im}(z) = \frac{2w}{y_1^2 + y_2^2} \in \mathcal{C}_a^2.$$

Thus  $\operatorname{Im}(z) \in (\mathcal{C}_a^2)^\times$  and  $\operatorname{Im}(z)^\dagger = -\operatorname{Re}(z)$  and  $\sigma(\operatorname{Im}(z)) = \omega(z) = f$  in  $\mathcal{C}_a^1$ . Replacing  $y_1$  by  $-y_1$  changes  $w$  to  $-w$ ; in this way we can arrange that  $w > 0$ .

**The property of  $\omega$ -freeness.** Let  $H \supseteq \mathbb{R}$  be a Liouville closed Hardy field. Note that then  $x \in H$  and  $\log f \in H$  for all  $f \in H^>$ . To express the property of  $\omega$ -freeness for  $H$  we introduce the ‘‘iterated logarithms’’  $\ell_\rho$ ; more precisely, transfinite recursion yields a sequence  $(\ell_\rho)$  in  $H^{>\mathbb{R}}$  indexed by the ordinals  $\rho$  less than some infinite limit ordinal  $\kappa$  as follows:  $\ell_0 = x$ , and  $\ell_{\rho+1} := \log \ell_\rho$ ; if  $\lambda$  is an infinite limit ordinal such that all  $\ell_\rho$  with  $\rho < \lambda$  have already been chosen, then we pick  $\ell_\lambda$  to be any element in  $H^{>\mathbb{R}}$  such that  $\ell_\lambda \prec \ell_\rho$  for all  $\rho < \lambda$ , if there is such an  $\ell_\lambda$ , while if there is no such  $\ell_\lambda$ , we put  $\kappa := \lambda$ . From  $(\ell_\rho)$  we obtain the sequences  $(\gamma_\rho)$  in  $H^>$  and  $(\lambda_\rho)$  in  $H$  as follows:

$$\gamma_\rho := \ell_\rho^\dagger, \quad \lambda_\rho := -\gamma_\rho^\dagger = -\ell_\rho^{\dagger\dagger} := -(\ell_\rho^{\dagger\dagger}).$$

Then  $\lambda_{\rho+1} = \lambda_\rho + \gamma_{\rho+1}$  and we have

$$\begin{aligned} \gamma_0 &= \ell_0^{-1}, & \gamma_1 &= (\ell_0 \ell_1)^{-1}, & \gamma_2 &= (\ell_0 \ell_1 \ell_2)^{-1}, \\ \lambda_0 &= \ell_0^{-1}, & \lambda_1 &= \ell_0^{-1} + (\ell_0 \ell_1)^{-1}, & \lambda_2 &= \ell_0^{-1} + (\ell_0 \ell_1)^{-1} + (\ell_0 \ell_1 \ell_2)^{-1}, \end{aligned}$$

and so on. Indeed,  $v(\gamma_\rho)$  is strictly increasing as a function of  $\rho$  and is cofinal in  $\Psi_H = \{v(f^\dagger) : f \in H, 0 \neq f \neq 1\}$ ; we refer to [2, Section 11.5] for this and some of what follows. Also,  $(\lambda_\rho)$  is a strictly increasing pc-sequence which is cofinal in  $\Lambda(H)$ ; see [2, Section 11.8] for the definition of the set  $\Lambda(H) \subseteq H$ , which is downward closed since  $H$  is Liouville closed. The latter also gives that  $H$  is  $\lambda$ -free as defined in [2, Section 11.6], equivalently,  $(\lambda_\rho)$  has no pseudolimit in  $H$ . The function  $\omega: H \rightarrow H$  is strictly increasing on  $\Lambda(H)$  and setting  $\omega_\rho := \omega(\lambda_\rho)$  we obtain a strictly increasing pc-sequence  $(\omega_\rho)$  which is cofinal in  $\omega(\Lambda(H)) = \omega(H)$ :

$$\omega_0 = \ell_0^{-2}, \quad \omega_1 = \ell_0^{-2} + (\ell_0 \ell_1)^{-2}, \quad \omega_2 = \ell_0^{-2} + (\ell_0 \ell_1)^{-2} + (\ell_0 \ell_1 \ell_2)^{-2},$$

and so on; see [2, Sections 11.7, 11.8] for this and some of what follows. Now  $H$  being  $\omega$ -free is equivalent to  $(\omega_\rho)$  having no pseudolimit in  $H$ . By [2, Corollary 11.8.30] the

pseudolimits of  $(\omega_\rho)$  in  $H$  are exactly the  $\omega \in H$  such that  $\omega(H) < \omega < \sigma(\Gamma(H))$ . Here the upward closed subset  $\Gamma(H)$  of  $H$  is given by

$$\Gamma(H) = \{a^\dagger : a \in H, a \succ 1\} = \{a \in H : a > \gamma_\rho \text{ for some } \rho\},$$

and  $\sigma$  is strictly increasing on  $\Gamma(H)$ . Thus  $H$  is not  $\omega$ -free if and only if there exists an  $\omega \in H$  such that  $\omega(H) < \omega < \sigma(\Gamma(H))$ .

We are now ready to prove the following:

**Theorem 3.1.** *Every Hardy field has an  $\omega$ -free Hardy field extension.*

*Proof.* It is enough to show that every maximal Hardy field is  $\omega$ -free. That reduces to showing that every non- $\omega$ -free Liouville closed Hardy field containing  $\mathbb{R}$  has a proper Hardy field extension. So assume  $H \supseteq \mathbb{R}$  is a Liouville closed Hardy field and  $H$  is not  $\omega$ -free. We shall construct a proper Hardy field extension of  $H$ . We have  $\omega \in H$  such that

$$\omega(H) < \omega < \sigma(\Gamma(H)).$$

Take  $a \in \mathbb{R}$  such that  $\omega$  is the germ of a function in  $\mathcal{C}_a^2$ , this function also to be denoted by  $\omega$ . With  $\omega$  in the role of  $f$  in the discussion preceding the statement of the theorem, we have  $\mathbb{R}$ -linearly independent solutions  $y_1, y_2 \in \mathcal{C}_a^2$  of the differential equation  $4Y'' + \omega Y = 0$  whose germs at  $+\infty$  (also denoted by  $y_1$  and  $y_2$ ) lie in  $\mathcal{C}^{<\infty}$ . Then the complex solution  $y = y_1 + y_2i$  is a unit of  $\mathcal{C}_a^2[i]$ , and so we have  $z := 2y^\dagger \in \mathcal{C}_a^1[i]$ . The germs of  $y$  and  $z$  at  $+\infty$  are also denoted by  $y$  and  $z$  and lie in  $\mathcal{C}^{<\infty}[i]$ . We shall prove that the elements  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$  of  $\mathcal{C}^{<\infty}$  generate a Hardy field extension  $K = H(\operatorname{Re}(z), \operatorname{Im}(z))$  of  $H$  with  $\omega = \sigma(\operatorname{Im}(z)) \in \sigma(K^\times)$ . We can assume that  $w := y_1y_2' - y_1'y_2 \in \mathbb{R}^>$ , so  $\operatorname{Im}(z)(t) > 0$  for all  $t \geq a$ .

We have  $\omega_\rho \rightsquigarrow \omega$ , with  $\omega - \omega_\rho \sim \gamma_{\rho+1}^2$  by [2, Lemma 11.7.1]. We set  $g_\rho := \gamma_\rho^{-1/2}$ , so  $2g_\rho^\dagger = \lambda_\rho = -\gamma_\rho^\dagger$ . For  $h \in H^\times$  we also have  $\omega(2h^\dagger) = -4h''/h$ , hence  $P := 4Y'' + \omega Y \in H\{Y\}$  gives

$$P(g_\rho) = g_\rho(\omega - \omega_\rho) \sim g_\rho\gamma_{\rho+1}^2,$$

and so with an eye towards using Lemma 2.12:

$$g_\rho^3 P(g_\rho) \sim g_\rho^4 \gamma_{\rho+1}^2 \sim \gamma_{\rho+1}^2 / \gamma_\rho^2 \asymp 1/\ell_{\rho+1}^2.$$

Thus with  $g := g_\rho = \gamma_\rho^{-1/2}$ ,  $\phi = g^{-2} = \gamma_\rho$  we have  $A_\rho \in \mathbb{R}^>$  such that

$$(3.1) \quad g^3 P_{\times g}^\phi(Y) = 4Y'' + g^3 P(g)Y, \quad |g^3 P(g)(t)| \leq A_\rho / \ell_{\rho+1}(t)^2, \text{ eventually.}$$

From  $P(y) = 0$  we get  $P_{\times g}^\phi(y/g) = 0$ , that is,  $y/g \in \mathcal{C}^{<\infty}[i]^\phi$  is a solution of  $4Y'' + g^3 P(g)Y = 0$ , with  $g^3 P(g) \in H \subseteq \mathcal{C}^{<\infty}$ . Now  $\ell'_{\rho+1} = \ell_\rho^\dagger = \phi$ , so the end of the previous section yields the isomorphism  $H^\phi \rightarrow H \circ \ell_{\rho+1}^{\text{inv}}$  of  $H$ -fields, where  $\ell_{\rho+1}^{\text{inv}}$  is the compositional inverse of  $\ell_{\rho+1}$ . Under this isomorphism the equation  $4Y'' + g^3 P(g)Y = 0$  corresponds to the equation

$$4Y'' + f_\rho Y = 0, \quad f_\rho := g^3 P(g) \circ \ell_{\rho+1}^{\text{inv}} \in H \circ \ell_{\rho+1}^{\text{inv}} \subseteq \mathcal{C}^{<\infty}.$$

The equation  $4Y'' + f_\rho Y = 0$  has the ‘‘real’’ solutions

$$y_{i,\rho} := (y_i/g) \circ \ell_{\rho+1}^{\text{inv}} \in \mathcal{C}^{<\infty} \circ \ell_{\rho+1}^{\text{inv}} = \mathcal{C}^{<\infty} \quad (i = 1, 2),$$

and the ‘‘complex’’ solution

$$y_\rho := y_{1,\rho} + y_{2,\rho}i = (y/g) \circ \ell_{\rho+1}^{\text{inv}},$$



which is a unit of the ring  $\mathcal{C}^{<\infty}[i]$ . We set  $z_\rho := 2y_\rho^\dagger \in \mathcal{C}^{<\infty}[i]$ . The bound in (3.1) gives

$$|f_\rho(t)| \leq A_\rho/t^2, \text{ eventually,}$$

which by Corollary 2.11 yields positive constants  $B_\rho, c_\rho$  such that

$$|z_\rho(t)| \leq B_\rho t^{c_\rho}, \text{ eventually.}$$

Using  $(\ell_{\rho+1}^{\text{inv}})' = (1/\ell'_{\rho+1}) \circ \ell_{\rho+1}^{\text{inv}}$  we obtain

$$z_\rho = 2((y/g)^\dagger \circ \ell_{\rho+1}^{\text{inv}}) \cdot (\ell_{\rho+1}^{\text{inv}})' = 2((y/g)^\dagger/\ell'_{\rho+1}) \circ \ell_{\rho+1}^{\text{inv}} = ((z - 2g^\dagger)/\ell'_{\rho+1}) \circ \ell_{\rho+1}^{\text{inv}}.$$

In combination with the eventual bound on  $|z_\rho(t)|$  this yields

$$\begin{aligned} \left| \frac{z(t) - 2g^\dagger(t)}{\ell'_{\rho+1}(t)} \right| &\leq B_\rho \ell_{\rho+1}(t)^{c_\rho} \text{ eventually, hence} \\ |z(t) - \lambda_\rho(t)| &\leq B_\rho \ell_{\rho+1}(t)^{c_\rho} \ell'_{\rho+1}(t) = B_\rho \ell_{\rho+1}(t)^{c_\rho} \gamma_\rho(t), \text{ eventually, so} \\ z(t) &= \lambda_\rho(t) + R_\rho(t), \quad |R_\rho(t)| \leq B_\rho \ell_{\rho+1}(t)^{c_\rho} \gamma_\rho(t), \text{ eventually.} \end{aligned}$$

We now use this last estimate with  $\rho + 1$  instead of  $\rho$ , together with

$$\lambda_{\rho+1} = \lambda_\rho + \gamma_{\rho+1}, \quad \ell_{\rho+1} \gamma_{\rho+1} = \gamma_\rho.$$

This yields

$$\begin{aligned} z(t) &= \lambda_\rho(t) + \gamma_{\rho+1}(t) + R_{\rho+1}(t) \text{ eventually, with} \\ |R_{\rho+1}(t)| &\leq B_{\rho+1} \ell_{\rho+2}(t)^{c_{\rho+1}} \gamma_{\rho+1}(t) \\ &= B_{\rho+1} (\ell_{\rho+2}(t)^{c_{\rho+1}}/\ell_{\rho+1}(t)) \gamma_\rho(t) \text{ eventually,} \\ \text{so } z(t) &= \lambda_\rho(t) + o(\gamma_\rho(t)) \text{ as } t \rightarrow \infty, \text{ and thus} \\ \text{Re}(z)(t) &= \lambda_\rho(t) + o(\gamma_\rho(t)), \quad \text{Im}(z)(t) = o(\gamma_\rho(t)), \text{ as } t \rightarrow \infty. \end{aligned}$$

Recall that  $(\lambda_\rho)$  is a strictly increasing divergent pc-sequence  $(\lambda_\rho)$  in  $H$  which is cofinal in  $\Lambda(H)$ . By the above,  $\lambda := \text{Re}(z) \in \mathcal{C}^{<\infty}$  satisfies  $\Lambda(H) < \lambda < \Delta(H)$ . This yields an ordered subfield  $H(\lambda)$  of  $\mathcal{C}^{<\infty}$ , which by Lemma 2.6 is an immediate valued field extension of  $H$  with  $\lambda_\rho \rightsquigarrow \lambda$ .

Pick functions in  $\mathcal{C}_a^0$  whose germs at  $+\infty$  are the elements  $\ell_\rho, \gamma_\rho, \lambda_\rho$  of  $H$ ; we denote these functions also by  $\ell_\rho, \lambda_\rho, \gamma_\rho$ . From  $\ell_\rho^\dagger = \gamma_\rho$  and  $\gamma_\rho^\dagger = -\lambda_\rho$  in  $H$  we obtain constants  $c_\rho, d_\rho \in \mathbb{R}^>$  such that for all  $t \geq a$ ,

$$\ell_\rho(t) = c_\rho \exp \left[ \int_a^t \gamma(s) ds \right], \quad \gamma_\rho(t) = d_\rho \exp \left[ - \int_a^t \lambda_\rho(s) ds \right].$$

Set  $\gamma := \text{Im}(z)$ , so  $\gamma^\dagger = -\lambda$ , and both  $\gamma$  and  $\lambda$  are already given as elements of  $\mathcal{C}_a^0$ . Since  $\gamma(t) > 0$  for all  $t \geq a$  we have a constant  $d \in \mathbb{R}^>$  such that for all  $t \geq a$ ,

$$\gamma(t) = d \exp \left[ - \int_a^t \lambda(s) ds \right].$$

The above estimate for  $\lambda = \text{Re}(z)$  gives

$$\lambda_\rho(t) < \lambda(t) < \lambda_\rho(t) + \gamma_\rho(t), \text{ eventually,}$$

so we have constants  $a_\rho, b_\rho \in \mathbb{R}$  such that

$$\int_a^t \lambda_\rho(s) ds < a_\rho + \int_a^t \lambda(s) ds < b_\rho + \int_a^t \lambda_\rho(s) ds + \int_a^t \gamma_\rho(s) ds, \text{ eventually,}$$

which by applying  $\exp(-*)$  yields

$$\frac{1}{d_\rho} \gamma_\rho(t) > \frac{1}{e^{a_\rho} d} \gamma(t) > \frac{c_\rho}{e^{b_\rho} d_\rho} \gamma_\rho(t) / \ell_\rho(t), \text{ eventually.}$$

Here the positive constant factors don't matter, since the valuation of  $\gamma_\rho$  is strictly increasing and that of  $\gamma_\rho / \ell_\rho = (1/\ell_\rho)'$  is strictly decreasing with  $\rho$ . Thus for all  $\rho$  we have  $\gamma_\rho > \gamma > (1/\ell_\rho)'$ , in  $C^{<\infty}$ . In view of Lemma 2.7 applied to  $H(\lambda)$ ,  $\gamma$  in the role of  $K$ ,  $f$  this yields an ordered subfield  $H(\lambda, \gamma)$  of  $C^{<\infty}$ . Moreover,  $\gamma$  is transcendental over  $H(\lambda)$  with  $\gamma^\dagger = -\lambda$ , and  $\gamma$  satisfies the second-order differential equation  $2yy'' - 3(y')^2 + y^4 - \omega y^2 = 0$  over  $H$  (obtained from the relation  $\sigma(\gamma) = \omega$  by multiplication with  $\gamma^2$ ). It follows that  $H(\lambda, \gamma)$  is closed under the derivation of  $C^{<\infty}$ , and hence  $H(\lambda, \gamma) = H\langle \lambda \rangle$  is a Hardy field.  $\square$

The proof also shows that every  $C^\infty$ -Hardy field has an  $\omega$ -free  $C^\infty$ -Hardy field extension, and the same with  $C^\omega$  instead of  $C^\infty$ .

REFERENCES

[1] M. Aschenbrenner, L. van den Dries, *Liouville closed H-fields*, J. Pure Appl. Algebra **197** (2005), 83–139.  
 [2] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries*, Ann. of Math. Stud. 195, Princeton Univ. Press, 2017.  
 [3] ———, *On numbers, germs, and transseries*, in: B. Sirakov et al. (eds.): *Proceedings of the International Congress of Mathematicians, Rio de Janeiro 2018*, vol. 2, pp. 19–42, World Scientific Publishing Co., Singapore, 2018.  
 [4] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.  
 [5] M. Boshernitzan, *An extension of Hardy's class L of "orders of infinity,"* J. Analyse Math. **39** (1981), 235–255.  
 [6] N. Bourbaki, *Fonctions d'une Variable Réelle*, Chapitre V, *Étude Locale des Fonctions*, Hermann, Paris, 1976.  
 [7] L. van den Dries, *An intermediate value property for first-order differential polynomials*, Quaderni di matematica **6** (2000), 95–105.  
 [8] G. H. Hardy, *Orders of Infinity*, 2nd ed., Cambridge Univ. Press, Cambridge, 1924.  
 [9] F. Hausdorff, *Die Graduierung nach dem Endverlauf*, Abh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **31** (1909), 295–334.  
 [10] J. van der Hoeven, *Transseries and Real Differential Algebra*, Lecture Notes in Math., vol. 1888, Springer-Verlag, New York, 2006.  
 [11] ———, *Transserial Hardy fields*, Astérisque **323** (2009), 453–487.  
 [12] A. Robinson, *On the real closure of a Hardy field*, in: G. Asser et al. (eds.), *Theory of Sets and Topology*, Deutscher Verlag der Wissenschaften, Berlin, 1972.  
 [13] M. Rosenlicht, *Hardy fields*, J. Math. Analysis and Appl. **93** (1983), 297–311.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, LOS ANGELES, CA 90095, U.S.A.

*E-mail address:* matthias@math.ucla.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, U.S.A.

*E-mail address:* vddries@math.uiuc.edu

ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE

*E-mail address:* vdhoeven@lix.polytechnique.fr