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# Matched asymptotics approach to the construction and justification of reduced graph models for 3D Maxwell's equations in networks of thin co-axial cables

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## Abstract

We consider electromagnetic wave propagation in domains constituted by thin coaxial cables (made of a dielectric material which surrounds a metallic inner-wire) and a small junction. The goal is to trim down 3D Maxwell's equations in this complicated geometry to a quantum graph (see [3]) in which, along each edge, one is reduced to compute the electrical potential and current by solving wave equations (the telegrapher's model) coupled by vertex conditions. In this work, using the method of matched asymptotics, we propose improved Kirchhoff conditions and we give a rigorous justification of such a model reduction.

**Keywords:** Maxwell's equations, telegrapher's equation, matched asymptotics, quantum graph.

## 1 The geometry

We consider a domain  $\Omega^\delta$ , with  $\delta > 0$ , which is homothetic to a (unbounded) reference domain namely

$$\Omega^\delta = \delta \Omega^1 \quad (1)$$

as described in Figure 1 where  $\Omega^1$  is the connected union of  $(L + 1)$  semi-infinite cables  $\Omega_\ell^1$  ( $\ell = 0 \dots L$ ) and a bounded junction  $J^1$  as illustrated by Figure 1. More precisely, each  $\Omega_\ell$  is isomorphic to  $S^\ell \times \mathbb{R}^+$ , where  $S^\ell$  is a non simply connected bounded domain of  $\mathbb{R}^2$  with one single hole. The "small" parameter  $\delta$  refers to the thinness of the propagation domain. When  $\delta \rightarrow 0$ ,  $\Omega^\delta$  converges to a graph  $\mathcal{G}$ , union of  $L$  half-lines  $D_\ell$ . In the following, we denote  $x_3^\ell \geq 0$  the abscissa along  $D_\ell$  and  $x_T^\ell = (x_1^\ell, x_2^\ell)$  associ-

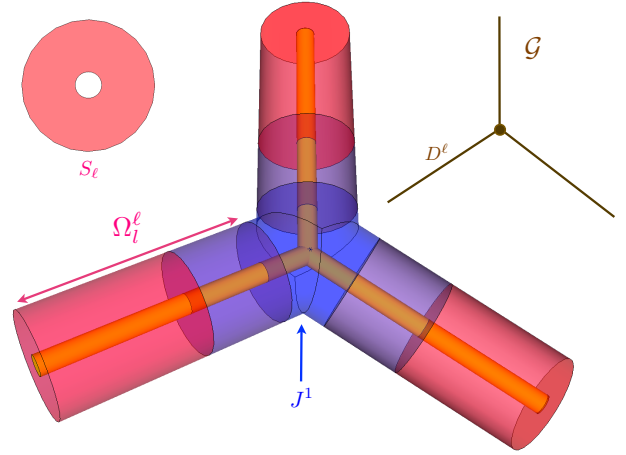


Figure 1: The reference domain  $\Omega^1$  (center) the limit graph  $\mathcal{G}$  (right) and a cross-section  $S_\ell$  (left)

ated transverse coordinates.

$$\Omega_\ell^\delta = \{(\delta x_T^\ell, x_3^\ell) \mid (x_T^\ell, x_3^\ell) \in \Omega_\ell^1\}$$

We are interested in the solution  $(E^\delta, H^\delta)$  of lossy 3D-Maxwell's equations in this domain, with constant coefficients for simplicity, and perfectly conducting boundary conditions along  $\partial\Omega^\delta$ . More precisely we wish to describe the behavior of this solution for small  $\delta$  from the solution of a 1D "effective model" on the limit graph.

## 2 The reduced model

We describe below only the behavior of the electromagnetic fields in the  $(L + 1)$  cables:

$$\begin{aligned} E^\delta(x_T^\ell, x_3^\ell, t) &\sim V_\ell^\delta(x_3^\ell, t) \nabla \varphi^\ell \left( \frac{x_T}{\delta} \right) \\ H^\delta(x_T^\ell, x_3^\ell, t) &\sim I_\ell^\delta(x_3^\ell, t) \nabla \psi^\ell \left( \frac{x_T}{\delta} \right) \end{aligned} \quad (2)$$

where the harmonic potentials  $\varphi^\ell$  and  $\psi^\ell$  are defined by elliptic problems in  $S^\ell$  (see [1]). The electrical potential  $V^\ell$  and current  $I^\ell$  are solutions of the telegrapher's equation ( $\partial_\ell \equiv \partial/\partial x_3^\ell$ ):

$$\begin{cases} (C^\ell \partial_t + G^\ell) \mathbf{V}_\ell^\delta + \partial_\ell \mathbf{I}_\ell^\delta = 0, \\ (L^\ell \partial_t + R^\ell) \mathbf{I}_\ell^\delta + \partial_\ell \mathbf{V}_\ell^\delta = 0, \end{cases} \quad \text{on } D_\ell \quad (3)$$

where  $C^\ell > 0, G^\ell \geq 0$  are explicitly given in term of  $\varphi^\ell$ , the permittivity  $\varepsilon$  and the electric conductivity  $\sigma_e$  while  $L^\ell > 0, R^\ell \geq 0$  are explicitly given in terms of  $\psi^\ell$ , the permeability  $\mu$  and the magnetic conductivity  $\sigma_m$ . The system has to be completed by vertex conditions.

At first order, these are the Kirchoff's laws

$$V_\ell^\delta(0, t) - V_0^\delta(0, t) = 0, \quad \sum_{\ell=0}^L I_\ell^\delta(0, t) = 0. \quad (4)$$

A better accuracy is obtained with second order conditions, namely **improved Kirchoff laws**

$$\begin{aligned} V_\ell^\delta(0, t) - V_0^\delta(0, t) &= \delta \sum_{\ell'=1}^L Z^{\ell, \ell'} \mathbf{I}_{\ell'}^\delta(0, t), \\ \sum_{\ell=1}^L I_\ell^\delta(0, t) + \delta Y V_0^\delta(0, t) &= 0. \end{aligned} \quad (5)$$

where the coefficient  $Y$  and the  $L \times L$  matrix  $\mathbf{Z} = (Z^{\ell, \ell'})$  are defined from the material properties of the medium and from 3D potentials  $\Phi$  and  $\{\Psi_\ell, 1 \leq \ell \leq L\}$  defined in the reference domain  $\Omega^1$ . These potentials are the solutions of elliptic equations in  $\Omega^1$  that are constrained to satisfy a specific non homogeneous behavior at infinity inside each cable  $\Omega^1$  (and, concerning the  $\Psi_\ell$ 's, non homogeneous jump conditions across  $L$  artificial cuts).

Note that the condition (4) only sees the structure of the limit graph while (5) also takes into account (partly) the geometry of the junction.

### 3 The method of analysis

The derivation of (3) and (4) or (5) relies on a preliminary asymptotic expansion of the solution. Since the problem is of multi-scale nature, a uniform asymptotic expansion in the whole domain is not possible. We use the method

of matched asymptotics, as in [2] for a simpler scalar case, which consists in looking for the electric field as follows:

- Far from the origin, for  $x_3^\ell > 0$ , we use the ansatz

$$E^\delta(x_T^\ell, x_3^\ell) = \sum_{p=0}^{\infty} \delta^p E_\ell^p\left(\frac{x_T}{\delta}, x_3^\ell\right) \quad (6)$$

where the fields  $E_\ell^p$  are defined in  $S_\ell \times \mathbb{R}^+$ .

- Close to the origin, we use the ansatz

$$E^\delta(\mathbf{x}) = \sum_{p=0}^{\infty} \delta^p E^p\left(\frac{\mathbf{x}}{\delta}\right) \quad (7)$$

where the fields  $E^p$  are defined in  $\Omega^1$ .

Our models are obtained only by looking at  $p = 0, 1$ . Using (6) leads to the construction of equation (3) (see [1]). To obtain (4) and (5), we need to express the fact that the two expansions (6) and (7) must match.

In addition, it is possible to obtain error estimates. More precisely, denoting  $(E_{app}^\delta, H_{app}^\delta)$  the right hand side of (2) with  $(V^\delta, I^\delta)$  defined as the solution of (3, 4) or (3, 5), one can show that, in appropriate energy norms

$$\|E_{app}^\delta - E^\delta\| \leq C \delta^k \|E^\delta\| \quad (8)$$

where  $k = 1$  for (3, 4) and  $k = 2$  for (3, 5), modulo, in this second case, a post-treatment which consists in adding a  $O(\delta)$  longitudinal component to the transverse electromagnetic field defined by the right hand side of (2).

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