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GEOMETRIC REALIZATIONS OF TAMARI INTERVAL LATTICES VIA CUBIC COORDINATES

CAMILLE COMBE

ABSTRACT. We introduce cubic coordinates, which are integer words encoding intervals in the Tamari lattices. Cubic coordinates are in bijection with interval-posets, themselves known to be in bijection with Tamari intervals. We show that in each degree the set of cubic coordinates forms a lattice, isomorphic to the lattice of Tamari intervals. Geometric realizations are naturally obtained by placing cubic coordinates in space, highlighting some of their properties. We consider the cellular structure of these realizations. Finally, we show that the poset of cubic coordinates is shellable.

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INTRODUCTION

The Tamari lattices are partial orders having extremely rich combinatorial and algebraic properties. These partial orders are defined on the set of binary trees and rely on the rotation operation [Tam62]. We are interested in the intervals of these lattices, meaning

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the pairs of comparable binary trees. Tamari intervals of size n also form a lattice. The number of these objects is given by a formula that was proved by Chapoton [Cha06]:

$$\frac{2(4n + 1)!}{(n + 1)!(3n + 2)!}. \quad (0.0.1)$$

Strongly linked with associahedra, Tamari lattices have been recently generalized in many ways [BPR12, PRV17]. In this process, the number of intervals of these generalized lattices have also been enumerated through beautiful formulas [BMFPR12, FPR17]. Many bijections between Tamari intervals of size n and other combinatorial objects are known. For instance, a bijection with planar triangulations is presented by Bernardi and Bonichon in [BB09]. It has been proved by Châtel and Pons that Tamari intervals are in bijection with interval-posets of the same size [CP15].

We provide in this paper a new bijection with Tamari intervals, which is inspired by interval-posets. More precisely, we first build two words of size n from the Tamari diagrams [Pal86] of a binary tree. If they satisfy a certain property of compatibility, we build a Tamari interval diagram from these two words. We show that Tamari interval diagrams and interval-posets are in bijection. Then we propose a new encoding of Tamari intervals, by building $(n - 1)$ -tuples of numbers from Tamari interval diagrams. We call these tuples cubic coordinates. This new encoding has two obvious virtues: it is very compact and it gives a way of comparing in a simple manner two Tamari intervals, through a fast algorithm. On the other hand, some properties of Tamari intervals translate nicely in the setting of cubic coordinates. For instance, synchronized Tamari intervals [FPR17] become cubic coordinates with no zero entry. Besides, cubic coordinates provide naturally a geometric realization of the lattice of Tamari intervals, by seeing them as space coordinates. Indeed, all cubic coordinates of size n can be placed in the space \mathbb{R}^{n-1} . By drawing their cover relations, we obtain an oriented graph. This gives us a realization of cubic coordinate lattices, which we call cubic realization. This realization leads us to many questions, in particular about the cells it contains. We characterize these cells in a combinatorial way, and we deduce a formula to compute a volume of the cubic realization in the geometrical sense. Another direction, more topological, involves the shellability of partial order. We show, drawing inspiration from the work of Björner and Wachs [BW96, BW97], that the cubic coordinates poset is EL-shellable, and as a consequence its associated complex is shellable.

This article is organized in three sections. The first section is dedicated to recalls of classical notions of combinatorics, and sets out the conventions used. Because of its key role in this work, the bijection between Tamari intervals and interval-posets is also recalled in this section. In the second section, we define the Tamari interval diagrams and show that their in bijection, size by size, with interval-posets. We define the cubic coordinates show that their sets are in bijection with the set of Tamari interval diagrams. Using the two bijections of the two previous parts, and after having provided the set of cubic coordinates with a partial order, we show that there is a poset isomorphism between the poset of cubic coordinates and the poset of Tamari intervals. As pointed out above, the poset of the cubic coordinates can then be realized geometrically. This cubic realization and the

cells that compose it are the object of the third section. For each cell, we then associate a synchronous cubic coordinate. By relying upon this particular cubic coordinate, we give a formula to compute the volume of the cubic realization. Finally, we extend the result of Björner and Wachs on the Tamari posets to the Tamari interval posets, by showing that the cubic coordinate posets are EL-shellable.

This article is a complete version of [Com19]. All the proofs are given and several new results are presented, such as the result of the EL-shellability of cubic coordinate posets.

General notations and conventions. Throughout this article, for all words u , we denote by u_i the i -th letter of u . For any integers i and j , $[i, j]$ denotes the set $\{i, i + 1, \dots, j\}$. For any integer i , $[i]$ denotes the set $[1, i]$. All sets considered in this article are finite.

1. TAMARI INTERVALS AND GENERAL PROPERTIES

In this first section we provide some basic notions of combinatorics and the conventions used afterwards. For this, we recall the definitions of posets, lattices, binary trees, Tamari intervals and interval-posets. Also, we recall the bijection given in [CP15].

1.1. Posets. A *partially ordered set*, commonly called *poset*, is a pair $(\mathcal{P}, \preceq_{\mathcal{P}})$ where \mathcal{P} is a set and $\preceq_{\mathcal{P}}$ is a partial order on \mathcal{P} .

A *lattice* is a poset $(\mathcal{L}, \preceq_{\mathcal{L}})$ satisfying, for all $x, y \in \mathcal{L}$, the two following conditions

- (i) the supremum $\vee_{\mathcal{L}}(x, y) := \min_{\preceq_{\mathcal{L}}} \{z \in \mathcal{L} : x \preceq_{\mathcal{L}} z \text{ et } y \preceq_{\mathcal{L}} z\}$ exists;
- (ii) the infimum $\wedge_{\mathcal{L}}(x, y) := \max_{\preceq_{\mathcal{L}}} \{z \in \mathcal{L} : z \preceq_{\mathcal{L}} x \text{ et } z \preceq_{\mathcal{L}} y\}$ exists.

The *poset of intervals* of a poset $(\mathcal{P}, \preceq_{\mathcal{P}})$ is the poset $(I(\mathcal{P}), \preceq_{I(\mathcal{P})})$ on the set $I(\mathcal{P}) := \{(x, y) \in \mathcal{P}^2 : x \preceq_{\mathcal{P}} y\}$ with the partial order $\preceq_{I(\mathcal{P})}$ defined by

$$(x, y) \preceq_{I(\mathcal{P})} (x', y') \text{ if and only if } x \preceq_{\mathcal{P}} x' \text{ and } y \preceq_{\mathcal{P}} y' \quad (1.1.1)$$

for all $(x, y), (x', y') \in I(\mathcal{P})$. Let us recall the following result.

Lemma 1.1.1. *If $(\mathcal{L}, \preceq_{\mathcal{L}})$ is a lattice then $(I(\mathcal{L}), \preceq_{I(\mathcal{L})})$ is a lattice.*

Proof. Let $(x, y), (x', y') \in I(\mathcal{L})$. To show that $(\vee_{\mathcal{L}}(x, x'), \vee_{\mathcal{L}}(y, y'))$ is a lattice it must be verified that $\vee_{\mathcal{L}}(x, x') \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$. By definition of the supremum one has $y \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$ and $y' \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$. Furthermore $x \preceq_{\mathcal{L}} y$ and $x' \preceq_{\mathcal{L}} y'$. Then $x \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$ and $x' \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$. In addition, $\vee_{\mathcal{L}}(x, x')$ is the minimal element of \mathcal{L} satisfying $x \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(x, x')$ and $x' \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(x, x')$. Thus $\vee_{\mathcal{L}}(x, x') \preceq_{\mathcal{L}} \vee_{\mathcal{L}}(y, y')$.

From the equation (1.1.1), one has

$$\begin{aligned} & \vee_{I(\mathcal{L})}((x, y), (x', y')) \\ &= \min_{\preceq_{I(\mathcal{L})}} \{(x'', y'') \in I(\mathcal{L}) : (x, y) \preceq_{I(\mathcal{L})} (x'', y'') \text{ and } (x', y') \preceq_{I(\mathcal{L})} (x'', y'')\} \\ &= \min_{\preceq_{I(\mathcal{L})}} \{(x'', y'') \in I(\mathcal{L}) : x \preceq_{\mathcal{L}} x'' \text{ and } y \preceq_{\mathcal{L}} y'' \text{ and } x' \preceq_{\mathcal{L}} x'' \text{ and } y' \preceq_{\mathcal{L}} y''\} \\ &= (\vee_{\mathcal{L}}(x, x'), \vee_{\mathcal{L}}(y, y')). \end{aligned} \quad (1.1.2)$$

The case of the infimum $\wedge_{I(\mathcal{L})}((x, y), (x', y')) = (\wedge_{\mathcal{L}}(x, x'), \wedge_{\mathcal{L}}(y, y'))$ is symmetrical. \square

Let $x, y \in \mathcal{L}$ such that $x \preceq_{\mathcal{L}} y$. The element y *covers* x for the partial order $\preceq_{\mathcal{L}}$ if and only if for all $z \in \mathcal{L}$ such that $x \preceq_{\mathcal{L}} z \preceq_{\mathcal{L}} y$ either $z = x$ or $z = y$. If y covers x then it is a *covering relation* for the partial order $\preceq_{\mathcal{L}}$. In the same way for $(x, y), (x', y') \in I(\mathcal{L})$ such that $(x, y) \preceq_{I(\mathcal{L})} (x', y')$, a covering relation for the partial order $\preceq_{I(\mathcal{L})}$ is defined.

1.2. Binary trees. In all this work, we call *binary tree* a planar, complete, and rooted binary tree. Recall that a binary tree T is either a *leaf* or an *internal node* attached through two edges to two binary trees called respectively *left subtree* and *right subtree* of T . One denotes by $\text{Ta}(n)$ the set of binary trees with n internal nodes. In the following, we use the usual conventions and notions of binary trees [Sta12]: *root, child, descendant, subtree, etc.* We draw the binary trees with the root at the top and the leaves at the bottom.

Let $T \in \text{Ta}(n)$. Each internal node of T is numbered recursively, starting with the left subtree, then the root, and ending with the right subtree. An example is given in Figure 1. This numbering then establishes a total order on the internal nodes of a binary tree called *infix order*. Afterwards, this numbering is used to refer to the internal nodes. The path following this numbering is called *infix traversal*.

The *canopy* of T is the word of size $n - 1$ on the alphabet $\{0, 1\}$ built by assigning to each leaf of T a letter as follows. Any leaf oriented to the left (resp. right) is labeled by 0 (resp. 1). The canopy of T is the word obtained by reading from left to right the labels thus established, forgetting the first and the last one. For instance, the binary tree in Figure 1 has for canopy the word 0110100.

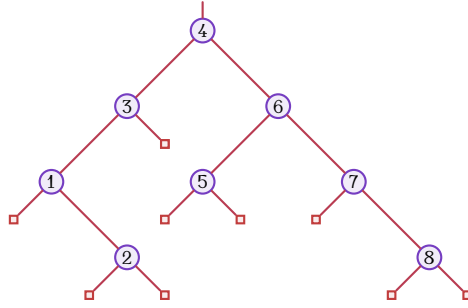


FIGURE 1. A binary tree of size 8 and the numbering of its nodes following the infix order.

Now let us recall a fundamental operation in binary trees, the *right rotation*. Let k and l be the indices in infix order of two nodes of a binary tree T , such that the node k is left child of the node l . Right rotation locally changes the tree T so that l becomes the right child of k (see Figure 2). Equivalently this means that $((A, B), C)$ becomes $(A, (B, C))$ where A, B and C are the subtrees shown in Figure 2.

1.3. Tamari intervals and interval-posets. Let $n \geq 0$ and $S, T \in \text{Ta}(n)$. We set $S \preceq_{\circlearrowright} T$ if and only if T can be obtained by successively applying one or more right rotations

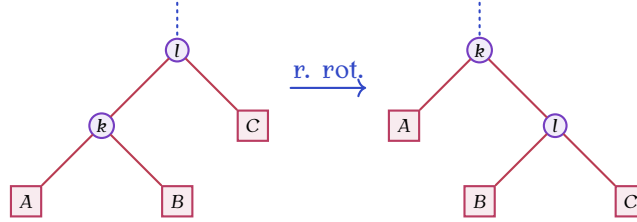


FIGURE 2. Right rotation of edge (k, l) in T (left), where A, B , and C are any subtrees.

in S . The set $\text{Ta}(n)$ endowed by the partial order \preceq_{\circ} is a lattice, known as *Tamari lattice* [Tam62].

A *Tamari interval* is formed by a pair of binary trees $[S, T]$ such that $S \preceq_{\circ} T$ and $S, T \in \text{Ta}(n)$. The size of a Tamari interval is the size of the binary trees involved. The set of Tamari intervals of size n is denoted by $I(\text{Ta})(n)$. The poset of Tamari intervals is the set of Tamari intervals endowed with the following partial order \preceq_{∞} . Let $n \geq 0$ and $[S, T], [S', T'] \in I(\text{Ta})(n)$, we set $[S, T] \preceq_{\infty} [S', T']$ if and only if $S \preceq_{\circ} S'$ and $T \preceq_{\circ} T'$. According to Lemma 1.1.1, the poset so defined is a lattice. Moreover $[S', T']$ covers $[S, T]$ if and only if

- (1) either S' is obtained by a single right rotation of an edge in S and $T' = T$;
- (2) or T' is obtained by a single right rotation of an edge in T and $S' = S$.

The set of interval-posets is known to be in bijection with the set of Tamari intervals [CP15]. Let us recall the definition of these objects and a part of the bijection in the broad outline which will be useful to us thereafter.

Let $n \geq 0$ and $X_n := \{x_1, \dots, x_n\}$ be a set of n symbols numbered from 1 to n . An *interval-poset* $P = (X_n, \triangleleft)$ of size n is a partial order \triangleleft on the set X_n such that

- (i) if $i < k$ and $x_k \triangleleft x_i$ then for all x_j such that $i < j < k$, one has $x_j \triangleleft x_i$;
- (ii) if $i < k$ and $x_i \triangleleft x_k$ then for all x_j such that $i < j < k$, one has $x_j \triangleleft x_k$.

The set of interval-posets of size n is denoted by $\text{IP}(n)$.

The two conditions (i) and (ii) of interval-posets are referred to as *interval-poset properties*. For any $i < j$, the relations $x_j \triangleleft x_i$ are known as *decreasing relations* and the relations $x_i \triangleleft x_j$ are known as *increasing relations*. The elements of X_n are called vertices. Contrary to the definition given in [CP15] where the vertices are the set $\{1, \dots, n\}$, we use in this article $\{x_1, \dots, x_n\}$ to simplify the distinction between the vertices and their positions. Thereafter the set X_n will be confused with the interval-poset P .

Another difference with some of the articles dealing with the interval-posets is the way in which they are graphically represented. For any $i < j$, if $x_j \triangleleft x_i$ and there is no x_k vertex such that $x_k \triangleleft x_i$ and $j < k$, then we draw an arrow with source x_j and goal x_i from below as shown in the example in Figure 3. Symmetrically, if $x_j \triangleleft x_k$ and $j < k$ and if there is no x_i such that $x_i \triangleleft x_k$ and $i < j$, then we draw an arrow with source x_j and goal x_k from above. We refer to this oriented graph with two types of arrows as

the *minimalist representation*. The closure for the interval-poset properties is given by adding the decreasing relations $x_j \triangleleft x_i$ for any relation $x_k \triangleleft x_i$ and by adding the increasing relations $x_j \triangleleft x_k$ for any relation $x_i \triangleleft x_k$, for any $i < j < k$. By taking the reflective closure and the closure for the interval-poset properties, an interval-poset is obtained from such a representation. The interest of the minimalist representation is later justified, in particular by Theorem 2.1.2. It is important to represent the decreasing relations and the increasing relations independently.

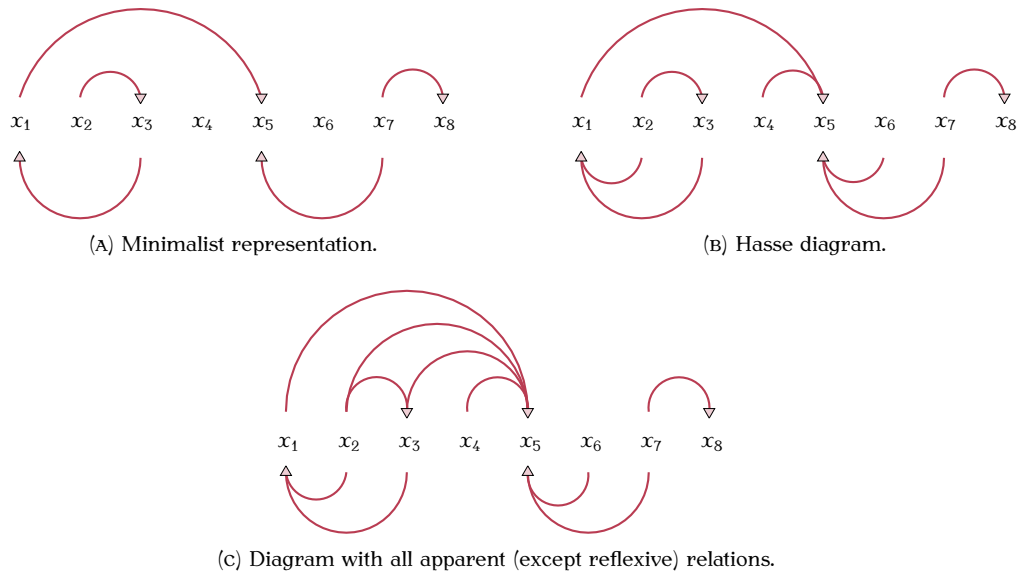
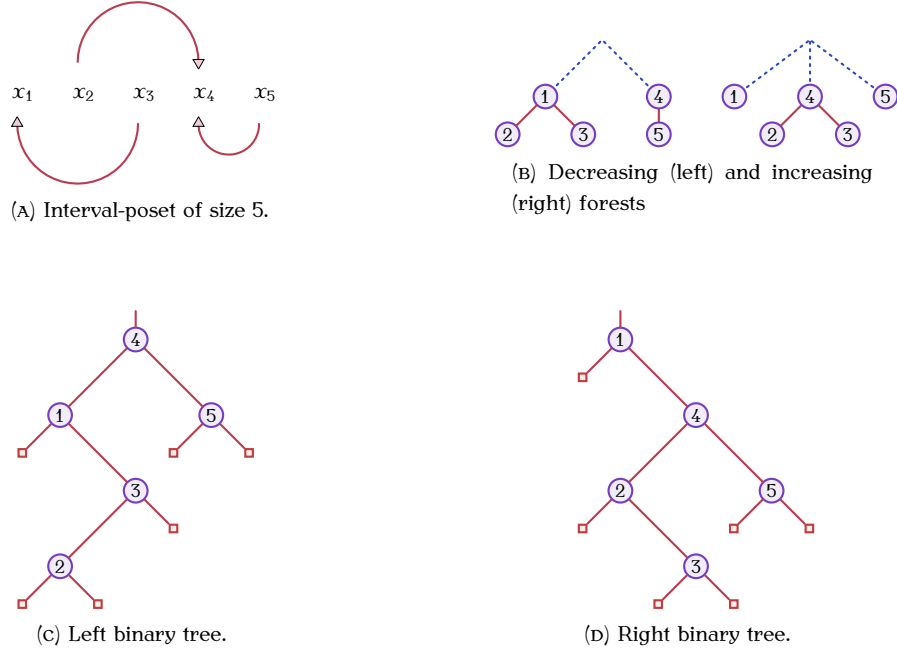


FIGURE 3. Different representations of an interval-poset of size 8.

In [CP15] a bijection ρ is built between the set of Tamari intervals and the set $\text{IP}(n)$. We recall here the passage from an interval-poset to a Tamari interval. Let $n \geq 0$ and $[S, T] \in \text{I}(\text{Ta})(n)$ and $P \in \text{IP}(n)$. The bijection ρ relates on the one hand the restriction of P to its decreasing relations with the binary tree S , and on the other hand the restriction of P to its increasing relations with the binary tree T .

Thus the restriction of P to its decreasing (resp. increasing) relations has a decreasing (resp. increasing) forest as Hasse diagram, where if $x_j \triangleleft x_i$ with $i < j$ (resp. $j < i$), then the node j is a descendant of the node i . Otherwise, it is placed to the right (resp. left) of the node i . To form the binary tree S (resp. T), then read the decreasing (resp. increasing) forest for the prefix transversal from right to left (resp. from left to right). If a node j is a descendant of a node i in the decreasing (resp. increasing) forest, then the node j becomes a right (resp. left) descendant of the node i in S (resp. T). Otherwise, it becomes the left (resp. right) descendant of the node i . The numbering of the binary trees thus obtained is exactly the infix order. Figure 4 gives an example of construction by the bijection ρ of a Tamari interval from an interval-poset of size 5.


 FIGURE 4. Construction of a Tamari interval from an interval-poset by ρ .

Let $n \geq 0$ and $P, P' \in \text{IP}(n)$ and $[S, T] := \rho(P)$, $[S', T'] := \rho(P')$. Let (\star) (resp. (\diamond)) the following condition: P' is obtained by adding (resp. removing) only decreasing (resp. increasing) relations of goal a vertex x_k in P , such that if only one of these decreasing (resp. increasing) relations is removed (resp. added), then either P is obtained or the object obtained is not an interval-poset.

Lemma 1.3.1. *The interval-posets P and P' satisfy (\star) (resp. (\diamond)) for the vertex x_k (resp. x_l) if and only if S' (resp. T') is obtained by a unique right rotation of the edge (k, l) in S (resp. T) and $T' = T$ (resp. $S' = S$).*

In other words $[S', T']$ covers $[S, T]$ if and only if P and P' satisfy either (\star) or (\diamond) .

Proof. Suppose P and P' satisfy (\star) for the vertex x_k . Therefore, P' has more decreasing relations of goal x_k than the vertex x_k in P . Suppose the vertices x_l and x_k are not related in P and are related in P' , with $k < l$. Then, by the interval-poset property (i), for any vertex of index k' such that $k < k' < l$, $x_{k'} \triangleleft x_k$. Moreover, if we remove only one of these decreasing relations, we obtain either P or an object that is no longer an interval-poset. This means that the number of descending relations added in P' is minimal, or equivalently, that the vertex x_l is closest to the vertex x_k such that x_l and x_k are not related in P and $k < l$. This case is depicted in Figure 6. By the bijection ρ , the adding of these decreasing relations of goal x_k in P leads to the decreasing forest induced by S' represented by Figure 5b. A unique right rotation is then made between the trees S and S' (see Figure 5a). Furthermore, since the increasing relations are unchanged between P and P' , the increasing forests induced by T and T' are the same, and thus $T' = T$.

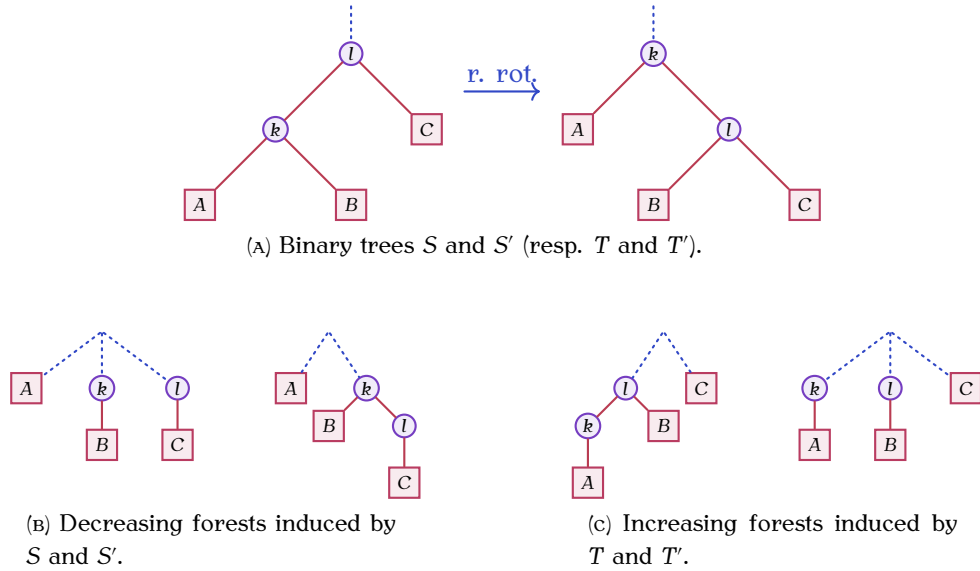


FIGURE 5. Right rotation of the edge (k, l) in the binary tree S (resp. T), where A, B and C are subtrees.

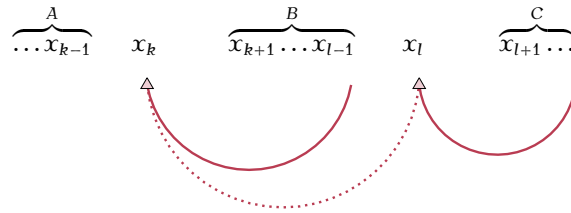


FIGURE 6. Interval-poset of the decreasing forest before (without dotted line) and after (with dotted line) the right rotation of the edge (k, l) , where A, B and C may be empty.

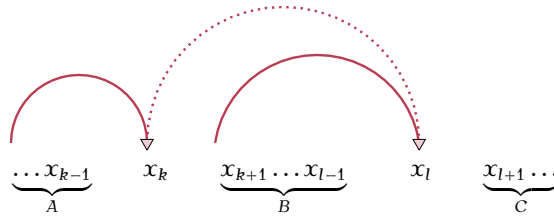


FIGURE 7. Interval-poset of the increasing forest before (with dotted lines) and after (without dotted lines) the right rotation of the edge (k, l) , where A, B and C may be empty.

Reciprocally, suppose that S' is obtained by a unique right rotation of the edge (k, l) in S and that $T' = T$. The case is depicted by Figure 5a, and the two decreasing forests induced by S and S' are depicted by Figure 5b. By the bijection ρ , we then obtain the interval-poset whose restriction to decreasing relations is shown by Figure 6. Since $T' = T$, the increasing relations of the interval-posets associated with $[S, T]$ and $[S', T']$ are the same. Finally, P is obtained by adding only decreasing relations of goal x_k in P . Furthermore, if only one of these relations is removed, then either P is obtained, or the object obtained is not an interval-poset. This means that P and P' satisfy (\star) .

Symmetrically, we show that P and P' satisfy (\diamond) for x_l if and only if T' is obtained by a unique right rotation of the edge (k, l) in T and $S' = S$. Figure 5c and Figure 7 depicts this case. \square

2. CUBIC COORDINATES AND TAMARI INTERVALS

The aim of this section is to build the poset of the cubic coordinates, then to establish the poset isomorphism between this poset and the poset of the Tamari intervals. To achieve this goal, we first define the Tamari interval diagrams based on the interval-posets. The cubic coordinates are then obtained from the Tamari interval diagrams.

2.1. Tamari interval diagrams. In this part, we recall the definition of Tamari diagrams [Pal86]. By checking a certain compatibility condition, we build the Tamari interval diagrams. Afterwards, we show that there is a bijection between the set of Tamari interval diagrams and the set of interval-posets of the same size.

Recall the definition of a Tamari diagram, as formulated in [BW97]. For any $n \geq 0$, a *Tamari diagram* is a word u of length n on the alphabet \mathbb{N} which satisfies the two following conditions

- (i) $0 \leq u_i \leq n - i$ for all $i \in [n]$;
- (ii) $u_{i+j} \leq u_i - j$ for all $i \in [n]$ and $j \in [0, u_i]$.

The size of a Tamari diagram is its number of letters. For instance, the fourteen Tamari diagrams of size 4 are

0000, 0010, 0100, 0200, 0210, 1000, 1010, 2000, 2100, 3000, 3010, 3100, 3200, 3210.

Let $n \geq 0$. The set of Tamari diagrams of size n is in bijection with the binary trees with n nodes. Indeed, one builds from a Tamari diagram u of size n a binary tree S recursively as follows. If $n = 0$, S is defined as the leaf. Otherwise, let i be the smallest position in u such that u_i is the maximum allowed value, namely $n - i$. Then $A := u_1 \dots u_{i-1}$ and $B := u_{i+1} \dots u_n$ are also Tamari diagrams. One forms S by grafting the binary trees obtained recursively by this process applied on A and on B to a new node. Reciprocally, for each internal node of index i of the tree S , labeled with an infix transversal, the value of the i -th letter of the corresponding Tamari diagram is given by the number of internal nodes in the right subtree of the node i . The complete demonstration is given in [Pal86].

In the literature, Tamari diagrams are also known as bracket vectors, objects inspired by the right parantheses introduced in [HT72] by Huang and Tamari.

Our aim is to encode a pair of binary trees of n nodes by two words of size n . In order to do that, we need to introduce dual Tamari diagrams. The first binary tree of the pair is encoded by its Tamari diagram and the second is encoded by its dual Tamari diagram.

For any $n \geq 0$, a *dual Tamari diagram* is a word v of length n on the alphabet \mathbb{N} which satisfies the two following conditions

- (i) $0 \leq v_i \leq i - 1$ for all $i \in [n]$;
- (ii) $v_{i-j} \leq v_i - j$ for all $i \in [n]$ et $j \in [0, v_i]$.

The size of a dual Tamari diagram is its number of letters. A word $v = v_1 \dots v_n$ is a dual Tamari diagram if and only if $v_n \dots v_1$ is a Tamari diagram.

The construction of the binary tree T is also recursive, except that it is the maximum position i in the dual Tamari diagram whose value is the highest allowed on that section of the word that should be chosen first. Similarly for the reciprocal, the procedure is identical, except that the value of the i -th letter in the dual Tamari diagram is given by the number of internal nodes in the left subtree of the node i in the tree T .

Note that the conditions for Tamari diagrams and dual Tamari diagrams imply that the last letter of a Tamari diagram is always equal to 0 and the first letter of a dual Tamari diagram is always equal to 0.

The graphical representation of a Tamari diagram of size n by needles and their diagonals provided in [Gir11] is a simple way to check the conditions of a Tamari diagram. Each letter in the Tamari diagram is represented by a needle whose height is equal to the value of the letter. At the top of each needle of position i and height h , a diagonal of slope -1 is drawn, which intersects the horizontal axis at the point $i + h$ (see Figure 8). The condition (ii) is then translated by the fact that no needle must cross a diagonal. The condition (i) is satisfied if, for any needle at position i , the abscissa point $i + h$ of the intersection with the diagonal is less than or equal to n .

Similarly, a graphical representation can be given for the dual Tamari diagram of size n . For each needle of position i and height h , a diagonal of slope 1 is drawn between the top of this needle and the abscissa point $i - h$. The condition (ii) is satisfied if no needle crosses a diagonal. The condition (i) is satisfied if for any needle at position i , the abscissa point $i - h$ of the intersection with the diagonal is greater than or equal to 1.

Let $n \geq 0$ and u be a Tamari diagram, and v be a dual Tamari diagram, both of size n . The u and v diagrams are *compatible* if for all $1 \leq i < j \leq n$ such that $u_i \geq j - i$ then $v_j < j - i$. If u and v are compatible, then the pair (u, v) is called *Tamari interval diagram*. The set of Tamari interval diagrams of size n is denoted by $\text{TID}(n)$.

In other words, a Tamari diagram u of size n and a dual Tamari diagram v of size n are compatible if for any needle of position i and height $v_i \neq 0$ in v (resp. $u_i \neq 0$ in u), there is no needle of position j and height greater than or equal to $i - j$ in u (resp. $j - i$ in v) with $i - v_i \leq j \leq i - 1$ (resp. $i + 1 \leq j \leq i + u_i$) and $i \in [n]$. For example, the two diagrams in Figure 8 are compatible. Figure 9 gives two other examples of two incompatible and two compatible diagrams. Thereafter, if u and v are compatible, we can also say that u and v satisfy the compatibility condition.

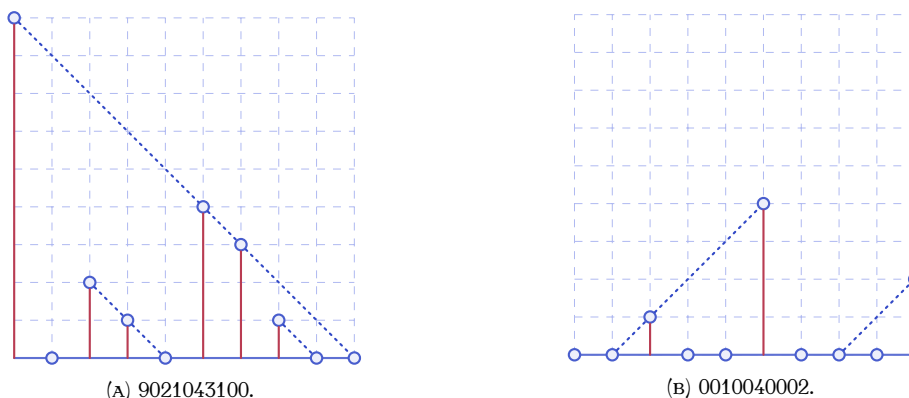


FIGURE 8. A Tamari diagram (left) and a dual Tamari diagram (right) of size 10.

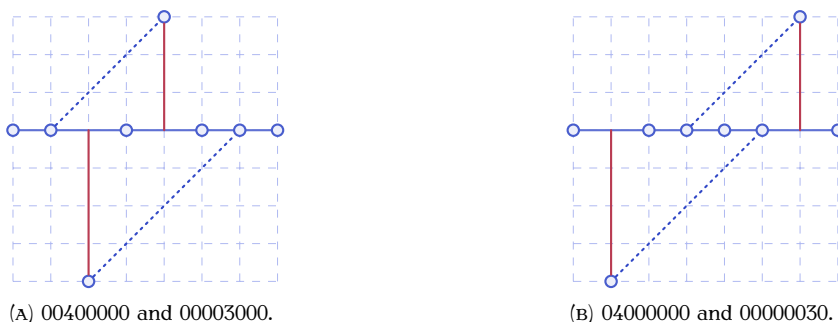


FIGURE 9. Two incompatible diagrams (left) and two compatible diagrams (right).

A graphical representation of the Tamari interval diagram is also possible, as shown in Figure 9. Figure 10 gives this representation for the two diagrams seen in Figure 8, where we have simply considered the symmetry relative to the abscissa axis of the Tamari diagram, and placed it under its dual. Thus, Tamari diagram u is drawn below (in blue) and his dual v is drawn above (in red). With such a representation, it is then easy to verify that u and v are compatible. Indeed, any needle of u that is below the diagonal linking the top of the needle in position j in v to the abscissa point $j - v_j$, has a diagonal that intersects the abscissa axis strictly before the position j . Symmetrically, any needles of v that are under a diagonal linking the top of the needle in position i in u to the abscissa point $i + u_i$, has a diagonal that intersects the abscissa axis strictly before the position i .

One consequence of the compatibility condition is that each needle of non-zero height in the dual Tamari diagram v is always preceded by a needle of u of zero height. Symmetrically, each non-zero height needle in the Tamari diagram u is always followed by a needle of v of zero height. In other words, for any $i \in [n]$, u_i and v_{i+1} can both be zero, but cannot both be non-zero.

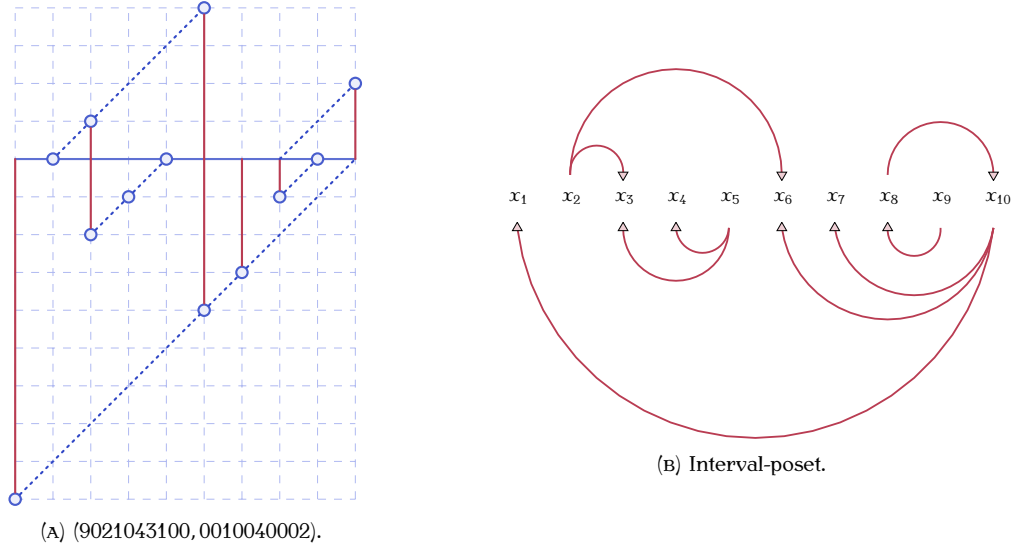


FIGURE 10. Tamari interval diagram of size 10 and its associated interval-poset.

Let $n \geq 0$ and χ be the map sending a Tamari interval diagram (u, v) of size n to the relation

$$(\{x_1, \dots, x_n\}, \triangleleft) \quad (2.1.1)$$

where $x_{i+l} \triangleleft x_i$ for all $i \in [n]$ and $0 \leq l \leq u_i$, and $x_{i-k} \triangleleft x_i$ for all $i \in [n]$ and $0 \leq k \leq v_i$.

Proposition 2.1.1. *For any $n \geq 0$, the map χ has values in $\text{IP}(n)$.*

Proof. Let $(u, v) \in \text{TID}(n)$ and $P := \chi(u, v)$. First, we show that \triangleleft is a partial order, then that interval-poset properties are satisfied.

- (1) By definition of χ one has $x_{i+l} \triangleleft x_i$ and $x_{i-k} \triangleleft x_i$ with $0 \leq l \leq u_i$ and $0 \leq k \leq v_i$ for all $x_i \in P$. Specifically, $x_i \triangleleft x_i$. This shows that P is reflexive.
- (2) Let x_i, x_j and x_k be vertices of P with $i < j < k$.
 - (a) Suppose that $x_j \triangleleft x_i$ and that $x_k \triangleleft x_j$. Then $x_j \triangleleft x_i$ implies that there is an integer $0 \leq i' \leq u_i$ such that $j = i + i'$. Therefore, by the condition (ii) of a Tamari diagram, $u_j = u_{i+i'} \leq u_i - i'$. Likewise, $x_k \triangleleft x_j$ implies that there is an integer $0 \leq j' \leq v_j$ such that $k = j + j'$. Still by the same condition, one has $u_k = u_{j+j'} \leq u_j - j'$. By using these two inequalities, we obtain that $u_i \geq u_k + i' + j'$. Since $i' + j' = k - i$, then we have $u_i \geq k - i$, which implies by definition of χ that $x_k \triangleleft x_i$ in P .
 - (b) Suppose that $x_j \triangleleft x_i$ and that $x_i \triangleleft x_k$. Therefore, $x_j \triangleleft x_k$ because $x_i \triangleleft x_k$ implies that for each vertex between x_i and x_k is in relation with x_k .
 - (c) Suppose that $x_i \triangleleft x_j$ and that $x_j \triangleleft x_k$. Then $x_i \triangleleft x_j$ implies that there is an integer $0 \leq i' \leq v_i$ such that $i = j - i'$. By the condition (ii) of a dual Tamari diagram, $v_i = v_{j-i'} \leq v_j - i'$. Likewise, $x_j \triangleleft x_k$ implies that there is an integer $0 \leq j' \leq v_j$ such that $j = k - j'$. By the same condition (ii), $v_j = v_{k-j'} \leq v_k - j'$.

By these two inequalities, one has $v_k \geq v_i + i' + j'$. Since $i' + j' = k - i$, one has $v_k \geq k - i$, which implies by definition of χ that $x_i \triangleleft x_k$ in P .

- (d) Suppose that $x_j \triangleleft x_k$ and that $x_k \triangleleft x_i$. Then $x_j \triangleleft x_i$ because $x_k \triangleleft x_i$ implies that all vertex between x_i and x_k is in relation with x_i .

This shows that P is transitive. Note that it is impossible to have the case $x_i \triangleleft x_k$ and $x_k \triangleleft x_j$ since P is the image of a Tamari interval diagram. Getting this case would contradict the fact that u and v are compatible. Similarly, the case $x_i \triangleleft x_j$ and $x_k \triangleleft x_i$ is impossible.

- (3) Let $i < j$ and x_i, x_j be vertices of P . Suppose that $x_j \triangleleft x_i$ and that $x_i \triangleleft x_j$. By definition of χ , $x_j \triangleleft x_i$ if and only if $u_i \geq j - i$. Likewise, $x_i \triangleleft x_j$ if and only if $v_j \geq j - i$. However, since u and v are compatible, this case is impossible. This shows that P is antisymmetric.
- (4) The definition of χ implies directly that P satisfied the interval-poset properties, namely that for all x_i, x_j and x_k vertices of P with $i < j < k$, if $x_k \triangleleft x_i$ then $x_j \triangleleft x_i$, and if $x_i \triangleleft x_k$ then $x_j \triangleleft x_k$.

Finally, P is an interval-poset. □

Theorem 2.1.2. *For any $n \geq 0$, the map $\chi : \text{TID}(n) \rightarrow \text{IP}(n)$ is bijective.*

Proof. Let $P \in \text{IP}(n)$ and let $(u, v) \in \mathbb{N}^n \times \mathbb{N}^n$ be a pair of words, such that for all $i \in [n]$,

$$u_i := \#\{x_j \in P : x_j \triangleleft x_i \text{ and } i < j\}; \quad (2.1.2)$$

$$v_j := \#\{x_i \in P : x_i \triangleleft x_j \text{ and } i < j\}. \quad (2.1.3)$$

Let us show that this pair of words (u, v) is a Tamari interval diagram and that its image by χ gives P .

- (1) Since P is an interval-poset, there are at most $n - i$ vertices of P in decreasing relation to x_i and at most $i - 1$ vertices of P in increasing relation to x_i for all $i \in [n]$. Therefore, the condition (i) of a Tamari diagram and (i) of a dual Tamari diagram are satisfied.
- (2) Let x_i and x_{i+j} be vertices of P such that $i \in [n]$ and $j \in [0, u_i]$. The fact that $u_i \geq j$ means according to the equation (2.1.2) that there are at least j vertices in decreasing relation to the vertex x_i , that is $x_{i+j} \triangleleft x_i$. Thus by transitivity of interval-posets, one has that for any $i + j \leq k \leq n$, if $x_k \triangleleft x_{i+j}$ then $x_k \triangleleft x_i$. Thus $u_{i+j} + j \leq u_i$, which satisfies the condition (ii) of a Tamari diagram. Symmetrically, the condition (ii) of a dual Tamari diagram is checked by considering x_i and x_{i-j} vertices of P such that $i \in [n]$ and $j \in [0, v_i]$.
- (3) Let $1 \leq i < j \leq n$ such that $u_i \geq j - i$. Suppose that $v_j \geq j - i$. The relation $u_i \geq j - i$ means that there are $j - i$ vertices of P in decreasing relation to x_i , meaning $x_j \triangleleft x_i$. Likewise, the relation $v_j \geq j - i$ means that $x_i \triangleleft x_j$. Both of these implications lead to a contradiction with the antisymmetric nature of interval-posets. Necessarily, we have $v_j < j - i$, namely u and v are compatible.

The pair (u, v) is a Tamari interval diagram of size n . Finally, it is clear that $\chi(u, v) = P$ by construction. The map χ is therefore surjective.

Let (u, v) and (u', v') be two Tamari interval diagrams of size n , such that $(u, v) \neq (u', v')$ and such that $\chi(u, v) := P$ and $\chi(u', v') := P'$. So there is at least one letter of (u, v) and (u', v') such that $u_i \neq u'_i$ or $v_i \neq v'_i$, for $i \in [n]$. Therefore the number of vertices of P in relation to the vertex x_i associated with the component u_i and v_i by χ is different from the number of vertices of P' in relation to the vertex x'_i associated with the component u'_i and v'_i by χ , that is $P \neq P'$. This shows that the map χ is injective. \square

The minimalist representation of the interval-posets defined in Section 1 allows a direct construction of the corresponding Tamari interval diagram. Indeed, let us consider the minimalist representation of an interval-poset P of size n . For any relation $x_j \triangleleft x_i$ (resp. $x_i \triangleleft x_j$) drawn, with $1 \leq i < j \leq n$, we set $u_i := j - i$ (resp. $v_j := j - i$). This forms a pair of words (u, v) which is the inverse image of P by χ .

An example is given by Figure 10, where a Tamari interval diagram and its interval-poset which is its image by χ are shown.

2.2. Construction of cubic coordinates. We describe in this part the set of cubic coordinates, and we show that there is a bijection between this set and the set of Tamari interval diagrams. We end this part with some properties of the cubic coordinates.

Let $n \geq 0$ and (u, v) be a Tamari interval diagram of size n . We build a $(n - 1)$ -tuple $(u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$ from the letters of (u, v) , by subtracting v_{i+1} from u_i for any $i \in [n]$. The resulting $(n - 1)$ -tuples can be characterized using Tamari interval diagram definition.

Let $n \geq 0$ and c be a $(n - 1)$ -tuple of components with value in \mathbb{Z} . The $(n - 1)$ -tuple c is a *cubic coordinate* if the pair (u, v) , where u is the word defined by $u_n := 0$ and for any $i \in [n - 1]$ by

$$u_i := \max(c_i, 0), \quad (2.2.1)$$

and v is the word defined by $v_1 := 0$ and for any $2 \leq i \leq n$ by

$$v_i := |\min(c_{i-1}, 0)|, \quad (2.2.2)$$

is a Tamari interval diagram. The size of a cubic coordinate is its number of components plus one. The set of cubic coordinates of size n is denoted by $\text{CC}(n)$.

For instance, the cubic coordinate of the Tamari interval diagram in Figure 10 is $(9, -1, 2, 1, -4, 4, 3, 1, -2)$.

Let us denote by ϕ the map which sends a cubic coordinate c to a Tamari interval diagram (u, v) .

Theorem 2.2.1. *For any $n \geq 0$, the map $\phi : \text{CC}(n) \rightarrow \text{TID}(n)$ is bijective.*

Proof. Let c and c' be two cubic coordinates of size n such that $c \neq c'$. Then there is a component c_i such that $c_i \neq c'_i$, with $i \in [n - 1]$. By the map ϕ , one has then $u_i \neq u'_i$ or $v_{i+1} \neq v'_{i+1}$, namely $(u, v) \neq (u', v')$. Which shows that the map ϕ is injective.

Let $(u, v) \in \text{TID}(n)$. Let $c := (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$, the $(n - 1)$ -tuple whose components are given by the difference between u_i and v_{i+1} for any $i \in [n - 1]$. Now if $u_i \neq 0$ then $v_{i+1} = 0$ for any $i \in [n - 1]$. Therefore $\phi(c) = (u, v)$, where (u, v) is indeed

a Tamari interval diagram by hypothesis. By definition of a cubic coordinate, one can conclude that $c \in \text{CC}(n)$. Which shows that the map ϕ is surjective. \square

Therefore, by the map ϕ it is possible to build a cubic coordinate from its Tamari interval diagram and reciprocally. Graphically, by simply shift the dual Tamari diagram to the left of one position and collect the height of the needles from left to right, putting a positive sign for the needles of the Tamari diagram and a negative sign for its dual, and forgetting the last needle of zero height. Reconstruct the needles of the Tamari diagram and its dual from the components of the cubic coordinate in the same way, and then shift the Tamari dual diagram to the right of one position.

Using the map χ we can then directly give the cubic coordinate of an interval-poset. In the same way that we shift the dual Tamari diagram one position to the left, we shift all the increasing relations of the interval-poset to the left of one vertex. Then, for each vertex x_i , we count the number of elements in increasing or decreasing relation of goal x_i , out of reflexive relation, for all $i \in [n - 1]$. These numbers become the components of positive sign if it is a decreasing relation, negative otherwise, of the cubic coordinate. As the increasing relations have been shifted, the number associated with the vertex x_n is always zero. This vertex is therefore forgotten for the cubic coordinate. In the same way, with each component of a cubic coordinate, we rebuild the increasing and decreasing relations on $n - 1$ vertices, then we shift the increasing relations to the right, in order to form the vertex x_n .

Lemma 2.2.2. *Let $n \geq 0$ and $c \in \text{CC}(n)$ such that there is a component $c_i \neq 0$, for $i \in [n - 1]$. Let c' the $(n - 1)$ -tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n - 1]$. Then c' is a cubic coordinate.*

Proof. Let $(u', v') := \phi(c')$ and (u'_i, v'_{i+1}) be the pair of letters corresponding to c'_i by the map ϕ . Since $c'_i = 0$ then $(u'_i, v'_{i+1}) = (0, 0)$. In order to show that c' is a cubic coordinate, we have to show that (u', v') is a Tamari interval diagram. This is equivalent to satisfying the conditions of a Tamari diagram, a dual Tamari diagram, and compatibility. Replace in (ii) of a Tamari diagram u_i with 0. The condition $u_{i+j} \leq u_i - j$ for any $i \in [n]$ and $j \in [0, u_i]$ becomes $0 \leq 0$ because j equals 0. Similarly, if we replace in (ii) of a dual Tamari diagram v_i by 0 then the condition $v_{i-j} \leq v_i - j$ for any $i \in [n]$ and $j \in [0, v_i]$ becomes $0 \leq 0$ for the same reason. Finally, we have to satisfy the condition of compatibility: for all $1 \leq i < j \leq n$, if $u_i \leq j - i$ then $v_j < j - i$. This condition is always true for $u_i = 0$ or for $v_j = 0$ because $j - i > 0$. Therefore, the $(n - 1)$ -tuple c' is a cubic coordinate. \square

Depending on the case, either the definition of cubic coordinates or the definition of Tamari interval diagrams is used, as it is done for the proof of Lemma 2.2.2. For example, the following results are stated for Tamari interval diagrams.

Let $n \geq 0$. A Tamari interval diagram (u, v) of size n is *synchronized* if either $u_i \neq 0$ or $v_{i+1} \neq 0$ for any $i \in [n - 1]$.

Likewise a cubic coordinate c of size n is synchronized if $c_i \neq 0$ for any $i \in [n - 1]$. The set of synchronized cubic coordinates of size n is denoted by $\text{CC}_s(n)$.

Recall that a Tamari interval $[S, T]$ is synchronized if and only if the binary trees S and T have the same canopy [FPR17]. The definition of the canopy is recalled in Section 1.

Proposition 2.2.3. *Let $n \geq 0$ and $(u, v) \in \text{TID}(n)$. The Tamari interval diagram (u, v) is synchronized if and only if $\rho(\chi(u, v))$ is a synchronized Tamari interval.*

Proof. Suppose that (u, v) is not synchronized, then there is an index $i \in [n - 1]$ such that $u_i = 0$ and $v_{i+1} = 0$. Let $P := \chi(u, v)$ be the interval-poset associated to (u, v) , and $[S, T] := \rho(\chi(u, v))$.

The letter u_i is equal to 0 if and only if there is no descending relation of goal x_i in P , namely if and only if the node i has no right child in the tree S (see Section 1 and [CP15]). Furthermore, since i cannot be equal to n , the node i cannot be the rightmost node in S . Therefore, it is a left child of the node $i + 1$. Then the right subtree of the node i is a leaf oriented to the right.

Symmetrically, $v_{i+1} = 0$ if and only if there is no increasing relation of goal x_{i+1} in P , namely if and only if the node $i + 1$ has no left child in the tree T . Since $i + 1$ is always different from 1, the node $i + 1$ cannot be the leftmost node in T , so the node $i + 1$ must be a right child of the node i . Therefore, the right subtree of the node i has a leaf oriented to the left as left subtree.

Finally, there is at least one letter of index i in the canopy of the tree S different from the canopy of the tree T , for the same index. However, two binary trees S and T are not synchronized if there is at least one letter of index i in the canopy of the tree S that is different from the letter of index i in the canopy of T . Therefore, the binary trees S and T are not synchronized if and only if (u, v) is not synchronized. \square

Recall that an interval-poset P of size $n \geq 3$ is *new* if

- (1) there is no decreasing relation of source x_n ;
- (2) there is no increasing relation of source x_1 ;
- (3) there is no relation $x_{i+1} \triangleleft x_{j+1}$ and $x_j \triangleleft x_i$ with $i < j$.

The definition of a new interval-poset is given in [Rog19].

For any $n \geq 3$, a Tamari interval diagram (u, v) of size n is *new* if the following conditions are satisfied

- (i) $0 \leq u_i \leq n - i - 1$ for all $i \in [n - 1]$;
- (ii) $0 \leq v_j \leq j - 2$ for all $j \in [2, n]$;
- (iii) $u_k < l - k - 1$ or $v_l < l - k - 1$ for all $k, l \in [n]$ such that $k + 1 < l$.

Proposition 2.2.4. *Let $n \geq 3$ and $(u, v) \in \text{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\chi(u, v)$ is a new interval-poset.*

Proof. Let us show that $P := \chi(u, v)$ is not new if and only if (u, v) is not new.

- (1) Suppose there is $x_n \triangleleft x_i$ with $i \in [n - 1]$. By Theorem 2.1.2, one has $u_i = \#\{x_j \in P : x_j \triangleleft x_i \text{ and } i < j\}$. Therefore for $\chi^{-1}(P)$ one has $u_i = n - i$. Which is the negation of (i) of a new Tamari interval diagram.

- (2) Suppose there is $x_1 \triangleleft x_j$ with $j \in [2, n]$. Then for $\chi^{-1}(P)$ one has $v_j = j - 1$ because $v_j = \#\{x_i \in P : x_i \triangleleft x_j \text{ and } i < j\}$. Which is the negation of (ii) of a new Tamari interval diagram.
- (3) Suppose there is one relation $x_{i+1} \triangleleft x_{j+1}$ and $x_j \triangleleft x_i$ with $i < j$. For $\chi^{-1}(P)$, it implies on the one hand $v_{j+1} \geq j - i$ and on the other hand $u_i \geq j - i$. Specifically, by setting $l := j + 1$ and $k := i$ one has $k + 1 < l$. Which is the negation of (iii) of a new Tamari interval diagram.

□

In [Rog19] it is shown that a Tamari interval is new if and only if the associated interval-poset is new. With Proposition 2.2.4 we get the following result.

Proposition 2.2.5. *Let $n \geq 3$ and $(u, v) \in \text{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\rho(\chi(u, v))$ is a new Tamari interval.*

Proposition 2.2.6. *Let $n \geq 3$ and $(u, v) \in \text{TID}(n)$. If (u, v) is synchronized then (u, v) is not new.*

Proof. If (u, v) is new, then $u_i < n - i$ for $i \in [n - 1]$, and $v_j < j - 1$ for $j \in [2, n]$. In particular, $u_{n-1} = 0$ and $v_2 = 0$. Which implies, since (u, v) is synchronized, that $u_1 \neq 0$ and $v_n \neq 0$. Furthermore, (u, v) is new if the condition (iii) of a Tamari interval diagram is satisfied. Specifically, for any $k \in [n - 2]$, either $u_k < 1$ or $v_{k+2} < 1$. Note (*) this condition. Assuming that $u_1 \neq 0$ one has either $u_2 \neq 0$ or $v_3 \neq 0$. By (*) the second choice is impossible, so $u_2 \neq 0$. By the same reasoning, for every $k \in [n - 2]$, $u_k \neq 0$. However, also by assumption $v_n \neq 0$. So one has $u_{n-2} \neq 0$ and $v_n \neq 0$ which is a contradiction with (*). □

2.3. Cubic coordinate lattices. Here, we endow the set of cubic coordinates with an order relation. Then we show that there is an isomorphism between this poset and the poset of Tamari intervals. The two bijections constructed in the first two parts of Section 2 allow us to establish this poset isomorphism.

Let $n \geq 0$ and $c, c' \in \text{CC}(n)$. We set that $c \preccurlyeq c'$ if and only if $c_i \leq c'_i$ for all $i \in [n - 1]$. Endowed with \preccurlyeq , the set $\text{CC}(n)$ is a poset called the *cubic coordinate poset*.

Let $[S, T], [S', T'] \in \text{I}(\text{Ta})(n)$. For the next results, let us denote $c := \psi([S, T])$, $c' := \psi([S', T'])$ and $(u, v) := \phi(c)$, $(u', v') := \phi(c')$, and $P := \chi(u, v)$, $P' := \chi(u', v')$.

Lemma 2.3.1. *If $[S', T']$ covers $[S, T]$ then there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preccurlyeq c'' \preccurlyeq c'$.*

Proof. In Section 1 we saw with Lemma 1.3.1 that $[S', T']$ covers $[S, T]$ if and only if P and P' satisfy either (\star) or (\diamond) . Recall that (\star) (resp. (\diamond)) is the following condition: P' is obtained by adding (resp. removing) only decreasing (resp. increasing) relations of goal a vertex x_i of P , such that if only one of these decreasing (resp. increasing) relations is removed (resp. added), then either P is obtained or the object obtained is not an interval-poset.

Let us assume that P and P' satisfy either (\star) or (\diamond) for the vertex x_i . Two cases are possible.

- (1) Suppose that P and P' satisfy (\star) , then since only decreasing relations are added in P' relative to P , only u' is modified in (u', v') relative to (u, v) . Furthermore, since P' is obtained by adding decreasing relations of goal x_i in P , then only the letter u'_i in u' is increased relative to u . Moreover, since the number of descending relations added in P is minimal, there cannot be Tamari interval diagram between (u, v) and (u', v') , and thus no cubic coordinate between c and c' . In the end, the image by ϕ^{-1} of (u', v') is the cubic coordinate c' with $c'_i = u'_i$ and $c'_j = c_j$ for any $j \neq i$.
- (2) Suppose that P and P' satisfy (\diamond) , then since only increasing relations are removed in P' relative to P , only v' is changed in (u', v') relative to (u, v) . Furthermore, since P' is obtained by removing increasing relations of goal x_i in P , then only the letter v'_i in v' is decreased relative to v . Adding the fact that the number of increasing relations removed in P is minimal, then only the component $c'_{i-1} = -v'_i$ of c' has increased relative to c .

In both cases, the implication is true. \square

Note that if there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preceq c'' \preceq c'$, then in particular c' covers c . Thus, Lemma 2.3.1 has the consequence that if $[S', T']$ covers $[S, T]$ then c' covers c .

Let us go back to the composition of bijections $\phi^{-1} \circ \chi^{-1}$. This composition associates to a pair of comparable binary trees $[S, T]$ a pair of words (u, v) such that u encodes the binary tree S and v encodes the binary tree T . Indeed, by this composition u (resp. v) is obtained by counting in S (resp. T) the number of left (resp. right) descendant of each internal node for the infix order. Now, if $[S, T] \preceq_{\infty} [S', T']$, then the interval $[S, T']$ is a Tamari interval because we always have $S \preceq_{\circ} S' \preceq_{\circ} T'$. This implies that the pair (u, v') is always a compatible pair of words. A direct consequence is the following lemma.

Lemma 2.3.2. *Let $n \geq 0$ and $c, c' \in \text{CC}(n)$. If $c \preceq c'$ then there is a cubic coordinate c'' such that $u'' = u$ and $v'' = v'$.*

For any $c, c' \in \text{CC}(n)$, let

$$D^-(c, c') := \{d : c_d \neq c'_d \text{ and } c'_d \leq 0\}, \quad (2.3.1)$$

and

$$D^+(c, c') := \{d : c_d \neq c'_d \text{ and } c_d \geq 0\}. \quad (2.3.2)$$

Now consider the case where c and c' share either their Tamari diagrams or their associated dual Tamari diagrams, then we have the two following lemmas.

Lemma 2.3.3. *Let $n \geq 0$ and $c, c' \in \text{CC}(n)$. If $c \preceq c'$ such that $u = u'$ and $D^-(c, c') \neq \emptyset$ then there is a cubic coordinate c'' different from c and c' such that $c \preceq c'' \preceq c'$.*

Proof. Let c'' be a $(n-1)$ -tuple such that this image (u'', v'') by ϕ is defined as follows: $u'' = u$ and for v'' we set $v''_i = v'_i$ and $v''_j = v_j$ for any $i \in [s]$ and $j \in [s+1, n]$ with $s \in D^-(c, c')$. Since $u'' = u$, the word u'' is a Tamari diagram. Furthermore, since c and c' are cubic coordinates, u and v are compatible and u' and v' are compatible. Therefore,

the only thing to check is that v'' is a dual Tamari diagram. The condition (i) is naturally satisfied. Since $c \leq c'$, the condition (ii) is satisfied because $v_k \geq v'_k$ for all $k \in [n]$. The $(n-1)$ -tuple c'' is a cubic coordinate. \square

Lemma 2.3.4. *Let $n \geq 0$ and $c, c' \in \text{CC}(n)$. If $c \preceq c'$ such that $v = v'$ and $D^+(c, c') \neq \emptyset$ then there is a cubic coordinate c'' different from c and c' such that $c \preceq c'' \preceq c'$.*

Proof. The proof is similar to the demonstration of Lemma 2.3.3 by choosing for the image (u'', v'') of c'' to set $v'' = v$ and $u''_i = u'_i$ and $u''_j = u_j$ for any $i \in [r]$ and $j \in [r+1, n]$ with $r \in D^+(c, c')$. \square

Lemma 2.3.5. *Let $n \geq 0$ and $c, c' \in \text{CC}(n)$. If $c \preceq c'$ then there is a chain*

$$(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c') \quad (2.3.3)$$

with s the number of different components between c and c' , such that between $c^{(k-1)}$ and $c^{(k)}$ only one component is different for any $k \in [s]$.

Proof. Suppose that $c \preceq c'$, it means that for all $i \in [n-1]$ one has $c_i \leq c'_i$. Let

$$D^-(c, c') := \{d_1, d_2, \dots, d_r\} \quad (2.3.4)$$

and

$$D^+(c, c') := \{d_{r+1}, d_{r+2}, \dots, d_s\}, \quad (2.3.5)$$

with $d_{k-1} < d_k$ for all $k \in [s]$. According to Lemma 2.3.2 there is a cubic coordinate $c^{(r)}$ such that $u^{(r)} = u$ and $v^{(r)} = v'$. Since between c and $c^{(r)}$ the positive components are the same, we can build from Lemma 2.3.3 a chain

$$(c = c^{(0)}, c^{(1)}, \dots, c^{(r-1)}, c^{(r)}) \quad (2.3.6)$$

where $c^{(k)}$ is obtained by replacing successively in c all the components $c_{d_1}, c_{d_2}, \dots, c_{d_k}$ by the components $c_{d_1}^{(r)}, c_{d_2}^{(r)}, \dots, c_{d_k}^{(r)}$, for all $k \in [r]$. Thus, we build a chain between c and $c^{(r)}$ by changing only one component from left to right between each $c^{(k-1)}$ and $c^{(k)}$ for all $k \in [r]$.

Note that the letters in the dual Tamari diagrams associated with $c^{(r)}$ and c' are the same, and the letters in the Tamari diagrams associated with $c^{(r)}$ and c are the same. In other words, $D^+(c, c') = D^+(c^{(r)}, c')$. Therefore, we build from Lemma 2.3.4 a chain

$$(c^{(r)}, c^{(r+1)}, \dots, c^{(s-1)}, c^{(s)} = c') \quad (2.3.7)$$

where $c^{(k)}$ is obtained by replacing successively in $c^{(r)}$ all the components $c_{d_{r+1}}, c_{d_{r+2}}, \dots, c_{d_k}$ by the components $c'_{d_{r+1}}, c'_{d_{r+2}}, \dots, c'_{d_k}$, for all $k \in [r+1, s]$. As before, we then obtain a chain between $c^{(r)}$ and c' by changing only one component from left to right between each $c^{(k-1)}$ and $c^{(k)}$ for all $k \in [r+1, s]$. \square

Let $\psi := \rho^{-1} \circ \chi^{-1} \circ \rho^{-1}$ be the map from the Tamari interval poset to the cubic coordinate poset $(\text{CC}(n), \preceq)$.

Theorem 2.3.6. *For any $n \geq 0$, the map ψ is a poset isomorphism.*

Proof. The map ψ is an isomorphism of posets if ψ and its inverse preserves the partial order. As these relations are transitive, Lemma 2.3.1 gives the direct implication. Suppose that $c \preceq c'$. According to Lemma 2.3.5 there is always a chain between c and c' such that the components are independently increasing one by one. So we can see what happens when we change only one component c_i by c'_i at any step between c and c' .

Obviously, if $c_i = c'_i$ then $u_i = u'_i$ and $v_{i+1} = v'_{i+1}$ and no changes are made between the corresponding binary tree pairs. Suppose that $c_i < c'_i$, then two cases are possible.

- (1) Suppose that c'_i is positive and c_i is positive or null. The image by ϕ of c and c' differ for the letter u_i , namely $c'_i = u'_i$ and $c_i = u_i$, and $v_{i+1} = v'_{i+1} = 0$. The difference of a letter u_i between (u, v) and (u', v') is directly translated by the map χ : the interval-poset P' has more decreasing relations of goal x_i than the vertex x_i in P . By the map ρ , it means that to go from the tree S to the tree S' at least one right rotation of the edge (i, j) is made, where j is the father of the node i in S .
- (2) Symmetrically, assume that c'_i is negative or null, then $c'_i = -v'_{i+1}$, $c_i = -v_{i+1}$ and $u_i = u'_i = 0$. By the map χ , the interval-poset P' has less decreasing relations of goal x_{i+1} than the vertex x_{i+1} in P . This implies by ρ that to pass from the tree T to the tree T' at least one right rotation of the edge $(k, i + 1)$ is made, where k is the right child of the node $i + 1$ in T .

In both cases $c \preceq c'$ implies that to get $[S', T']$ only right rotations in the tree S and in the tree T can be made. Therefore $[S, T] \preceq_{\infty} [S', T']$.

The map ψ is an isomorphism of posets. □

Let us denote by \prec the covering relation of the poset $(CC(n), \preceq)$.

Proposition 2.3.7. *Let $n \geq 0$ and $c, c' \in CC(n)$. The cubic coordinate c' covers c if and only if there is a unique different component c_i between c and c' such that $c_i < c'_i$.*

Proof. It is a consequence of Theorem 2.3.6 and Lemma 2.3.1. □

The following diagram provides a summary of the applications used in Section 2. Recall that $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$, therefore this diagram of poset isomorphisms is commutative.

$$\begin{array}{ccc}
 \text{TID}(n) & \xrightarrow{\chi} & \text{IP}(n) \\
 \uparrow \phi & & \downarrow \rho \\
 \text{CC}(n) & \xleftarrow{\psi} & \text{I(Ta)}(n)
 \end{array} \tag{2.3.8}$$

A consequence of the poset isomorphism ψ is that the order dimension [MP90, Tro02] of the poset of Tamari intervals is at most $n - 1$.

3. CUBIC REALIZATIONS AND GEOMETRIC PROPERTIES

In this section, we give a very natural geometrical realization for the lattices of cubic coordinates. After defining the cells of this realization, we give some properties related to them. Finally, we show that the lattice of the cubic coordinates is EL-shellable.

3.1. Cubic realizations and cells. Theorem 2.3.6 provides a simpler translation of the order relation between two Tamari intervals. We provide the geometrical realization induced by this order relation which is natural for cubic coordinates. In a combinatorial way we study the cells formed by this realization.

For any Tamari interval, we place in space \mathbb{R}^{n-1} its corresponding cubic coordinate. The geometrical realization of the poset $(CC(n), \preceq)$ is then constructed as follows. Two cubic coordinates c and c' are connected by an arrow oriented from c to c' if and only if c' covers c . This geometrical realization is called *cubic realization*.

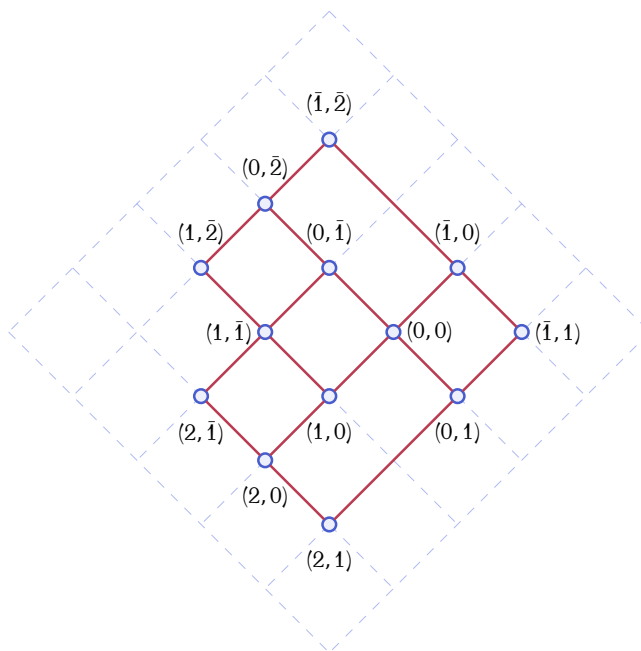


FIGURE 11. Cubic realization of $(CC(3), \preceq)$.

Figure 11 is the cubic realisation of $(CC(3), \preceq)$, where the elements of $CC(3)$ are the vertices and the arrows are the covering relations oriented to the covering elements. Figure 12 is the cubic realization of $(CC(4), \preceq)$. In these drawings the negative sign components are denoted with a bar.

In algebraic topology, to define the tensor products of A_∞ -algebras, we use a cell complex called the *diagonal of the associahedron*. This complex has notably been studied by Loday [Lod11], by Sanedlidze and Umble [SU04] or by Markl and Shnider [MS06]. More recently, there is a description of this object in the article [MTTV19]. The realization of this complex seems to be identical to the cubic realization, with some continuous deformation made.

In Figure 11 and Figure 12, we notice that a "cellular" organization appears. Thanks to the cubic coordinates, a combinatorial definition of these cells is provided. The aim is to have a better understanding of the realization of the cubic coordinate posets, as a geometrical object.

Let $n \geq 0$ and $c \in \text{CC}(n)$. Suppose there is $c' \in \text{CC}(n)$ such that $c'_i > c_i$ and $c'_j = c_j$ for all $j \neq i$ with $i, j \in [n-1]$. The *minimal increasing map* of \uparrow_i is defined by

$$\uparrow_i(c) := (c_1, \dots, c_{i-1}, \uparrow c_i, c_{i+1}, \dots, c_{n-1}), \quad (3.1.1)$$

such that $c < \uparrow_i(c)$ and $c_i < \uparrow c_i \leq c'_i$. This map \uparrow_i allows us to select one covering cubic coordinate of c in particular. In the following, it is said that $\uparrow_i(c)$ is the *minimal increasing* of c for the component c_i .

Let $n \geq 0$ and $c \in \text{CC}(n)$, and $(u, v) := \phi(c)$. If $\uparrow c_i$ is positive then the letter u_i increases and becomes equal to $\uparrow c_i$ and v_{i+1} is equal to 0. Then, we set $\uparrow u_i := \uparrow c_i$. If $\uparrow c_i$ is negative or null then v_{i+1} decreases and becomes equal to $|\uparrow c_i|$ and u_i is equal to 0. Then, we set $\downarrow v_{i+1} := \uparrow c_i$.

Lemma 3.1.1. *Let $n \geq 0$ and $c \in \text{CC}(n)$, and $i \in [n-1]$ such that $\uparrow_i(c)$ is well-defined. Then,*

- (i) *if $c_i < 0$ then $\uparrow c_i \leq 0$;*
- (ii) *if $c_i \geq 0$ then $\uparrow c_i > 0$.*

Proof. Let us show the first implication, the second being obvious because the minimal increasing map always strictly increases a component. Let $c_i < 0$. Suppose by the absurd that $\uparrow c_i > 0$. Let us then note c' the $(n-1)$ -tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n-1]$. By Lemma 2.2.2 c' is a cubic coordinate. Clearly, $c \preceq c' \preceq \uparrow_i(c)$, with the three distinct elements. Which is impossible by definition of the minimal increasing map. \square

Let $n \geq 0$ and $c \in \text{CC}(n)$. The cubic coordinate c is a *minimal-cellular* if for any $i \in [n-1]$, $\uparrow_i(c)$ is well-defined. In other words, a cubic coordinate of size n is a minimal-cellular if it has $n-1$ covers.

Lemma 3.1.2. *Let $n \geq 0$ and c be a minimal-cellular of size n and $i \in [n-1]$. If*

$$c' = \uparrow_{i+1}(\uparrow_{i+2}(\dots(\uparrow_{n-1}(c))\dots)), \quad (3.1.2)$$

is well-defined then $\uparrow_i(c')$ is well-defined.

Proof. Suppose that (3.1.2) is satisfied for $i+1$. Let us show that $\uparrow_i(c')$ is also well-defined. Then two cases are possible for c_i .

Suppose that $c_i < 0$. In this case, consider c'' the $(n-1)$ -tuple obtained from c' by replacing the c_i component by 0. This $(n-1)$ -tuple c'' is a cubic coordinate by Lemma 2.2.2. Since $c_i < 0$ one has $c' \preceq c''$. If c'' is a cover for c' then $c'' = \uparrow_i(c')$. Otherwise, it is always possible to find another cubic coordinate c''' between c' and c'' such that $c'' = \uparrow_i(c')$. In both cases, $\uparrow_i(c')$ is well-defined.

Suppose that $c_i \geq 0$. Let $(u, v) := \phi(c)$, then $c_i = u_i$. The minimal increasing of c' for u_i can lead to three different cases due to the two conditions of a Tamari diagram and the compatibility condition.

- (1) If there is an index j such that $1 \leq i < j \leq n$ and $\downarrow v_j \geq j-i$ then $v_j \geq j-i$ because $\downarrow v_j < v_j$. By the compatibility condition that implies $u_i < j-i$. Moreover, since c

is assumed minimal-cellular, $u_i < j - i - 1$, so that u_i can be increased in c . This inequality remains true for c' .

- (2) If there is an index h such that $1 \leq i - h \leq u_h$ then $u_i \leq u_h - i + h$ by the condition (ii) of a Tamari diagram. This remains true in c' because components with index smaller than i remain unchanged between c and c' . Furthermore, since c is minimal-cellular then $u_i < u_h - i + h$. This property remains true for c' .
- (3) If there is an index k such that $1 \leq i < k \leq n$ then by (i) of a Tamari diagram, $\uparrow u_k \leq n - k$.

Let us build a $(n - 1)$ -tuple c'' different from c' only for component c_i and let us see what choices are available for u_i .

- (a) Suppose there is a j satisfying (1) and there is no h satisfying (2) in c' . In this case, we set $u_i := j - i - 1$. The compatibility condition is satisfied because $u_i < j - i$. Furthermore, since c' is assumed to be well-defined, all conditions in a Tamari diagram and a dual Tamari diagram are satisfied for c'' . Our candidate c'' is therefore a cubic coordinate.
- (b) Suppose there is a h satisfying (2) and there is no j satisfying (1) in c' . Then we set $u_i := u_h - i + h$. The condition (ii) of a Tamari diagram is thus satisfied for u_i . Also, by the condition (i) of a Tamari diagram, $u_h \leq n - h$ which implies $u_i \leq n - i$. Finally, the compatibility condition is also satisfied because it was assumed that there was no j satisfying (1). The tuple c'' is thus a cubic coordinate.
- (c) Suppose there is a j and a h satisfying (1) and (2) in c' . In this case, we set $u_i := \min\{u_h - i + h, j - i - 1\}$. The tuple c'' is then a cubic coordinate by the two previous cases.
- (d) Otherwise, we set $u_i := n - i$. The tuple c'' is a cubic coordinate.

In all four cases, the existence of a k satisfying (3) has no influence. Indeed, in (a) $\uparrow u_k$ is increased by $\downarrow v_j$ and is thus lower than $u_i = j - i - 1$ in c'' . In (b) $\uparrow u_k$ is increased by u_h and is thus lower than $u_i = u_h - i + h$ in c'' . In (c) $\uparrow u_k$ is increased by either $\downarrow v_j$ or u_h . Finally in (d) since $\uparrow u_k \leq n - k$ and $n - k < n - i$ one has $\uparrow u_k < n - i$.

In any case, for u_i fixed in c'' , either there is a $\uparrow u_i$ such that $0 < \uparrow u_i < u_i$ and $\uparrow_i(c')$ is well-defined, otherwise $\uparrow_i(c') = c''$. \square

Let $n \geq 0$ and c be a minimal-cellular of size n and c' a cubic coordinate of size n . The cubic coordinate c' is the *corresponding maximal-cellular* of c if

$$c' = \uparrow_1 (\uparrow_2 (\dots (\uparrow_{n-1} (c)) \dots)). \quad (3.1.3)$$

For instance $c = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular, and its corresponding maximal-cellular is $c' = (1, 0, 2, 0, -4, 3, 2, 0, -2)$. By Lemma 3.1.2 such an element does exist. Note that performing the minimal increasing of c in a different order does not always result in the corresponding maximal-cellular. This observation can already be made on the two pentagons of Figure 11.

Let $n \geq 0$ and c^m be a minimal-cellular of size n and c^M be its corresponding maximal-cellular. The pair (c^m, c^M) is called *cell*, and is denoted by $\langle c^m, c^M \rangle$. The size of the cell is the size of c^m .

A consequence of Lemma 3.1.1 is that for any cell $\langle c^m, c^M \rangle$ of size n , for all $i \in [n-1]$,

- (i) if $c_i^m < 0$ then $c_i^M \leq 0$;
- (ii) if $c_i^m \geq 0$ then $c_i^M > 0$.

Theorem 3.1.3. *Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n , and c be a $(n-1)$ -tuple such that all component c_i is equal either to c_i^m or to c_i^M , for all $i \in [n-1]$. Then c is a cubic coordinate.*

Proof. If all the components of c are equal to those of c^m (resp. to those of c^M), then c is a cubic coordinate. Suppose this is not the case, meaning that c has components of c^m and c^M .

Let us note (u_i^m, v_{i+1}^m) (resp. (u_i^M, v_{i+1}^M)) the pair of letters corresponding to c_i^m (resp. c_i^M) and (u_i, v_{i+1}) the one corresponding to c_i for any $i \in [n-1]$. By hypothesis on c^m and c^M the letter u_i which is equal to u_i^m or u_i^M satisfies $0 \leq u_i \leq n-i$ for any $i \in [n]$. Similarly, the letter v_i which is equal to v_i^m or v_i^M satisfies $0 \leq v_i \leq i-1$ for any $i \in [n]$. Let us show that c satisfies the condition (ii) of a Tamari diagram, the condition (ii) of a dual Tamari diagram and the compatibility condition.

- (1) Let us show that for any choice of letters u_i and u_{i+j} with $i \in [n]$ and $j \in [0, u_i]$ one has $u_{i+j} \leq u_i - j$.
 - * If u_i and u_{i+j} are equal respectively to u_i^m and to u_{i+j}^m (resp. to u_i^M and to u_{i+j}^M) then the condition (ii) of a Tamari diagram is satisfied because c^m (resp. c^M) is a cubic coordinate.
 - * Suppose that $u_i = u_i^M$ and $u_{i+j} = u_{i+j}^m$. By definition of c^M one has $u_{i+j}^m < u_{i+j}^M$. However $u_{i+j}^M \leq u_i^M - j$ because c^M is a cubic coordinate. Therefore the condition (ii) of a Tamari diagram is satisfied.
 - * Suppose that $u_i = u_i^m$ and $u_{i+j} = u_{i+j}^M$. Let $c' = \uparrow_{i+j} (\uparrow_{i+j+1} (\dots (\uparrow_{n-1} (c^m)) \dots))$. According to Lemma 3.1.2 c' is a cubic coordinate such that $c'_i = u_i^m$ and $c'_{i+j} = u_{i+j}^M$. Since the condition (ii) of a Tamari diagram is satisfied for c' , it must also be satisfied for c .
- (2) The condition (ii) of a dual Tamari diagram is satisfied with the same arguments given for the three previous cases, applied to the dual Tamari diagram v .
- (3) Rather than showing the compatibility condition as it is stated, let us show the contrapositive. That is, for every $1 \leq i < j \leq n$ such that $v_j \geq j-i$, let us show that $u_i < j-i$.
 - * Clearly, if u_i and v_j are equal to u_i^m and v_j^m (resp. to u_i^M and v_j^M) then the compatibility condition is satisfied.
 - * Suppose that $u_i = u_i^m$ and $v_j = v_j^M$. If $v_j^M \geq j-i$ then for c^m one has $v_j^m \geq j-i$ because $v_j^M < v_j^m$. Since c^m is a cubic coordinate, this implies that $u_i^m < j-i$.
 - * Suppose that $u_i = u_i^M$ and $v_j = v_j^m$. If $v_j^m \geq j-i$ then for all $k \in [i, j-1]$, $u_k^m < j-k$ because c^m is a cubic coordinate and then satisfies the compatibility condition. Moreover since c^m is minimal-cellular each component can be minimally increased independently of the others, thus $u_k^m < j-k-1$ for all $k \in [i, j-1]$. For the same reason $u_{i+h} < u_i - h$ for all $h \in [0, u_i]$. These two reasons imply that if one builds the cubic coordinate $c' = \uparrow_i (\uparrow_{i+1} (\dots (\uparrow_{n-1} (c^m)) \dots))$

then by definition of the minimal increasing map one has $c'_i = u'_i < j - i$, because at worst, the minimal increasing map sends u_i^m to $j - i - 1$. However, by definition of c^M one has $u_i^M = u'_i$, that is $u_i^M < j - i$. Therefore the compatibility condition between u^M and v_j^m is satisfied for c .

Thus, for all choices of letters of u and v one has that c is a cubic coordinate. \square

One of the direct consequences of Theorem 3.1.3 is that for every cell $\langle c^m, c^M \rangle$, at least 2^{n-1} cubic coordinates belong to this cell. This theorem also implies that the maximal-cellular covers $n - 1$ cubic coordinates in $(\text{CC}(n), \preccurlyeq)$.

3.2. Cells properties. We now have a definition of cells. In addition, we know that each cell contains at least 2^{n-1} elements. In this part, we show that it is possible to associate bijectively each cell to a synchronized cubic coordinate. Finally, we deduce a formula to compute the volume of the cubic realization.

Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n and γ be the map defined by

$$\gamma(c_i^m, c_i^M) := \begin{cases} c_i^m & \text{if } c_i^m < 0, \\ c_i^M & \text{if } c_i^m \geq 0, \end{cases} \quad (3.2.1)$$

for all $i \in [n - 1]$. Note that the components returned by the map γ are never zero. Let denote by (u_i^m, v_{i+1}^m) (resp. (u_i^M, v_{i+1}^M)) the pair of letters corresponding to c_i^m (resp. c_i^M) by the map ϕ , for any $i \in [n - 1]$. Then the map γ becomes

$$\gamma(c_i^m, c_i^M) := \begin{cases} -v_{i+1}^m & \text{if } c_i^m < 0, \\ u_i^M & \text{if } c_i^m \geq 0. \end{cases} \quad (3.2.2)$$

Let Γ be the map defined by

$$\Gamma(\langle c^m, c^M \rangle) := (\gamma(c_1^m, c_1^M), \gamma(c_2^m, c_2^M), \dots, \gamma(c_{n-1}^m, c_{n-1}^M)). \quad (3.2.3)$$

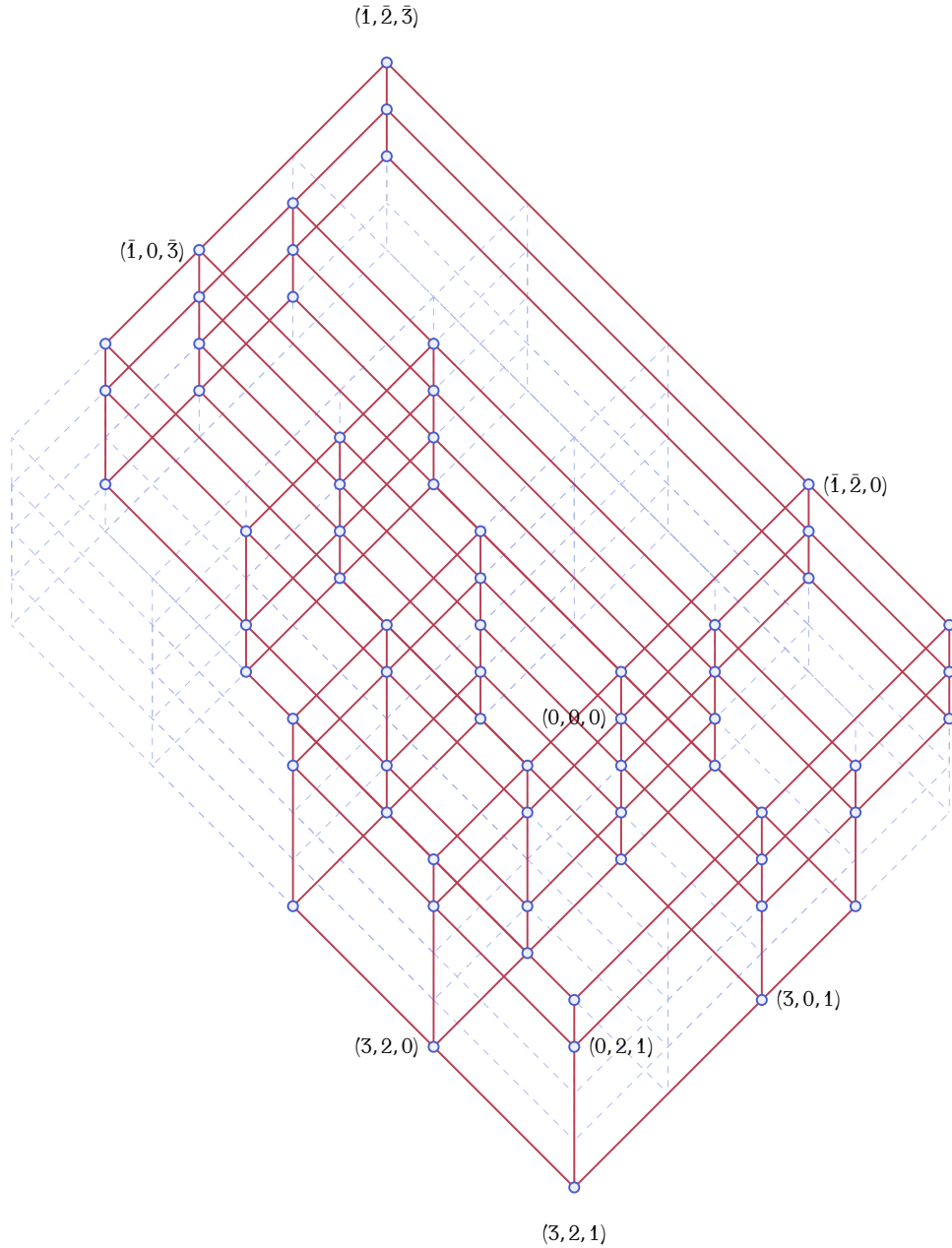
For instance, the cell $\langle (0, -1, 1, -1, -5, 0, 1, -1, -3), (1, 0, 2, 0, -4, 3, 2, 0, -2) \rangle$ is sent by Γ to $(1, -1, 2, -1, -5, 3, 2, -1, -3)$.

Theorem 3.2.1. *For any $n \geq 0$, the map Γ is a bijection from the set of cells of size n to $\text{CC}_s(n)$.*

Proof. The components of $\Gamma(\langle c^m, c^M \rangle)$ belong to either c^m or c^M . In both cases, it is a non-zero component. According to Theorem 3.1.3, $\Gamma(\langle c^m, c^M \rangle)$ is therefore a cubic coordinate of size n . Moreover, this cubic coordinate is synchronized because none of its components is null.

Let $\langle c^m, c^M \rangle$ and $\langle e^m, e^M \rangle$ be two cells of size n such that $\Gamma(\langle c^m, c^M \rangle) = \Gamma(\langle e^m, e^M \rangle)$. Let us note (u_i^m, v_{i+1}^m) (resp. (u_i^M, v_{i+1}^M)) the pair of letters corresponding to c_i^m (resp. c_i^M) and (x_i^m, y_{i+1}^m) (resp. (x_i^M, y_{i+1}^M)) the pair of letters corresponding to e_i^m (resp. e_i^M) by the map ϕ , for all $i \in [n - 1]$.

Γ is injective if $c_i^m = e_i^m$ (resp. $c_i^M = e_i^M$) for any $i \in [n - 1]$. Suppose that $\Gamma(\langle c^m, c^M \rangle) = \Gamma(\langle e^m, e^M \rangle)$ is equivalent to stating that for all $i \in [n - 1]$, $\gamma(c_i^m, c_i^M) = \gamma(e_i^m, e_i^M)$. Two cases are then to be considered, either $\gamma(c_i^m, c_i^M) = u_i^M$ or $\gamma(c_i^m, c_i^M) = -v_{i+1}^m$. By definition of the map γ , no other case is possible.

FIGURE 12. Cubic realization of $(CC(4), \preceq)$.

- (1) Suppose that $\gamma(c_i^m, c_i^M) = u_i^M$ with $i \in [n - 1]$.
- ★ In this case, $\gamma(e_i^m, e_i^M) = x_i^M$ and $u_i^M = x_i^M$. Moreover, since $u_i^M \neq 0$ (resp. $x_i^M \neq 0$), then necessarily $v_{i+1}^M = 0$ (resp. $y_{i+1}^M = 0$). Therefore $c_i^M = e_i^M$.
 - ★ Let us show that $c_i^m = e_i^m$. The fact that $u_i^M > 0$ (resp. $x_i^M > 0$) implies by Lemma 2.2.2 that $0 \leq u_i^m < u_i^M$ and $v_{i+1}^m = 0$ (resp. $0 \leq x_i^m < x_i^M$ and $y_{i+1}^m = 0$).

Thus one has $v_{i+1}^m = y_{i+1}^m$. So it remains to be shown that $u_i^m = x_i^m$. Suppose by the absurd that $u_i^m < x_i^m$. By definition of the minimal increasing map, one has $x_i^m < x_i^M$. This implies, in addition to the hypothesis that $x_i^M = u_i^M$, that $u_i^m < x_i^m < u_i^M$. Let $c = \uparrow_{i+1} (\dots (\uparrow_{n-1} (c^m)) \dots)$ and $e = \uparrow_{i+1} (\dots (\uparrow_{n-1} (e^m)) \dots)$. By Lemma 3.1.2 c and e are both cubic coordinate. By construction $c_j = u_j^m$ (resp. $e_j = x_j^m$) for all $j \in [i]$ and $c_k = c_k^M$ (resp. $e_k = e_k^M$) for all $k \in [i+1, n-1]$. Now let c' be a tuple such that $c'_i = x_i^m$ and $c'_j = c_j$ for all $j \neq i$. Let us show that c' is a cubic coordinate. Let (u, v) and (u', v') be the two pairs of words corresponding respectively to c and c' . Since only one positive letter changes between c and c' , the words v and v' are the same. Furthermore, since c is a cubic coordinate, the word v is in particular a dual Tamari diagram. Therefore v' is also a dual Tamari diagram. On the other hand, for any $k \in [i+1, n-1]$ one has $u'_k = u_k^M$ by definition of a maximal-cellular. However, by hypothesis $u_k^M = x_k^M$. Since the cubic coordinate e is in particular a Tamari diagram, the fact that $u'_k = x_k^M$ for any $k \in [i+1, n-1]$ means that u' is also a Tamari diagram. Finally, since $\uparrow_i (c)$ is a cubic coordinate by Lemma 3.1.2, it satisfies in particular the compatibility condition, with $\uparrow c_i = u_i^M$ by definition of a maximal-cellular. This condition remains satisfied if the letter u_i^M is decreased to the letter x_i^m . Therefore, c' satisfies the compatibility condition and is a cubic coordinate. We have built a cubic coordinate c' distinct from c and $\uparrow_i (c)$ such that $c \preceq c' \preceq \uparrow_i (c)$, which is impossible according to the definition of the minimal increasing map.

- (2) Suppose that $\gamma(c_i^m, c_i^M) = -v_{i+1}^m$. In this case $\gamma(e_i^m, e_i^M) = -y_{i+1}^m$ and $v_{i+1}^m = y_{i+1}^m$. By rephrasing the arguments of the case (1) for the dual, we show that $c_i^m = e_i^m$ and $c_i^M = e_i^M$.

This shows that the map Γ is injective.

Now let us show that the cardinal of the set of cells of size n is equal to the cardinal of $\text{CC}(n)$. Recall that the set of cells of size n is exactly the set of minimal-cellular of size n . Moreover, it is also the set of cubic coordinates which are covered by exactly $n-1$ elements in $(\text{CC}(n), \preceq)$. Furthermore, by the poset isomorphism ψ we know that these elements are the Tamari intervals having $n-1$ elements covering in the poset of Tamari intervals. In [Cha18] Chapoton shows that the set of these Tamari intervals has the same cardinal as the set of synchronized Tamari intervals (see Theorem 2.1 and Theorem 2.3 from [Cha18]). Finally, Proposition 2.2.3 allows us to conclude that the cardinal of $\text{CC}_s(n)$ and the cardinal of the set of cells of size n are equal. Thus, the map Γ is bijective. \square

Thereafter, we also use the *opposite cubic coordinate* of the synchronized cubic coordinate. It is given by the map $\bar{\gamma}$ defined by

$$\bar{\gamma}(c_i^m, c_i^M) := \begin{cases} c_i^M & \text{if } c_i^m < 0, \\ c_i^m & \text{if } c_i^m \geq 0, \end{cases} \quad (3.2.4)$$

for all $i \in [n-1]$. Then $\bar{\Gamma}$ is defined by

$$\bar{\Gamma}(\langle c^m, c^M \rangle) := (\bar{\gamma}(c_1^m, c_1^M), \bar{\gamma}(c_2^m, c_2^M), \dots, \bar{\gamma}(c_{n-1}^m, c_{n-1}^M)). \quad (3.2.5)$$

By Theorem 3.1.3, $\bar{\Gamma}(\langle c^m, c^M \rangle)$ is a cubic coordinate belonging to $\langle c^m, c^M \rangle$, called *opposite cubic coordinate*. For the synchronized cubic coordinate c associated with $\langle c^m, c^M \rangle$ by Γ , note c^{op} the opposite cubic coordinate. All the components of c^{op} are different from those of c , and these differences are maximal. For any synchronized cubic coordinate c , such a cubic coordinate c^{op} always exists and is unique.

Note that the map Γ only returns the positive components of c^M and the negative components of c^m . Conversely, the map $\bar{\Gamma}$ returns the positive components of c^m and the negative components of c^M . We already know that the latter combination is always possible for any comparable cubic coordinates according to Lemma 2.3.2. On the other hand this is not the case for the first mentioned combination.

Now let us take a closer look at the geometry of the cubic realization. We already know that there are at least 2^{n-1} cubic coordinates forming an outline of each cell. The following notions will allow us to say more.

Let $n \geq 0$ and $\epsilon \in \{-1, 1\}^{n-1}$, and $c \in \text{CC}(n)$. A *region* of c is the set

$$\mathcal{R}_\epsilon(c) := \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : x_i < c_i \text{ if } \epsilon_i = -1, x_i > c_i \text{ otherwise}\}. \quad (3.2.6)$$

The cubic coordinate c is *external* if there is $\epsilon \in \{-1, 1\}^{n-1}$ such that $\text{CC}(n) \cap \mathcal{R}_\epsilon(c) = \emptyset$. The region $\mathcal{R}_\epsilon(c)$ is then *empty*. Otherwise c is *internal*.

Proposition 3.2.2. *Let $n \geq 0$ and $c \in \text{CC}(n)$. If c is internal then $\phi(c)$ is a new Tamari interval diagram.*

Proof. Instead, let us show that if $\phi(c)$ is not new, then c is external. Let us note (u_i, v_{i+1}) the pair of letters corresponding to c_i by the map ϕ for $i \in [n-1]$.

Tamari interval diagram $\phi(c)$ is not new if there is

- (1) either $i \in [n-1]$ such that $u_i = n-i$;
- (2) or $j \in [2, n]$ such that $v_j = j-1$;
- (3) or $k, l \in [n]$ such that $u_k = l-k-1$ and $v_l = l-k-1$ with $k+1 < l$.

Suppose there is i satisfying (1) then there cannot be a cubic coordinate c' such that $c'_i > c_i$ because by definition of a Tamari diagram $c'_i \leq n-i$. Similarly, if we assume that there is j satisfying (2) then there cannot be a cubic coordinate c' such that $c'_{j-1} < c_{j-1}$ because by definition of a dual Tamari diagram, $c'_{j-1} \geq 1-j$. If (3) is satisfied, then there cannot be a cubic coordinate c' such that $c'_k > c_k$ and $c'_{l-1} < c_{l-1}$. Indeed, if the letters u_k and v_l are increased in c then the compatibility condition is contradicted, so the result cannot be a cubic coordinate. Since in each case at least one region is empty, c is external. \square

Proposition 3.2.3. *Let $n \geq 0$ and $c \in \text{CC}_s(n)$. Then c is external.*

Proof. By Proposition 2.2.6 we know that if c is synchronized then $\phi(c)$ is not new. Now, we just saw from Proposition 3.2.2 that if $\phi(c)$ is not new, then c is external. \square

Proposition 3.2.4. *Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n . Then there is no cubic coordinate c such that $c_i^m < c_i < c_i^M$ for all $i \in [n-1]$.*

Proof. Suppose there is such a cubic coordinate c . By Lemma 3.1.1 we know that if $c_i^m < 0$ then $c_i^M \geq 0$ and if $c_i^m \geq 0$ then $c_i^M > 0$. However, since $c_i^m < c_i < c_i^M$ then c_i is different from 0. In the end, if such a cubic coordinate c exists, it would be synchronized. But then, there would be a cubic coordinate both synchronized and internal by hypothesis. This is impossible according to Proposition 3.2.3. \square

We showed with Theorem 3.1.3 that each cell contains at least 2^{n-1} cubic coordinates. These cubic coordinates are linked by arrows representing the coverage relationships. With Proposition 3.2.4, we can then conclude that each cell is a hypercube of dimension $n - 1$ in cubic realization, having only vertices on its border.

Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n . The volume of this cell satisfies

$$V(\langle c^m, c^M \rangle) = \prod_{i=1}^{n-1} (c_i^M - c_i^m). \quad (3.2.7)$$

Note c^0 the cubic coordinate such that $c_i^0 = 0$ for any $i \in [n - 1]$. To compute the volume of a cell $\langle c^m, c^M \rangle$ from the synchronized cubic coordinate associated by Γ , we must first compute the volume of the hypercube formed by c^0 and this synchronized cubic coordinate. Let us summarize the data we have so far.

By Lemma 3.1.1, any cell is included in a region of the c^0 cubic coordinate. This means that no cell can intersect hyperplanes passing through the origin c^0 .

According to Lemma 2.2.2, for any cubic coordinate, replacing any component by 0 gives a cubic coordinate. In other words, for any cubic coordinate c , there are $n - 1$ cubic coordinates related to c which are its projections on the hyperplanes of the origin. We can then generate a hypercube with the cubic coordinate c^0 and any other cubic coordinate c . The hypercube thus formed, denoted by $\mathcal{H}(c)$, then depends on c and is of dimension less than or equal to $n - 1$ and can be non-empty, in the sense that it can contain other hypercubes of the same dimension.

By the map Γ the components of the synchronized cubic coordinate of cell $\langle c^m, c^M \rangle$ are the greatest in absolute value between c^m and c^M . Therefore, in the cell $\langle c^m, c^M \rangle$ it is the furthest cubic coordinate from c^0 . Thus the hypercube generated by c^0 and the synchronized cubic coordinate contains the cell $\langle c^m, c^M \rangle$. This hypercube is therefore of dimension $n - 1$.

Let $n \geq 0$ and $c \in \text{CC}_s(n)$. Let us note $\mathcal{V}(c)$ the *extended synchronized volume* of c defined by

$$\mathcal{V}(c) := \prod_{i=1}^{n-1} |c_i|. \quad (3.2.8)$$

The extended synchronized volume computes the volume of the hypercube formed by c^0 and the synchronized cubic coordinate c associated with $\langle c^m, c^M \rangle$.

Let $x \in \mathbb{R}^{n-1}$ and $\langle c^m, c^M \rangle$ be a cell of size n . The element x belongs to $\langle c^m, c^M \rangle$ if $c_i^m \leq x_i \leq c_i^M$ for all $i \in [n - 1]$.

Let us endow the set $\text{CC}_s(n)$ with the partial order \preceq_s such that for $c, c' \in \text{CC}_s(n)$ one has $c' \preceq_s c$ if and only if c'_i and c_i are of the same sign and $|c'_i| \leq |c_i|$ for any $i \in [n - 1]$.

Lemma 3.2.5. *Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n , and c be the synchronized cubic coordinate associated with it by Γ , and $x \in \mathcal{H}(c)$. If $x \notin \langle c^m, c^M \rangle$ then there is $c' \in \text{CC}_s(n)$ different of c such that $c' \preceq_s c$ and $x \in \mathcal{H}(c')$.*

Proof. Let c^{op} be the opposite cubic coordinate of c . Since $x \notin \langle c^m, c^M \rangle$ and $x \in \mathcal{H}(c)$, then necessarily $c^{op} \neq c^0$. For the same reasons, there is an index i such that $|x_i| < |c_i^{op}|$ where $c_i^{op} \neq 0$. Let us build $\nabla_i c$ the $(n-1)$ -tuple such that $\nabla_i c_i = c_i^{op}$ and $\nabla_i c_j = c_j$ for all $j \neq i$. According to Theorem 3.1.3, $\nabla_i c$ is a cubic coordinate and belongs to the cell $\langle c^m, c^M \rangle$. Also, $\nabla_i c$ is a synchronized cubic coordinate which satisfies $\nabla_i c \preceq_s c$ and which is different from c . We can then associate to $\nabla_i c$ a cell, which is strictly included in $\mathcal{H}(c)$. Then x belongs to the hypercube associated with $\nabla_i c$. \square

Proposition 3.2.6. *Let $n \geq 0$ and $c \in \text{CC}_s(n)$. Then*

$$\mathcal{H}(c) = \bigsqcup_{c' \preceq_s c} \Gamma^{-1}(c'). \quad (3.2.9)$$

Proof. The inclusion of $\mathcal{H}(c)$ in $\bigsqcup_{c' \preceq_s c} \Gamma^{-1}(c')$ is a consequence of Lemma 3.2.5. The reciprocal inclusion is obvious. \square

Let $n \geq 0$ and $c \in \text{CC}_s(n)$, and let $\bar{\mathcal{V}}(c)$ be the *synchronized volume* of c defined by

$$\bar{\mathcal{V}}(c) := \mathcal{V}(c) - \sum_{\substack{c' \preceq_s c \\ c' \neq c}} \bar{\mathcal{V}}(c'). \quad (3.2.10)$$

The equation (3.2.10) is a Möbius inversion [Sta12].

Proposition 3.2.7. *Let $n \geq 0$ and $\langle c^m, c^M \rangle$ be a cell of size n , be the synchronized cubic coordinate associated with it by Γ . Then*

$$V(\langle c^m, c^M \rangle) = \bar{\mathcal{V}}(c). \quad (3.2.11)$$

Proof. Since the volume of $\mathcal{H}(c)$ is given by $\mathcal{V}(c)$, the equality (3.2.11) is a consequence of Proposition 3.2.6 and the equation (3.2.10). \square

3.3. EL-shellability. To each finite poset \mathcal{P} , it is possible to relate a simplicial complex $\Delta(\mathcal{P})$ whose k -faces correspond to the k -length chains in \mathcal{P} . The properties of the simplicial complex $\Delta(\mathcal{P})$ such as purity, shellability, or homotopy type are strongly linked to the poset \mathcal{P} . In [BW96] and [BW97], Björner and Wachs generalize the method of labeling the covering relations of graded posets to the case of ungraded posets. In particular, they show in [BW97] that Tamari poset is EL-shellable, and thus the associated simplicial complex is shellable. We show in this section that this is also the case for the poset of cubic coordinates.

Let $(\mathcal{P}, \preceq_{\mathcal{P}})$ be a poset and $(\Lambda, \preceq_{\Lambda})$ be a poset, and $\lambda : \preceq_{\mathcal{P}} \rightarrow \Lambda$ be a map. For any saturated chain $(x^{(1)}, \dots, x^{(k)})$ of \mathcal{P} , by a slight abuse of notation, we set

$$\lambda(x^{(1)}, \dots, x^{(k)}) := \left(\lambda(x^{(1)}, x^{(2)}), \dots, \lambda(x^{(k-1)}, x^{(k)}) \right). \quad (3.3.1)$$

We say that a saturated chain of \mathcal{P} is *λ -increasing* (resp. *λ -weakly decreasing*) if its image by λ is an increasing (resp. weakly decreasing) word relative to the partial order \preceq_{Λ} . We

say also that a saturated chain $(x^{(1)}, \dots, x^{(k)})$ of \mathcal{P} is λ -smaller than a saturated chain $(y^{(1)}, \dots, y^{(k)})$ of \mathcal{P} if the image by λ of $(x^{(1)}, \dots, x^{(k)})$ is smaller than the image by λ of $(y^{(1)}, \dots, y^{(k)})$ for the lexicographic order induced by \preceq_Λ . The map λ is an *EL-labeling* of \mathcal{P} if there exist such a poset Λ and a map λ such that for any $x, y \in \mathcal{P}$ satisfying $x \preceq_{\mathcal{P}} y$, there is exactly one λ -increasing saturated chain from x to y which is minimal among all saturated chains from x to y for the order on saturated chains just described. The poset \mathcal{P} is *EL-shellable* [BW96, BW97] if \mathcal{P} is bounded and admits an EL-labeling.

The EL-shellability of a poset \mathcal{P} implies several topological and order theoretical properties of the associated order complex $\Delta(\mathcal{P})$ built from \mathcal{P} . Recall that the elements of this simplicial complex are all the chains of \mathcal{P} . For instance, one of the consequences for \mathcal{P} for having at most one λ -weakly decreasing chain between any pair of its elements is that the Möbius function of \mathcal{P} takes values in $\{-1, 0, 1\}$. In an equivalent way, the simplicial complex associated with each open interval of \mathcal{P} is either contractile or has the homotopy type of a sphere [BW97].

For the sequel, we set Λ as the poset \mathbb{Z}^3 wherein elements are ordered lexicographically. Let $(c, c') \in \prec$ such that $c_i < c'_i$ for $i \in [n-1]$ and let $\lambda : \prec \rightarrow \mathbb{Z}^3$ be the map defined by

$$\lambda(c, c') := (\varepsilon, i, c_i), \quad (3.3.2)$$

$$\text{where } \varepsilon := \begin{cases} -1 & \text{if } c_i < 0, \\ 1 & \text{else.} \end{cases}$$

Theorem 3.3.1. *For any $n \geq 0$, the map λ is an EL-labeling of $CC(n)$. Moreover, there is at most one λ -weakly decreasing chain between any pair of elements of $CC(n)$.*

Proof. Let $c, c' \in CC(n)$ such that $c \preceq c'$. By Lemma 2.3.5, there is a chain

$$(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c') \quad (3.3.3)$$

with s the number of different components between c and c' , such that between $c^{(k-1)}$ and $c^{(k)}$ only one component is different for any $k \in [s]$.

Recall that the chain (3.3.3) is obtained by considering

$$D^-(c, c') := \{d : c_d \neq c'_d \text{ and } c'_d \leq 0\} = \{d_1, d_2, \dots, d_r\} \quad (3.3.4)$$

and

$$D^+(c, c') := \{d : c_d \neq c'_d \text{ and } c_d \geq 0\} = \{d_{r+1}, d_{r+2}, \dots, d_s\}, \quad (3.3.5)$$

with $d_{k-1} < d_k$ for all $k \in [s]$. the chain (3.3.3) is then the concatenation of two chains, the first one between c and $c^{(r)}$

$$(c = c^{(0)}, c^{(1)}, \dots, c^{(r-1)}, c^{(r)}) \quad (3.3.6)$$

where $c^{(k)}$ is obtained by replacing successively in c all the components $c_{d_1}, c_{d_2}, \dots, c_{d_k}$ by the components $c_{d_1}^{(r)}, c_{d_2}^{(r)}, \dots, c_{d_k}^{(r)}$, for all $k \in [r]$, and the second chain between $c^{(r)}$ and c'

$$(c^{(r)}, c^{(r+1)}, \dots, c^{(s-1)}, c^{(s)} = c') \quad (3.3.7)$$

where $c^{(k)}$ is obtained by replacing successively in $c^{(r)}$ all the components $c_{d_{r+1}}, c_{d_{r+2}}, \dots, c_{d_k}$ by the components $c'_{d_{r+1}}, c'_{d_{r+2}}, \dots, c'_{d_k}$, for all $k \in [r+1, s]$, with the observation that $D^+(c, c') = D^+(c^{(r)}, c')$.

Since in this chain only one component differs between two cubic coordinates $c^{(k-1)}$ and $c^{(k)}$ for all $k \in [s]$, the saturated chain can be constructed by considering all the cubic coordinates between them. Besides, since the chain between c and c' is obtained by changing only one component from left to right between each cubic coordinates, then this saturated chain is λ -increasing for the lexicographic order induced by (3.3.2). Let us note this chain μ .

Moreover, any other choice of saturated chain between c and c' implies choosing, at a certain step k , a greater label for the lexicographical order than the label (ε, k, c_k) of μ , and then having to choose the label (ε, k, c'_k) afterwards. Thus, the saturated chain μ is unique and is λ -smaller.

If there is a saturated chain λ -weakly decreasing between c and c' , then it is obtained by first replacing successively in c the components $c_{d_s}, c_{d_{s-1}}, \dots, c_{d_k}$ by the components $c'_{d_s}, c'_{d_{s-1}}, \dots, c'_{d_k}$ for any $k \in [r+1, s]$, with $D^+(c, c') := \{d_{r+1}, d_{r+2}, \dots, d_s\}$. Then, by replacing successively in the cubic coordinate thus obtained the components $c_{d_r}, c_{d_{r-1}}, \dots, c_{d_k}$ by $c^{(r)}_{d_r}, c^{(r)}_{d_{r-1}}, \dots, c^{(r)}_{d_k}$ for any $k \in [r]$, with $D^-(c, c') := \{d_1, d_2, \dots, d_r\}$. To summarize, if a saturated chain λ -weakly decreasing exist between c and c' , it is built by first changing the different and positive components between c and c' from right to left, and then changing the different and negative components between c and c' from right to left. For the same reason that any saturated λ -increasing chain is unique for any interval, if it exists, the λ -weakly decreasing chain is also unique. \square

For instance, in Figure 11, the λ -increasing saturated chain between $(-1, -2)$ and $(2, 1)$ is the chain

$$((-1, -2), (0, -2), (0, -1), (0, 0), (1, 0), (2, 0), (2, 1)), \quad (3.3.8)$$

and

$$\lambda((-1, -2), \dots, (2, 1)) = ((-1, 1, -1), (-1, 2, -2), (-1, 2, -1), (1, 1, 0), (1, 1, 1), (1, 2, 0)). \quad (3.3.9)$$

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