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STATIONARY SOLUTIONS TO COAGULATION-FRAGMENTATION EQUATIONS

PHILIPPE LAURENÇOT

ABSTRACT. Existence of stationary solutions to the coagulation-fragmentation equation is shown when the coagulation kernel K and the overall fragmentation rate a are given by $K(x, y) = x^\alpha y^\beta + x^\beta y^\alpha$ and $a(x) = x^\gamma$, respectively, with $0 \leq \alpha \leq \beta \leq 1$, $\alpha + \beta \in [0, 1)$, and $\gamma > 0$. The proof requires two steps: a dynamical approach is first used to construct stationary solutions under the additional assumption that the coagulation kernel and the overall fragmentation rate are bounded from below by a positive constant. The general case is then handled by a compactness argument.

1. INTRODUCTION

The coagulation-fragmentation equation is a mean-field model describing the time evolution of the size distribution function f of a system of particles increasing their size by pairwise merging or reducing it by splitting, no matter being lost during these processes. Denoting the coagulation kernel, the overall fragmentation rate, and the daughter distribution function by K , a , and b , respectively, the coagulation-fragmentation equation reads

$$\partial_t f = \mathcal{C}f + \mathcal{F}f, \quad (t, x) \in (0, \infty)^2, \quad (1.1a)$$

$$f(0) = f^{in}, \quad x \in (0, \infty), \quad (1.1b)$$

where the coagulation term $\mathcal{C}f$ is given by

$$\mathcal{C}f(x) := \frac{1}{2} \int_0^x K(y, x-y) f(x-y) f(y) dy - \int_0^\infty K(x, y) f(x) f(y) dy, \quad x > 0, \quad (1.1c)$$

and the fragmentation term $\mathcal{F}f$ by

$$\mathcal{F}f(x) := -a(x) f(x) + \int_x^\infty a(y) b(x, y) f(y) dy, \quad x > 0. \quad (1.1d)$$

The first term in (1.1c) accounts for the formation of particles of size $x > 0$ as a consequence of the merging of two smaller particles with respective sizes $y \in (0, x)$ and $x - y$. The second term in (1.1c) and the first term in (1.1d) describe the depletion of particles of size $x > 0$ due to coalescence with other particles and fragmentation, respectively. Finally, the breakup of a particle of size $y > x$ produces fragments of various sizes ranging in $(0, y)$, including fragments of size x according to the

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distribution $b(x, y)$ as indicated by the second term in (1.1d). We further assume that there is no loss of matter during the breakage process, which amounts to require that b satisfies

$$\int_0^y xb(x, y) dx = y, \quad y > 0, \quad \text{and} \quad b(x, y) = 0, \quad x > y > 0. \quad (1.2)$$

Since there is also no loss of matter during coalescence, the total mass of the system is expected to be invariant throughout time evolution; that is,

$$\int_0^\infty xf(t, x) dx = \int_0^\infty xf(0, x) dx, \quad t \geq 0. \quad (1.3)$$

Though this property may fail to be true when, either the coagulation is too strong compared to the fragmentation, a phenomenon known as *gelation*, or the overall fragmentation rate a is unbounded as $x \rightarrow 0$, a phenomenon known as *shattering*, both are excluded in the forthcoming analysis and we refer to [10, 11, 16, 17, 21, 22] and [3, 13, 23], respectively, for detailed information on these issues.

Our interest in this paper is rather related to the possible balance between coagulation and fragmentation, which are competing mechanisms. Indeed, the latter increases the number of particles and reduces the mean size of particles, while the former acts in the opposite direction. It is then of interest to figure out the outcome of this competition and, in particular, whether it could lead to stationary solutions. This is the issue we aim at investigating herein.

The first example of coagulation-fragmentation equation featuring steady state solutions is the case of constant coefficients [1]

$$\begin{aligned} \partial_t f(t, x) = & \int_0^x [f(t, x-y)f(t, y) - A_0 f(t, x)] dy \\ & - 2 \int_0^\infty [f(t, x)f(t, y) - A_0 f(t, x+y)] dy, \quad (t, x) \in (0, \infty)^2, \end{aligned} \quad (1.4)$$

which is obtained with the choice

$$K(x, y) = 2, \quad a(x) = A_0 x, \quad b(x, y) = \frac{2}{y}, \quad 0 < x < y, \quad (1.5)$$

in (1.1). For any $z > 0$, the function Q_z defined by $Q_z(x) := A_0 e^{x \ln z}$, $x > 0$, is a stationary solution to (1.4) and Q_z has finite total mass if and only if $z \in (0, 1)$. The example (1.5) is actually a particular case of coagulation and fragmentation coefficients satisfying the so-called *detailed balance condition*: there are a non-negative symmetric function F defined on $(0, \infty)^2$ and a non-negative function Q defined on $(0, \infty)$ such that

$$a(x) = \frac{1}{2} \int_0^x F(x_*, x - x_*) dx_*, \quad a(y)b(x, y) = F(x, y - x), \quad 0 < x < y, \quad (1.6a)$$

$$K(x, y)Q(x)Q(y) = F(x, y)Q(x + y), \quad (x, y) \in (0, \infty)^2. \quad (1.6b)$$

Note that we recover (1.5) from (1.6) by setting $F \equiv 2A_0$ and $Q \equiv A_0$. Thanks to (1.6), the equation (1.1) reads

$$\begin{aligned} \partial_t f(t, x) = & \frac{1}{2} \int_0^x [K(x-y, y)f(t, x-y)f(t, y) - F(y, x-y)f(t, x)] dy \\ & - \int_0^\infty [K(x, y)f(t, x)f(t, y) - F(x, y)f(t, x+y)] dy, \quad (t, x) \in (0, \infty)^2, \end{aligned} \quad (1.7)$$

and $Q_z : x \mapsto Q(x)e^{x \ln z}$ is a stationary solution to (1.7) for all $z \in (0, \infty)$. Whether Q_z has finite total mass then depends on both the value of z and the integrability properties of Q . We refer to [5, 6, 18, 20] for a more detailed account on the various situations that may happen.

Coagulation and fragmentation coefficients satisfying the detailed balance condition (1.6) are however far from being generic and different approaches have to be designed to investigate the existence of stationary solutions to (1.1) when (1.6) fails to hold. When the coagulation and fragmentation coefficients are given by

$$K(x, y) = k_0 + k_1(x + y), \quad a(x) = A_0 x, \quad b(x, y) = \frac{2}{y}, \quad 0 < x < y, \quad (1.8)$$

the existence of a stationary solution to (1.1) having total mass $\varrho > 0$ is proved in [9] for all $\varrho > 0$, the proof relying on a fixed point argument performed on the stationary version of (1.1a). It uses in an essential way the specific form of the coefficients and does not seem to extend to handle more general cases. Uniqueness and local stability of steady states are also established in [9]. In the same vein but with a completely different approach, a complete description of stationary solutions to (1.1) is obtained in [7, Theorem 5.1 & Remark 5.2] when

$$K(x, y) = k_0(xy)^{\lambda/2}, \quad a(x) = A_0 x^{\lambda/2}, \quad b(x, y) = \frac{2}{y}, \quad 0 < x < y, \quad (1.9)$$

for some $\lambda \in [0, 2]$, $k_0 > 0$, and $A_0 > 0$. Two steps are needed to obtain this result: first, when $\lambda = 0$, $k_0 = 2$, and $A_0 = 1$, given an integrable stationary solution f to (1.1), its Bernstein transform

$$U(s) := \int_0^\infty (1 - e^{-sx}) f(x) dx, \quad s \geq 0,$$

solves the integro-differential equation

$$U(s)^2 + U(s) = \frac{2}{s} \int_0^s U(r) dr, \quad s > 0, \quad U(0) = 0. \quad (1.10)$$

This equation turns out to have an explicit solution U_\star which is the Bernstein transform of a non-negative function $f_\star \in L^1((0, \infty), (1+x)dx)$ satisfying

$$\int_0^\infty f_\star(x) dx = \int_0^\infty x f_\star(x) dx = 1, \quad (1.11)$$

and any solution U to (1.10) is a dilation of U_* ; that is, there is $\mu > 0$ such that $U(s) = U_*(\mu s)$ for $s \geq 0$. Moreover,

$$f_*(x) \underset{x \rightarrow 0}{\sim} \frac{x^{-2/3}}{\Gamma(1/3)} \quad \text{and} \quad f_*(x) \underset{x \rightarrow \infty}{\sim} \frac{9}{8} \frac{x^{-3/2}}{\Gamma(1/2)} e^{-4x/27} . \quad (1.12)$$

In particular, f_* features an integrable singularity as $x \rightarrow 0$. To handle the case $\lambda > 0$ in (1.9), it suffices to note that, if f is a stationary solution to (1.1) corresponding to coagulation and fragmentation rates given by (1.8) for some $\lambda \in [0, 2]$, $k_0 > 0$, and $A_0 > 0$, then $x \mapsto k_0 x^{\lambda/2} f(x)/2A_0$ is a stationary solution to (1.1) corresponding to coagulation and fragmentation rates given by (1.8) with $\lambda = 0$, $k_0 = 2$, and $A_0 = 1$. Consequently, there is $\mu > 0$ such that

$$f(x) = \frac{2A_0\mu}{k_0} x^{-\lambda/2} f_*(\mu x) , \quad x \in (0, \infty) . \quad (1.13)$$

It readily follows from (1.12) and (1.13) that f also features a singularity as $x \rightarrow 0$ which is not integrable if $\lambda > 2/3$. However, the total mass of f is finite for all $\lambda \in [0, 2]$. Stability of stationary solutions is also investigated in [7] when $\lambda = 0$, $k_0 = 2$, and $A_0 = 1$.

The just described results only deal with very specific coagulation and fragmentation coefficients, and the approaches used in both cases exploit their particular structure. They are thus rather unlikely to extend to a wider setting. As far as we know, the only result handling a fairly general class of coagulation and fragmentation coefficients is to be found in [12], the coagulation and fragmentation coefficients being given by

$$K(x, y) = x^{-\alpha} y^\beta + x^\beta y^{-\alpha} , \quad (x, y) \in (0, \infty)^2 , \quad (1.14a)$$

and

$$a(x) = a_0 x^\gamma , \quad b(x, y) = \frac{1}{y} B\left(\frac{x}{y}\right) , \quad 0 < x < y , \quad (1.14b)$$

where

$$(\alpha, \beta) \in [0, 1]^2 , \quad \beta - \alpha \in [0, 1) , \quad \gamma \geq 0 , \quad a_0 > 0 , \quad (1.14c)$$

and

$$B \text{ is a non-negative function in } L^1((0, 1), (z + z^{-2\alpha}) dz) . \quad (1.14d)$$

Assuming further that $(\beta, \gamma) \neq (1, 0)$ and $(\alpha, \gamma) \neq (0, 0)$, the existence of a non-negative stationary solution to (1.1) with total mass ϱ is shown in [12, Theorem 4.1] for all $\varrho > 0$. Furthermore, this stationary solution belongs to $L^1((0, \infty), x^m dx)$ for all $m \geq -2\alpha$ and, under the additional assumption that $B \in L^\infty(0, 1)$, it belongs to $L^p(0, \infty)$ for all $p \in [1, \infty)$. The approach developed to prove this result is of a completely different nature and actually relies on a dynamical approach. Roughly speaking, the basic idea is to find a suitable functional setting in which the initial value problem (1.1) is well-posed, along with a closed and convex set \mathcal{Z} which is compact for the associated topology and is positively invariant for the dynamical system associated to (1.1) (in the sense that $f(t) \in \mathcal{Z}$ for all $t > 0$ as soon as $f(0) \in \mathcal{Z}$). If a fixed point theorem is available in this functional setting, then a classical argument guarantees the existence of at least one stationary solution, see [2, Theorem 16.5], [15, Proof of Theorem 5.2], and [12, Theorem 1.2], for instance. Though this method merely gives the existence of a steady state solution without any information on uniqueness

or stability, it is far more flexible than the previous ones and we shall partially employ it in the forthcoming analysis. Let us mention that it is also the cornerstone of the construction of mass-conserving self-similar solutions to the coagulation equation [12, 14, 25].

According to the previous description, no result on the existence of steady state solutions seems to be available for the classical coagulation kernel

$$K(x, y) = K_0 (x^\alpha y^\beta + x^\beta y^\alpha) , \quad (x, y) \in (0, \infty)^2 , \quad (1.15a)$$

with

$$0 \leq \alpha \leq \beta \leq 1 , \quad \lambda := \alpha + \beta \in [0, 1) , \quad (1.15b)$$

and the purpose of this paper is to fill this gap for a rather large class of fragmentation coefficients. More precisely, we assume that there are

$$\gamma > 0 , \quad a_0 > 0 , \quad p_0 > 1 , \quad (1.16a)$$

and a non-negative function

$$B \in L^1((0, 1), z dz) \cap L^{p_0}(0, 1) , \quad \int_0^1 z B(z) dz = 1 , \quad (1.16b)$$

such that

$$a(x) = a_0 x^\gamma , \quad b(x, y) = \frac{1}{y} B\left(\frac{x}{y}\right) , \quad 0 < x < y . \quad (1.16c)$$

Note that the class of coagulation kernels (1.15) includes the sum kernels corresponding to $\alpha = 0$ and $\beta = \lambda \in [0, 1)$ and the product kernels corresponding to $\alpha = \beta = \lambda/2 \in [0, 1/2)$. The constraint on B in (1.16b) stems from the conservation of matter (1.2) during fragmentation events. Examples of daughter distribution functions satisfying (1.16b) include the power-law breakup distribution

$$B(z) = B_{1,\nu}(z) := (\nu + 2)z^\nu , \quad z \in (0, 1) , \quad \nu > -1 , \quad (1.17)$$

and the parabolic breakup distribution

$$B(z) = B_{2,\nu}(z) := (\nu + 2)(\nu + 1)z^{\nu-1}(1 - z) , \quad z \in (0, 1) , \quad \nu > 0 . \quad (1.18)$$

Indeed, $B_{1,\nu}$ given by (1.17) satisfies (1.16b) for any $p_0 > 1$ when $\nu \geq 0$ and for any $p_0 \in (1, 1/|\nu|)$ when $\nu \in (-1, 0)$. Similarly, $B_{2,\nu}$ given by (1.18) satisfies (1.16b) for any $p_0 > 1$ when $\nu \geq 1$ and $p_0 \in (1, 1/(1 - \nu))$ when $\nu \in (0, 1)$.

Before stating the main result, let us introduce some notation. Throughout the paper, for $m \in \mathbb{R}$, we set

$$X_m := L^1((0, \infty), x^m dx) , \quad M_m(h) := \int_0^\infty x^m h(x) dx , \quad h \in X_m , \quad (1.19)$$

and denote the positive cone of X_m by X_m^+ . We also denote the space X_m endowed with its weak topology by $X_{m,w}$.

Theorem 1.1. *Assume that the coagulation and fragmentation coefficients satisfy (1.15) and (1.16). Given $\varrho > 0$ there exists at least a stationary (weak) solution $\varphi \in X_1^+$ to (1.1) with the following properties:*

(s1) $M_1(\varphi) = \varrho$;

(s2) there are $p_1 \in (1, p_0)$ and $m_1 \in (\lambda, 1)$ such that

$$\varphi \in L^{p_1}((0, \infty), x^{m_1+\gamma} dx) \cap \bigcap_{m>\lambda} X_m ;$$

(s3) for all $\vartheta \in \Theta_1 := \{h \in W^{1,\infty}(0, \infty) : h(0) = 0\}$,

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) [\vartheta(x+y) - \vartheta(x) - \vartheta(y)] \varphi(x)\varphi(y) \, dy dx \\ &= \int_0^\infty a(y)\varphi(y) \left[\vartheta(y) - \int_0^y \vartheta(x)b(x, y) \, dx \right] \, dy . \end{aligned}$$

It is worth pointing out here that Theorem 1.1 **(s2)** does not exclude a non-integrable singularity of φ as $x \rightarrow 0$, a situation which may indeed occur, as we shall see below. This feature is not encountered for the coagulation and fragmentation coefficients given by (1.14) and considered in [12] when $\alpha < 0$, as the unboundedness of the coagulation kernel for small sizes implies the vanishing of the stationary solution as $x \rightarrow 0$. This possible singular behaviour for small sizes is actually the main difficulty to be overcome in the analysis carried out below and requires a more involved approach, which we describe now.

The proof of Theorem 1.1 is carried out in two steps. We fix $\varrho > 0$. Using the dynamical approach already alluded to, given $\varepsilon \in (0, 1)$, we first construct a stationary solution $\varphi_\varepsilon \in X_1$ to

$$\begin{aligned} \partial_t f &= \mathcal{C}_\varepsilon f + \mathcal{F}_\varepsilon f , & (t, x) &\in (0, \infty)^2 , \\ f(0) &= f^{in} , & x &\in (0, \infty) , \end{aligned} \tag{1.20}$$

satisfying $M_1(\varphi_\varepsilon) = \varrho$, where the coagulation and fragmentation operators \mathcal{C}_ε and \mathcal{F}_ε are given by (1.1c) with $K_\varepsilon := K + 2\varepsilon K_0$ instead of K and (1.1d) with $a_\varepsilon := a + a_0\varepsilon^2$ instead of a , respectively. For this choice of coagulation and fragmentation coefficients, we actually build a closed convex and sequentially weakly compact subset \mathcal{Z}_ε of X_1 such that solutions to (1.20) starting from an initial condition in \mathcal{Z}_ε remain in \mathcal{Z}_ε for all positive times. Recalling that, according to the Dunford-Pettis theorem, sequential weak compactness in X_1 requires to prevent concentration and escape of mass for small and large sizes, finding \mathcal{Z}_ε amounts to derive time-independent estimates in $X_{m_0} \cap X_m \cap L^{p_2}(0, \infty)$ for some suitably chosen $m_0 < 1 < m$ and $p_2 > 1$. While some of the moment estimates can be obtained directly for $\varepsilon = 0$ (Section 2.1), it does not seem to be possible to derive uniform integrability estimates without the positive lower bounds on K_ε and a_ε (Section 2.2). Besides the construction of \mathcal{Z}_ε (Section 3.2), we also show the well-posedness of (1.20) in Section 3.1, as well as the continuous dependence of solutions to (1.20) in $X_{1,w}$ with respect to the initial condition (Section 3.3). To justify rigorously the computations performed in Section 2, an additional approximation is needed and we shall actually work with truncated versions of K_ε and a_ε . Thanks to this analysis, it remains to apply [12, Theorem 1.2] to obtain the existence of a stationary solution $\varphi_\varepsilon \in \mathcal{Z}_\varepsilon$ to (1.20) (Section 3.4). To complete the proof of Theorem 1.1, we are left with taking the limit $\varepsilon \rightarrow 0$. To this end, we realize that, since we have paid special attention to the dependence on ε of the estimates derived in

Section 2, there is a sequentially weakly compact subset \mathcal{Z} in X_1 such that $\mathcal{Z}_\varepsilon \subset \mathcal{Z}$ for all $\varepsilon \in (0, 1)$, see Section 3.5. Consequently, there are $\varphi \in \mathcal{Z}$ and a subsequence $(\varphi_{\varepsilon_k})_{k \geq 1}$ of $(\varphi_\varepsilon)_{\varepsilon \in (0, 1)}$ such that $\varphi_{\varepsilon_k} \rightharpoonup \varphi$ in X_1 . We finally combine this convergence with the properties of \mathcal{Z} and $(\varphi_{\varepsilon_k})_{k \geq 1}$ to prove that φ is a stationary weak solution to (1.1) as described in Theorem 1.1 (Section 3.5).

Theorem 1.1 only provides the finiteness of the moments of φ of order larger than λ and thus does not provide much information on its behaviour for small sizes. In fact, the small size behaviour described in Theorem 1.1 **(s2)** does not seem to be accurate. Indeed, formal asymptotics indicate that, if φ is a stationary weak solution to (1.1) satisfying the properties **(s1)**-**(s3)** stated in Theorem 1.1 and

$$\varphi(x) \sim Ax^{-\tau} \text{ as } x \rightarrow 0 \quad (1.21a)$$

for some $A > 0$ and $\tau > 0$, then τ can be identified and depends on the values of α , β , γ , and possibly on B . Specifically,

– if $\gamma > \alpha$, then

$$\tau = \alpha + 1 + m_\star < \alpha + 1, \quad (1.21b)$$

where m_\star is defined in (1.22) below;

– if $\gamma = \alpha < \beta$, then

$$\tau = \alpha + 1; \quad (1.21c)$$

– if $\gamma = \alpha = \beta$ and $B = B_{1,\nu}$, see (1.17), then

$$\tau = \alpha + \frac{2}{\nu + 3} < \alpha + 1; \quad (1.21d)$$

– if $\gamma < \alpha$, then

$$\tau = \lambda + 1 - \gamma. \quad (1.21e)$$

In particular, the prediction (1.21d) perfectly agrees with (1.13) when $\gamma = \alpha = \beta = \lambda/2 \in [0, 1/2)$ and $\nu = 0$ ($B = B_{1,0}$). On the one hand, (1.21) implies that φ may have a non-integrable singularity as $x \rightarrow 0$ and, in particular, it is not expected to belong to X_α when $\gamma < \alpha$. On the other hand, different behaviours are predicted in (1.21), which vary according to the sign of $\gamma - \alpha$, and seem to be sensitive to the behaviour of $B(z)$ as $z \rightarrow 0$ when $\gamma = \alpha = \beta$. We shall not attempt a complete proof of (1.21) herein but, as a first step in that direction, we provide additional integrability properties of φ which complies with (1.21).

Proposition 1.2. *Consider $\varrho > 0$ and let φ be a stationary weak solution to (1.1) satisfying the properties **(s1)**-**(s3)** stated in Theorem 1.1.*

(m1) *If $\gamma > \alpha$, then $\varphi \in X_m$ for any $m > \alpha + m_\star$, where*

$$m_\star := \inf \{ m \in \mathbb{R} : B \in L^1((0, 1), z^m dz) \} \leq \frac{1 - p_0}{p_0} < 0. \quad (1.22)$$

Moreover, if $m_\star > -\infty$ and $B \notin L^1((0, 1), z^{m_\star} dz)$, then $\varphi \notin X_{\alpha+m_\star}$;

(m2) *if $\gamma = \alpha < \beta$, then $\varphi \in X_m$ for any $m \geq \beta$;*

(m3) *if $\gamma = \alpha = \beta$, then $\varphi \in X_m$ for any $m \geq \alpha$;*

(m4) if $\gamma < \alpha$, then $\varphi \in X_m$ for any $m > \lambda - \gamma$.

The proof of Proposition 1.2 is carried out in Section 4 and relies on the choice of suitable test functions in Theorem 1.1 **(s3)**. Comparing (1.21) and Proposition 1.2 reveals that the properties **(m2)** and **(m3)** are not optimal. Improving Proposition 1.2 so that it matches (1.21) in these cases seems to require a finer analysis which we have yet been unable to set up. We however hope to return to this problem in the near future.

2. A TRUNCATED APPROXIMATION

Let $\varrho > 0$ and assume that K , a , and b are coagulation and fragmentation coefficients satisfying (1.15) and (1.16). Also, let f^{in} be an initial condition satisfying

$$f^{in} \in X_0^+ \cap X_{2+\gamma} \quad \text{with} \quad M_1(f^{in}) = \varrho . \quad (2.1)$$

We now introduce the approximation to (1.1) we are going to work with in this section. Besides requiring a positive lower bound on the coagulation kernel and the overall fragmentation rate as already mentioned, we also truncate both of them as in [12]. Specifically, we fix a positive integer $j \geq 2$ and a positive real number $\varepsilon \in (0, 1)$ and set

$$K_{j,\varepsilon}(x, y) := 2\varepsilon K_0 + K(\min\{x, j\}, \min\{y, j\}) , \quad (x, y) \in (0, \infty)^2 , \quad (2.2)$$

$$a_{j,\varepsilon}(x) := a_0(\min\{x, j\}^\gamma + \varepsilon^2) , \quad x \in (0, \infty) . \quad (2.3)$$

Since $K_{j,\varepsilon}$ and $a_{j,\varepsilon}$ are bounded, we may proceed as in [4, 12, 26, 28] to show, by a Banach fixed point argument in $X_0 = L^1(0, \infty)$, that there is a unique non-negative strong solution

$$f_{j,\varepsilon} \in C^1([0, \infty); X_0)$$

to the coagulation-fragmentation equation

$$\partial_t f_{j,\varepsilon} = \mathcal{C}_{j,\varepsilon} f_{j,\varepsilon} + \mathcal{F}_{j,\varepsilon} f_{j,\varepsilon} , \quad (t, x) \in (0, \infty)^2 , \quad (2.4a)$$

$$f_{j,\varepsilon}(0) = f^{in} , \quad x \in (0, \infty) , \quad (2.4b)$$

where the coagulation and fragmentation operators $\mathcal{C}_{j,\varepsilon}$ and $\mathcal{F}_{j,\varepsilon}$ are given by (1.1c) with $K_{j,\varepsilon}$ instead of K and (1.1d) with $a_{j,\varepsilon}$ instead of a , respectively. A first consequence of (2.4a) is that, for $t \geq 0$ and $\vartheta \in L^\infty(0, \infty)$,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \vartheta(x) f_{j,\varepsilon}(t, x) \, dx &= \frac{1}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) \chi_\vartheta(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - \int_0^\infty a_{j,\varepsilon}(y) N_\vartheta(y) f_{j,\varepsilon}(t, y) \, dy , \end{aligned} \quad (2.5)$$

where

$$\chi_\vartheta(x, y) := \vartheta(x + y) - \vartheta(x) - \vartheta(y) , \quad (x, y) \in (0, \infty)^2 , \quad (2.6a)$$

$$N_\vartheta(y) := \vartheta(y) - \int_0^y \vartheta(x) b(x, y) \, dx , \quad y \in (0, \infty) . \quad (2.6b)$$

Owing to (1.16c), an alternative formula for N_ϑ reads

$$N_\vartheta(y) = \vartheta(y) - \int_0^1 \vartheta(yz)B(z) \, dz, \quad y \in (0, \infty). \quad (2.6c)$$

For the particular choice $\vartheta(x) = \vartheta_m(x) := x^m$, $x > 0$, for some $m \in \mathbb{R}$, we set $\chi_m := \chi_{\vartheta_m}$ and $N_m := N_{\vartheta_m}$ for simplicity.

Owing to the boundedness of $K_{j,\varepsilon}$ and $a_{j,\varepsilon}$ and the integrability (1.16b) of B over $(0, 1)$, we infer from (2.5) by an approximation argument that $f_{j,\varepsilon}$ is mass-conserving; that is, $f_{j,\varepsilon} \in L^\infty((0, \infty), X_1)$ and

$$M_1(f_{j,\varepsilon}(t)) = \varrho, \quad t \geq 0. \quad (2.7)$$

Moreover, a similar approximation argument allows us to show that, if $f^{in} \in X_m$ for some $m > 1$, then $f_{j,\varepsilon} \in L^\infty((0, T), X_m)$ for any $T > 0$. We shall refine this result in the next section.

We now derive several estimates for the family $\{f_{j,\varepsilon} : j \geq 2, \varepsilon \in (0, 1)\}$, which do not depend on $j \geq 2$. We also pay special attention to the dependence on $\varepsilon \in (0, 1)$, if any. Throughout this section, C and C_i , $i \geq 1$, denote positive constants which depend only on K_0 , α , β , a_0 , γ , B , and ϱ . Dependence upon additional parameters will be indicated explicitly. For further use, we set

$$\mathfrak{b}_m := \int_0^1 z^m B(z) \, dz \text{ for } m > m_\star \text{ and } \mathcal{B}_p^p := \int_0^1 B(z)^p \, dz \text{ for } p \in [1, p_0], \quad (2.8a)$$

which are finite by (1.16b) and (1.22), and satisfy

$$\mathfrak{b}_m < 1 \iff m > 1 \quad (2.8b)$$

due to (1.16b). Also, Young's inequality and (1.15) entail that

$$K(x, y) \leq K_0 (x^\lambda + y^\lambda), \quad (x, y) \in (0, \infty)^2. \quad (2.9)$$

2.1. Moment Estimates. For $m \in \mathbb{R}$ we set

$$\mathcal{M}_{m,j,\varepsilon} := \sup_{t \geq 0} \{M_m(f_{j,\varepsilon}(t))\} \in [0, \infty] \quad (2.10)$$

and begin with the behaviour of $f_{j,\varepsilon}$ for large sizes.

Lemma 2.1. *Let $m \geq 2$ and assume that $f^{in} \in X_m$. There is a positive constant $\mu_m \geq \Gamma(m+1)\varrho^m$ depending only on K_0 , α , β , a_0 , γ , B , ϱ , and m such that*

$$\mathcal{M}_{m,j,\varepsilon} \leq \max\{M_m(f^{in}), \mu_m\}.$$

Proof. We first recall that there is $c_m > 0$ depending only on m such that

$$\chi_m(x, y) \leq c_m (xy^{m-1} + x^{m-1}y), \quad (x, y) \in (0, \infty)^2, \quad (2.11)$$

see [5, Lemma 2.3 (ii)] for instance. Let $t > 0$. We infer from (2.5) with $\vartheta = \vartheta_m$, (2.8), (2.11), and the symmetry of K that

$$\begin{aligned} \frac{d}{dt} M_m(f_{j,\varepsilon}(t)) &\leq \frac{c_m}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) (x^{m-1}y + xy^{m-1}) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - (1 - \mathfrak{b}_m) \int_0^\infty x^m a_{j,\varepsilon}(x) f_{j,\varepsilon}(t, x) \, dx \\ &\leq c_m \int_0^\infty \int_0^\infty xy^{m-1} K_{j,\varepsilon}(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - a_0(1 - \mathfrak{b}_m) \int_0^\infty x^m \min\{x, j\}^\gamma f_{j,\varepsilon}(t, x) \, dx . \end{aligned}$$

On the one hand, by (2.7),

$$\begin{aligned} \int_0^\infty x^m \min\{x, j\}^\gamma f_{j,\varepsilon}(t, x) \, dx &\geq \int_1^\infty x^m \min\{x, j\}^\gamma f_{j,\varepsilon}(t, x) \, dx \\ &\geq \int_1^\infty x^m f_{j,\varepsilon}(t, x) \, dx \\ &= M_m(f_{j,\varepsilon}(t)) - \int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx \\ &\geq M_m(f_{j,\varepsilon}(t)) - \varrho . \end{aligned}$$

On the other hand, it follows from (2.7) and Hölder's and Young's inequalities that

$$\begin{aligned} \int_0^\infty \int_0^\infty xy^{m-1} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx &\leq \varrho M_{m-1}(f_{j,\varepsilon}(t)) \\ &\leq \varrho M_m(f_{j,\varepsilon}(t))^{(m-2)/(m-1)} M_1(f_{j,\varepsilon}(t))^{1/(m-1)} \\ &\leq \frac{a_0(1 - \mathfrak{b}_m)}{8c_m K_0} M_m(f_{j,\varepsilon}(t)) + C(m) . \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^\infty \int_0^\infty xy^{m-1} \min\{y, j\}^\lambda f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx &\leq \varrho M_{m+\lambda-1}(f_{j,\varepsilon}(t)) \\ &\leq \varrho M_m(f_{j,\varepsilon}(t))^{(m+\lambda-2)/(m-1)} M_1(f_{j,\varepsilon}(t))^{(1-\lambda)/(m-1)} \\ &\leq \frac{a_0(1 - \mathfrak{b}_m)}{4c_m K_0} M_m(f_{j,\varepsilon}(t)) + C(m) , \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty xy^{m-1} \min\{x, j\}^\lambda f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\
 & \leq M_{1+\lambda}(f_{j,\varepsilon}(t)) M_{m-1}(f_{j,\varepsilon}(t)) \\
 & \leq \varrho M_m(f_{j,\varepsilon}(t))^{(m+\lambda-2)/(m-1)} M_1(f_{j,\varepsilon}(t))^{(m-\lambda)/(m-1)} \\
 & \leq \frac{a_0(1 - \mathfrak{b}_m)}{4c_m K_0} M_m(f_{j,\varepsilon}(t)) + C(m) .
 \end{aligned}$$

Collecting the previous inequalities and using (2.9), we obtain

$$\begin{aligned}
 \frac{d}{dt} M_m(f_{j,\varepsilon}(t)) & \leq \frac{2+\varepsilon}{4} a_0(1 - \mathfrak{b}_m) M_m(f_{j,\varepsilon}(t)) + C(m) \\
 & \quad - a_0(1 - \mathfrak{b}_m) (M_m(f_{j,\varepsilon}(t)) - \varrho) \\
 & \leq -\frac{a_0}{4} (1 - \mathfrak{b}_m) M_m(f_{j,\varepsilon}(t)) + C(m) .
 \end{aligned}$$

Integrating the previous differential inequality gives

$$M_m(f_{j,\varepsilon}(t)) \leq e^{-a_0(1-\mathfrak{b}_m)t/4} M_m(f^{in}) + \frac{4C(m)}{a_0(1 - \mathfrak{b}_m)} (1 - e^{-a_0(1-\mathfrak{b}_m)t/4})$$

for $t \geq 0$. Therefore,

$$M_m(f_{j,\varepsilon}(t)) \leq \max \left\{ M_m(f^{in}), \frac{4C(m)}{a_0(1 - \mathfrak{b}_m)} \right\} , \quad t \geq 0 ,$$

from which Lemma 2.1 follows. □

From now on, we fix a positive real number

$$\sigma > \max \{1, \varrho, \mu_2, \mu_{2+\gamma}\} \tag{2.12a}$$

such that

$$\max \{M_2(f^{in}), M_{2+\gamma}(f^{in})\} \leq \sigma . \tag{2.12b}$$

A first consequence of (2.7), (2.12), Lemma 2.1, and Hölder's inequality is that

$$\mathcal{M}_{1+\gamma,j,\varepsilon} \leq \sigma \text{ and } \mathcal{M}_{2,j,\varepsilon} \leq \sigma . \tag{2.13}$$

Next, owing to (2.7), (2.12), and (2.13), another application of Hölder's inequality provides a similar bound for moments of order $m \in (1, 2)$, which we report now.

Corollary 2.2. *For $m \in (1, 2)$,*

$$\mathcal{M}_{m,j,\varepsilon} \leq \sigma .$$

We next turn to the behaviour for small sizes and, to this end, derive estimates for moments of order smaller than one.

Lemma 2.3. *Let $m \in (\lambda, 1)$. There is $\mu_m \geq \Gamma(m+1)\varrho^m$ depending only $K_0, \alpha, \beta, a_0, \gamma, B, \varrho$, and m such that*

$$\mathcal{M}_{m,j,\varepsilon} \leq \max\{M_m(f^{in}), \mu_m + \sigma\} .$$

Proof. Let $m \in (\lambda, 1)$ and $t > 0$. We first argue as in [14, Lemma 3.1] to estimate the contribution of the coagulation term to the time evolution of $M_m(f_{j,\varepsilon})$, see also [4, Lemma 8.2.12]. More precisely, since $j \geq 2$, $\chi_m \leq 0$, and $K_{j,\varepsilon}(x, y) \geq 2K_0(xy)^{\lambda/2}$ for $(x, y) \in (0, 1)^2$, we obtain

$$\begin{aligned} & -\frac{1}{2K_0} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) \chi_m(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ & \geq \frac{1}{2K_0} \int_0^1 \int_0^1 [x^m + y^m - (x+y)^m] K_{j,\varepsilon}(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ & \geq P_{j,\varepsilon}(t) := \int_0^1 \int_0^1 [x^m + y^m - (x+y)^m] (xy)^{\lambda/2} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx . \end{aligned} \quad (2.14)$$

Since $m < 1$, it follows from the convexity of $x \mapsto x^{m-1}$ that, for $(x, y) \in (0, \infty)^2$,

$$\begin{aligned} x^m + y^m - (x+y)^m &= x [x^{m-1} - (x+y)^{m-1}] + y [y^{m-1} - (x+y)^{m-1}] \\ &\geq 2(1-m)xy(x+y)^{m-2} . \end{aligned}$$

Therefore,

$$P_{j,\varepsilon}(t) \geq 2(1-m) \int_0^1 \int_0^1 (x+y)^{m-2} (xy)^{(2+\lambda)/2} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx .$$

Introducing

$$x_i := i^{-2/(m-\lambda)} \quad \text{and} \quad P_{j,\varepsilon}(t, i) := \int_{x_{i+1}}^{x_i} x^{(2+\lambda)/2} f_{j,\varepsilon}(t, x) \, dx , \quad i \geq 1 ,$$

we further obtain

$$\begin{aligned} P_{j,\varepsilon}(t) &\geq 2(1-m) \sum_{i=1}^\infty \int_{x_{i+1}}^{x_i} \int_{x_{i+1}}^{x_i} (x+y)^{m-2} (xy)^{(2+\lambda)/2} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\geq 2^{m-1} (1-m) \sum_{i=1}^\infty x_i^{m-2} P_{j,\varepsilon}(t, i)^2 . \end{aligned} \quad (2.15)$$

It next follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx &= \sum_{i=1}^\infty \int_{x_{i+1}}^{x_i} x^m f_{j,\varepsilon}(t, x) \, dx \leq \sum_{i=1}^\infty x_{i+1}^{(2m-2-\lambda)/2} P_{j,\varepsilon}(t, i) \\ &\leq \left(\sum_{i=1}^\infty x_{i+1}^{2m-2-\lambda} x_i^{2-m} \right)^{1/2} \left(\sum_{i=1}^\infty x_i^{m-2} P_{j,\varepsilon}(t, i)^2 \right)^{1/2} . \end{aligned} \quad (2.16)$$

Since

$$x_{i+1}^{2m-2-\lambda} x_i^{2-m} \leq (2i)^{2(2+\lambda-2m)/(m-\lambda)} i^{-2(2-m)/(m-\lambda)} = 4^{(2+\lambda-2m)/(m-\lambda)} i^{-2} ,$$

the series in the right-hand side of (2.16) converges and we deduce from (2.15) and (2.16) that

$$P_{j,\varepsilon}(t) \geq C_1(m) \left(\int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx \right)^2. \quad (2.17)$$

Furthermore, as

$$\begin{aligned} M_m(f_{j,\varepsilon}(t)) &= \int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx + \int_1^\infty x^m f_{j,\varepsilon}(t, x) \, dx \\ &\leq \int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx + \int_1^\infty x f_{j,\varepsilon}(t, x) \, dx \\ &\leq \int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx + \varrho \end{aligned}$$

by (2.7), we infer from Young's inequality that

$$\left(\int_0^1 x^m f_{j,\varepsilon}(t, x) \, dx \right)^2 \geq \frac{M_m(f_{j,\varepsilon}(t))^2}{2} - \varrho^2. \quad (2.18)$$

Combining (2.14), (2.17), and (2.18) provides the existence of two positive constants $C_2(m)$ and $C_3(m)$ such that

$$\frac{1}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) \chi_m(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \leq C_2(m) - C_3(m) M_m(f_{j,\varepsilon}(t))^2. \quad (2.19)$$

Consequently, recalling that $\mathbf{b}_m > 1$ by (2.8) as $m < 1$, it follows from (2.5) with $\vartheta = \vartheta_m$, (2.13), (2.19), and Young's inequality that

$$\begin{aligned} \frac{d}{dt} M_m(f_{j,\varepsilon}(t)) &= \frac{1}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) \chi_m(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad + a_0 (\mathbf{b}_m - 1) \int_0^\infty x^m (\varepsilon^2 + \min\{x, j\}^\gamma) f_{j,\varepsilon}(t, x) \, dx \\ &\leq C_2(m) - C_3(m) M_m(f_{j,\varepsilon}(t))^2 + a_0 \mathbf{b}_m M_{m+\gamma}(f_{j,\varepsilon}(t)) \\ &\quad + a_0 \mathbf{b}_m \varepsilon^2 M_m(f_{j,\varepsilon}(t)) \\ &\leq C_2(m) - C_3(m) M_m(f_{j,\varepsilon}(t))^2 + \frac{a_0 \mathbf{b}_m \gamma}{\gamma + 1 - m} M_{\gamma+1}(f_{j,\varepsilon}(t)) \\ &\quad + a_0 \mathbf{b}_m \left[\frac{1 - m}{\gamma + 1 - m} + 1 \right] M_m(f_{j,\varepsilon}(t)) \\ &\leq C_4(m) [1 + M_m(f_{j,\varepsilon}(t)) + \sigma] - C_3(m) M_m(f_{j,\varepsilon}(t))^2. \end{aligned}$$

As

$$M_m(f_{j,\varepsilon}(t)) \leq \frac{C_3(m)}{2C_4(m)} M_m(f_{j,\varepsilon}(t))^2 + \frac{C_4(m)}{2C_3(m)},$$

we finally obtain

$$\frac{d}{dt}M_m(f_{j,\varepsilon}(t)) \leq -C_4(m)M_m(f_{j,\varepsilon}(t)) + C_4(m) \left[1 + \frac{C_4(m)}{C_3(m)} + \sigma \right], \quad t \geq 0.$$

Integrating the previous differential inequality gives

$$M_m(f_{j,\varepsilon}(t)) \leq e^{-C_4(m)t}M_m(f^{in}) + \left[1 + \frac{C_4(m)}{C_3(m)} + \sigma \right] (1 - e^{-C_4(m)t}), \quad t \geq 0.$$

Therefore,

$$M_m(f_{j,\varepsilon}(t)) \leq \max \left\{ M_m(f^{in}), 1 + \frac{C_4(m)}{C_3(m)} + \sigma \right\}, \quad t \geq 0,$$

from which Lemma 2.3 follows. \square

The next step is devoted to the derivation of additional estimates for small sizes but now with a strong dependence on ε .

Lemma 2.4. *There is $\mu_0 \geq 1$ depending only on K_0 , a_0 , B , and ϱ such that*

$$\mathcal{M}_{0,j,\varepsilon} \leq \max \left\{ M_0(f^{in}), \sigma + \frac{\mu_0}{\varepsilon} \right\}.$$

Proof. It follows from (2.5) with $\vartheta \equiv 1$, (2.8), (2.13), and Young's inequality that, for $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}M_0(f_{j,\varepsilon}(t)) &= -\frac{1}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x,y) f_{j,\varepsilon}(t,x) f_{j,\varepsilon}(t,y) \, dy dx \\ &\quad - a_0(1 - \mathfrak{b}_0) \int_0^\infty (\varepsilon^2 + \min\{x,j\}^\gamma) f_{j,\varepsilon}(t,x) \, dx \\ &\leq -\varepsilon K_0 M_0(f_{j,\varepsilon}(t))^2 + a_0 \mathfrak{b}_0 \varepsilon^2 M_0(f_{j,\varepsilon}(t)) + a_0 \mathfrak{b}_0 M_\gamma(f_{j,\varepsilon}(t)) \\ &\leq -\varepsilon K_0 M_0(f_{j,\varepsilon}(t))^2 + \frac{\gamma a_0 \mathfrak{b}_0}{1 + \gamma} M_{\gamma+1}(f_{j,\varepsilon}(t)) \\ &\quad + a_0 \mathfrak{b}_0 \left(\frac{1}{1 + \gamma} + 1 \right) M_0(f_{j,\varepsilon}(t)) \\ &\leq -\varepsilon K_0 M_0(f_{j,\varepsilon}(t))^2 + a_0 \mathfrak{b}_0 \sigma + 2a_0 \mathfrak{b}_0 M_0(f_{j,\varepsilon}(t)). \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$4a_0 \mathfrak{b}_0 M_0(f_{j,\varepsilon}(t)) \leq \varepsilon K_0 M_0(f_{j,\varepsilon}(t))^2 + \frac{4a_0^2 \mathfrak{b}_0^2}{\varepsilon K_0}.$$

Hence

$$\frac{d}{dt}M_0(f_{j,\varepsilon}(t)) + 2a_0 \mathfrak{b}_0 M_0(f_{j,\varepsilon}(t)) \leq a_0 \mathfrak{b}_0 \sigma + \frac{4a_0^2 \mathfrak{b}_0^2}{\varepsilon K_0}, \quad t \geq 0.$$

Integrating this differential inequality, we find

$$\begin{aligned} M_0(f_{j,\varepsilon}(t)) &\leq M_0(f^{in})e^{-2a_0b_0t} + \left(\frac{\sigma}{2} + \frac{2a_0b_0}{\varepsilon K_0} \right) (1 - e^{-2a_0b_0t}) \\ &\leq \max \left\{ M_0(f^{in}), \sigma + \frac{2a_0b_0}{\varepsilon K_0} \right\} \end{aligned}$$

for $t \geq 0$, as claimed. \square

The previous result actually extends to some moments of negative order.

Lemma 2.5. *Let $m \in (m_*, 0)$ and set*

$$\varepsilon_{m,\sigma} := \frac{1}{\sigma} \min \left\{ 1, \frac{K_0 \varrho^2}{4a_0 b_m} \right\}, \quad (2.20)$$

where m_* and σ are defined in (1.22) and (2.12), respectively. There is $\mu_m > 0$ depending only on K_0 , a_0 , ϱ , B , and m such that, if $f^{in} \in X_m$ and $\varepsilon \in (0, \varepsilon_{m,\sigma})$, then

$$\mathcal{M}_{m,j,\varepsilon} \leq \max \left\{ M_m(f^{in}), \mu_m \sigma^2 \varepsilon^{-(\gamma+2-2m)/\gamma} \right\}.$$

We may also assume that $\mu_m \geq \Gamma(m+1)\varrho^m$ when $m > -1$.

Proof. For $\delta \in (0, 1)$, we set $\vartheta_{m,\delta}(x) := (x + \delta)^m$, $x > 0$, and notice that

$$\chi_{\vartheta_{m,\delta}}(x, y) \leq -(x + \delta)^m \leq 0, \quad (x, y) \in (0, \infty)^2.$$

Let $\varepsilon \in (0, \varepsilon_{m,\sigma})$ and $t > 0$. We infer from (2.5) with $\vartheta = \vartheta_{m,\delta}$ that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) dx &\leq -\varepsilon K_0 \int_0^\infty \int_0^\infty (x + \delta)^m f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) dy dx \\ &\quad + \int_0^\infty a_{j,\varepsilon}(y) f_{j,\varepsilon}(t, y) \int_0^y \vartheta_{m,\delta}(x) b(x, y) dx dy. \end{aligned}$$

On the one hand, by (2.7), (2.13), and the Cauchy-Schwarz inequality,

$$\varrho^2 = M_1(f_{j,\varepsilon}(t))^2 \leq M_0(f_{j,\varepsilon}(t)) M_2(f_{j,\varepsilon}(t)) \leq \sigma M_0(f_{j,\varepsilon}(t)),$$

so that

$$\begin{aligned} U_{j,\varepsilon}(t) &:= \int_0^\infty \int_0^\infty (x + \delta)^m f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) dy dx \\ &= M_0(f_{j,\varepsilon}(t)) \int_0^\infty (x + \delta)^m f_{j,\varepsilon}(t, x) dx \\ &\geq \frac{\varrho^2}{\sigma} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) dx. \end{aligned}$$

On the other hand, we infer from (1.16c), (2.13), and the negativity of m that

$$\begin{aligned}
V_{j,\varepsilon}(t) &:= \int_0^\infty a_{j,\varepsilon}(y) f_{j,\varepsilon}(t, y) \int_0^y \vartheta_{m,\delta}(x) b(x, y) \, dx dy \\
&= \int_0^\infty a_{j,\varepsilon}(y) f_{j,\varepsilon}(t, y) \int_0^1 (yz + \delta)^m B(z) \, dz dy \\
&\leq \int_0^\infty a_{j,\varepsilon}(y) f_{j,\varepsilon}(t, y) \int_0^1 (yz + \delta z)^m B(z) \, dz dy \\
&\leq a_0 \mathbf{b}_m \int_0^\infty (x + \delta)^m (\varepsilon^2 + \min\{x, j\}^\gamma) f_{j,\varepsilon}(t, x) \, dx .
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^\infty (x + \delta)^m \min\{x, j\}^\gamma f_{j,\varepsilon}(t, x) \, dx &\leq \varepsilon^2 \int_0^{\varepsilon^{2/\gamma}} (x + \delta)^m f_{j,\varepsilon}(t, x) \, dx \\
&\quad + \varepsilon^{2(m-1)/\gamma} \int_{\varepsilon^{2/\gamma}}^\infty x^{\gamma+1} f_{j,\varepsilon}(t, x) \, dx \\
&\leq \varepsilon^2 \int_0^\infty (x + \delta)^m f_{j,\varepsilon}(t, x) \, dx + \varepsilon^{2(m-1)/\gamma} \mathcal{M}_{\gamma+1, j, \varepsilon} \\
&\leq \varepsilon^2 \int_0^\infty (x + \delta)^m f_{j,\varepsilon}(t, x) \, dx + \sigma \varepsilon^{2(m-1)/\gamma}
\end{aligned}$$

by (2.13), we further obtain

$$V_{j,\varepsilon}(t) \leq a_0 \mathbf{b}_m \left(2\varepsilon^2 \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx + \sigma \varepsilon^{2(m-1)/\gamma} \right) .$$

Collecting the previous estimates and using the definition (2.20) of $\varepsilon_{m,\sigma}$ lead us to the differential inequality

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx &\leq -\frac{\varepsilon K_0 \varrho^2}{\sigma} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx \\
&\quad + 2a_0 \mathbf{b}_m \varepsilon^2 \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx + a_0 \mathbf{b}_m \sigma \varepsilon^{2(m-1)/\gamma} \\
&\leq 2a_0 \mathbf{b}_m \varepsilon (\varepsilon - 2\varepsilon_{m,\sigma}) \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx + a_0 \mathbf{b}_m \sigma \varepsilon^{2(m-1)/\gamma} \\
&\leq -2a_0 \mathbf{b}_m \varepsilon \varepsilon_{m,\sigma} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx + a_0 \mathbf{b}_m \sigma \varepsilon^{2(m-1)/\gamma} .
\end{aligned}$$

After integration with respect to time, we end up with

$$\begin{aligned} \int_0^\infty \vartheta_{m,\delta}(x) f_{j,\varepsilon}(t, x) \, dx &\leq e^{-2a_0 \mathfrak{b}_m \varepsilon \varepsilon_{m,\sigma} t} \int_0^\infty \vartheta_{m,\delta}(x) f^{in}(x) \, dx \\ &\quad + \frac{\sigma \varepsilon^{-(\gamma+2-2m)/\gamma}}{2\varepsilon_{m,\sigma}} (1 - e^{-2a_0 \mathfrak{b}_m \varepsilon \varepsilon_{m,\sigma} t}) \\ &\leq \max \{ M_m(f^{in}), \mu_m \sigma^2 \varepsilon^{-(\gamma+2-2m)/\gamma} \} , \quad t \geq 0 . \end{aligned}$$

Since the right-hand side of the previous inequality does not depend on $\delta \in (0, 1)$ and is finite, we may pass to the limit as $\delta \rightarrow 0$ and thereby complete the proof of Lemma 2.5. \square

Remark 2.6. *It is worth mentioning here that the positivity of γ is only used in the proof of Lemma 2.5.*

2.2. Integrability Estimates. We now turn to weighted L^p -estimates and actually derive two different estimates, one depending on ε but not on t , and the other one depending on t but not on ε . For $m \geq 0$, $p \geq 1$, and $h \in L^p((0, \infty), x^m dx)$, we set

$$L_{m,p}(h) := \int_0^\infty x^m |h(x)|^p \, dx . \quad (2.21)$$

Lemma 2.7. *Consider $m \in (\lambda, 1)$ and $p \in (1, p_0]$ satisfying*

$$1 < p < \frac{m+1}{\lambda+1} \quad \text{and} \quad p \leq \frac{m+\gamma}{\gamma} , \quad (2.22)$$

and assume that $f^{in} \in L^p((0, \infty), x^m dx)$. Then

$$L_{m,p}(f_{j,\varepsilon}(t)) \leq \max \left\{ L_{m,p}(f^{in}), \frac{S_{j,\varepsilon}(m, p)}{\varepsilon^2} \right\} \quad (2.23)$$

and

$$\frac{1}{t} \int_0^t \int_0^\infty x^m \min\{x, j\}^\gamma (f_{j,\varepsilon}(s, x))^p \, dx ds \leq \frac{1}{a_0 t} L_{m,p}(f^{in}) + S_{j,\varepsilon}(m, p) , \quad (2.24)$$

where

$$S_{j,\varepsilon}(m, p) := 2^p \mathcal{B}_p^p \left(\mathcal{M}_{(m+1+\gamma-p)/p, j, \varepsilon}^p + \mathcal{M}_{(m+1+\gamma p-p)/p, j, \varepsilon}^p + \varepsilon^2 \mathcal{M}_{(m+1-p)/p, j, \varepsilon}^p \right)$$

and \mathcal{B}_p is defined in (2.8a).

Proof. We first note that (1.16a) and (2.22) ensure that

$$\frac{m+1+\gamma p-p}{p} \geq \frac{m+1+\gamma-p}{p} \geq \frac{m+1-p}{p} > \lambda ,$$

so that $S_{j,\varepsilon}(m, p)$ is well-defined and finite by Lemma 2.3.

Let $t > 0$. We first deal with the contribution of the coagulation term. As already observed in [4, 8, 18, 24], the sublinearity of $x \mapsto x^m$ and the monotonicity of $x \mapsto K_{j,\varepsilon}(x, y)$ for all $y > 0$ allow us to show that this contribution is negative. Indeed, it follows from the inequality

$$(x + y)^m \leq x^m + y^m, \quad (x, y) \in (0, \infty)^2,$$

the symmetry of $K_{j,\varepsilon}$, and Fubini's theorem that

$$\begin{aligned} P_{j,\varepsilon}(t) &:= p \int_0^\infty x^m (f_{j,\varepsilon}(t, x))^{p-1} (\mathcal{C}_{j,\varepsilon} f_{j,\varepsilon})(t, x) \, dx \\ &= \frac{p}{2} \int_0^\infty \int_0^\infty (x + y)^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x + y))^{p-1} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - p \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dy dx \\ &\leq \frac{p}{2} \int_0^\infty \int_0^\infty (x^m + y^m) K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x + y))^{p-1} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - p \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dy dx \\ &= p \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x + y))^{p-1} f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - p \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dy dx \end{aligned}$$

We next deduce from the convexity inequality

$$pU^{p-1}V \leq (p-1)U^p + V^p, \quad (U, V) \in [0, \infty)^2,$$

that

$$\begin{aligned} P_{j,\varepsilon}(t) &\leq (p-1) \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x + y))^p f_{j,\varepsilon}(t, y) \, dy dx \\ &\quad - (p-1) \int_0^\infty \int_0^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dy dx \\ &\leq (p-1) \int_0^\infty \int_y^\infty (x - y)^m K_{j,\varepsilon}(x - y, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dx dy \\ &\quad - (p-1) \int_0^\infty \int_y^\infty x^m K_{j,\varepsilon}(x, y) (f_{j,\varepsilon}(t, x))^p f_{j,\varepsilon}(t, y) \, dx dy. \end{aligned}$$

Now, the monotonicity of $x \mapsto x^m$ and $x \mapsto K_{j,\varepsilon}(x, y)$ implies that

$$(x - y)^m K_{j,\varepsilon}(x - y, y) \leq x^m K_{j,\varepsilon}(x, y), \quad 0 < y < x.$$

Consequently,

$$P_{j,\varepsilon}(t) \leq 0 . \quad (2.25)$$

Concerning the contribution of the fragmentation term, it reads

$$\begin{aligned} Q_{j,\varepsilon}(t) &:= p \int_0^\infty x^m (f_{j,\varepsilon}(t, x))^{p-1} (\mathcal{F}_{j,\varepsilon} f_{j,\varepsilon})(t, x) \, dx \\ &= -pa_0 \Lambda_j(f_{j,\varepsilon}(t)) - pa_0 \varepsilon^2 L_{m,p}(f_{j,\varepsilon}(t)) + R_{j,\varepsilon}(t) , \end{aligned} \quad (2.26)$$

where

$$\Lambda_j(f_{j,\varepsilon}(t)) := \int_0^\infty x^m \min\{x, j\}^\gamma (f_{j,\varepsilon}(t, x))^p \, dx$$

and

$$\begin{aligned} R_{j,\varepsilon}(t) &:= p \int_0^\infty a_{j,\varepsilon}(y) f_{j,\varepsilon}(t, y) \int_0^y x^m b(x, y) (f_{j,\varepsilon}(t, x))^{p-1} \, dx dy \\ &= pa_0 \int_0^\infty \min\{y, j\}^\gamma y^{-1} f_{j,\varepsilon}(t, y) \int_0^y x^m B(xy^{-1}) (f_{j,\varepsilon}(t, x))^{p-1} \, dx dy \\ &\quad + pa_0 \varepsilon^2 \int_0^\infty y^{-1} f_{j,\varepsilon}(t, y) \int_0^y x^m B(xy^{-1}) (f_{j,\varepsilon}(t, x))^{p-1} \, dx dy . \end{aligned}$$

We infer from Hölder's inequality that

$$\begin{aligned} &\int_0^y x^m B(xy^{-1}) (f_{j,\varepsilon}(t, x))^{p-1} \, dx \\ &= \int_0^y x^{m/p} \min\{x, j\}^{-\gamma(p-1)/p} B(xy^{-1}) x^{m(p-1)/p} \min\{x, j\}^{\gamma(p-1)/p} (f_{j,\varepsilon}(t, x))^{p-1} \, dx \\ &\leq \left(\int_0^y x^m \min\{x, j\}^{-\gamma(p-1)} [B(xy^{-1})]^p \, dx \right)^{1/p} \left(\int_0^y x^m \min\{x, j\}^\gamma (f_{j,\varepsilon}(t, x))^p \, dx \right)^{(p-1)/p} . \end{aligned}$$

Since

$$\begin{aligned} &\left(\int_0^y x^m \min\{x, j\}^{-\gamma(p-1)} [B(xy^{-1})]^p \, dx \right)^{1/p} \\ &\leq \left(\int_0^y (x^{m-\gamma(p-1)} + x^m) [B(xy^{-1})]^p \, dx \right)^{1/p} \\ &\leq y^{(m+1-\gamma(p-1))/p} \left(\int_0^1 z^{m-\gamma(p-1)} B(z)^p \, dz \right)^{1/p} \\ &\quad + y^{(m+1)/p} \left(\int_0^1 z^m B(z)^p \, dz \right)^{1/p} , \end{aligned}$$

we further obtain

$$\begin{aligned} & \int_0^y x^m B(xy^{-1}) (f_{j,\varepsilon}(t, x))^{p-1} dx \\ & \leq y^{(m+1-\gamma(p-1))/p} \left(\int_0^1 z^{m-\gamma(p-1)} B(z)^p dz \right)^{1/p} \Lambda_j(f_{j,\varepsilon}(t))^{(p-1)/p} \\ & \quad + y^{(m+1)/p} \left(\int_0^1 z^m B(z)^p dz \right)^{1/p} \Lambda_j(f_{j,\varepsilon}(t))^{(p-1)/p} . \end{aligned}$$

Similarly, by Hölder's inequality,

$$\begin{aligned} & \int_0^y x^m B(xy^{-1}) (f_{j,\varepsilon}(t, x))^{p-1} dx \\ & = \int_0^y x^{m/p} B(xy^{-1}) x^{m(p-1)/p} (f_{j,\varepsilon}(t, x))^{p-1} dx \\ & \leq \left(\int_0^y x^m [B(xy^{-1})]^p dx \right)^{1/p} \left(\int_0^y x^m (f_{j,\varepsilon}(t, x))^p dx \right)^{(p-1)/p} \\ & \leq y^{(m+1)/p} \left(\int_0^1 z^m B(z)^p dz \right)^{1/p} [L_{m,p}(f_{j,\varepsilon}(t))]^{(p-1)/p} . \end{aligned}$$

Since $0 \leq m - \gamma(p-1) \leq m$ and $p \in [1, p_0]$ by (2.22), we infer from (2.8a) that

$$\int_0^1 z^m B(z)^p dz \leq \int_0^1 z^{m-\gamma(p-1)} B(z)^p dz \leq \mathcal{B}_p^p < \infty .$$

Gathering the above estimates and using Young's inequality, we end up with

$$\begin{aligned} R_{j,\varepsilon}(t) & \leq pa_0 \mathcal{B}_p M_{(m+1+\gamma-p)/p}(f_{j,\varepsilon}(t)) \Lambda_j(f_{j,\varepsilon}(t))^{(p-1)/p} \\ & \quad + pa_0 \mathcal{B}_p M_{(m+1+\gamma p-p)/p}(f_{j,\varepsilon}(t)) \Lambda_j(f_{j,\varepsilon}(t))^{(p-1)/p} \\ & \quad + pa_0 \varepsilon^2 \mathcal{B}_p M_{(m+1-p)/p}(f_{j,\varepsilon}(t)) [L_{m,p}(f_{j,\varepsilon}(t))]^{(p-1)/p} \\ & \leq \frac{p-1}{2} a_0 \Lambda_j(f_{j,\varepsilon}(t)) + 2^{p-1} a_0 \mathcal{B}_p^p \mathcal{M}_{(m+1+\gamma-p)/p,j,\varepsilon}^p \\ & \quad + \frac{p-1}{2} a_0 \Lambda_j(f_{j,\varepsilon}(t)) + 2^{p-1} a_0 \mathcal{B}_p^p \mathcal{M}_{(m+1+\gamma p-p)/p,j,\varepsilon}^p \\ & \quad + (p-1) a_0 \varepsilon^2 L_{m,p}(f_{j,\varepsilon}(t)) + a_0 \varepsilon^2 \mathcal{B}_p^p \mathcal{M}_{(m+1-p)/p,j,\varepsilon}^p . \end{aligned} \tag{2.27}$$

We then deduce from (2.26) and (2.27) that

$$Q_{j,\varepsilon}(t) \leq -a_0 [\Lambda_j(f_{j,\varepsilon}(t)) + \varepsilon^2 L_{m,p}(f_{j,\varepsilon}(t))] + a_0 S_{j,\varepsilon}(m, p) . \tag{2.28}$$

Combining (2.4), (2.25), and (2.28) leads us to the differential inequality

$$\frac{d}{dt} L_{m,p}(f_{j,\varepsilon}(t)) + a_0 [\Lambda_j(f_{j,\varepsilon}(t)) + \varepsilon^2 L_{m,p}(f_{j,\varepsilon}(t))] \leq a_0 S_{j,\varepsilon}(m, p) \tag{2.29}$$

for $t > 0$. We first infer from (2.29) that, for $t > 0$,

$$\frac{d}{dt}L_{m,p}(f_{j,\varepsilon}(t)) + a_0\varepsilon^2L_{m,p}(f_{j,\varepsilon}(t)) \leq a_0S_{j,\varepsilon}(m,p) .$$

Hence, after integration with respect to time,

$$\begin{aligned} L_{m,p}(f_{j,\varepsilon}(t)) &\leq e^{-a_0\varepsilon^2t}L_{m,p}(f^{in}) + \frac{S_{j,\varepsilon}(m,p)}{\varepsilon^2} \left(1 - e^{-a_0\varepsilon^2t}\right) \\ &\leq \max \left\{ L_{m,p}(f^{in}), \frac{S_{j,\varepsilon}(m,p)}{\varepsilon^2} \right\} , \end{aligned}$$

from which (2.23) follows. We also infer from (2.29) that, for $t > 0$,

$$\frac{d}{dt}L_{m,p}(f_{j,\varepsilon}(t)) + a_0\Lambda_j(f_{j,\varepsilon}(t)) \leq a_0S_{j,\varepsilon}(m,p) .$$

Integrating with respect to time and using the non-negativity of $L_{m,p}(f_{j,\varepsilon}(t))$, we obtain

$$a_0 \int_0^t \Lambda_j(f_{j,\varepsilon}(s)) \, ds \leq L_{m,p}(f^{in}) + a_0tS_{j,\varepsilon}(m,p)$$

for $t > 0$. Dividing the above inequality by a_0t gives (2.24). \square

Combining the outcome of Lemma 2.5 and Lemma 2.7 leads to an ε -dependent L^p -estimate for $(f_{j,\varepsilon})_{j \geq 2}$ for a suitable value of p .

Corollary 2.8. *Let $m_0 \in (m_*, 0)$, $m_1 \in (\lambda, 1)$, and $p_1 \in (1, p_0)$ be such that*

$$1 < p_1 < \frac{m_1 + 1}{\lambda + 1} \quad \text{and} \quad p_1 \leq \frac{m_1 + \gamma}{\gamma} , \tag{2.30}$$

and assume that $f^{in} \in X_{m_0} \cap L^{p_1}((0, \infty), x^{m_1} dx)$. For $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$ and $t \geq 0$,

$$\begin{aligned} L_{0,p_2}(f_{j,\varepsilon}(t)) &\leq \max \left\{ M_{m_0}(f^{in}), \mu_{m_0}\sigma^2\varepsilon^{-(\gamma+2-2m_0)/\gamma} \right\} \\ &\quad + \max \left\{ L_{m_1,p_1}(f^{in}), \frac{S_{j,\varepsilon}(m_1, p_1)}{\varepsilon^2} \right\} , \end{aligned}$$

where

$$p_2 := \frac{m_1}{m_1 - m_0} + p_1 \frac{|m_0|}{m_1 - m_0} \in (1, p_1) .$$

Proof. Since

$$\frac{m_1}{m_1 - m_0}m_0 + \frac{|m_0|}{m_1 - m_0}m_1 = 0 ,$$

we infer from Young's inequality that, if $h \in X_{m_0} \cap L^{p_1}((0, \infty), x^{m_1} dx)$, then $h \in L^{p_2}(0, \infty)$ and

$$\begin{aligned} L_{0,p_2}(h) &= \|h\|_{p_2}^{p_2} = \int_0^\infty (x^{m_0} |h(x)|)^{m_1/(m_1-m_0)} (x^{m_1} |h(x)|)^{|m_0|/(m_1-m_0)} dx \\ &\leq \frac{m_1}{m_1-m_0} \int_0^\infty x^{m_0} |h(x)| dx + \frac{|m_0|}{m_1-m_0} \int_0^\infty x^{m_1} |h(x)|^{p_1} dx \\ &\leq M_{m_0}(|h|) + L_{m_1,p_1}(h) . \end{aligned} \quad (2.31)$$

Now, consider $t \geq 0$. As $\varepsilon \in (0, \varepsilon_{m_0,\sigma})$ and p_1 satisfies (2.30), Corollary 2.8 readily follows from Lemma 2.5 (with $m = m_0$), Lemma 2.7 (with $(m, p) = (m_1, p_1)$), and (2.31) (with $h = f_{j,\varepsilon}(t)$). \square

2.3. Time Equicontinuity. The last estimate to be derived in this section provides the time equicontinuity of the sequence $(f_{j,\varepsilon})_{j \geq 2}$ in $L^1(0, \infty)$, which is needed later to apply a variant of the Arzelà-Ascoli theorem.

Lemma 2.9. *There is a positive constant $C_5 > 0$ such that*

$$\|\partial_t f_{j,\varepsilon}(t)\|_1 \leq C_5 (\sigma + \mathcal{M}_{0,j,\varepsilon}^2) , \quad t \geq 0 .$$

Proof. Let $t > 0$. It follows from (2.4a), (2.9), and Fubini's theorem that

$$\begin{aligned} \|\partial_t f_{j,\varepsilon}(t)\|_1 &\leq \frac{3}{2} \int_0^\infty \int_0^\infty K_{j,\varepsilon}(x, y) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) dy dx \\ &\quad + (1 + \mathfrak{b}_0) \int_0^\infty a_{j,\varepsilon}(x) f_{j,\varepsilon}(t, x) dx \\ &\leq \frac{3K_0}{2} \int_0^\infty \int_0^\infty (x^\lambda + y^\lambda + 2\varepsilon) f_{j,\varepsilon}(t, x) f_{j,\varepsilon}(t, y) dy dx \\ &\quad + a_0(1 + \mathfrak{b}_0) \int_0^\infty (x^\gamma + \varepsilon^2) f_{j,\varepsilon}(t, x) dx \\ &\leq 3K_0 [M_\lambda(f_{j,\varepsilon}(t))M_0(f_{j,\varepsilon}(t)) + M_0(f_{j,\varepsilon}(t))^2] \\ &\quad + a_0(1 + \mathfrak{b}_0) [M_\gamma(f_{j,\varepsilon}(t)) + M_0(f_{j,\varepsilon}(t))] . \end{aligned}$$

We then infer from (2.7), (2.12a), (2.13), and the inequalities

$$x^\lambda \leq 1 + x , \quad x^\gamma \leq 1 + x^{1+\gamma} , \quad x \geq 0 ,$$

that

$$\begin{aligned} \|\partial_t f_{j,\varepsilon}(t)\|_1 &\leq 3K_0 [M_0(f_{j,\varepsilon}(t))M_1(f_{j,\varepsilon}(t)) + 2M_0(f_{j,\varepsilon}(t))^2] \\ &\quad + a_0(1 + \mathfrak{b}_0) [M_{\gamma+1}(f_{j,\varepsilon}(t)) + 2M_0(f_{j,\varepsilon}(t))] \\ &\leq 3K_0 [\varrho^2 + 3\mathcal{M}_{0,j,\varepsilon}^2] + a_0(1 + \mathfrak{b}_0) (\sigma + 1 + \mathcal{M}_{0,j,\varepsilon}^2) \\ &\leq [3K_0(\varrho + 3) + 2a_0(1 + \mathfrak{b}_0)] (\sigma + \mathcal{M}_{0,j,\varepsilon}^2) , \end{aligned}$$

and the proof is complete. \square

3. STATIONARY SOLUTIONS BY A DYNAMICAL APPROACH: $\varepsilon \in (0, 1)$

In this section, we fix $\varepsilon \in (0, 1)$ and study the coagulation-fragmentation equation (1.20) with coagulation kernel K_ε and overall fragmentation rate a_ε given by

$$K_\varepsilon(x, y) = K(x, y) + 2\varepsilon K_0, \quad a_\varepsilon(x) = a(x) + a_0\varepsilon^2, \quad (x, y) \in (0, \infty)^2; \quad (3.1)$$

that is,

$$\partial_t f = \mathcal{C}_\varepsilon f + \mathcal{F}_\varepsilon f, \quad (t, x) \in (0, \infty)^2, \quad (3.2a)$$

$$f(0) = f^{in}, \quad x \in (0, \infty), \quad (3.2b)$$

where the coagulation and fragmentation operators \mathcal{C}_ε and \mathcal{F}_ε are defined in (1.20).

Several results are established in this section. We begin with the well-posedness of (3.2) for a suitable class of initial conditions, the existence of solutions being obtained by passing to the limit as $j \rightarrow \infty$ in (2.4) (Section 3.1). We also establish the continuity of the solutions to (3.2) with respect to the initial condition for the weak topology of X_1 (Section 3.3) and construct an invariant set for the dynamics of (3.2) (Section 3.2). Combining the outcome of this analysis with a consequence of Tychonov's fixed point theorem provides the existence of a stationary solution to (3.2a) (Section 3.4). The estimates derived in the previous section are of course at the heart of the proofs of the results of this section.

We fix

$$m_0 \in (m_*, 0) \cap (-1, 0), \quad m_1 \in (\lambda, 1), \quad p_1 \in (1, p_0), \quad (3.3a)$$

such that

$$1 < p_1 < \frac{m_1 + 1}{\lambda + 1} \quad \text{and} \quad p_1 \leq \frac{m_1 + \gamma}{\gamma}. \quad (3.3b)$$

We recall that (3.3) implies that

$$m_2 \in (\lambda, 1) \quad \text{and} \quad m_2 < \frac{m_1 + 1 + \gamma - p_1}{p_1} < \frac{m_1 + 1 + \gamma p_1 - p_1}{p_1} \leq 1 + \gamma, \quad (3.4a)$$

where

$$m_2 := \frac{m_1 + 1 - p_1}{p_1} < 1. \quad (3.4b)$$

We also fix $\varrho > 0$ and $\sigma > 0$ satisfying

$$\sigma > \max\{1, \varrho, \mu_2, \mu_{2+\gamma}\}, \quad (3.5)$$

recalling that μ_m is defined in Lemma 2.1 for $m \geq 2$.

We next define a subset \mathcal{Y}_ε of X_1^+ as follows: $h \in \mathcal{Y}_\varepsilon$ if and only if

$$h \in X_1^+ \cap X_{m_0} \cap X_{2+\gamma}, \quad M_1(h) = \varrho, \quad (3.6a)$$

$$\max\{M_2(h), M_{2+\gamma}(h)\} \leq \sigma, \quad M_{m_2}(h) \leq \mu_{m_2} + \sigma, \quad (3.6b)$$

$$M_0(h) \leq \sigma + \mu_0\varepsilon^{-1}, \quad M_{m_0}(h) \leq \mu_{m_0}\sigma^2\varepsilon^{-(\gamma+2-2m_0)/\gamma}, \quad (3.6c)$$

$$L_{m_1, p_1}(h) \leq \sigma_1\varepsilon^{-2}, \quad (3.6d)$$

where

$$\sigma_1 := 2^{p_1} \mathcal{B}_{p_1}^{p_1} [2\sigma^{p_1} + 3(\mu_{m_2} + \sigma)^{p_1}] \quad (3.7)$$

and \mathcal{B}_{p_1} is defined in (2.8a).

3.1. Well-posedness of (3.2). We begin with the well-posedness of (3.2) in \mathcal{Y}_ε , along with several estimates for its solutions.

Proposition 3.1. *Consider $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$ and $f^{in} \in \mathcal{Y}_\varepsilon$, recalling that*

$$\varepsilon_{m_0, \sigma} = \frac{1}{\sigma} \min \left\{ 1, \frac{K_0 \varrho^2}{4a_0 \mathfrak{b}_{m_0}} \right\}$$

is defined in (2.20) with \mathfrak{b}_{m_0} given by (2.8a). There is a unique weak solution

$$\Psi_\varepsilon(\cdot, f^{in}) = f_\varepsilon \in C([0, \infty), X_0^+) \cap C([0, \infty), X_{1,w})$$

to (3.2) which satisfies

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \vartheta(x) f_\varepsilon(t, x) \, dx &= \frac{1}{2} \int_0^\infty \int_0^\infty K_\varepsilon(x, y) \chi_\vartheta(x, y) f_\varepsilon(t, x) f_\varepsilon(t, y) \, dy dx \\ &\quad - \int_0^\infty a_\varepsilon(y) N_\vartheta(y) f_\varepsilon(t, y) \, dy , \end{aligned} \quad (3.8)$$

for all $t \geq 0$ and $\vartheta \in L^\infty(0, \infty)$, the functions χ_ϑ and N_ϑ being defined in (2.6), and possesses the following properties:

$$M_1(f_\varepsilon(t)) = \varrho , \quad t \geq 0 , \quad (3.9a)$$

$$\sup_{t \geq 0} M_m(f_\varepsilon(t)) \leq \sigma , \quad m \in (1, 2 + \gamma] , \quad (3.9b)$$

$$\sup_{t \geq 0} M_m(f_\varepsilon(t)) \leq \max \{ M_m(f^{in}), \sigma + \mu_m \} , \quad m \in (\lambda, 1) , \quad (3.9c)$$

$$\sup_{t \geq 0} M_0(f_\varepsilon(t)) \leq \sigma + \mu_0 \varepsilon^{-1} , \quad (3.10a)$$

$$\sup_{t \geq 0} M_{m_0}(f_\varepsilon(t)) \leq \mu_{m_0} \sigma^2 \varepsilon^{-(\gamma+2-2m_0)/\gamma} , \quad (3.10b)$$

$$\sup_{t \geq 0} L_{m_1, p_1}(f_\varepsilon(t)) \leq \sigma_1 \varepsilon^{-2} , \quad (3.11a)$$

$$\sup_{t \geq 0} L_{0, p_2}(f_\varepsilon(t)) \leq \kappa_\varepsilon := \mu_{m_0} \sigma^2 \varepsilon^{-(\gamma+2-2m_0)/\gamma} + \sigma_1 \varepsilon^{-2} , \quad (3.11b)$$

and

$$\frac{1}{t} \int_0^t L_{m_1 + \gamma, p_1}(f_\varepsilon(s)) \, ds \leq \frac{1}{a_0 t} L_{m_1, p_1}(f^{in}) + \sigma_1 , \quad t > 0 . \quad (3.12)$$

Moreover, if $f^{in} \in X_m$ for some $m > 2 + \gamma$, then $f_\varepsilon \in L^\infty((0, \infty), X_m)$ and

$$\sup_{t \geq 0} M_m(f_\varepsilon(t)) \leq \max\{M_m(f^{in}), \mu_m\}, \quad (3.13)$$

the constant μ_m being defined in Lemma 2.1.

Proof. Step 1: Existence. Let $j \geq 2$ and recall that $f_{j,\varepsilon}$ is the strong solution to the coagulation-fragmentation equation (2.4), see Section 2. Since $f^{in} \in \mathcal{Y}_\varepsilon$, it follows from (2.7) that

$$M_1(f_{j,\varepsilon}(t)) = \varrho, \quad t \geq 0, \quad j \geq 2, \quad (3.14)$$

and from (2.12), (3.5), Lemma 2.1, and Corollary 2.2 that

$$\sup_{t \geq 0} M_m(f_{j,\varepsilon}(t)) \leq \sigma, \quad m \in (1, 2 + \gamma], \quad j \geq 2. \quad (3.15)$$

Next, (2.12), (3.5), (3.6b), and Lemma 2.3 guarantee that

$$\sup_{t \geq 0} M_{m_2}(f_{j,\varepsilon}(t)) \leq \mu_{m_2} + \sigma, \quad j \geq 2, \quad (3.16)$$

while, since $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$, we deduce from (2.12), (3.5), (3.6c), Lemma 2.4, and Lemma 2.5 that

$$\sup_{t \geq 0} M_0(f_{j,\varepsilon}(t)) \leq \sigma + \mu_0 \varepsilon^{-1}, \quad j \geq 2, \quad (3.17)$$

$$\sup_{t \geq 0} M_{m_0}(f_{j,\varepsilon}(t)) \leq \mu_{m_0} \sigma^2 \varepsilon^{-(\gamma+2-2m_0)/\gamma}, \quad j \geq 2. \quad (3.18)$$

Finally, by (3.4), (3.15), and Hölder's and Young's inequalities,

$$\begin{aligned} M_{(m+1+\gamma-p_1)/p_1}(f_{j,\varepsilon}(t))^{p_1} &\leq \frac{\gamma}{p_1(1+\gamma-m_2)} M_{1+\gamma}(f_{j,\varepsilon}(t))^{p_1} \\ &\quad + \frac{p_1(1+\gamma-m_2)-\gamma}{p_1(1+\gamma-m_2)} M_{m_2}(f_{j,\varepsilon}(t))^{p_1} \\ &\leq \sigma^{p_1} + M_{m_2}(f_{j,\varepsilon}(t))^{p_1}, \end{aligned}$$

and

$$\begin{aligned} M_{(m+1+\gamma p_1-p_1)/p_1}(f_{j,\varepsilon}(t))^{p_1} &\leq \frac{\gamma}{1+\gamma-m_2} M_{1+\gamma}(f_{j,\varepsilon}(t))^{p_1} + \frac{1-m_2}{1+\gamma-m_2} M_{m_2}(f_{j,\varepsilon}(t))^{p_1} \\ &\leq \sigma^{p_1} + M_{m_2}(f_{j,\varepsilon}(t))^{p_1} \end{aligned}$$

for $t \geq 0$ and $j \geq 2$, so that, using also (3.7) and (3.16),

$$\begin{aligned} S_{j,\varepsilon}(m_1, p_1) &= 2^{p_1} \mathcal{B}_{p_1}^{p_1} \sup_{t \geq 0} M_{(m+1+\gamma-p_1)/p_1}(f_{j,\varepsilon}(t))^{p_1} \\ &\quad + 2^{p_1} \mathcal{B}_{p_1}^{p_1} \sup_{t \geq 0} M_{(m+1+\gamma p_1-p_1)/p_1}(f_{j,\varepsilon}(t))^{p_1} \\ &\quad + 2^{p_1} \mathcal{B}_{p_1}^{p_1} \varepsilon^2 \sup_{t \geq 0} M_{m_2}(f_{j,\varepsilon}(t))^{p_1} \\ &\leq 2^{p_1} \mathcal{B}_{p_1}^{p_1} [2\sigma^{p_1} + 3(\mu_{m_2} + \sigma)^{p_1}] = \sigma_1. \end{aligned} \quad (3.19)$$

Combining (2.23), (3.6d), and (3.19), we conclude that

$$\sup_{t \geq 0} L_{m_1, p_1}(f_{j, \varepsilon}(t)) \leq \sigma_1 \varepsilon^{-2}, \quad j \geq 2. \quad (3.20)$$

A straightforward consequence of (3.3b), (3.5), (3.9), (3.18), (3.20), and Corollary 2.8 is the bound

$$\sup_{t \geq 0} L_{0, p_2}(f_{j, \varepsilon}(t)) \leq \mu_{m_0} \sigma^2 \varepsilon^{-(\gamma+2-2m_0)/\gamma} + \sigma_1 \varepsilon^{-2} = \kappa_\varepsilon, \quad j \geq 2. \quad (3.21)$$

Now, introducing the set

$$\mathcal{W}_\varepsilon := \left\{ \begin{array}{l} h \in X_{m_0} \cap X_{2+\gamma} \cap L^{p_2}(0, \infty) : \\ M_{2+\gamma}(h) \leq \sigma, \max\{M_{m_0}(h), L_{0, p_2}(h)\} \leq \kappa_\varepsilon \end{array} \right\}, \quad (3.22)$$

it readily follows from (3.15), (3.18), and (3.21) that

$$f_{j, \varepsilon}(t) \in \mathcal{W}_\varepsilon, \quad t \geq 0, \quad j \geq 2, \quad (3.23)$$

while the Dunford-Pettis theorem ensures that

$$\mathcal{W}_\varepsilon \text{ is a relatively sequentially weakly compact subset of } X_m \quad (3.24)$$

for any $m \in (m_0, 2 + \gamma)$, and in particular of X_0 . Moreover, it follows from (3.17) and Lemma 2.9 that, for $0 \leq t_1 \leq t_2$ and $j \geq 2$,

$$\|f_{j, \varepsilon}(t_2) - f_{j, \varepsilon}(t_1)\|_1 \leq \int_{t_1}^{t_2} \|\partial_t f_{j, \varepsilon}(t)\|_1 dt \leq C_5 \left[\sigma + (\sigma + \mu_0 \varepsilon^{-1})^2 \right] (t_2 - t_1). \quad (3.25)$$

Consequently, $(f_{j, \varepsilon})_{j \geq 2}$ is equicontinuous at each $t \geq 0$ for the norm-topology of $L^1(0, \infty)$, and thus it is also equicontinuous for the weak topology of $L^1(0, \infty)$. This property, along with (3.23) and the relative compactness (3.24) of \mathcal{W}_ε , allows us to apply a variant of the Arzelà-Ascoli theorem [27, Theorem A.3.1] to conclude that there are a subsequence of $(f_{j, \varepsilon})_{j \geq 2}$ (possibly depending on ε but not relabeled) and $f_\varepsilon \in C([0, \infty), X_{0, w})$ such that

$$f_{j, \varepsilon} \longrightarrow f_\varepsilon \text{ in } C([0, T], X_{0, w}) \text{ for all } T > 0. \quad (3.26)$$

A first consequence of (3.26) is that $f_\varepsilon(t) \in X_0^+$ for all $t \geq 0$. It next follows from (3.14), (3.15), (3.17), (3.18), and (3.26) by a weak lower semicontinuity argument that f_ε satisfies (3.9b), (3.10a), (3.10b), and

$$M_1(f_\varepsilon(t)) \leq \varrho, \quad t \geq 0.$$

A similar argument allows us to deduce (3.9c) from Lemma 2.3 and (3.26). We then combine the just established property (3.9b) with (3.15) and (3.26) to improve the convergence (3.26) to

$$f_{j, \varepsilon} \longrightarrow f_\varepsilon \text{ in } C([0, T], X_{1, w} \cap X_{\gamma, w}) \text{ for all } T > 0. \quad (3.27)$$

Recalling (3.14), we readily infer from (3.27) that f_ε satisfies the mass conservation (3.9a). We employ again weak lower semicontinuity arguments to deduce (3.11) and

$$\frac{1}{t} \int_0^t \int_0^R x^{m+\gamma} (f_\varepsilon(s, x))^p dx ds \leq \frac{1}{a_0 t} L_{m_1, p_1}(f_\varepsilon^{in}) + \sigma_1, \quad t > 0, \quad R \geq 1, \quad (3.28)$$

from (2.23), (2.24), (3.3b), (3.6d), (3.19), (3.20), (3.21), and (3.26). As the right-hand side of (3.28) does not depend on R , we may let $R \rightarrow \infty$ in (3.28) and use Fatou's lemma to obtain (3.12).

Now, owing to (1.15), (1.16), (3.26), and (3.27), we may proceed as in [26], see also [4, 10, 12, 19], to deduce from (2.5) that f_ε is a weak solution to (3.2), in the sense that it satisfies (3.8). Furthermore, we may argue as in the proof of Lemma 2.9 with the help of (3.9a), (3.9b), and (3.10a) to show that $\partial_t f_\varepsilon(t)$ belongs to X_0 for any $t \geq 0$ and satisfies

$$\|\partial_t f_\varepsilon(t)\|_1 \leq C_5 \left[\sigma + (\sigma + \mu_0 \varepsilon^{-1})^2 \right], \quad t \geq 0, \quad (3.29)$$

the constant C_5 being defined in Lemma 2.9.

Step 2: Uniqueness. It is a consequence of [4, Theorem 8.2.55] (with $\ell(x) = 1 + x^{\max\{1, \gamma\}}$, $x > 0$, and $\zeta = 1$), see also [12].

Step 3: Higher moments. Finally, if $f^{in} \in X_m$ for some $m > 2 + \gamma$, then the proof of (3.13) relies on a weak lower semicontinuity argument as that of (3.9b) and follows from (3.26) and Lemma 2.1. \square

3.2. Invariant Set. As a consequence of the various estimates derived in Proposition 3.1, we construct a subset \mathcal{Z}_ε of \mathcal{Y}_ε which is left invariant by Ψ_ε . Specifically, $h \in \mathcal{Z}_\varepsilon$ if and only if

$$h \in \mathcal{Y}_\varepsilon \cap \bigcap_{m > 2 + \gamma} X_m, \quad (3.30a)$$

$$M_m(h) \leq \mu_m, \quad m > 2 + \gamma, \quad (3.30b)$$

$$M_m(h) \leq \sigma, \quad m \in (1, 2 + \gamma], \quad (3.30c)$$

$$M_m(h) \leq \sigma + \mu_m, \quad m \in (\lambda, 1). \quad (3.30d)$$

Proposition 3.2. *Consider $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$ and $f^{in} \in \mathcal{Z}_\varepsilon$. Then $\Psi_\varepsilon(t, f^{in}) \in \mathcal{Z}_\varepsilon$ for all $t \geq 0$.*

Proof. Set $f_\varepsilon := \Psi_\varepsilon(\cdot, f^{in})$ and consider $t > 0$. We first deduce from (3.9a), (3.9b) (with $m = 2$ and $m = 2 + \gamma$), (3.9c) (with $m = m_2$), (3.10), and (3.11a) that $f_\varepsilon(t) \in \mathcal{Y}_\varepsilon$. In addition, $f_\varepsilon(t) \in X_m$ for all $m > 2 + \gamma$ and satisfies (3.30b) by (3.13), while (3.30c) and (3.30d) follow from (3.9b) and (3.9c), respectively. \square

3.3. Dynamical System in $X_{1,w}$. We go on with the continuity properties of the map $f^{in} \mapsto \Psi_\varepsilon(\cdot, f^{in})$ defined in Proposition 3.1 and actually show that Ψ_ε is a dynamical system on \mathcal{Y}_ε for the weak topology of X_1 .

Proposition 3.3. *Consider $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$, $f^{in} \in \mathcal{Y}_\varepsilon$, and a sequence $(f_n^{in})_{n \geq 1}$ of initial conditions in \mathcal{Y}_ε such that*

$$f_n^{in} \rightharpoonup f^{in} \text{ in } X_1. \quad (3.31)$$

Then, for any $T > 0$,

$$\Psi_\varepsilon(\cdot, f_n^{in}) \longrightarrow \Psi_\varepsilon(\cdot, f^{in}) \text{ in } C([0, T], X_{1,w}).$$

Proof. For $n \geq 1$ we put $f_{\varepsilon,n} := \Psi_\varepsilon(\cdot, f_n^{in})$. On the one hand, it follows from (3.9b), (3.10b), and (3.11b) that

$$f_{\varepsilon,n}(t) \in \mathcal{W}_\varepsilon, \quad t \geq 0, \quad n \geq 1, \quad (3.32)$$

recalling that the set \mathcal{W}_ε is defined in (3.22). On the other hand, let $0 \leq t_1 < t_2$ and $n \geq 1$. We infer from (3.29) that

$$\|f_{\varepsilon,n}(t_2) - f_{\varepsilon,n}(t_1)\|_1 \leq C_5 \left[\sigma + (\sigma + \mu_0 \varepsilon^{-1})^2 \right] (t_2 - t_1).$$

Combining this estimate with (3.9b) gives, for $R > 0$,

$$\begin{aligned} \int_0^\infty x |f_{\varepsilon,n}(t_2, x) - f_{\varepsilon,n}(t_1, x)| \, dx &\leq R \|f_{\varepsilon,n}(t_2) - f_{\varepsilon,n}(t_1)\|_1 \\ &\quad + \frac{1}{R} \int_R^\infty x^2 (f_{\varepsilon,n}(t_2, x) + f_{\varepsilon,n}(t_1, x)) \, dx \\ &\leq C_5 \left[\sigma + (\sigma + \mu_0 \varepsilon^{-1})^2 \right] (t_2 - t_1) + \frac{2\sigma}{R}. \end{aligned}$$

Now, taking $R = 1/\sqrt{t_2 - t_1}$ in the previous inequality, we end up with

$$\int_0^\infty x |f_{\varepsilon,n}(t_2, x) - f_{\varepsilon,n}(t_1, x)| \, dx \leq \left\{ C_5 \left[\sigma + (\sigma + \mu_0 \varepsilon^{-1})^2 \right] + 2\sigma \right\} \sqrt{t_2 - t_1}.$$

Consequently, the sequence $(f_{\varepsilon,n})_{n \geq 1}$ is equicontinuous at each $t \geq 0$ for the norm-topology of X_1 and thus also for the weak topology of X_1 . Recalling (3.24) and (3.32), we are again in a position to use the variant of the Arzelà-Ascoli theorem stated in [27, Theorem A.3.1] to deduce that there are $F_\varepsilon \in C([0, \infty), X_{1,w})$ and a subsequence $(f_{\varepsilon,n_k})_{k \geq 1}$ of $(f_{\varepsilon,n})_{n \geq 1}$ (possibly depending on ε) such that

$$f_{\varepsilon,n_k} \longrightarrow F_\varepsilon \quad \text{in } C([0, T], X_{1,w}) \quad (3.33)$$

for any $T > 0$. Since f_{ε,n_k} satisfies (3.9), (3.10), (3.11), (3.12), and (3.33) for $k \geq 1$, we can argue as in Step 1 of the proof of Proposition 3.1 to establish that F_ε is a weak solution to (3.2) with initial condition f^{in} and also satisfies (3.9), (3.10), (3.11), and (3.12), along with

$$f_{\varepsilon,n_k} \longrightarrow F_\varepsilon \quad \text{in } C([0, T], X_{0,w} \cap X_{\gamma,w})$$

for any $T > 0$. The uniqueness assertion in Proposition 3.1 then guarantees that $F_\varepsilon = \Psi_\varepsilon(\cdot, f^{in})$.

A consequence of the above analysis is that $\Psi_\varepsilon(\cdot, f^{in})$ is the only cluster point of the sequence $(f_{\varepsilon,n})_{n \geq 1}$ in the space $C([0, T], X_{1,w})$, whatever the value of $T > 0$. Together with the compactness of $(f_{\varepsilon,n})_{n \geq 1}$, this observation ensures that it is the whole sequence $(f_{\varepsilon,n})_{n \geq 1}$ which converges to $\Psi_\varepsilon(\cdot, f^{in})$ in $C([0, T], X_{1,w})$ for any $T > 0$, thereby completing the proof of Proposition 3.3. \square

3.4. Stationary Solution to (3.2). Thanks to the outcome of Sections 3.1-3.3, we are now in a position to prove the existence of at least one stationary weak solution φ_ε to the coagulation-fragmentation equation (3.2) for $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$, along with some estimates on φ_ε which will be needed in Section 3.5 to carry out the limit $\varepsilon \rightarrow 0$.

Theorem 3.4. *For $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$, the coagulation-fragmentation equation (3.2a) has a stationary weak solution $\varphi_\varepsilon \in \mathcal{Z}_\varepsilon \cap L^{p_1}((0, \infty), x^{m_1 + \gamma} dx)$ satisfying*

$$\frac{1}{2} \int_0^\infty \int_0^\infty K_\varepsilon(x, y) \chi_\vartheta(x, y) \varphi_\varepsilon(x) \varphi_\varepsilon(y) \, dy dx = \int_0^\infty a_\varepsilon(x) N_\vartheta(x) \varphi_\varepsilon(x) \, dx \quad (3.34)$$

for all $\vartheta \in L^\infty(0, \infty)$ and

$$L_{m_1 + \gamma, p_1}(\varphi_\varepsilon) \leq \sigma_1, \quad (3.35)$$

the constant σ_1 being defined in (3.7).

Proof. Let $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$. By Propositions 3.1 and 3.3, Ψ_ε is a dynamical system on \mathcal{Y}_ε for the weak topology of X_1 and, according to Proposition 3.2, the subset \mathcal{Z}_ε of \mathcal{Y}_ε is invariant under the action of Ψ_ε ; that is, $\Psi_\varepsilon(t, \mathcal{Z}_\varepsilon) \subset \mathcal{Z}_\varepsilon$ for all $t \geq 0$. Since $x \mapsto \varrho^{-1} e^{-x/\varrho}$ belongs to \mathcal{Z}_ε , the set \mathcal{Z}_ε is a non-empty convex and closed subset of X_1 . In addition, owing to the Dunford-Pettis theorem, \mathcal{Z}_ε is a sequentially weakly compact subset of X_1 . Thanks to these properties, we infer from [12, Theorem 1.2] that there is $\varphi_\varepsilon \in \mathcal{Z}_\varepsilon$ such that $\Psi_\varepsilon(t, \varphi_\varepsilon) = \varphi_\varepsilon$ for all $t \geq 0$. In other words, φ_ε is a stationary solution to (3.2) as described in Proposition 3.1, and the weak formulation (3.34) readily follows from (3.8). We also deduce from (3.12) that, for $t > 0$,

$$L_{m_1 + \gamma, p_1}(\varphi_\varepsilon) = \frac{1}{t} \int_0^t L_{m_1 + \gamma, p_1}(\varphi_\varepsilon) \, ds \leq \frac{1}{a_0 t} L_{m_1, p_1}(\varphi_\varepsilon) + \sigma_1.$$

Letting $t \rightarrow \infty$ in the above inequality gives (3.35) and completes the proof of Theorem 3.4. \square

3.5. Proof of Theorem 1.1. We are left with investigating the limit $\varepsilon \rightarrow 0$ (if any) of the family $(\varphi_\varepsilon)_{\varepsilon \in (0, \varepsilon_{m_0, \sigma})}$ of stationary weak solutions to (1.20) constructed in Theorem 3.4. To this end, we first observe that, since $\varphi_\varepsilon \in \mathcal{Z}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$, it satisfies

$$M_1(\varphi_\varepsilon) = \varrho, \quad \varepsilon \in (0, \varepsilon_{m_0, \sigma}), \quad (3.36)$$

$$M_m(\varphi_\varepsilon) \leq \mu_m, \quad m > 2 + \gamma, \quad \varepsilon \in (0, \varepsilon_{m_0, \sigma}), \quad (3.37)$$

$$M_m(\varphi_\varepsilon) \leq \sigma, \quad m \in (1, 2 + \gamma], \quad \varepsilon \in (0, \varepsilon_{m_0, \sigma}), \quad (3.38)$$

$$M_m(\varphi_\varepsilon) \leq \sigma + \mu_m, \quad m \in (\lambda, 1), \quad \varepsilon \in (0, \varepsilon_{m_0, \sigma}), \quad (3.39)$$

and

$$L_{m_1 + \gamma, p_1}(\varphi_\varepsilon) \leq \sigma_1, \quad \varepsilon \in (0, \varepsilon_{m_0, \sigma}), \quad (3.40)$$

see the definition (3.30) of \mathcal{Z}_ε . We claim that these estimates guarantee that

$$(\varphi_\varepsilon)_{\varepsilon \in (0, \varepsilon_{m_0, \sigma})} \text{ is relatively sequentially weakly compact in } X_m \text{ for any } m > \lambda. \quad (3.41)$$

Indeed, let E be a measurable subset of $(0, \infty)$ with finite measure and $R > 1$. We infer from Hölder's inequality that, for $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$,

$$\begin{aligned} \int_E x^m \varphi_\varepsilon(x) \, dx &\leq \int_0^{1/R} x^m \varphi_\varepsilon(x) \, dx + \int_{1/R}^R x^m \mathbf{1}_E(x) \varphi_\varepsilon(x) \, dx \\ &\quad + \int_R^\infty x^m \varphi_\varepsilon(x) \, dx \\ &\leq R^{(\lambda-m)/2} \int_0^{1/R} x^{(m+\lambda)/2} \varphi_\varepsilon(x) \, dx + R^m |E|^{(p_1-1)/p_1} \left(\int_{1/R}^R \varphi_\varepsilon(x)^{p_1} \, dx \right)^{1/p_1} \\ &\quad + R^{-2-\gamma} \int_R^\infty x^{m+2+\gamma} \varphi_\varepsilon(x) \, dx \\ &\leq R^{(\lambda-m)/2} M_{(m+\lambda)/2}(\varphi_\varepsilon) + R^{(mp_1+m_1+\gamma)/p_1} |E|^{(p_1-1)/p_1} L_{m_1+\gamma, p_1}(\varphi_\varepsilon)^{1/p_1} \\ &\quad + R^{-2-\gamma} M_{m+2+\gamma}(\varphi_\varepsilon) . \end{aligned}$$

We now infer from (3.37), (3.38), (3.39), and (3.40) that

$$\int_E x^m \varphi_\varepsilon(x) \, dx \leq A_{m, \sigma} \left(R^{(\lambda-m)/2} + R^{(mp_1+m_1+\gamma)/p_1} |E|^{(p_1-1)/p_1} + R^{-2-\gamma} \right) , \quad (3.42)$$

with

$$A_{m, \sigma} := \sup_{\varepsilon \in (0, \varepsilon_{m_0, \sigma})} \left\{ M_{(m+\lambda)/2}(\varphi_\varepsilon) \right\} + \sigma_1^{1/p_1} + \mu_{m+2+\gamma} < \infty .$$

Introducing

$$\eta_{X_m}(\delta) := \sup \left\{ \int_E x^m \varphi_\varepsilon(x) \, dx : |E| < \delta , \varepsilon \in (0, \varepsilon_{m_0, \sigma}) \right\} , \quad \delta \in (0, 1) ,$$

we deduce from (3.42) that

$$\eta_{X_m}(\delta) \leq A_{m, \sigma} \left(R^{(\lambda-m)/2} + R^{(mp_1+m_1+\gamma)/p_1} \delta^{(p_1-1)/p_1} + R^{-2-\gamma} \right) .$$

Hence, since $p_1 > 1$,

$$\limsup_{\delta \rightarrow 0} \eta_{X_m}(\delta) \leq A_{m, \sigma} \left(R^{(\lambda-m)/2} + R^{-2-\gamma} \right) .$$

We finally let $R \rightarrow \infty$ to conclude that

$$\lim_{\delta \rightarrow 0} \eta_{X_m}(\delta) = 0 . \quad (3.43)$$

Similarly, for $\varepsilon \in (0, \varepsilon_{m_0, \sigma})$ and $R > 1$, it follows from (3.37) that

$$\int_R^\infty x^m \varphi_\varepsilon(x) \, dx \leq R^{-2-\gamma} \mu_{m+2+\gamma} ,$$

and thus

$$\lim_{R \rightarrow \infty} \sup_{\varepsilon \in (0, \varepsilon_{m_0, \sigma})} \left\{ \int_R^\infty x^m \varphi_\varepsilon(x) \, dx \right\} = 0 . \quad (3.44)$$

The claim (3.41) is then a consequence of (3.43), (3.44), and the Dunford-Pettis theorem.

We now infer from (3.41) and the reflexivity of $L^{p_1}((0, \infty), x^{m_1+\gamma} dx)$ that there are a subsequence $(\varphi_{\varepsilon_k})_{k \geq 1}$ of the family $(\varphi_\varepsilon)_{\varepsilon \in (0, \varepsilon_{m_0, \sigma})}$ and

$$\varphi \in X_1^+ \cap L^{p_1}((0, \infty), x^{m_1+\gamma} dx) \cap \bigcap_{m > \lambda} X_m \quad (3.45)$$

such that

$$\begin{aligned} \varphi_{\varepsilon_k} &\rightharpoonup \varphi \text{ in } X_m, & m > \lambda, \\ \varphi_{\varepsilon_k} &\rightharpoonup \varphi \text{ in } L^{p_1}((0, \infty), x^{m_1+\gamma} dx). \end{aligned} \quad (3.46)$$

A straightforward consequence of (3.36) and (3.46) (with $m = 1$) is that

$$M_1(\varphi) = \varrho. \quad (3.47)$$

Let us now check that φ is a stationary weak solution to (1.1), as described in Theorem 1.1 (s3). To this end, we consider $\vartheta \in \Theta_1$ and first note that

$$|\chi_\vartheta(x, y)| \leq 2\|\vartheta'\|_\infty \min\{x, y\}, \quad (x, y) \in (0, \infty)^2, \quad (3.48)$$

and

$$|N_\vartheta(x)| \leq 2\|\vartheta'\|_\infty x, \quad x > 0, \quad (3.49)$$

by (1.16c) and (1.16b).

Let us begin with the coagulation term. By (3.36), (3.39), and Hölder's inequality,

$$\begin{aligned} &\left| 2\varepsilon_k K_0 \int_0^\infty \int_0^\infty \chi_\vartheta(x, y) \varphi_{\varepsilon_k}(x) \varphi_{\varepsilon_k}(y) dy dx \right| \\ &\leq 4\varepsilon_k K_0 \|\vartheta'\|_\infty M_{(\lambda+1)/2}(\varphi_{\varepsilon_k}) M_{(1-\lambda)/2}(\varphi_{\varepsilon_k}) \\ &\leq 4\varepsilon_k K_0 \|\vartheta'\|_\infty M_{(\lambda+1)/2}(\varphi_{\varepsilon_k}) M_1(\varphi_{\varepsilon_k})^{(1-\lambda)/2} M_0(\varphi_{\varepsilon_k})^{(1+\lambda)/2} \\ &\leq 4\varepsilon_k K_0 \|\vartheta'\|_\infty (\sigma + \mu_{(\lambda+1)/2}) \varrho^{(1-\lambda)/2} M_0(\varphi_{\varepsilon_k})^{(1+\lambda)/2}. \end{aligned}$$

Since $\varphi_{\varepsilon_k} \in \mathcal{Z}_{\varepsilon_k} \subset \mathcal{Y}_{\varepsilon_k}$, we further deduce from (3.6c) that

$$\begin{aligned} &\left| 2\varepsilon_k K_0 \int_0^\infty \int_0^\infty \chi_\vartheta(x, y) \varphi_{\varepsilon_k}(x) \varphi_{\varepsilon_k}(y) dy dx \right| \\ &\leq 4\varepsilon_k K_0 \|\vartheta'\|_\infty (\sigma + \mu_{(\lambda+1)/2}) \varrho^{(1-\lambda)/2} (\sigma + \mu_0 \varepsilon_k^{-1})^{(1+\lambda)/2} \\ &\leq 4K_0 \|\vartheta'\|_\infty (\sigma + \mu_{(\lambda+1)/2}) \varrho^{(1-\lambda)/2} (\sigma + \mu_0)^{(1+\lambda)/2} \varepsilon_k^{(1-\lambda)/2}. \end{aligned}$$

Consequently,

$$\lim_{k \rightarrow \infty} 2\varepsilon_k K_0 \int_0^\infty \int_0^\infty \chi_\vartheta(x, y) \varphi_{\varepsilon_k}(x) \varphi_{\varepsilon_k}(y) dy dx = 0. \quad (3.50)$$

Next, by (3.48),

$$\frac{|\chi_\vartheta(x, y)|}{x^{(2\beta+1-\lambda)/2} y^{(2\alpha+1-\lambda)/2}} \leq 2\|\vartheta'\|_\infty, \quad (x, y) \in (0, \infty),$$

and, since

$$\chi_{\vartheta}(x, y)x^{\alpha}y^{\beta}\varphi_{\varepsilon_k}(x)\varphi_{\varepsilon_k}(y) = \frac{\chi_{\vartheta}(x, y)}{x^{(2\beta+1-\lambda)/2}y^{(2\alpha+1-\lambda)/2}}x^{(1+\lambda)/2}\varphi_{\varepsilon_k}(x)y^{(1+\lambda)/2}\varphi_{\varepsilon_k}(y) ,$$

it follows from (3.46) (with $m = (1 + \lambda)/2$) that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_0^{\infty} \int_0^{\infty} \chi_{\vartheta}(x, y)x^{\alpha}y^{\beta}\varphi_{\varepsilon_k}(x)\varphi_{\varepsilon_k}(y) \, dydx \\ &= \lim_{k \rightarrow \infty} \int_0^{\infty} \int_0^{\infty} \frac{\chi_{\vartheta}(x, y)}{x^{(2\beta+1-\lambda)/2}y^{(2\alpha+1-\lambda)/2}}x^{(1+\lambda)/2}\varphi_{\varepsilon_k}(x)y^{(1+\lambda)/2}\varphi_{\varepsilon_k}(y) \, dydx \\ &= \int_0^{\infty} \int_0^{\infty} \frac{\chi_{\vartheta}(x, y)}{x^{(2\beta+1-\lambda)/2}y^{(2\alpha+1-\lambda)/2}}x^{(1+\lambda)/2}\varphi(x)y^{(1+\lambda)/2}\varphi(y) \, dydx \\ &= \int_0^{\infty} \int_0^{\infty} \chi_{\vartheta}(x, y)x^{\alpha}y^{\beta}\varphi(x)\varphi(y) \, dydx . \end{aligned} \tag{3.51}$$

Similarly,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_0^{\infty} \int_0^{\infty} \chi_{\vartheta}(x, y)x^{\beta}y^{\alpha}\varphi_{\varepsilon_k}(x)\varphi_{\varepsilon_k}(y) \, dydx \\ &= \int_0^{\infty} \int_0^{\infty} \chi_{\vartheta}(x, y)x^{\beta}y^{\alpha}\varphi(x)\varphi(y) \, dydx . \end{aligned} \tag{3.52}$$

For the fragmentation term, it readily follows from (3.36) and (3.49) that

$$\left| a_0\varepsilon_k^2 \int_0^{\infty} N_{\vartheta}(x)\varphi_{\varepsilon_k}(x) \, dx \right| \leq 2a_0\varepsilon_k^2 \|\vartheta'\|_{\infty} M_1(\varphi_{\varepsilon_k}) = 2a_0\varepsilon_k^2 \|\vartheta'\|_{\infty} \varrho .$$

Hence,

$$\lim_{k \rightarrow \infty} a_0\varepsilon_k^2 \int_0^{\infty} N_{\vartheta}(x)\varphi_{\varepsilon_k}(x) \, dx = 0 . \tag{3.53}$$

We finally infer from (3.46) (with $m = 1 + \gamma$) and (3.49) that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_0^{\infty} x^{\gamma} N_{\vartheta}(x)\varphi_{\varepsilon_k}(x) \, dx = \lim_{k \rightarrow \infty} \int_0^{\infty} \frac{N_{\vartheta}(x)}{x} x^{1+\gamma}\varphi_{\varepsilon_k}(x) \, dx \\ &= \int_0^{\infty} \frac{N_{\vartheta}(x)}{x} x^{1+\gamma}\varphi(x) \, dx = \int_0^{\infty} x^{\gamma} N_{\vartheta}(x)\varphi(x) \, dx . \end{aligned} \tag{3.54}$$

Collecting (3.50), (3.51), (3.52), (3.53), and (3.54) allows us to take the limit $\varepsilon_k \rightarrow 0$ in (3.34) and conclude that φ is a stationary weak solution to (1.1) in the sense of Theorem 1.1 **(s3)**. Recalling (3.45) and (3.47), we have shown that φ satisfies the properties **(s1)**-**(s3)** stated in Theorem 1.1.

4. SMALL SIZE BEHAVIOUR

This section is devoted to the proof of Proposition 1.2. The starting point is the finiteness of some moments of order lower than λ when $\gamma \geq \alpha$.

Lemma 4.1. *Let $\rho > 0$ and consider a stationary weak solution φ to (1.1) satisfying the properties (s1)-(s3) stated in Theorem 1.1.*

- If $\gamma > \alpha$, then $\varphi \in X_\alpha$;
- If $\gamma = \alpha$, then $\varphi \in X_\beta$.

Proof. For $\delta \in (0, 1)$, we set $\zeta_{0,\delta}(x) = x \max\{x, \delta\}^{-1}$, $x > 0$. Then $\zeta_{0,\delta} \in \Theta_1$ and satisfies

$$\begin{aligned} - \chi_{\zeta_{0,\delta}}(x, y) &\geq \mathbf{1}_{(\delta, \infty)^2}(x, y), & (x, y) \in (0, \infty)^2, \\ - N_{\zeta_{0,\delta}}(x) &\leq \mathbf{b}_0 \mathbf{1}_{(\delta, \infty)}(x), & x > 0. \end{aligned}$$

It then follows from Theorem 1.1 (s3) that

$$\begin{aligned} K_0 \left(\int_\delta^\infty x^\alpha \varphi(x) dx \right) \left(\int_\delta^\infty y^\beta \varphi(y) dy \right) &= \frac{1}{2} \int_\delta^\infty \int_\delta^\infty K(x, y) \varphi(x) \varphi(y) dy dx \\ &\leq -\frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \chi_{\zeta_{0,\delta}}(x, y) \varphi(x) \varphi(y) dy dx \\ &= - \int_0^\infty a(x) N_{\zeta_{0,\delta}}(x) \varphi(x) dx \leq a_0 \mathbf{b}_0 \int_\delta^\infty x^\gamma \varphi(x) dx. \end{aligned} \tag{4.1}$$

- If $\gamma > \alpha$, then we infer from Theorem 1.1 (s2) and Hölder's inequality that

$$\begin{aligned} \int_\delta^\infty x^\gamma \varphi(x) dx &\leq \left(\int_\delta^\infty x^\alpha \varphi(x) dx \right)^{1/(1+\gamma-\alpha)} \left(\int_\delta^\infty x^{1+\gamma} \varphi(x) dx \right)^{(\gamma-\alpha)/(1+\gamma-\alpha)} \\ &\leq M_{1+\gamma}(\varphi)^{(\gamma-\alpha)/(1+\gamma-\alpha)} \left(\int_\delta^\infty x^\alpha \varphi(x) dx \right)^{1/(1+\gamma-\alpha)} \end{aligned}$$

and

$$\begin{aligned} \left(\int_\delta^\infty y \varphi(y) dy \right)^{2-\beta} &\leq \left(\int_\delta^\infty y^\beta \varphi(y) dy \right) \left(\int_\delta^\infty y^2 \varphi(y) dy \right)^{1-\beta} \\ &\leq M_2(\varphi)^{1-\beta} \int_\delta^\infty y^\beta \varphi(y) dy. \end{aligned}$$

Combining (4.1) and the above inequalities gives

$$\begin{aligned} M_2(\varphi)^{\beta-1} \left(\int_\delta^\infty y \varphi(y) dy \right)^{2-\beta} \left(\int_\delta^\infty x^\alpha \varphi(x) dx \right)^{(\gamma-\alpha)/(1+\gamma-\alpha)} \\ \leq \frac{a_0 \mathbf{b}_0}{K_0} M_{1+\gamma}(\varphi)^{(\gamma-\alpha)/(1+\gamma-\alpha)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \left(\int_{\delta}^{\infty} y\varphi(y) \, dy \right)^{(2-\beta)(1+\gamma-\alpha)/(\gamma-\alpha)} \int_{\delta}^{\infty} x^{\alpha}\varphi(x) \, dx \\ & \leq M_{1+\gamma}(\varphi) \left(\frac{a_0 \mathfrak{b}_0 M_2(\varphi)^{1-\beta}}{K_0} \right)^{(1+\gamma-\alpha)/(\gamma-\alpha)}. \end{aligned}$$

Owing to Theorem 1.1 **(s1)** and the positivity of ϱ , we can take the limit $\delta \rightarrow 0$ in the previous inequality to deduce that $\varphi \in X_{\alpha}$.

– If $\gamma = \alpha$, then (4.1) gives, since $\varphi \not\equiv 0$ by Theorem 1.1 **(s1)**,

$$\int_{\delta}^{\infty} y^{\beta}\varphi(y) \, dy \leq \frac{a_0 \mathfrak{b}_0}{K_0}$$

for δ small enough, which obviously implies that $\varphi \in X_{\beta}$ after taking the limit $\delta \rightarrow 0$. \square

Proof of Proposition 1.2. First, the integrability properties **(m2)** and **(m3)** stated in Proposition 1.2 readily follow from Lemma 4.1 and Theorem 1.1 **(s2)** by interpolation.

(m1): $\gamma > \alpha$. Consider $m \in (m_{\star}, 0)$ and recall that $\mathfrak{b}_m \in (1, \infty)$ by (1.22) and (2.8b). We first observe that, since $\gamma > \alpha$, $\beta \in [\alpha, 1)$, $\varphi \in X_{\alpha} \cap X_{1+\gamma}$, and $\varphi \not\equiv 0$ by (1.15b), Theorem 1.1, and Lemma 4.1, one has

$$0 < M_{\beta}(\varphi) < \infty \text{ and } M_{\gamma}(\varphi) < \infty. \quad (4.2)$$

This implies that there is $\delta_0 \in (0, 1)$ such that

$$r_{\delta} := \left(\frac{K_0}{2a_0 \mathfrak{b}_m} \int_{\delta}^{\infty} y^{\beta}\varphi(y) \, dy \right)^{1/(\gamma-\alpha)} > \delta, \quad \delta \in [0, \delta_0). \quad (4.3)$$

Next, for $\delta \in (0, \delta_0)$, we define the function $\zeta_{m,\delta}$ by $\zeta_{m,\delta}(x) := x \max\{x, \delta\}^{m-1}$, $x > 0$, and note that $\zeta_{m,\delta}$ belongs to Θ_1 . Moreover, since $m < 0$,

– for $(x, y) \in (\delta, \infty)^2$,

$$-\chi_{\zeta_{m,\delta}}(x, y) = x^m + y^m - (x + y)^m \geq x^m;$$

– for $(x, y) \in (\delta, \infty) \times (0, \delta)$,

$$-\chi_{\zeta_{m,\delta}}(x, y) = x^m + y\delta^{m-1} - (x + y)^m \geq 0;$$

– for $(x, y) \in (0, \delta) \times (\delta, \infty)$,

$$-\chi_{\zeta_{m,\delta}}(x, y) = x\delta^{m-1} + y^m - (x + y)^m \geq 0;$$

– for $(x, y) \in (0, \delta)^2$ such that $x + y > \delta$,

$$\begin{aligned} -\chi_{\zeta_{m,\delta}}(x, y) &= x\delta^{m-1} + y\delta^{m-1} - (x + y)^m \\ &\geq (x + y) [\delta^{m-1} - (x + y)^{m-1}] \geq 0; \end{aligned}$$

– for $(x, y) \in (0, \delta)^2$ such that $x + y < \delta$,

$$-\chi_{\zeta_{m,\delta}}(x, y) = x\delta^{m-1} + y\delta^{m-1} - (x + y)\delta^{m-1} = 0 .$$

Also, by (1.16c) and (1.16b),

– for $x \in (0, \delta)$,

$$-N_{\zeta_{m,\delta}}(x) = \delta^{m-1} \int_0^x yb(y, x) \, dy - \delta^{m-1}x = 0 ;$$

– for $x > \delta$,

$$\begin{aligned} -N_{\zeta_{m,\delta}}(x) &= \int_0^\delta y\delta^{m-1}b(y, x) \, dy + \int_\delta^x y^mb(y, x) \, dy - x^m \\ &\leq \int_0^x y^mb(y, x) \, dy = \mathbf{b}_m x^m . \end{aligned}$$

We infer from Theorem 1.1 **(s3)** and the previous inequalities that

$$\begin{aligned} K_0 \int_\delta^\infty \int_\delta^\infty x^{\alpha+m}y^\beta\varphi(x)\varphi(y) \, dydx &\leq K_0 \int_0^\infty \int_0^\infty \chi_{\zeta_{m,\delta}}(x, y)x^\alpha y^\beta\varphi(x)\varphi(y) \, dydx \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \chi_{\zeta_{m,\delta}}(x, y)K(x, y)\varphi(x)\varphi(y) \, dydx = a_0 \int_0^\infty x^\gamma N_{\zeta_{m,\delta}}(x)\varphi(x) \, dx \\ &\leq a_0 \mathbf{b}_m \int_\delta^\infty x^{\gamma+m}\varphi(x) \, dx . \end{aligned}$$

Therefore,

$$K_0 \left(\int_\delta^\infty y^\beta\varphi(y) \, dy \right) \int_\delta^\infty x^{\alpha+m}\varphi(x) \, dx \leq a_0 \mathbf{b}_m \int_\delta^\infty x^{\gamma+m}\varphi(x) \, dx . \quad (4.4)$$

Now, since $\gamma > \alpha$, it follows from (4.2) and (4.3) that

$$\begin{aligned} a_0 \mathbf{b}_m \int_\delta^\infty x^{\gamma+m}\varphi(x) \, dx &\leq a_0 \mathbf{b}_m r_\delta^{\gamma-\alpha} \int_\delta^{r_\delta} x^{\alpha+m}\varphi(x) \, dx + a_0 \mathbf{b}_m r_\delta^m \int_{r_\delta}^\infty x^\gamma\varphi(x) \, dx \\ &\leq a_0 \mathbf{b}_m r_\delta^{\gamma-\alpha} \int_\delta^\infty x^{\alpha+m}\varphi(x) \, dx + a_0 \mathbf{b}_m r_\delta^m M_\gamma(\varphi) . \end{aligned}$$

Combining this inequality with (4.3) and (4.4) gives

$$\frac{K_0}{2} \left(\int_\delta^\infty y^\beta\varphi(y) \, dy \right) \int_\delta^\infty x^{\alpha+m}\varphi(x) \, dx \leq a_0 \mathbf{b}_m r_\delta^m M_\gamma(\varphi) .$$

Thanks to (4.2), we may let $\delta \rightarrow 0$ in the above inequality and use Fatou's lemma to find

$$\frac{K_0 M_\beta(\varphi)}{2} \int_0^\infty x^{\alpha+m}\varphi(x) \, dx \leq a_0 \mathbf{b}_m r_0^m M_\gamma(\varphi) .$$

Hence, $\varphi \in X_{\alpha+m}$ for any $m \in (m_\star, 0)$ which, together with Theorem 1.1 **(s2)** and an interpolation argument implies that $\varphi \in X_{\alpha+m}$ for any $m > m_\star$.

To prove the second assertion in **(m1)** when $m_\star > -\infty$ and $\mathfrak{b}_{m_\star} = \infty$, we argue by contradiction and assume that $\varphi \in X_{\alpha+m_\star}$. Then, owing to (1.15b) and the assumption $\gamma > \alpha$,

$$\overline{M} := \max \{M_{\alpha+m_\star}(\varphi), M_{\beta+m_\star}(\varphi), M_\alpha(\varphi), M_\beta(\varphi), M_{\gamma+m_\star}(\varphi)\} < \infty . \quad (4.5)$$

Consider next $R > 1$. Since $\mathfrak{b}_{m_\star} = \infty$, there is $\delta_R \in (0, 1)$ such that

$$\int_{\sqrt{\delta}}^1 z^{m_\star} B(z) dz \geq R , \quad \delta \in (0, \delta_R) . \quad (4.6)$$

Fix $\delta \in (0, \delta_R)$. It follows from the negativity of m_\star and the definition of $\zeta_{m_\star, \delta}$ that

$$0 \leq -\chi_{\zeta_{m_\star, \delta}}(x, y) \leq x^{m_\star} + y^{m_\star} , \quad (x, y) \in (0, \infty)^2 ,$$

and

$$-N_{\zeta_{m_\star, \delta}}(x) \geq 0 , \quad x > 0 ,$$

while (4.6) entails that, for $x > \sqrt{\delta}$,

$$-N_{\zeta_{m_\star, \delta}}(x) \geq \left(\int_{\delta/x}^1 z^{m_\star} B(z) dz - 1 \right) x^{m_\star} \geq \left(\int_{\sqrt{\delta}}^1 z^{m_\star} B(z) dz - 1 \right) x^{m_\star} \geq (R-1)x^{m_\star} .$$

Since $\zeta_{m_\star, \delta} \in \Theta_1$, we infer from (4.5), Theorem 1.1 **(s3)**, and the previous inequalities that

$$\begin{aligned} a_0(R-1) \int_{\sqrt{\delta}}^{\infty} x^{\gamma+m_\star} \varphi(x) dx &\leq - \int_{\sqrt{\delta}}^{\infty} a(x) N_{\zeta_{m_\star, \delta}}(x) \varphi(x) dx \\ &\leq - \int_0^{\infty} a(x) N_{\zeta_{m_\star, \delta}}(x) \varphi(x) dx \\ &= -\frac{1}{2} \int_0^{\infty} \int_0^{\infty} K(x, y) \chi_{\zeta_{m_\star, \delta}}(x, y) \varphi(x) \varphi(y) dy dx \\ &= -K_0 \int_0^{\infty} \int_0^{\infty} x^\alpha y^\beta \chi_{\zeta_{m_\star, \delta}}(x, y) \varphi(x) \varphi(y) dy dx \\ &\leq K_0 [M_{\alpha+m_\star}(\varphi) M_\beta(\varphi) + M_{\beta+m_\star}(\varphi) M_\alpha(\varphi)] \\ &\leq 2K_0 \overline{M}^2 . \end{aligned}$$

Hence, using again (4.5),

$$a_0 R \int_{\sqrt{\delta}}^{\infty} x^{\gamma+m_\star} \varphi(x) dx \leq a_0 M_{\gamma+m_\star}(\varphi) + 2K_0 \overline{M}^2 \leq a_0 \overline{M} + 2K_0 \overline{M}^2 .$$

Taking the limit $\delta \rightarrow 0$ gives

$$a_0 R M_{\gamma+m_\star}(\varphi) \leq a_0 \overline{M} + 2K_0 \overline{M}^2 .$$

The above inequality being valid for all $R > 1$, we let $R \rightarrow \infty$ to conclude that $M_{\gamma+m_\star}(\varphi) = 0$; that is, $\varphi \equiv 0$, which contradicts Theorem 1.1 **(s1)**.

(m4): $\alpha > \gamma$. As in the proof of Lemma 2.3, we use a decomposition technique in the spirit of [14, Lemma 3.1] and [4, Lemma 8.2.12], along with a truncation procedure, to estimate the

contribution of the coagulation term. More precisely, for $m \in (\lambda - \gamma, \lambda)$, we deduce from (1.15b) and the assumption $\alpha > \gamma > 0$ that

$$0 \leq \lambda - 2\alpha < \lambda - 2\gamma < m - \gamma < m < \lambda < 1 .$$

We define

$$\omega := 2/(m + \gamma - \lambda) > 0 , \quad y_i := i^{-\omega} , \quad i \geq 1 ,$$

and set $\zeta_i(x) := x \max\{x, y_i\}^{m-\gamma}$, $x > 0$, $i \geq 2$. Clearly, $\zeta_i \in \Theta_1$ for all $i \geq 2$ and we infer from the convexity and monotonicity of $x \mapsto x^{m-\gamma-1}$ that,

– for $(x, y) \in (y_i, \infty)^2$,

$$\begin{aligned} -\chi_{\zeta_i}(x, y) &= x^{m-\gamma} + y^{m-\gamma} - (x+y)^{m-\gamma} \\ &= x [x^{m-\gamma-1} - (x+y)^{m-\gamma-1}] + y [y^{m-\gamma-1} - (x+y)^{m-\gamma-1}] \\ &\geq 2(1 + \gamma - m)xy(x+y)^{m-\gamma-2} ; \end{aligned}$$

– for $(x, y) \in (y_i, \infty) \times (0, y_i)$,

$$\begin{aligned} -\chi_{\zeta_i}(x, y) &= x^{m-\gamma} + yy_i^{m-\gamma-1} - (x+y)^{m-\gamma} \\ &= x [x^{m-\gamma-1} - (x+y)^{m-\gamma-1}] + y [y_i^{m-\gamma-1} - (x+y)^{m-\gamma-1}] \geq 0 ; \end{aligned}$$

– for $(x, y) \in (0, y_i) \times (y_i, \infty)$,

$$-\chi_{\zeta_i}(x, y) = xy_i^{m-\gamma-1} + y^{m-\gamma} - (x+y)^{m-\gamma} \geq 0 ;$$

– for $(x, y) \in (0, y_i)^2$ such that $x+y > y_i$,

$$\begin{aligned} -\chi_{\zeta_i}(x, y) &= xy_i^{m-\gamma-1} + yy_i^{m-\gamma-1} - (x+y)^{m-\gamma-1} \\ &\geq (x+y) [y_i^{m-\gamma-1} - (x+y)^{m-\gamma-1}] \geq 0 ; \end{aligned}$$

– for $(x, y) \in (0, y_i)^2$ such that $x+y < y_i$,

$$-\chi_{\zeta_i}(x, y) = xy_i^{m-\gamma-1} + yy_i^{m-\gamma-1} - (x+y)y_i^{m-\gamma-1} = 0 .$$

Also, by (1.16c) and (1.16b),

– for $x \in (0, y_i)$,

$$-N_{\zeta_i}(x) = y_i^{m-\gamma-1} \int_0^x yb(y, x) dy - y_i^{m-\gamma-1}x = 0 ;$$

– for $x > y_i$,

$$\begin{aligned} -N_{\zeta_i}(x) &= \int_0^{y_i} yy_i^{m-\gamma-1}b(y, x) dy + \int_{y_i}^x y^{m-\gamma}b(y, x) dy - x^{m-\gamma} \\ &\leq \int_0^x y^{m-\gamma}b(y, x) dy = \mathfrak{b}_{m-\gamma}x^{m-\gamma} . \end{aligned}$$

Let $I \geq 2$. Since

$$(xy)^{\lambda/2} \leq \frac{1}{2} (x^\alpha y^\beta + x^\beta y^\alpha) = \frac{K(x, y)}{2K_0}, \quad (x, y) \in (0, \infty)^2,$$

we deduce from Theorem 1.1 (**s3**) and the above properties of ζ_I , χ_{ζ_I} , and N_{ζ_I} that

$$\begin{aligned} & K_0(1 + \gamma - m) \int_{y_I}^{\infty} \int_{y_I}^{\infty} (xy)^{(\lambda+2)/2} (x+y)^{m-\gamma-2} \varphi(x)\varphi(y) \, dydx \\ & \leq \frac{1 + \gamma - m}{2} \int_{y_I}^{\infty} \int_{y_I}^{\infty} xyK(x, y)(x+y)^{m-\gamma-2} \varphi(x)\varphi(y) \, dydx \\ & \leq -\frac{1}{2} \int_{y_I}^{\infty} \int_{y_I}^{\infty} K(x, y)\chi_{\zeta_I}(x, y)\varphi(x)\varphi(y) \, dydx \\ & \leq -\frac{1}{2} \int_0^{\infty} \int_0^{\infty} K(x, y)\chi_{\zeta_I}(x, y)\varphi(x)\varphi(y) \, dydx \\ & = -\int_0^{\infty} a(x)N_{\zeta_I}(x)\varphi(x) \, dx \\ & \leq a_0 \mathfrak{b}_{m-\gamma} \int_{y_I}^{\infty} x^m \varphi(x) \, dx. \end{aligned} \tag{4.7}$$

Next, $(y_I, 1) = \bigcup_{1 \leq i \leq I-1} (y_{i+1}, y_i)$, so that

$$\begin{aligned} & \int_{y_I}^{\infty} \int_{y_I}^{\infty} (xy)^{(\lambda+2)/2} (x+y)^{m-\gamma-2} \varphi(x)\varphi(y) \, dydx \\ & \geq \int_{y_I}^1 \int_{y_I}^1 (xy)^{(\lambda+2)/2} (x+y)^{m-\gamma-2} \varphi(x)\varphi(y) \, dydx \\ & \geq \sum_{i=1}^{I-1} \int_{y_{i+1}}^{y_i} \int_{y_{i+1}}^{y_i} (xy)^{(\lambda+2)/2} (x+y)^{m-\gamma-2} \varphi(x)\varphi(y) \, dydx \\ & \geq 2^{m-\gamma-2} \sum_{i=1}^{I-1} y_i^{m-\gamma-2} J_i^2, \end{aligned} \tag{4.8}$$

where

$$J_i := \int_{y_{i+1}}^{y_i} x^{(\lambda+2)/2} \varphi(x) \, dx, \quad i \geq 1.$$

Next, since $m < (\lambda + 2)/2$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
 \int_{y_I}^1 x^m \varphi(x) \, dx &= \sum_{i=1}^{I-1} \int_{y_{i+1}}^{y_i} x^m \varphi(x) \, dx \leq \sum_{i=1}^{I-1} y_{i+1}^{(2m-\lambda-2)/2} J_i \\
 &\leq \left(\sum_{i=1}^{I-1} y_{i+1}^{2m-\lambda-2} y_i^{\gamma+2-m} \right)^{1/2} \left(\sum_{i=1}^{I-1} y_i^{m-\gamma-2} J_i^2 \right)^{1/2} \\
 &\leq 2^{\omega(\lambda+2-2m)/2} \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{1/2} \left(\sum_{i=1}^{I-1} y_i^{m-\gamma-2} J_i^2 \right)^{1/2}.
 \end{aligned} \tag{4.9}$$

We then infer from (4.8) and (4.9) that there is $c_1(m) > 0$ depending only on $K_0, \alpha, \beta, a_0, \gamma, B, \varrho$, and m such that

$$\begin{aligned}
 K_0(1 + \gamma - m) \int_{y_I}^{\infty} \int_{y_I}^{\infty} (xy)^{(\lambda+2)/2} (x+y)^{m-\gamma-2} \varphi(x) \varphi(y) \, dy dx \\
 \geq c_1(m) a_0 \mathfrak{b}_{m-\gamma} \left(\int_{y_I}^1 x^m \varphi(x) \, dx \right)^2.
 \end{aligned} \tag{4.10}$$

In addition, since $m < 1$, we infer from Theorem 1.1 (s1) that

$$\begin{aligned}
 \left(\int_{y_I}^{\infty} x^m \varphi(x) \, dx \right)^2 &\leq 2 \left(\int_{y_I}^1 x^m \varphi(x) \, dx \right)^2 + 2 \left(\int_1^{\infty} x \varphi(x) \, dx \right)^2 \\
 &\leq 2 \left(\int_{y_I}^1 x^m \varphi(x) \, dx \right)^2 + 2\varrho^2.
 \end{aligned} \tag{4.11}$$

Collecting (4.7), (4.10), and (4.11) and using the Cauchy-Schwarz inequality, we end up with

$$\begin{aligned}
 \left(\int_{y_I}^{\infty} x^m \varphi(x) \, dx \right)^2 &\leq \frac{2}{c_1(m)} \int_{y_I}^{\infty} x^m \varphi(x) \, dx + 2\varrho^2 \\
 &\leq \frac{1}{2} \left(\int_{y_I}^{\infty} x^m \varphi(x) \, dx \right)^2 + \frac{2}{c_1(m)^2} + 2\varrho^2.
 \end{aligned}$$

Hence,

$$\int_{y_I}^{\infty} x^m \varphi(x) \, dx \leq \frac{2}{c_1(m)} (1 + c_1(m)^2 \varrho^2)^{1/2}.$$

The above inequality being valid for any $I \geq 2$ with a right-hand side which does not depend on $I \geq 2$, we may take the limit $I \rightarrow \infty$ to conclude that $\varphi \in X_m$ and complete the proof of Proposition 1.2. \square

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