Advances of Implicit Description Techniques in Modelling and Control of Switched Systems

Moisés E. Bonilla, Michel Malabre, Vadim Azhmyakov

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Author	Family Name	**Bonilla**
	Particle	
	Given Name	**E. Moisés**
	Prefix	
	Suffix	
	Role	
	Division	
	Organization	CINVESTAV-IPN, Control Automático, UMI 3175 CINVESTAV -CNRS, A.P. 14-740
	Address	07000, México City, México
	Email	mbonilla@cinvestav.mx
Corresponding Author	Family Name	**Malabre**
	Particle	
	Given Name	**Michel**
	Prefix	
	Suffix	
	Role	
	Division	
	Organization	CNRS, LS2N (Laboratoire des Sciences du Numérique de Nantes) UMR 6004, B.P. 92101
	Address	44321 Nantes, Cedex 03, France
	Email	Michel.Malabre@ls2n.fr
Author	Family Name	**Azhmyakov**
	Particle	
	Given Name	**Vadim**
	Prefix	
	Suffix	
	Role	
	Division	Department of Mathematical Science
	Organization	Universidad EAFIT
	Address	Medellin, Colombia
	Email	vazhmyako@eafit.edu.co

Abstract

Our contribution is devoted to a constructive overview of the implicit system approach in modern control of switched dynamic models. We study a class of non-stationary autonomous switched systems and formally establish the existence of solution. We next incorporate the implicit systems approach into our consideration. At the beginning of the contribution, we also develop a specific system example that is used

for illustrations of various system aspects that we consider. Our research involves among others a deep examination of the reachability property in the framework of the implicit system framework that we propose. Based on this methodology, we finally propose a resulting robust control design for the switched systems under consideration and the proposed control strategy is implemented in the context of the illustrative example.

Chapter 7
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E. Moisés Bonilla, Michel Malabre and Vadim Azhmyakov

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Notation

Let us first introduce the necessary notation used in this manuscript.

- Script capitals $\mathscr{V}$, $\mathscr{W}$, ... denote finite-dimensional linear spaces with elements $v, w, \ldots$. The dimension of a space $\mathscr{V}$ is denoted by $\dim(\mathscr{V})$, $\mathscr{V} \approx \mathscr{W}$ stands for $\dim(\mathscr{V}) = \dim(\mathscr{W})$. Moreover, in the case $\mathscr{V} \subset \mathscr{W}$, $\frac{\mathscr{W}}{\mathscr{V}}$ or $\mathscr{W} \big/ \mathscr{V}$ stands for the quotient space $\mathscr{W}$ modulo $\mathscr{V}$. The direct sum of independent spaces is written as $\oplus$. $X^{-1}\mathscr{V}$ stands for the inverse image of the subspace $\mathscr{V}$ by the linear transformation X. Given a linear transformation $X : \mathscr{V} \to \mathscr{W}$, the expression $\mathrm{Im}X = X\mathscr{V}$ denotes

E. M. Bonilla
CINVESTAV-IPN, Control Automático, UMI 3175 CINVESTAV -CNRS,
A.P. 14-740, 07000 México City, México
e-mail: mbonilla@cinvestav.mx

M. Malabre (✉)
CNRS, LS2N (Laboratoire des Sciences du Numérique de Nantes)
UMR 6004, B.P. 92101, 44321 Nantes Cedex 03, France
e-mail: Michel.Malabre@ls2n.fr

V. Azhmyakov
Department of Mathematical Science, Universidad EAFIT, Medellin, Colombia
e-mail: vazhmyako@eafit.edu.co

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its image and Ker X denotes the corresponding kernel. In the case $\mathscr{V} \approx \mathscr{W}$, we write $X : \mathscr{V} \leftrightarrow \mathscr{W}$. Given a space $\mathscr{X} = \mathscr{S} \oplus \mathscr{T}$, the natural projection on $\mathscr{S}$ along $\mathscr{T}$ is denoted as $P : \mathscr{X} \to \mathscr{S}//\mathscr{T}$. A zero-dimensional subspace is denoted by $\{0\}$, and the identity operator is I. $e_i \in \mathbb{R}^n$ stands for the vector whose i-th entry is equal to 1 and the other ones are equal to 0. $\{e_k, \ldots e_\ell\}$ stands for the subspace generated by the vectors $e_k, \ldots e_\ell$.

- Additionally $\mathbb{R}^+$, $\mathbb{R}^{+*}$, $\mathbb{Z}^+$ and $\mathbb{N}$ stand for sets of non-negative real numbers, positive real numbers, non-negative integers and correspondingly for positive integers (the natural numbers), respectively. The notations $\mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{V}\right)$ and $\mathcal{L}^\infty\left(\mathbb{R}^+,\ \mathscr{V}\right)$ are used for the space of infinitely differentiable functions and space of bounded functions from $\mathbb{R}^+$ to $\mathscr{V}$.

7.1 Introduction

We review recent contributions related to the implicit linear systems and to the corresponding modelling approaches. We mainly analyse here the effective control design schemes for some specific classes of complex systems, namely, for dynamic systems with switches. First, let us mention the celebrated *implicit systems representation* proposed by Rosenbrock [39]. It was developed in the context of a specific generalization of proper linear systems (see also [28]).

Recall that an *implicit representation* $\Sigma^{imp}(E, A, B, C)$ is a set of differential and algebraic equations of the generic form[1]:

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu \ \text{ and } \ y = Cx, \ \ \forall\, t \geq 0, \tag{7.1}$$

where $E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$ and $C : \mathscr{X}_d \to \mathscr{Y}$ are linear maps. The spaces $\mathscr{X}_d \approx \mathbb{R}^n$, $\underline{\mathscr{X}}_{eq} \approx \mathbb{R}^{n_{eq}}$, $\mathscr{U} \approx \mathbb{R}^m$ and $\mathscr{Y} \approx \mathbb{R}^p$ are usually called the "*descriptor*","*equation*" and the "*input*" and the "*output*" spaces, respectively. In [5], it was shown that under the condition $n_{eq} \leq n$ one can constructively describe a linear system with an internal Variable Structure. However, in case $n_{eq} < n$, when the system under consideration is solvable, solutions are generally non-unique. In some sense, there is an additional degree of freedom in (7.1) that can finally incorporate (by an implicit way) a *structure variation*. In [8], a non-square implicit description was effectively used for modelling and control of various classes of linear systems. This effective control approach also includes systems with internal switches. Moreover, the necessary and sufficient conditions for a unique dynamic system behaviour (expressed in terms of the overall implicit model) are developed. These conditions imply existence of the system parts which are associated with the common internal dynamic equation and also with the algebraic constraints. The last one are "*controlled*" (in an hidden way) by the degree of freedom. It was also shown how to include the variable internal structure representation into the common square

[1] In the case where there is no output equation $y = Cx$, we simply write $\Sigma^{imp}(E, A, B)$.

implicit descriptions for an (A, E, B) invariant subspace generated by the kernel of the generic output map. The above-mentioned embedding makes it possible to get an unobservable variable internal structure. As a consequence of this effect, a proper closed-loop system with a controllable pre-specified structure was obtained.

In [13], we have taken advantage of the results obtained in [8] for a particular model (a class) of the so-called "*time-dependent, autonomous switched systems*" [29]. In [10], the authors propose a specific variable structure decoupling control strategy based on the celebrated ideal proportional and derivative (PD) feedback. Moreover, our contribution [14] is dedicated to a proper practical approximation of the ideal PD feedback mentioned above. This control strategy "rejects" in some sense the given variable structure and makes it possible to establish the stability property (stabilization) of both implicit control strategies. In [15], the authors have tackled the descriptor variable observation problem for implicit descriptions having *column minimal indices* blocks. In this paper, two concrete design procedures are considered, namely, the *(i) Linear descriptor observers* approach (based on the fault detection techniques) and the *(ii) Indirect variable descriptor observers* technique. The last one is based on the finite time structure detection methodology. In the first design scheme, the observer is composed of the celebrated Beard-Jones filter which makes it possible to observe the existing degree of freedom in rectangular implicit representations. Since this observation is accomplished by a pole-zero cancellation, this technique is reserved to minimum phase systems. The second idea from the contribution mentioned above is based on an adaptive structure detection. This technique is implemented in finite time and guarantees avoiding of possible stability problems (due to the temporarily unstable closed-loop systems into the detection procedure [11]).

Our contribution is organized as follows. In Sect. 7.2, we review a class of *time-dependent autonomous switched systems* which can be studied in combination with the newly developed approach of the *linear time-invariant implicit systems* [8, 13]. In Sect. 7.3, we review some properties of the *rectangular implicit representations* [8, 12]. Section 7.4 is devoted to the important structural property associated with the system reachability [6, 8, 9, 12, 16]. Section 7.5 includes the control strategy development for the *rectangular implicit representations* when the descriptor variable is available [8, 14]. In Sect. 7.6, we present some numerical simulations. Section 7.7 summarizes our contribution.

7.2 Time-Dependent Autonomous Switched Systems

In [13], taking into consideration the analytic results obtained in [8], an important class of the so-called *time-dependent autonomous switched systems* [29, 40] has been considered. It can be formally represented by the following *state-space representation* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\bar{x} = A_{q_i}\bar{x} + Bu \text{ and } y = C_{q_i}\bar{x}, \tag{7.2}$$

where u and y are the input and output variables. Here, q_i are elements of a finite set of indexes (locations) $\mathscr{Q}$. The system remains in location

$$q_i \in \mathscr{Q} = \{q_1, \ldots, q_\eta |\, q_i \in \mathbb{R}^\mu,\ i \in \{1, \ldots, \eta\}\}$$

for all time instants $t \in [T_{i-1}, T_i)$, and some $i \in \mathbb{N}$,

$$T_{i-1} \in \mathfrak{T}\left\{T_i \in \mathbb{R}^+ \,\middle|\, T_0 = 0,\ T_{i-1} < T_i\ \forall i \in \mathbb{N},\ \text{with } \lim_{i\to\infty} T_i = \infty\right\}.$$

The matrices A_{q_i} and C_{q_i} have here a particular structure (cf. [33, 42]):

$$A_{q_i} = \overline{A}_0 + \overline{A}_1\overline{D}(q_i) \text{ and } C_{q_i} = \overline{C}_0 + \overline{C}_1\overline{D}(q_i), \tag{7.3}$$

where $B \in \mathbb{R}^{\bar{n}\times m}$, $\overline{A}_0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\overline{C}_0 \in \mathbb{R}^{p\times\bar{n}}$, $\overline{A}_1 \in \mathbb{R}^{\bar{n}\times\hat{n}}$, $\overline{C}_1 \in \mathbb{R}^{p\times\hat{n}}$ and $\overline{D}(q_i) \in \mathbb{R}^{\hat{n}\times\bar{n}}$, $i \in \{1, \ldots, \eta\}$. Moreover, $\overline{A}_1$ and B are monic, C_1 and $\overline{D}(q_i)$ are epic, and $\overline{D}(0) = 0$.

Let us first introduce an illustrative example which will be used along with this chapter.

7.2.1 Example (Part 1)

Consider (7.2) and (7.3) with the following state-space matrices ($q = (\alpha, \beta)$):

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \overline{A}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \overline{C}_0 = \begin{bmatrix} 0\ 2 \end{bmatrix},\ \overline{C}_1 = \begin{bmatrix} 1 \end{bmatrix},\ \overline{D}(q) = \begin{bmatrix} \alpha\ \beta \end{bmatrix},$$
$$q \in \{q_1, q_2, q_3\},\ q_1 = (-1, -1),\ q_2 = (-1, 0),\ q_3 = (-1, -2). \tag{7.4}$$

Here

$$A_{q_i} = \begin{bmatrix} \alpha & (1+\beta) \\ (1+\alpha) & \beta \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_{q_i} = \begin{bmatrix} \alpha\ (\beta+2) \end{bmatrix}. \tag{7.5}$$

For a pair (α, β) we have an expression of the transfer function ($i \in \{1, 2, 3\}$)

$$F_{q_i}(\mathrm{s}) = C_{q_i}(\mathrm{sI} - A_{q_i})^{-1}B = \frac{(\beta+2)\mathrm{s} - \alpha}{(\mathrm{s}+1)(\mathrm{s} - (1+\alpha+\beta))}. \tag{7.6}$$

For the possible three index values $q \in \mathcal{Q}$, we obtain the corresponding three dynamic behaviours [37, 44, 45]:

$$\mathfrak{B}^{\infty}_{q_1} = \left\{(u(\cdot), y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_i, \mathbb{R}^2\right) \middle| \left[-1 \middle| (\mathrm{d}/\mathrm{dt}+1)\right]\left[\frac{u}{y}\right] = 0\right\}$$

$$\mathfrak{B}^{\infty}_{q_2} = \left\{(u(\cdot), y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_j, \mathbb{R}^2\right) \middle| \left[-(2\mathrm{d}/\mathrm{dt}+1) \middle| (\mathrm{d}/\mathrm{dt}+1)(\mathrm{d}/\mathrm{dt})\right]\left[\frac{u}{y}\right] = 0\right\}$$

$$\mathfrak{B}^{\infty}_{q_3} = \left\{(u(\cdot), y(\cdot)) \in \mathcal{C}^{\infty}\left(\mathcal{I}_k, \mathbb{R}^2\right) \middle| \left[-1 \middle| (\mathrm{d}/\mathrm{dt}+1)(\mathrm{d}/\mathrm{dt}+2)\right]\left[\frac{u}{y}\right] = 0\right\} \quad (7.7)$$

associated with the disjoints $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}_k \in \left\{\mathcal{I}_\tau = [T_{\tau-1}, T_\tau) \subset \mathbb{R}^+ | \ \tau \in \mathbb{N},\ T_{\tau-1} \in \mathfrak{T}\right\}$, $i, j, k \in \mathbb{N}$.

Taking into consideration the dynamic "behaviour" determined by Eq. (7.7), one can interpret it as a result of formally different state-space representations. However, one can show that it is a consequence of the same system represented by (7.2), (7.3) and by (7.4). The evident change of its internal structure is caused by the pole-zero cancellation. The last one generates uncontrollable and/or unobservable modes, see Fig. 7.1.

Comparing Fig. 7.1 with the dynamic behaviours (7.7), one can conclude that the lack of order of $\mathfrak{B}^{\infty}_{q_1}$ is due to an unobservable mode $\overline{\mathcal{O}}$. In fact, one could carefully handle the unobservable subspace for getting a desired internal structure. Indeed, in [13] we have taken an advantage of the particular structure (7.3) in the sense of handling the unobservable subspace. This approach also includes the consequent changes of structure inside a known set of models. Let us refer in that connection to the so-called *ladder systems* which consider irreducible factors of order 1 over $\mathbb{R}$, irreducible factors of order 2 over $\mathbb{R}$ and lead/lag compensation networks [7, 13].

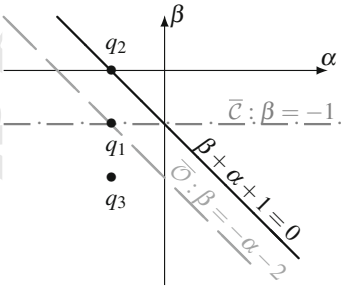


Fig. 7.1 Structural properties of the *state-space representation* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$ with matrices (7.5). The characteristic polynomial is $\det(\mathrm{s}\,\mathrm{I} - A_{q_i}) = (\mathrm{s}+1)\big(\mathrm{s} - (1+\alpha+\beta)\big)$, the uncontrollable and unobservable modes, $\overline{\mathcal{C}}$ and $\overline{\mathcal{O}}$, are $\det\left[\begin{array}{c|c} 0 & (1+\beta) \\ 1 & \beta \end{array}\right] = -(1+\beta)$ and $\det\begin{bmatrix} \alpha & (\beta+2) \\ (\alpha^2+(\alpha+1)(\beta+2)) & (\alpha(\beta+1)+\beta(\alpha+2)) \end{bmatrix} = -(2+\alpha+\beta)^2$, respectively

As it is shown in the next sections, the particular matrices' structure (7.3) of the *time-dependent autonomous switched systems* (7.2) enables its representation by *time-independent linear rectangular implicit descriptions* (7.1). The corresponding control here is given in the form of static or dynamic descriptor variable feedbacks.

7.3 Implicit Systems

Let us come back to the *time-dependent autonomous switched systems* described by (7.2) and (7.3). We next define the *descriptor variable* $x = \left[\bar{x}^T \ \hat{x}^T\right]^T$, where $\hat{x} = -\overline{D}(q_i)\bar{x}$, and get the following expression:

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} x = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} x. \tag{7.8}$$

From (7.8), we easily deduce that all the possible structure variations of (7.2) and (7.3) share the same dynamics represented by the *rectangular implicit representation*:

$$\begin{bmatrix} \mathrm{I} & 0 \end{bmatrix} \mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \end{bmatrix} x + Bu \ \text{ and } \ y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} x\,. \tag{7.9}$$

So, if there is a static or dynamic descriptor variable feedback controlling the *rectangular implicit representation* (7.9), it also controls (7.2) and (7.3).

We next review some properties of the *rectangular implicit representations*. These useful properties make it easy to study the *time-dependent autonomous switched systems* in the theoretic framework of *linear time-invariant implicit systems theory*.

7.3.1 *Existence of Solution*

Let us begin with some formal definitions of the *implicit representation* Σ^{imp} (E, A, B, C) of the generic form (cf. (7.1)).

Definition 7.1 (*Implicit representation*) An *implicit representation*, Σ^{imp} (E, A, B, C), is a *set of differential and algebraic equations* of the form (7.1), where

Hypothesis $\boldsymbol{H}1$. $\mathrm{Ker}\, B = \{0\}$ and $\mathrm{Im}\, C = \mathscr{Y}$.
Hypothesis $\boldsymbol{H}2$. $\mathrm{Im}\begin{bmatrix} E & A & B \end{bmatrix} = \underline{\mathscr{X}}_{eq}$.

Definition 7.2 (*Input/descriptor system* [12, 24, 37]) An implicit representation Σ^{imp} (E, A, B),

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu \ \ \forall\, t \geq 0, \tag{7.10}$$

is called an *input/descriptor system*, when for all initial condition $x_0 \in \mathscr{X}_d$, there exists at least one solution $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{U} \times \mathscr{X}_d\right)$, such that $x(0) = x_0$. The *input/descriptor system* is completely defined by a triple: $\Sigma_{i/d} = \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d,\ \mathfrak{B}_{[E,A,B]}\right)$, with behaviour:

$$\mathfrak{B}_{[E,A,B]} = \left\{(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{U} \times \mathscr{X}_d\right) \;\middle|\; \left[\,(E\mathrm{d}/\mathrm{dt} - A)\ -B\,\right]\begin{bmatrix} x \\ u \end{bmatrix} = 0\right\}. \tag{7.11}$$

At this point, it is important to clarify what exactly means the qualitative expression "*there exists at least one solution*".

Let us first review the existence of solution for two conceptually crucial points:

1. *given any initial condition* and
2. *for all admissible inputs*.

7.3.1.1 Existence of Solution Given Any Initial Condition

A. Behavioural approach

Following Hautus [20] and Hautus and Silverman [21], Geerts [18] generalized the solvability results of [19]. An important advantage of this generalization is the natural way of definition. It is based on the distributional framework [41] and, moreover, considers the usual time domain associated with the ordinary differential equations. This fact constitutes the real starting point of the so-called behavioural approach [37]. Geerts introduced the following definition for the distributional version[2] of the *implicit representation* (7.10)[3]: $pEx = Ax + Bu + Ex_0$.

Definition 7.3 (*C-solvability in the function sense* [18]) Given the *solution set*, $S_C(x_0, u) \stackrel{\text{def}}{=} \left\{ x \in \mathscr{C}^{n_d}_{\text{imp}} \;\middle|\; [p\,E\,-\,A]\,x = Bu + Ex_0 \right\}$, the *implicit representation* (7.10) is *C-solvable in the function sense* if $\forall x_0 \in \mathscr{X}_d\ \exists\ u \in \mathscr{C}^m_{\text{sm}} : S_C(x_0, u) \cap \mathscr{C}^n_{\text{sm}} \neq \emptyset$. Given the *"consistent initial conditions set"*, $\mathcal{I}_C \stackrel{\text{def}}{=} \left\{ z_0 \in \mathscr{X}_d \mid\ \exists u \in \mathscr{C}^m_{\text{sm}}\ \exists x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}} : x(0^+) = z_0 \right\}$, and the *"weakly consistent initial conditions set"*, $\mathcal{I}^w \stackrel{\text{def}}{=} \left\{ z_0 \in \mathscr{X}_d \mid\ \exists u \in \mathscr{C}^m_{\text{sm}}\ \exists x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}} \right\}$, a point $x_0 \in \mathscr{X}_d$ is called *C-consistent* if $x_0 \in \mathcal{I}_C$ and *weakly C-consistent* if $x_0 \in \mathcal{I}^w$.

The *C-solvability in the function sense* is concerned with solutions only composed of some arbitrarily often differentiable ordinary functions. The two notions of consistency, *C-consistent* and *weakly C-consistent*, lead to *smooth solutions*, namely,

[2] Geerts [18] considered the linear combinations of impulsive and smooth distributions, with μ coordinates, denoted by $\mathscr{C}^\mu_{\text{imp}}$, as the signal sets. The set $\mathscr{C}^\mu_{\text{imp}}$ is a subalgebra and it can be decomposed as $\mathscr{C}^\mu_{\text{p-imp}} \oplus \mathscr{C}^\mu_{\text{sm}}$, where $\mathscr{C}^\mu_{\text{p-imp}}$ and $\mathscr{C}^\mu_{\text{sm}}$ denote the subalgebras of pure impulses and smooth distributions, respectively [41]. The unit element of this subalgebra is the Dirac delta distribution, δ. Any linear combination of δ and its distributional derivatives $\delta^{(\ell)}$, $\ell > 1$, is called impulsive.

[3] Ex_0 stands for $Ex_0\delta$, $x_0 \in \mathscr{X}_d$ being the initial condition, and pEx stands for $\delta^{(1)} * Ex$ ($*$ denotes convolution); if pEx is smooth and $E\dot{x}$ stands for the distribution that can be identified with the ordinary derivative, $E\mathrm{d}x/\mathrm{d}t$, then $pEx = E\dot{x} + Ex_{0^+}$.

with no impulsions, but the *C-consistency* avoids jumps at the origin, namely, the *smooth solutions* are continuous on the left. Note that the *weakly C-consistent* case enables jumps at the origin.

Geerts in [18] characterized the existence of solutions for every initial condition in his main result (see Corollary 3.6, Proposition 4.2 and Theorem 4.5). Hereafter, we summarize some results concerning *smooth solutions*, together with their geometric equivalences (see [12] for details).

Theorem 7.1 (C-solvability in the function sense [18]) *If* ***H****2 is fulfilled, then the* implicit representation *(7.10) is* C-solvable in the function sense *if and only if* $\mathcal{I}^w = \mathscr{X}_d$*, namely, if and only if* $\operatorname{Im} E + \mathbf{A}\operatorname{Ker} E + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}$*, i.e. if and only if*

$$E\mathscr{V}^*_{\mathscr{X}_d} = \operatorname{Im} E. \tag{7.12}$$

Moreover, the initial conditions will be C-consistent, $\mathcal{I}_C = \mathscr{X}_d$*, if and only if* $\operatorname{Im} E + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}$*, i.e. if and only if*

$$E\mathscr{V}^*_{\mathscr{X}_d} + \operatorname{Im} B = \underline{\mathscr{X}}_{eq}. \tag{7.13}$$

$\mathscr{V}^*_{\mathscr{X}_d}$ is the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$ [30, 43],

$$\mathscr{V}^*_{\mathscr{X}_d} \overset{\text{def}}{=} \sup\{\mathscr{V} \subset \mathscr{X}_d \mid A\mathscr{V} \subset E\mathscr{V} + \operatorname{Im} B\}, \tag{7.14}$$

which is the limit of the following algorithm:

$$\mathscr{V}^0_{\mathscr{X}_d} = \mathscr{X}_d, \quad \mathscr{V}^{\mu+1}_{\mathscr{X}_d} = A^{-1}\left(E\mathscr{V}^{\mu}_{\mathscr{X}_d} + \operatorname{Im} B\right). \tag{7.15}$$

B. Viability approach

In order to study the reachability problem for implicit systems, Frankowska in [16] introduced the specific set-valued map (the set of all admissible velocities) $\mathbf{F} : \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}(Ax + \operatorname{Im} B) = \{v \in \mathscr{X} \mid Ev \in Ax + \operatorname{Im} B\}$, and considered the generic differential inclusion:

$$\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x), \quad \text{where } x(0) = x_0. \tag{7.16}$$

Frankowska [16] showed that the solutions of (7.10) and (7.16) are the same. Additionally, the meaning of a viable solution was constructively clarified. The largest subspace of such viable solutions is given as follows.

Definition 7.4 (*Viability kernel* [2, 16]) An absolutely continuous function, $x : \mathbb{R}^+ \to \mathscr{X}_d$, is called a *trajectory* of (7.16), if $x(0) = x_0$ and $\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x)$ for almost every $t \in \mathbb{R}^+$, that is to say, if there exists a measurable function, $u : \mathbb{R}^+ \to \mathscr{U}$, such that $x(0) = x_0$ and $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$ for almost every $t \in \mathbb{R}^+$.

Let $\mathscr{K}$ be a subspace[4] of $\mathscr{X}_d$. A trajectory x of (7.16) is called *viable in* $\mathscr{K}$, if $x(t) \in \mathscr{K}$ for all $t \geq 0$. The set of such trajectories is called *the set of viable solutions in* $\mathscr{K}$. The subspace $\mathscr{K}$ is called *a viability domain of* $\mathbf{F}$, if for all $x \in \mathscr{K}$: $\mathbf{F}(x) \cap \mathscr{K} \neq \emptyset$. The subspace $\mathscr{K}$ is called *the viability kernel of* (7.16) when it is the largest viability domain of $\mathbf{F}$.

Theorem 7.2 (Viability kernel [2]) *The supremal* (A, E, B)*-invariant subspace contained in* $\mathscr{X}_d$, $\mathscr{V}^*_{\mathscr{X}_d}$, *is the viability kernel of* $\mathscr{X}_d$ *for the set-valued map,* $\mathbf{F} : \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}(Ax + \operatorname{Im} B)$. *Moreover, for all* $x_0 \in \mathscr{V}^*_{\mathscr{X}_d}$, *there exists a trajectory,* $x \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{V}^*_{\mathscr{X}_d}\right)$, *solution of (7.10), satisfying* $x(0) = x_0$.

A singular system is "strict" *when the viability kernel coincides with the whole descriptor space,* $\mathscr{X}_d$, *namely,*

$$\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{X}_d. \tag{7.17}$$

As we have shown in Theorem 7.1, the specific condition (7.13) implies that for any initial condition $\lim_{t\to 0^+} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one solution pair $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$ of (7.10).

7.3.1.2 Existence of Solution for All Admissible Inputs

When an implicit representation Σ^{imp} (E, A, B) has a solution for all admissible inputs, it is simply called *solvable*.

Definition 7.5 (*Solvable representation* [4]) The *implicit representation* (7.10) is called *solvable*, if for any $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, there exists at least one trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{X}_d)$ solution of $\left[E\mathrm{d}/\mathrm{d}t - A\right]x(t) = Bu(t), \forall\, t \geq 0$.

Lemma 7.1 (Existence of solution [25, 27, 28]) *The* implicit representation *(7.10) admits at least one solution for all* $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$ *if and only if*

$$\operatorname{rang}\left[\lambda E - A \;\; B\right] = \operatorname{rang}\left[\lambda E - A\right], \quad \textit{for almost all} \;\; \lambda \in \mathbb{C} \tag{7.18}$$

if and only if

$$\operatorname{Im} B \subset \operatorname{Im}(\lambda E - A), \quad \textit{for almost any} \;\; \lambda \in \mathbb{C} \tag{7.19}$$

if and only if

$$\operatorname{Im} B \subset \mathcal{B}^*_1 + \mathcal{B}^*_2, \tag{7.20}$$

[4]We restrict our discussion to subspaces of finite-dimensional vector spaces. In [16] and in [2], these definitions are stated in the more general framework of closed sets of normed vector spaces.

where $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ are the limits of the following geometric algorithms:

$$\mathcal{B}_1^0 = \underline{\mathscr{X}}_{eq}\,, \quad \mathcal{B}_1^{\mu+1} = EA^{-1}\mathcal{B}_1^{\mu} \tag{7.21}$$

$$\mathcal{B}_2^0 = \{0\}\,, \quad \mathcal{B}_2^{\mu+1} = AE^{-1}\mathcal{B}_2^{\mu}. \tag{7.22}$$

Corollary 7.1 (Existence of solution [8]) *The following statements hold true:*

1. *If the geometric condition*

$$\operatorname{Im} A + \operatorname{Im} B \subset \operatorname{Im} E \tag{7.23}$$

holds, then the implicit representation *(7.10) admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, and for any initial condition $\lim_{t\to 0^+} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$ solution of (7.10).*
2. *If the geometric condition*

$$\operatorname{Im} E + \operatorname{Im} A = \underline{\mathscr{X}}_{eq} \tag{7.24}$$

holds, then the implicit representation *(7.10) admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$.*

Indeed, (7.23) implies $\mathcal{B}_1^* = \operatorname{Im} E$ and $\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d$, and (7.24) implies (7.19).

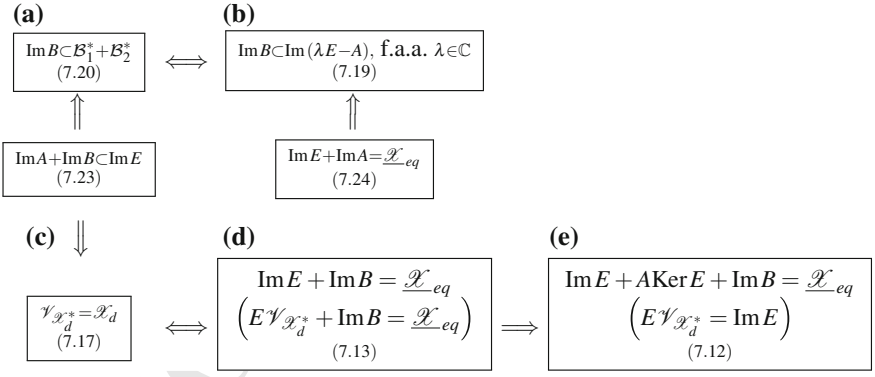


Fig. 7.2 Connexions between the notions of existence of solution. **a**, **b** Conditions of existence of at least one solution for all admissible inputs of Lebret [25]. **c** Condition of viable solution of Aubin and Frankowska [2] or smooth solution (without any jump) of Özçaldiran and Haliloğlu [36]. **d** Condition of Geerts [18] guaranteeing that the set of consistent initial conditions equals the whole space. **e** Condition of *C-solvability in the function sense* of Geerts [18] or the condition of Przyluski and Sosnowski [38] guaranteeing that the set of initial conditions of smooth solutions (with possible jumps) equals the whole space, or the impulse controllability condition of Ishihara and Terra [23], or the impulse-mode controllability with arbitrary initial conditions of Hou [22]

In Fig. 7.2, we compare the solvability conditions of Geerts (7.12) and (7.13), based on a distributional framework, with the solvability condition of Frankowska (7.17), based on a viability approach; as well as the solvability conditions of Lebret [25], (7.19) and (7.20).

7.3.2 Proper Implicit Representations

We now are interested in the proper linear systems in the presence of internal switches, which can be represented and controlled by means of *implicit representations*. Let us first introduce some basic definitions and present the necessary analytic results which naturally lead to *implicit representations* given by a proper linear system with internal structure variations.

Definition 7.6 (*Regularity* [17]) A pencil $[\lambda E - A]$, with $\lambda \in \mathbb{C}$, is called *regular* if it is square and its determinant is not the zero polynomial. An *implicit representation* $\Sigma^{imp}\ (E, A, B, C)$ is called *regular* if its pencil $[\lambda E - A]$ is regular.

Definition 7.7 (*Internal properness* [1, 3]) An *implicit representation* $\Sigma^{imp}\ (E, A, B, C)$ is called *internally proper* if its pencil $[\lambda E - A]$ is proper, namely, if its pencil is regular and has no infinite elementary divisor greater than 1. In other words, there is no derivative action in the system dynamics.

It is common knowledge that an *implicit representation* is completely characterized by the canonical Kronecker form of its pencil $[\lambda E - A]$, with $\lambda \in \mathbb{C}$. Usually, there are four possible types of suitable blocks [17]:

1. *Finite elementary divisors (fed)*, as for example, $[\lambda E_{fed} - A_{fed}] = \begin{bmatrix} (\lambda - \alpha) & 1 \\ 0 & (\lambda - \alpha) \end{bmatrix}$. The *fed* corresponds to the proper part of the system (integral actions), and it was geometrically characterized by Wong [46] and Bernhard [3].
2. *Infinite elementary divisors (ied)*, as for example, $[\lambda E_{ied} - A_{ied}] = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. The *ied* corresponds to the non-proper part of the system (time-derivative actions), and it was geometrically characterized by Armentano [1].
3. *Minimal column indices (mci)*, as for example, $[\lambda E_{mci} - A_{mci}] = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix}$. The *mci* corresponds to the existence of a certain degree of freedom (more variables than equations), and it was geometrically characterized by Armentano [1].
4. *Minimal row indices (mri)*, as for example, $[\lambda E_{mri} - A_{mri}] = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \\ 0 & 1 \end{bmatrix}$. The *mri* is related with the existence of algebraic and differential constraints on the external signals. For example, an admissible input has to satisfy some given algebraic and differential equations. Clearly, [1] geometrically characterized the *mri*.

Example 7.1 Let us consider the following *implicit representation*:

$$[E\mathrm{d}/\mathrm{dt} - A]x - Bu = \left[\begin{array}{c|cc} \boxed{1}_{ied} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\boxed{\begin{bmatrix} \mathrm{d}/\mathrm{dt} & 1 \end{bmatrix}}_{mci}} \end{array}\right] \begin{bmatrix} x_{ied} \\ \bar{x}_{mci} \\ \hat{x}_{mci} \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} u = 0,$$

$$y - Cx = y - \left[\begin{array}{c|cc} 1 & 2 & -1 \end{array}\right] \begin{bmatrix} x_{ied} \\ \bar{x}_{mci} \\ \hat{x}_{mci} \end{bmatrix} = 0, \tag{7.25}$$

and let us suppose that the degree of freedom satisfies the algebraic equation:

$$Dx = \left[\begin{array}{c|cc} a & b & c \end{array}\right] \begin{bmatrix} x_{ied} \\ \bar{x}_{mci} \\ \hat{x}_{mci} \end{bmatrix} = 0, \tag{7.26}$$

1. if $\left[\begin{array}{c|cc} a & b & c \end{array}\right] = \left[\begin{array}{c|cc} 1 & 1 & 0 \end{array}\right]$, we then get the non-proper external behaviour:

$$y(t) = \mathrm{d}u(t)/\mathrm{d}t, \tag{7.27}$$

2. if $\left[\begin{array}{c|cc} a & b & c \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & -1 \end{array}\right]$, we then get the proper external behaviour:

$$y(t) = \mathrm{e}^{-t}\bar{x}_c(0) + \int_0^t \mathrm{e}^{-(t-\tau)}u(\tau)\mathrm{d}\tau - u(t). \tag{7.28}$$

As we can see from the analysis realized above the existence of the degree of freedom can lead to a non-proper solution. This fact implies the necessity to add some specific geometric conditions on the degree of freedom in order that proper solutions are guaranteed.

Let us consider an implicit representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, where $\mathbb{E} : \mathscr{X}_d \to \underline{\mathscr{X}}_g$, $\mathbb{A} : \mathscr{X}_d \to \underline{\mathscr{X}}_g$, $\mathbb{B} : \mathscr{U} \to \underline{\mathscr{X}}_g$, $C : \mathscr{X}_d \to \mathscr{Y}$, such that the following hypotheses are satisfied:

Hypothesis $\boldsymbol{H}3$. $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ satisfies the standard assumptions **H**1 and **H**2, namely,

$$\mathrm{Ker}\mathbb{B} = \{0\}, \quad \mathrm{Im}\, C = \mathscr{Y} \text{ and } \mathrm{Im}\left[\begin{array}{ccc} \mathbb{E} & \mathbb{A} & \mathbb{B} \end{array}\right] = \underline{\mathscr{X}}_g. \tag{7.29}$$

Hypothesis $\boldsymbol{H}4$. $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, which is implied by (cf. Corollary 7.1 (7.24)):

$$\mathrm{Im}\,\mathbb{E} + \mathrm{Im}\,\mathbb{A} = \underline{\mathscr{X}}_g. \tag{7.30}$$

Hypothesis $\boldsymbol{H}5$. The differential part of $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, say $\Sigma^{ir}(E, A, B)$: $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$ ($E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$, $\underline{\mathscr{X}}_{eq} \subset$

$\underline{\mathscr{X}}_g$), admits at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, and for any initial condition $\lim_{t\to 0^+} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d\right)$ solution of $\Sigma^{ir}(E, A, B)$, which is implied by (cf. Corollary 7.1: (7.23))

$$\operatorname{Im} A + \operatorname{Im} B \subset \operatorname{Im} E. \tag{7.31}$$

Keeping in mind assumptions (7.29), (7.30) and (7.31), the implicit representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ can be expressed as follows:

$$\underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{\mathbb{E}} \frac{\mathrm{d}}{\mathrm{dt}} x = \underbrace{\begin{bmatrix} A \\ D \end{bmatrix}}_{\mathbb{A}} x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathbb{B}} u \quad \text{and} \quad y = Cx. \tag{7.32}$$

Lemma 7.2 (Σ^{ig} internally proper [8]) *Let us consider the implicit global representation $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, (7.32), satisfying the standard assumptions (7.29), and the solvability assumptions (7.30) and (7.31). Then, (7.32) is internally proper if and only if*

$$\operatorname{Ker} D \oplus \operatorname{Ker} E = \mathscr{X}_d. \tag{7.33}$$

Let us note that assumptions (7.29.c) and (7.30) are equivalent to

$$\underline{\mathscr{X}}_g = \operatorname{Im} E \oplus \operatorname{Im} D. \tag{7.34}$$

Let us introduce the following implicit representations definitions.

Definition 7.8 (*Rectangular implicit representation*) An implicit representation $\Sigma^{ir}(E, A, B, C)$,

$$E \frac{\mathrm{d}}{\mathrm{dt}} x = Ax + Bu \quad \text{and} \quad y = Cx, \tag{7.35}$$

where the matrices E and A have more columns than rows, and the solvability condition (7.31) is satisfied, is called *implicit rectangular representation*.

Definition 7.9 (*Algebraic constraint*) An *algebraic constraint* is a set of algebraic equations independent of the input variable, $\Sigma^{alc}(0, D, 0)$:

$$0 = Dx, \tag{7.36}$$

where $D : \mathscr{X}_d \to \underline{\mathscr{X}}_{alc}$ is a linear map and the finite-dimensional space, $\underline{\mathscr{X}}_{alc}$, is called the algebraic constraint space.

Definition 7.10 (*Global implicit representation*) If we gather the *implicit rectangular representation* (7.35) with the *algebraic constraint* (7.36), which describes the degree of freedom, we get the following *global implicit representation*, $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$:

$$\mathbb{E}\tfrac{\mathrm{d}}{\mathrm{dt}}x = \mathbb{A}x + \mathbb{B}u \ \text{ and } \ y = Cx$$
$$\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ \mathbb{A} = \begin{bmatrix} A \\ D \end{bmatrix}, \ \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{7.37}$$

The Cartesian product, $\underline{\mathscr{X}}_g \overset{\text{def}}{=} \underline{\mathscr{X}}_{eq} \times \underline{\mathscr{X}}_{cal}$, is the *space of global equations*. We shall assume that (7.34) is satisfied.

7.3.3 Switched Systems

Let us note that the above-mentioned conditions (7.33) and (7.34) imply the following important statement: there exist bases in $\mathscr{X}_d$ and also in $\underline{\mathscr{X}}_g$ such that (7.32) takes the specific form ($\overline{D}(q_i)$ is a variable matrix with respect to the location $q_i \in \mathscr{Q}$):

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \ \text{ and } \ y = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}, \tag{7.38}$$

and defining: $\begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ \overline{D}(q_i) & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}$, we get

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} (\overline{A}_0 + \overline{A}_1 \overline{D}(q_i)) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} (\overline{C}_0 + \overline{C}_1 \overline{D}(q_i)) & -\overline{C}_1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix}, \tag{7.39}$$

which coincides with *time-dependent autonomous switched systems* (7.2) with the particular structure (7.3).

Remark 7.1 (Implicit representation of switched systems) The *time-dependent autonomous switched systems* (7.2), with the particular structure (7.3), are described by the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ (7.37), where the linear maps $E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{X}_{eq}$, $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$, $C : \mathscr{X}_d \to \mathscr{Y}$ and $D_i : \mathscr{X}_d \to \underline{\mathscr{X}}_{alc}$ are equal to

$$E = \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix}, \ A = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \end{bmatrix}, \ C = \begin{bmatrix} \overline{C}_0 & -\overline{C}_1 \end{bmatrix}, \ D_i = \begin{bmatrix} \overline{D}(q_i) & \mathrm{I} \end{bmatrix}. \tag{7.40}$$

- The fixed structure of all $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$, which is active for a particular D_i, is described by the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ (7.1).
- The degree of freedom is characterized by the *algebraic constraints* $\Sigma^{alc}(0, D_i, 0)$ (7.36).
- Since $\dim \underline{\mathscr{X}}_{eq} < \dim \mathscr{X}_d$, there then exists a degree of freedom.

- Since $\mathrm{Im}\, A + \mathrm{Im}\, B \subset \mathrm{Im}\, E = \underline{\mathscr{X}}_{eq}$, $\Sigma^{ir}(E, A, B, C)$ admits at least one solution for all $u(\cdot) \in \mathcal{C}^{\infty}\left(\mathbb{R}^{+}, \mathscr{U}\right)$, and for any initial condition $\lim_{t \to 0^{+}} x(t) = x_0 \in \mathscr{X}_d$, there exists at least one trajectory $(u, x) \in \mathcal{C}^{\infty}\left(\mathbb{R}^{+}, \mathscr{U} \times \mathscr{X}_d\right)$ solution of $\Sigma^{ir}(E, A, B)$.
- Since $\mathrm{Im}\, D_i = \underline{\mathscr{X}}_{alc}$ and $\underline{\mathscr{X}}_{eq} \times \underline{\mathscr{X}}_{alc} \approx \mathscr{X}_d$ for all $i \in \{1, \ldots, \eta\}$ then the $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ have unique solutions for any $i \in \{1, \ldots, \eta\}$.
- Since $\mathrm{Ker} D_i \oplus \mathrm{Ker} E = \mathscr{X}_d$ for all $i \in \{1, \ldots, \eta\}$ then the *implicit global representations* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ are proper.

7.3.4 Example (Part 2)

Let us continue Example 7.2.1

A. Global implicit representation

The *state-space representation* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$, (7.2) and (7.5), is also represented by the following *global implicit representation* (cf. (7.39)):

$$\begin{array}{c} \begin{array}{ccc} & \mathrm{Ker}\, E & \\ & \longleftrightarrow & \end{array} \\ \begin{array}{c} E\mathrm{Ker}\, D \\ \updownarrow \\ D\mathrm{Ker}\, E \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \end{array} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{array}{c} \begin{array}{c} \mathrm{Ker}\, D \\ \longleftrightarrow \end{array} \\ \left[\begin{array}{cc|c} \alpha & (1+\beta) & 0 \\ (1+\alpha) & \beta & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \end{array} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|c} \alpha & (\beta+2) & -1 \end{array}\right] \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} \tag{7.41}$$

Let us note that

- $\mathrm{Im}\, A + \mathrm{Im}\, B \subset \mathrm{Im}\, E$, $\underline{\mathscr{X}}_g = E\mathrm{Ker} D \oplus D\mathrm{Ker} E = \mathrm{Im}\, E \oplus \mathrm{Im}\, D$ and $\mathscr{X}_d = \mathrm{Ker} D \oplus \mathrm{Ker} E$ then the *implicit global representation* (7.41) is externally proper (cf. Lemma 7.2).
- The part limited to $\mathrm{Ker} E$ and $D\mathrm{Ker} E$ is algebraically redundant.
- The part of the *implicit global representation* (7.41) limited to $\mathrm{Ker} D$, in the domain, and to $E\mathrm{Ker} D$, in the co-domain, which matrices are depicted with continuous lines, coincides with the state representations (7.2) and (7.5).
- The upper part of the *implicit global representation* (7.41) is an *implicit rectangular representation*, in which matrices are depicted with dashed lines. This part explicitly contains the changes in the behaviour which are due to the switches in the α and β parameters.
- The lower part of the *implicit global representation* (7.41) is an *algebraic constraint* which includes the components of the descriptor variable which are always zero.

B. Fixed structure

Premultiplying (7.41) by $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right]$, and defining: $x = \left[\begin{array}{c} \bar{x} \\ \hat{x} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline -\alpha & -\beta & 1 \end{array}\right] \left[\begin{array}{c} \bar{x} \\ \tilde{x} \end{array}\right]$, we get the *implicit global representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ (cf. (7.37)):

$$\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \frac{\mathrm{d}}{\mathrm{dt}} x = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & -1 \\ \hline \alpha & \beta & 1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] u \quad \text{and} \quad y = \left[\begin{array}{c|c|c} 0 & 2 & -1 \end{array}\right] x. \tag{7.42}$$

In the upper part of the *implicit global representation* (7.42), we get the *rectangular implicit representation* $\Sigma^{ir}(E, A, B, C)$ (cf. (7.35)):

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \frac{\mathrm{d}}{\mathrm{dt}} x = \left[\begin{array}{ccc} 0 & 1 & -1 \\ 1 & 0 & -1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u \quad \text{and} \quad y = \left[\begin{array}{ccc} 0 & 2 & -1 \end{array}\right] x. \tag{7.43}$$

In the lower part of the *implicit global representation* (7.42), we get the *algebraic constraint* $\Sigma^{alc}(0, D_i, 0)$ (cf. (7.36)):

$$0 = \left[\begin{array}{ccc} \alpha & \beta & 1 \end{array}\right] x. \tag{7.44}$$

C. Kronecker normal form

In order to better understand how the internal structure variation is acting in the implicit representations $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ and $\Sigma^{ir}(E, A, B, C)$, let us obtain their respective Kronecker normal forms.

C.1 Kronecker normal forms of $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$:

- If $\beta = -1$:

$$G_{ig1}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig1} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda - \alpha)}_{fed} & 0 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}. \tag{7.45}$$

- If $\beta \neq -1$ and $\alpha + \beta = -2$:

$$G_{ig2}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig2} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda + 1)}_{fed} & 1 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}. \tag{7.46}$$

- If $\beta \neq -1$ and $\alpha + \beta \neq -2$:

$$G_{ig3}[\lambda\mathbb{E} - \mathbb{A}_i]D_{ig3} = \begin{bmatrix} \boxed{1}_{ied} & 0 & 0 \\ 0 & \boxed{(\lambda - 1 - \alpha - \beta)}_{fed} & 0 \\ 0 & 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix}, \quad (7.47)$$

where $G_{ig1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, $D_{ig1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & (1-\alpha) & 1 \end{bmatrix}$, $G_{ig2} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ (1+\beta) & -(1+\beta) & 0 \end{bmatrix}$, $D_{ig2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \frac{1}{1+\beta} \\ 1 & 2 & \frac{\beta}{1+\beta} \end{bmatrix}$, $G_{ig3} = \begin{bmatrix} 0 & 0 & -1 \\ \left(1 - \frac{1+\beta}{2+\alpha+\beta}\right) & \frac{1+\beta}{2+\alpha+\beta} & 1 \\ (1+\beta) & -(1+\beta) & 0 \end{bmatrix}$ and $D_{ig3} = \begin{bmatrix} 0 & 1 & \frac{1}{2+\alpha+\beta} \\ 0 & 1 & \left(\frac{1}{2+\alpha+\beta} - \frac{1}{(1+\beta)}\right) \\ 1 & -(\alpha+\beta) & \left(\frac{\beta}{1+\beta} - \frac{\alpha+\beta}{2+\alpha+\beta}\right) \end{bmatrix}$.

C.2 Kronecker normal form of $\Sigma^{ir}(E, A, B, C)$:

$$G_{ir}[\lambda E - A]D_{ir} = \begin{bmatrix} \boxed{\lambda \ 1}_{mci} & 0 \\ 0 \ 0 & \boxed{(\lambda + 1)}_{fed} \end{bmatrix} \quad (7.48)$$

where $G_{ir} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $D_{ir} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Remark 7.2 (Internal structure variation) When we split the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, (7.41), *via* the *global implicit representation* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$, (7.42), into the *rectangular implicit representation* $\Sigma^{ir}(E, A, B, C)$, (7.43), and the *algebraic constraint* $\Sigma^{alc}(0, D_k, 0)$, (7.44), we get the common structure of the system which is described by $\Sigma^{ir}(E, A, B, C)$ (7.43).

When comparing the Kronecker normal forms, (7.45), (7.46) and (7.47), of the pencils associated with (7.42), with the Kronecker normal form, (7.48), of the pencil associated with (7.43), we realize that the **variable internal structure** of the *global implicit representation* (7.42) is taken into account by the **fixed block** *minimal column index* of the Kronecker normal form (7.48), $\boxed{\lambda \ 1}_{mci}$, associated with the *rectangular implicit representation* (7.43).

7.4 Reachability

Reachability is the most important concept studied in System Theory, since it characterizes the set of vectors which can be reached from the origin, in a finite time, following trajectories, solutions of the system. For *state-space representations* $\Sigma^{state}(A, B)$, $\mathrm{d}x/\mathrm{d}t = Ax + Bu$, this set of vectors is geometrically characterized by the reachability subspace (see, for example, [47]):

$$\mathscr{R}^* = \langle A \mid \mathrm{Im}\, B \rangle \stackrel{\mathrm{def}}{=} \mathrm{Im}\, B + A\mathrm{Im}\, B + \cdots + A^{n-1}\mathrm{Im}\, B, \tag{7.49}$$

and the trajectories are generated by the external control input, u.

For the case of *implicit representations—/* $\Sigma^{imp}(E, A, B)$, $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$, where E and A are square but $[\lambda E - A]$ is not necessarily invertible, Özçaldiran extended his geometric characterization of reachability by considering the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$ [34, 35]:

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}_{\mathscr{X}_d^*} \cap \mathscr{S}^*_{\mathscr{X}_d}, \tag{7.50}$$

where $\mathscr{V}^*_{\mathscr{X}_d}$ is the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$ (7.14), computed by (7.15), and $\mathscr{S}^*_{\mathscr{X}_d}$ is the infimal (E, A, B)-invariant subspace associated with $\mathrm{Im}\, B$,

$$\mathscr{S}^*_{\mathscr{X}_d} = \inf \left\{ \mathscr{S} \subset \mathscr{X}_d \;\middle|\; \mathscr{S} = E^{-1}\left(A\mathscr{S} + \mathrm{Im}\, B\right) \right\}, \tag{7.51}$$

which is the limit of the algorithm:

$$\mathscr{S}^0_{\mathscr{X}_d} = \mathrm{Ker}E\ , \quad \mathscr{S}^{\mu+1}_{\mathscr{X}_d} = E^{-1}\left(A\mathscr{S}^{\mu}_{\mathscr{X}_d} + \mathrm{Im}\, B\right). \tag{7.52}$$

The geometric characterization of $\mathscr{R}^*_{\mathscr{X}_d}$, given by (7.50), (7.14) and (7.51), is a nice generalization of the classical state-space characterization (7.49). Indeed, for $\Sigma^{state}(A, B) = \Sigma^{imp}(\mathrm{I}, A, B)$: $\mathscr{V}^*_{\mathscr{X}} = \mathscr{X}$ and $\mathscr{S}^*_{\mathscr{X}} = \langle A \mid \mathrm{Im}\, B \rangle$. Thus, it would appear quite natural that for the more general representations $\Sigma^{imp}(E, A, B)$, with E and A not necessarily square, the reachability would be also characterized by $\mathscr{R}^*_{\mathscr{X}_d}$.

The trueness of this conjecture was established by Frankowska [16] using tools of differential inclusions. But, as enhanced later on, this reachability concept needed to be further determined in order to discriminate the action of an effective external control input from an internal degree of freedom.

Indeed, the trajectories generated by $\Sigma^{imp}(E, A, B)$ depend on the initial conditions, $x(0)$, and not only on the external control input, but also possibly on internal degrees of freedom, which are completely free and unknown. Since the system $\Sigma^{imp}(E, A, B)$ represented by (7.10) has more unknowns than equations, when a solution does exist, this is, in general, non-unique. The possible resulting trajectories can be studied within so-called *viability domains*, see Frankowska [16].

7.4.1 $\mathscr{R}^*_{\mathscr{X}_d}$: Reachable Subspace

Frankowska formally defined reachability as follows:

Definition 7.11 (*Reachability* [16]) The implicit representation (7.10) is called *reachable* if for any possible $x_0, x_1 \in \mathscr{X}_d$ and for any time $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot)$, solution of (7.10), such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

And using tools of differential inclusions, she proved for the more general case.

Theorem 7.3 (Reachability [2, 16]) *For any $t_1 > t_0 \geq 0$, the reachable subspace of (7.10) at time t_1, starting from any initial value $x(t_0)$, is equal to $\mathscr{R}^*_{\mathscr{X}_d}$. Moreover, $\mathscr{R}^*_{\mathscr{X}_d}$ is the largest subspace such that for any x_0, $x_1 \in \mathscr{R}^*_{\mathscr{X}_d}$ and any $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{R}^*_{\mathscr{X}_d}\right)$, solution of (7.10), with $x(t_0) = x_0$ and $x(t_1) = x_1$.*

Note that the reachability Definition 7.11 requires no explicit control action!

In order to have a better understanding of Frankowska's reachability concept, let us decompose the *descriptor* and *equation* spaces in function of the supremal (A, E, B)-invariant subspace contained in $\mathscr{X}_d$, $\mathscr{V}^*_{\mathscr{X}_d}$, and of the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$, $\mathscr{R}^*_{\mathscr{X}_d}$.

In the third Lemma of [12], it is proved that there exist some complementary subspaces, $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{B}_\mathcal{C}$ and $\mathscr{R}_\mathcal{C}$, such that

$$\begin{aligned} \mathscr{X}_d &= \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{X}_1, \\ \mathscr{V}^*_{\mathscr{X}_d} &= \mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{X}_2, \\ \mathscr{R}^*_{\mathscr{X}_d} &= \mathscr{R}_\mathcal{C} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E), \end{aligned} \tag{7.53}$$

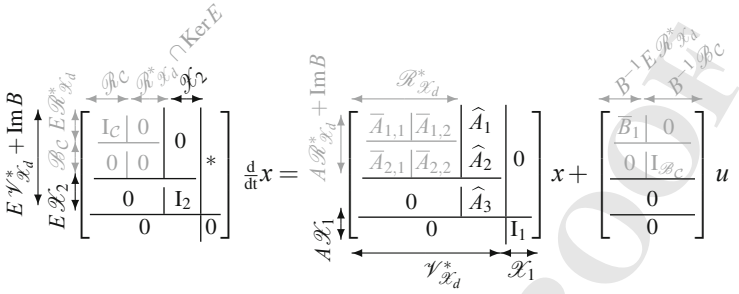
$$\begin{aligned} \underline{\mathscr{X}}_{eq} &= (E\mathscr{V}^*_{\mathscr{X}_d} + \mathrm{Im}\,B) \oplus A\mathscr{X}_1, \\ E\mathscr{V}^*_{\mathscr{X}_d} + \mathrm{Im}\,B &= (A\mathscr{R}^*_{\mathscr{X}_d} + \mathrm{Im}\,B) \oplus E\mathscr{X}_2, \\ A\mathscr{R}^*_{\mathscr{X}_d} + \mathrm{Im}\,B &= E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_\mathcal{C}, \end{aligned} \tag{7.54}$$

$$\mathscr{U} = B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \oplus B^{-1}\mathscr{B}_\mathcal{C}, \tag{7.55}$$

satisfying

$$\begin{aligned} &\mathscr{R}_\mathcal{C} \approx E\mathscr{R}^*_{\mathscr{X}_d}, \quad \mathscr{X}_2 \approx E\mathscr{X}_2, \quad \mathscr{X}_1 \approx A\mathscr{X}_1, \\ &\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E, \quad \mathrm{Im}\,B \cap E\mathscr{V}^*_{\mathscr{X}_d} = \mathrm{Im}\,B \cap E\mathscr{R}^*_{\mathscr{X}_d}. \end{aligned} \tag{7.56}$$

Given the geometric decompositions (7.53), (7.54) and (7.55), the implicit representation (7.10) takes the following form (recall (7.56)):



(7.57)

In the third Lemma of [12], it is also proved that

$$E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \;\middle|\; \mathrm{Im}\begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix} \right\rangle. \tag{7.58}$$

- $E\mathscr{R}^*_{\mathscr{X}_d}$ has the form of the classical state reachable subspace (7.49).
- $E\mathscr{R}^*_{\mathscr{X}_d}$ is handled by two actions: *(i)* the input action, *via* $\mathrm{Im}\,\overline{B}_1$, and *(ii)* the internal degree of freedom action, *via* $\mathrm{Im}\,\overline{A}_{1,2}$.
- The pair $\left(\overline{A}_{1,1}, \begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix}\right)$ is reachable in the classical state sense.

Example 7.2 Let us consider the following implicit representation, which is constituted by a *minimal column index* and has no input actions:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u. \tag{7.59}$$

Let us compute its reachability subspace $\mathscr{R}^*_{\mathscr{X}_d}$: From algorithms (7.15) and (7.52), we get

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} = \{e_1,\ e_2\} = \mathscr{X}_d,$$

and also (cf. (7.53), (7.54) and (7.55)):

$$\begin{gathered} \mathrm{Im}\,E = E\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{R}_{\mathcal{C}} = \{e_1\},\quad \mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E = \{e_2\}, \\ \mathscr{B}_{\mathcal{C}} = \mathrm{Im}\,B = B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} = B^{-1}\mathscr{B}_{\mathcal{C}} = \{0\}. \end{gathered}$$

The matrices involved in (7.57) are

$$\overline{A}_{1,1} = 0,\quad \overline{A}_{1,2} = 1,\quad \overline{A}_{2,1} = \overline{A}_{2,2} = \emptyset,\quad \overline{B}_1 = 0,\quad \mathrm{I}_{\mathcal{C}} = 1,\quad \mathrm{I}_{\mathscr{B}_{\mathcal{C}}} = \emptyset.$$

- (7.59) is reachable: $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d}$.
- (7.59) has no inputs actions: $\mathrm{Im}\, B = \{0\}$.
- (7.59) is handled by its internal degree of freedom: $E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \;\middle|\; \mathrm{Im}\begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix} \right\rangle = \langle 0 \mid \mathrm{Im}\begin{bmatrix} 1 & 0 \end{bmatrix} \rangle$.

7.4.2 External Reachability

In order to avoid the pathologies illustrated in the previous example, in [6] we have introduced the concept of *external reachability*.

Definition 7.12 (*External reachability* [6]) The implicit representation (7.10) is called *externally reachable* (by $P.D.$ feedback) if

- It is *reachable*.
- The spectrum of $\lambda(E - BF_d) - (A + BF_p)$ can be freely assigned by the selection of $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t$.

Theorem 7.4 (External reachability [6]) *(7.10) is* externally reachable *(by $P.D.$ feedback) if and only if*

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d \tag{7.60}$$

$$\dim\left(\frac{\mathrm{Im}\, B}{E\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Im}\, B}\right) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E\right). \tag{7.61}$$

- To prove this, Theorem, Bonilla, and Malabre [6] have used tools from Kronecker theory.
- Theorem 7.4 is the combination of the notion of *reachability* by Frankowska [16] and the notion of *unicity of the descriptor variable solution* by Lebret [25]
- Indeed, if there exists a proportional and derivative feedback of the descriptor variable which insures the unicity of the descriptor variable, no internal degree of freedom will be present. This implies that the trajectory of the descriptor variable is compulsorily due to an action of the external control input.

Example 7.3 Let us consider again the implicit representation (7.59) of Example 7.2. For that example, we have computed $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, $\mathscr{B}_{\mathcal{C}} = \{0\}$ and $\mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker}E = \{e_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.4 is not satisfied.

This means that there exists no external control input $u \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathscr{U}\right)$, to control the whole descriptor variable, x of system (7.59).

Example 7.4 If we add an effective input action to (7.59), say

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u, \tag{7.62}$$

we get from algorithms (7.15) and (7.52): $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} = \{e_1,\ e_2\} = \mathscr{X}_d$, which imply $\mathrm{Im}\, B = \mathrm{Im}\, E = E\mathscr{R}^*_{\mathscr{X}_d} = \mathrm{Im}\, B \cap E\mathscr{R}^*_{\mathscr{X}_d} = \{e_1\} \approx \mathbb{R}^1$ and $\mathscr{R}^*_{\mathscr{X}_d} \cap \mathrm{Ker}\, E = \{e_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.4 is still not satisfied. This means that there exists no external control input $u \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathscr{U}\right)$ able to control the whole descriptor variable, x, of system (7.62).

However, we would like to control, at least partly, systems with representations like (7.62).

7.4.3 *Externally Assignable Output Dynamics*

In order to partly control implicit representations with an internal degree of freedom, like (7.62), we have introduced in [9] the concept of *external output dynamics assignment*.

Definition 7.13 (*External output dynamics assignment* [9]) The implicit representation (7.10) has an *assignable external output dynamics* when there exists a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$ such that the closed-loop system is *externally reachable*.

Theorem 7.5 (External output dynamics assignment [9]) *The implicit representation* $\Sigma^{imp}(E, A, B, C)$, *(7.1), has an* assignable external output dynamics *if and only if*

$$\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{X}_d \tag{7.63}$$

$$\dim\left(\frac{\mathrm{Im}\, B}{E\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Im}\, B}\right) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathrm{Ker}\, E\right) - \dim\left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B\right), \tag{7.64}$$

$\mathscr{V}^*$ is the supremal (A, E, B) invariant subspace contained in $\mathrm{Ker}\, C$ [30, 31],

$$\mathscr{V}^* \stackrel{\mathrm{def}}{=} \sup\{\mathscr{V} \subset \mathrm{Ker}\, C \mid A\mathscr{V} \subset E\mathscr{V} + \mathrm{Im}\, B\}, \tag{7.65}$$

which is the limit of the following algorithm:

$$\mathscr{V}^0 = \mathscr{X}_d\ ,\quad \mathscr{V}^{\mu+1} = \mathrm{Ker}\, C \cap A^{-1}(E\mathscr{V}^\mu + \mathrm{Im}\, B). \tag{7.66}$$

$\mathcal{V}^*$ characterizes the supremal part of the implicit representation $\Sigma^{imp}\,(E, A, B, C)$ which can be made unobservable when using a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, namely, for all derivative feedback $F_d : \mathcal{X}_d \to \mathcal{U}$ there exists a proportional feedback $F_p : \mathcal{X}_p \to \mathcal{U}$, such that $(A + BF_p)\mathcal{V}^* \subset (E - BF_d)\mathcal{V}^*$. The set feedback pairs (F_p, F_d) satisfying this geometric inclusion is noted as $\mathbf{F}(\mathcal{V}^*)$.

Condition (7.64) has been established by Lebret [25] to guarantee unicity of the output.

Example 7.5 For the implicit representation (7.62) of Example 7.4, let us add an output equation:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} a & b \end{bmatrix} x, \tag{7.67}$$

with $a^2 + b^2 \neq 0$.

From (7.67), (7.15) and (7.52), we get $\mathscr{R}^*_{\mathcal{X}_d} = \mathcal{X}_d$, $\mathrm{Im}\, B = \mathrm{Im}\, E = E\mathcal{V}^*_{\mathcal{X}_d} \cap \mathrm{Im}\, B = \{e_1\} \approx \mathbb{R}^1$ and $\mathcal{V}^*_{\mathcal{X}_d} \cap \mathrm{Ker} E = \{e_2\} \approx \mathbb{R}^1$, then $E^{-1}\mathrm{Im}\, B = \mathcal{X}_d$.

From (7.67) and (7.66), follows that $\mathcal{V}^* = \mathrm{Ker} C = \{be_1 - ae_2\} \approx \mathbb{R}^1$. Hence, Theorem 7.5 is satisfied and there exists a $P.D.$ feedback, $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, such that the output dynamics of the closed-loop system is externally reachable, like, for example,

$$u = \begin{bmatrix} (1-a) & -b \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} a & (b-1) \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + u_r$$

obtaining in this way:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u_r \quad \text{and} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix},$$

where $\begin{bmatrix} \bar{\xi} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}$.

7.4.4 Example (Part 3)

Let us come back to the *rectangular implicit representation* (7.43), which comes from the *implicit global representation* (7.42) of the *switched system* of Sects. 7.2.1 and 7.3.4, described by (7.2) and (7.5), or by (7.41).

From (7.43), (7.15) and (7.52), we get $\mathrm{Im}\, A + \mathrm{Im}\, B \subset \mathrm{Im}\, E = \underline{\mathcal{X}}_{eq}$, $\mathcal{V}^*_{\mathcal{X}_d} = \mathscr{R}^*_d = \mathcal{X}_d$, $E\mathcal{V}^*_{\mathcal{X}_d} = \mathrm{Im}\, E$ and $E\mathcal{V}^*_{\mathcal{X}_d} \cap \mathrm{Im}\, B = \mathrm{Im}\, B$. Also, $\mathrm{Ker} E = \{e_3\} \approx \mathbb{R}^1$ and $E^{-1}\mathrm{Im}\, B = \{e_2, e_3\}$.

From (7.43) and (7.66), follows that $\mathcal{V}^* = \mathrm{Ker} C = \{e_1, e_2\}$, then $\mathcal{V}^* \cap E^{-1}\mathrm{Im}\, B = \{e_2\} \approx \mathbb{R}^1$.

Hence, Theorem 7.5 is satisfied and there then exists a $P.D.$ feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + u_r$, such that the output dynamics of the closed-loop system is externally reachable.

7.5 Control

Lebret and Loiseau [26] have extended the famous Morse Canonical Form [32] to the general case of *implicit descriptions*. In that paper, Lebret and Loiseau have completely characterized the internal structure of the *implicit descriptions*. With respect to the *minimal column indices*, which are responsible for the variation of the internal structure, they have distinguished two kinds of blocks, namely,

Blocks L_{q_i}: These blocks characterize the *degree of freedom* which are observable at the output. For reachable representations, $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, their number is characterized as follows:

$$\text{card}\left\{L_{q(i)}; q_i \geq 1\right\} = \dim\left(\frac{\text{Ker}\, E}{\mathscr{V}^* \cap \text{Ker}\, E}\right). \tag{7.68}$$

Blocks L_{σ_i}: These blocks characterize the *degree of freedom* which are unobservable at the output. For reachable representations, $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, their number is characterized as follows:

$$\text{card}\left\{L_{\sigma(i)}; \sigma_i \geq 1\right\} = \dim\left(\mathscr{V}^* \cap \text{Ker}\, E\right). \tag{7.69}$$

The internal structure variation will then be unobservable at the output if there exists a pair $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, such that

$$\mathscr{V}^* \supset \text{Ker}(E - BF_d^*). \tag{7.70}$$

7.5.1 *Decoupling of the Variable Structure*

In [8], we have introduced the *variable structure decoupling problem*.

Problem 7.1 (*Variable structure decoupling* [8]) Let us consider the *global implicit representation* $\Sigma^{ig}(\mathbf{E}, \mathbf{A}, \mathbf{B}, C)$, (7.37), such that the solvability assumptions, (7.31) and (7.34), and the internal properness condition, (7.33), are satisfied.

Under which geometric conditions does there exist a $P.D.$ feedback, $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$, for the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$, (7.35), such that the external behaviour of the closed-loop system is time-invariant with prespecified dynamics?

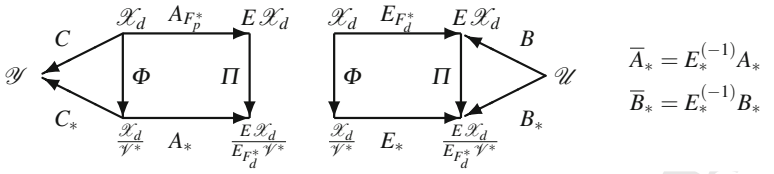


Fig. 7.3 Maps induced by $A_{F_p^*}$ and $E_{F_d^*}$. Φ and Π are canonical projections. The map E_* is invertible and $E_*^{(-1)}$ is its inverse

Theorem 7.6 (Variable structure decoupling [8]) *If the* implicit rectangular representation $\Sigma^{ir}(E, A, B, C)$, *(7.35), satisfies (7.31), (7.34), (7.33) and* [5]

$$\dim(\mathrm{Ker}E) \leq \dim\left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\,B\right), \tag{7.71}$$

there then exists a P.D. feedback, $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$, *such that the internal variable structure of the closed-loop system* implicit rectangular representation $\Sigma^{ir}(E_{F_d^*}, A_{F_p^*}, B, C)$ *is made unobservable, namely,*

$$\mathscr{V}^* \supset \mathrm{Ker}(E - BF_d^*),$$

where $(F_p^*, F_d^*) \in \boldsymbol{F}(\mathscr{V}^*)$.

Moreover, $\Sigma^{ir}(E_{F_d^*}, A_{F_p^*}, B, C)$ *is externally equivalent*[6] *to the* state-space representation $\Sigma^{state}(\overline{A}_*, \overline{B}_*, C_*)$, *where* $E_{F_d^*} = E - BF_d^*$ *and* $A_{F_p^*} = A + BF_p^*$, *and* $\overline{A}_*$, $\overline{B}_*$ *and* C_*, *are induced by* $A_{F_p^*}$ *and* $E_{F_d^*}$ *as it is shown in Fig. 7.3.*

Furthermore, if $\Sigma^{ir}(E, A, B, C)$, *(7.35), satisfies*

$$\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{X}_d,$$

then Σ^{state} $(\overline{A}_*, \overline{B}_*, C_*)$ *is controllable (reachable), namely,* $\left\langle \overline{A}_* \,\middle|\, \mathrm{Im}\,\overline{B}_* \right\rangle = \mathscr{X}_d \big/ \mathscr{V}^*$.

For proving Theorem 7.6, in [8] we have done the following geometric decompositions:

$$\begin{aligned} \mathscr{X}_d &= \left(\mathscr{V}^* + E^{-1}\mathrm{Im}\,B\right) \oplus \mathscr{X}_0, \\ \mathscr{V}^* &= \mathscr{X}_{\mathscr{V}^*} \oplus \left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\,B\right), \\ E^{-1}\mathrm{Im}\,B &= \left(\left(\mathscr{V}^* \cap E^{-1}\mathrm{Im}\,B\right) + \mathrm{Ker}E\right) \oplus \mathscr{X}_3, \\ \mathrm{Ker}E &= (\mathscr{V}^* \cap \mathrm{Ker}E) \oplus \mathscr{X}_E, \end{aligned} \tag{7.72}$$

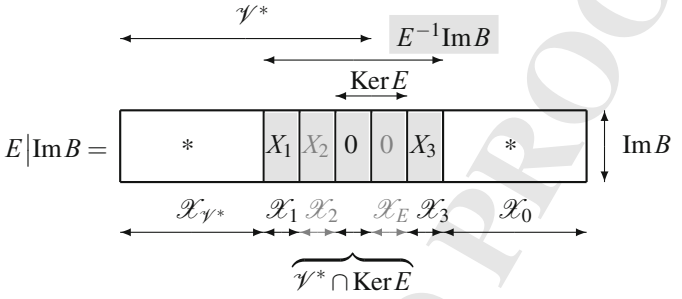
[5] Let us note that (7.31) implies that (cf. Fig. 7.2): $E\mathscr{V}_{\mathscr{X}_d^*} + \mathrm{Im}\,B = \underline{\mathscr{X}_{eq}}$ and $\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d$, hence (7.64) takes the form (7.71).

[6] Recall that two representations are externally equivalent when the sets of all possible trajectories for their external signals (here u and y) are identical (see [37, 44, 45]).

and we have shown that (7.71) implies that there exist two complementary subspaces, $\mathscr{X}_1$ and $\mathscr{X}_2$, such that

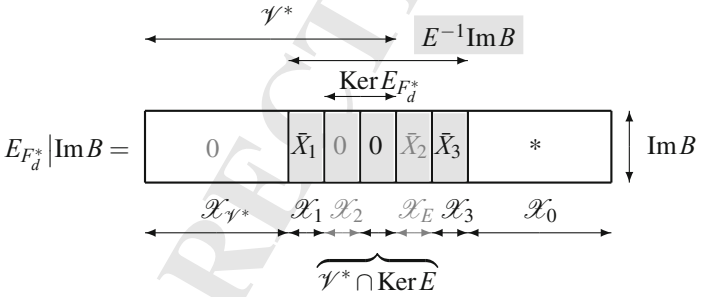
$$\mathscr{V}^* \cap E^{-1}\mathrm{Im}\, B = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus (\mathscr{V}^* \cap \mathrm{Ker}\, E) \;\text{ and }\; \mathscr{X}_2 \approx \mathscr{X}_E. \tag{7.73}$$

Hence, under the bases (7.72) and (7.73), the map E restricted to $\mathrm{Im}\, B$ takes the following form:



(7.74)

Now, in view of the isomorphism $\mathscr{X}_2 \approx \mathscr{X}_E$, for satisfying (7.70) we only have to move the zero block of $\mathscr{X}_E$ to $\mathscr{X}_2$ by means of the derivative action F_d^*, namely,

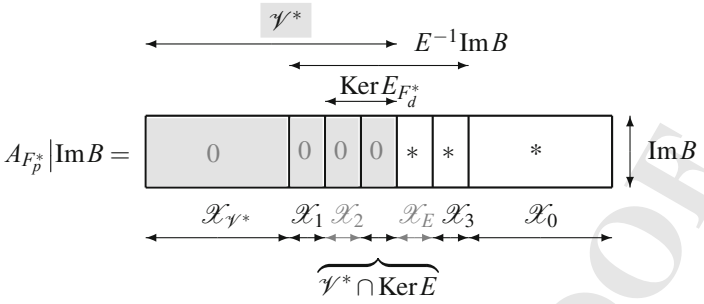


(7.75)

After having chosen F_d^*, we shall select F_p^*, such that $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, namely,

$$(A + BF_p)\mathscr{V}^* \subset (E - BF_d)\mathscr{V}^*. \tag{7.76}$$

Hence,



(7.77)

7.5.2 Example (Part 4)

Let us now verify if the *rectangular implicit representation* (7.43) satisfies the geometric conditions of Theorem 7.6: $\mathscr{R}_d^* = \mathscr{X}_d$, $\mathscr{V}^* = \{e_1, -e_2 - 2e_3\}$, $\mathrm{Ker}\,E = \{e_3\} \approx \mathbb{R}^1$ and $E^{-1}\mathrm{Im}\,B = \{e_2, e_3\}$; hence, $\mathscr{V}^* \cap E^{-1}\mathrm{Im}\,B = \{-e_2 - 2e_3\} \approx \mathbb{R}^1$, which implies (7.71). There then exists $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$ making unobservable the structure variation.

In order to satisfy (7.70), the derivative part of the control law has to contain the term $\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$. Indeed, $\mathrm{Ker}(E - BF_d^*) = \{-e_2 - 2e_3\} \subset \mathscr{V}^*$.

In order to satisfy (7.76), the proportional part of the control law has to contain the term[7] $\begin{bmatrix} -1 & -2/\tau & (1 + 1/\tau) \end{bmatrix}$, where τ is a positive real number. Indeed, $(A + BF_p^*)\mathscr{V}^* = \{e_1\} = (E - BF_d^*)\mathscr{V}^*$.

Thus, the proportional and derivative feedback is

$$u^* = \begin{bmatrix} -1 & -2/\tau & (1 + 1/\tau) \end{bmatrix} x + \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \mathrm{d}x/\mathrm{d}t + \begin{bmatrix} 1/\tau \end{bmatrix} u_r. \tag{7.78}$$

Applying the control law (7.78) to system (7.42), we get the closed-loop system:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{array}\right] \frac{\mathrm{d}}{\mathrm{d}t}\xi = \left[\begin{array}{ccc} 0 & 1 & \\ \hline 0 & 0 & \boxed{-1/\tau} \\ \alpha & -(\beta+2) & -1 \end{array}\right] \xi + \left[\begin{array}{c} 0 \\ \hline \boxed{1/\tau} \\ 0 \end{array}\right] u_r,$$
$$y^* = \begin{bmatrix} 0 & 0 & \boxed{1} \end{bmatrix} \xi \tag{7.79}$$

[7] Since Theorem 7.5 is satisfied, one can also assign the output dynamics.

where $\xi = T^{-1}x$, $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$. In [14], we show that the necessary stability region is[8]

$$\mathcal{R}^*_{\mathcal{NSC}}(\alpha, \beta) = \big\{(\alpha, \beta)\big|\ \alpha \cdot (\beta + 2) < 0\big\} \cup \big\{(\alpha, \beta)\big|\ \beta = -2\ \&\ \alpha \neq 0\big\}, \quad (7.80)$$

and that the sufficient stability region is[9]

$$\mathcal{R}^*_{\mathcal{SSC}}(\alpha, \beta) = \big\{(\alpha, \beta)\big|\ \alpha \cdot (\beta + 2) < 0,\ \underline{\beta} \leq |\,\beta + 2\,| \leq \overline{\beta},\ \underline{\alpha} \leq |\,\alpha\,| \leq \overline{\alpha}\big\} \cup \big\{(\alpha, \beta)\big|\ \beta = -2,\ \underline{\alpha} \leq |\,\alpha\,| \leq \overline{\alpha}\big\}, \quad (7.81)$$

where $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$ and $\overline{\beta}$ are some given real numbers $0 < \underline{\alpha} \leq \overline{\alpha}$ and $0 < \underline{\beta} \leq \overline{\beta}$.

7.5.3 Rejection of the Variable Structure

Since the implementation of the "pure" derivative-based actions is not practically feasible, we have to generate a proper filter with the aim to approximate the external behaviour of the ideal non-proper controller.

In [14], we have considered the following problem.

Problem 7.2 (*Variable structure rejection* [14]) Let us consider a *global implicit representation* $\Sigma^{ig}(\mathbf{E}, \mathbf{A}, \mathbf{B}, C)$, (7.37), such that the solvability assumptions, (7.31) and (7.34), and the internal properness condition, (7.33), are satisfied, and $Ex(t)$ is continuous for all $t \geq 0$. Let us consider the $P.D.$ feedback

$$u^* = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t + u_r \quad (7.82)$$

which constitutes a solution of Problem 7.1, where the feedback pair $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$ was chosen as it is indicated in Theorem 7.6.

Find a proper approximation of the ideal control law (7.82) such that the closed-loop system is BIBO-stable and moreover, for a given $\delta > 0$

$$\big|\, y(t) - y^*(t)\,\big| \leq \delta \qquad \forall\, t \geq t^*(\delta), \quad (7.83)$$

where $t^*(\delta)$ is a fixed transient time, y^* is the output for the ideal control law (7.82) and y is the output associated with the proper approximation of (7.82).

[8] This region is obtained from $\det \begin{bmatrix} \mathrm{s} & -1 & -1 \\ 0 & 0 & (\mathrm{s} + 1/\tau) \\ -\alpha & (\beta + 2) & 1 \end{bmatrix} = -((\beta + 2)\mathrm{s} - \alpha)(\mathrm{s} + 1/\tau)$.

[9] This region is obtained following the methodology of [42], namely, we solve two Lyapunov equations for the two cases: *(i)* $\beta \neq -2$ and *(ii)* $\alpha \neq 0$ (with $\beta = -2$), with a common positive definite matrix P.

Theorem 7.7 (Variable structure rejection [14]) *Under the same conditions like in Theorem 7.6 and the additional assumptions*

Hypothesis ***H****6.* $u_r \in \mathcal{C}^\infty\left(\mathbb{R}^+,\ \mathbb{R}^m\right)$ *and* $\mathrm{d}^2 u_r/\mathrm{d}t^2 \in \mathcal{L}^\infty\left(\mathbb{R}^+,\ \mathbb{R}^m\right)$,

Hypothesis ***H****7. The matrix* $\overline{A}_*$, *defined in Theorem 7.6 is Hurwitz,*

Hypothesis ***H****8. Given* $\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_\ell \in \mathcal{Q}$, $g = \left[\begin{array}{ccc} g_1 & \cdots & g_\ell \end{array}\right]^T$, $g_1, \ldots, g_\ell \in \mathbb{R}^+$, *the locations* $q \in \mathcal{Q}$ *belong to the convex set*

$$\overline{\mathcal{Q}}_{\bar{q}_0}(g) = \left\{ q \in \mathcal{Q} \,\middle|\, q = \bar{q}_0 + \sum_{j=1}^{\ell} \gamma_{(i,j)} g_j \bar{q}_j \right\},$$

where for each $\left[T_{i-1}, T_i\right)$, *the value of* $\gamma_{(i,j)}$ *takes constant values in the closed subset of* $\mathbb{R}$: $\left[0,\ 1\right]$,

we now consider the following proper approximation of the ideal control law (7.82):

$$\begin{aligned} \mathrm{d}\bar{x}/\mathrm{d}t &= -(1/\varepsilon)\bar{x} + (1/\varepsilon) F_d^* x, \\ u &= -(1/\varepsilon)\bar{x} + \left((1/\varepsilon) F_d^* + F_p^*\right) x + u_r, \end{aligned} \tag{7.84}$$

where $\varepsilon > 0$. *If for a given pair* $(\varepsilon, \overline{A}_*)$, *there exists a nonempty convex* sufficient stability condition region $\mathcal{R}^{\#}_{SSC}(q;\varepsilon)$ *contained in the stability region of the ideal solution,* $\mathcal{R}^{*}_{SSC}(q)$, *for which the linear combination* $\overline{X} + \Gamma\overline{\Delta}_0$, *of the matrices coming from*

$$\left[\mathrm{s}\left[\begin{array}{cc} E & 0 \\ -BF_d^* & \varepsilon \mathrm{I} \end{array}\right] - \left[\begin{array}{cc} A_{F_p^*} & \mathrm{I} \\ 0 & -\mathrm{I} \\ \hline D_i & 0 \end{array}\right]\right] \approx \left[\begin{array}{cc} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{array}\right], \tag{7.85}$$

is a Hurwitz matrix and moreover, there exist constant positive definite matrices $\overline{P}$, $\overline{Q}_0$ *such that*

$$\begin{aligned} \left(\overline{X} + \Gamma\overline{\Delta}_0\right)^T \overline{P} + \overline{P}\left(\overline{X} + \Gamma\overline{\Delta}_0\right) &= -\overline{Q}_0, \\ \lambda_{\min}\left(\overline{Q}_0\right) + \sum_{j=1}^{\ell} g_j \lambda_{\min}\left(\overline{Q}_j\right) &> 0, \end{aligned} \tag{7.86}$$

then Problem 7.2 has a solution. Here we denote $\overline{Q}_j = \left(\Gamma\overline{\Delta}_j\right)^T \overline{P} + \overline{P}\left(\Gamma\overline{\Delta}_j\right)$, $j \in \{1, \ldots, \ell\}$.

7.5.4 *Example (Part 5)*

The proper approximation of the ideal control law (7.78) is (cf. (7.84))

$$\begin{aligned} \mathrm{d}\bar{x}/\mathrm{d}t &= -(1/\varepsilon)\bar{x} + \begin{bmatrix} 0 & -1/\varepsilon & 1/\varepsilon \end{bmatrix} x \\ u &= -(1/\varepsilon)\bar{x} + \begin{bmatrix} -1 & -(2/\tau + 1/\varepsilon) & (1 + 1/\tau + 1/\varepsilon) \end{bmatrix} x + \begin{bmatrix} 1/\tau \end{bmatrix} u_r . \end{aligned} \tag{7.87}$$

ε is a positive number which tunes the precision of the approximation.

Equation (7.85) takes the form (cf. (7.42) and (7.78), with $x = T\xi$)

$$\left[\mathrm{s} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 1 & 1 & 0 & \varepsilon \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1/\tau & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline \alpha & -(\beta+2) & 1 & 0 & 0 \end{array}\right] \right] \approx \begin{bmatrix} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{bmatrix}$$

$$\overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/\varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ (1/\varepsilon + 1/\tau) \\ 0 \\ 1/(\varepsilon\tau) \end{bmatrix}, \quad \overline{\Delta}_i = \begin{bmatrix} \alpha & -(\beta+2) & 0 & 0 \end{bmatrix}.$$

If in our example we put $\tau = 4$ and $\varepsilon = 1/4$, we then get

$$\overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 17/4 \\ 0 \\ 1 \end{bmatrix},$$

and $\phi_q^{\#} = \det \begin{bmatrix} \mathrm{sI} - \overline{X} & \Gamma \\ -\overline{\Delta}_i & -\mathrm{I} \end{bmatrix} = -(\mathrm{s}+4)\big(\mathrm{s}^3 + (17\beta/4 - \alpha + 17/2)\mathrm{s}^2 + (\beta - 17\alpha/4 + 2)\mathrm{s} - \alpha\big)$. Thus the necessary stability region is

$$\begin{aligned} \mathcal{R}^{\#}_{\mathcal{NSC}}(\alpha, \beta; \varepsilon = 1/4) = \Big\{ (\alpha, \beta) \big|\ & \alpha < 0, \quad 17\beta/4 - \alpha + 17/2 > 0 \\ & \beta^2 + (4 - 305\alpha/68)\,\beta + (4\alpha/17 + 4\,(17\alpha/4 - 2)\,(\alpha - 17/2)/17) > 0 \Big\}, \end{aligned}$$

thus $q_1, q_2, q_3 \in \mathcal{R}^{\#}_{\mathcal{NSC}}(-1, \beta; \varepsilon = 1/4) = \big\{ (\alpha, \beta) \big|\ -(\beta+2) < 0.1775 \big\}$ (see (7.4) and Fig. 7.1). Then $(\underline{\alpha}, \underline{\beta}) = (1, 0)$ and $(\overline{\alpha}, \overline{\beta}) = (1, 2)$. We assume that (*c.f.* **H**8)

$$\big(\alpha, -(\beta+2)\big) \in \Big\{(-1,\beta) \in \mathscr{Q} \big|\ (-1,-(\beta+2)) = (-1,0) + 0\gamma_1(-1,0) + 2\gamma_2(0,-1),\ \ \gamma_1,\gamma_2 \in [0,1]\Big\}.$$

Then,

$$\overline{\Delta}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix},\ \ \overline{\Delta}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ \ \overline{\Delta}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\overline{X} + \Gamma\overline{\Delta}_0 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -17/4 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma\overline{\Delta}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 17/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ \ \Gamma\overline{\Delta}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 17/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Choosing $\overline{Q}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we get

$$\overline{P} = \begin{bmatrix} 3.7404 & -0.5000 & 0.7480 & -1.1154 \\ -0.5000 & 1.1154 & 0.0620 & -0.5000 \\ 0.7480 & 0.0620 & 0.3120 & -0.2633 \\ -1.1154 & -0.5000 & -0.2633 & 2.7404 \end{bmatrix},\ \sigma\{\overline{P}\} = \begin{Bmatrix} 0.1237 \\ 0.7815 \\ 2.3950 \\ 4.6079 \end{Bmatrix},$$

$$\overline{Q}_1 = \begin{bmatrix} 1.0000 & 3.7404 & 0.7482 & -0.5000 \\ 3.7404 & 0 & 0 & 0 \\ 0.7482 & 0 & 0 & 0 \\ -0.5000 & 0 & 0 & 0 \end{bmatrix},\quad \sigma\{\overline{Q}_1\} = \begin{Bmatrix} 4.3795 \\ -3.3795 \\ 0 \\ 0 \end{Bmatrix},$$

$$\overline{Q}_2 = \begin{bmatrix} 0 & 0.5000 & 0 & 0 \\ 0.5000 & 7.4808 & 0.7482 & -0.5000 \\ 0 & 0.7482 & 0 & 0 \\ 0 & -0.5000 & 0 & 0 \end{bmatrix},\quad \sigma\{\overline{Q}_2\} = \begin{Bmatrix} 7.6199 \\ 0.0000 \\ -0.1391 \\ 0 \end{Bmatrix}.$$

Then

$$\lambda_{\min}\left(\overline{Q}_0\right) + (\overline{\alpha} - \underline{\alpha})\lambda_{\min}\left(\overline{Q}_1\right) + (\overline{\beta} - \underline{\beta})\lambda_{\min}\left(\overline{Q}_2\right) = 1 + 0(-3.3795) + 2(-0.1391) = 0.7218 > 0.$$

The last condition implies that the stability condition (7.86) is satisfied.

7.6 Numerical Simulation

We made a MATLAB® numerical simulation:

> *"Start time" = 0.0, "Stop time" = 150, "Type" = "Variable–Step", "Solver" = "ode45 Domand–Prince", "Max step size" = "auto", "Relative tolerance" = 1e^{-4}, "Min step size" = "auto", "Absolute tolerance" = "auto", "Initial step size" = "auto", "Consecutive min step size violations allowed" = 1, "States shape preservation" = "Disable all", et "Zero crossing control" = "Disable all".*

The behaviours, $\mathfrak{B}^{\infty}_{q_i}$, take place as follows (recall (7.7)): In the time interval $[0, 50)$ takes place $\mathfrak{B}^{\infty}_{q_1}$. In the time interval $[50, 100)$ takes place $\mathfrak{B}^{\infty}_{q_2}$. In the time interval $[100, 150]$ takes place $\mathfrak{B}^{\infty}_{q_3}$.

We apply the proper approximation (7.87) of the ideal control law (7.78), with the choice: $\tau = 4$ and $\varepsilon = 0.25$. We assume that we do not have access to the descriptor variable x, so we use the following descriptor variable observer synthesized in [15] (see equations (3.35) and (3.19) in [15]):

$$\begin{aligned} \mathrm{d}\hat{x}_c/\mathrm{d}t &= \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ \hat{x}_\ell &= \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} -1 \end{bmatrix} y, \end{aligned} \tag{7.88}$$

where

$$\begin{bmatrix} \bar{x}_c \\ \bar{x}_\ell \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right]^{-1} x. \tag{7.89}$$

The reference u_r has been chosen as follows (see Definition 2.4.5–[37]):

$$\begin{aligned} \phi(t) &= \begin{cases} \mathrm{e}^{-\frac{1}{1-(t')^2}}, & t \in A = (\frac{1}{6}, \frac{2}{6}),\ t' = 12t - 3 \\ -\mathrm{e}^{-\frac{1}{1-(t'')^2}}, & t \in B = (\frac{4}{6}, \frac{5}{6}),\ t'' = -12t + 9 \\ 0 \quad , & t \in \mathbb{R} \backslash (A \cup B) \end{cases} \\ r(t) &= \int_0^t \left(\sum_{i=0}^{3} (-1)^i \phi(\frac{2}{75}\sigma - i) \right) \mathrm{d}\sigma, \quad t \in [0, 150]. \end{aligned}$$

The model matching error is computed as follows:

$$\left| y(t) - y^*(t) \right| = \left| y(t) - \int_0^t \mathrm{e}^{-\frac{1}{\tau}(t-\sigma)} u_r(\sigma) \mathrm{d}(\sigma) \right|.$$

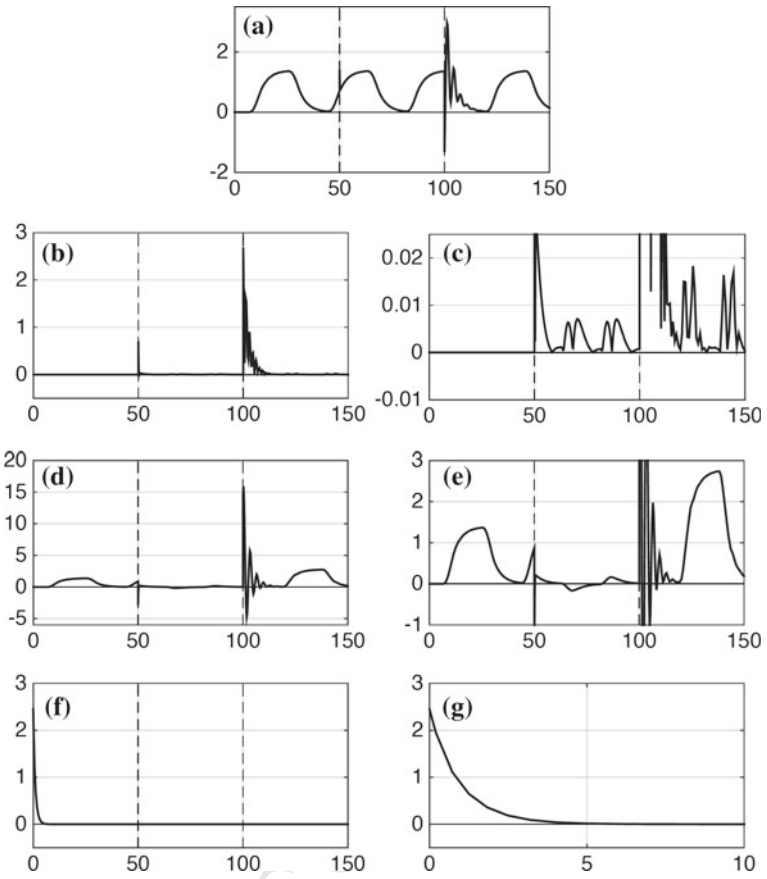


Fig. 7.4 Simulation results of Control with observation of the descriptor variable. **a** Output, y. **b** and **c** Model matching error, $|\, y(t) - y^*(t)\,|$. **d** and **e** Control input, u. **f** and **g** Observation error, $\left\| \hat{x} - x \right\|_2$

In Fig. 7.4, we show the numerical simulations for this minimum phase case. In order to appreciate the performance of the remainder generator, in this simulation, we have set the initial condition: $\bar{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

7.7 Summary

In Sect. 7.2, we have shown how to use the *linear implicit systems theory* in order to model and control, in an efficient way, a class of complex systems, namely, *time-varying, autonomous, switched systems*. Thanks to a simple example, we have shown

that the internal structure variations can take into account wide situations with varying parameters, such as, among others: (1) the relative degree, (2) the system gain and (3) the values of the finite zeros.

In Sect. 7.3, we have shown how linear time-invariant implicit systems theory can be efficiently used to model a certain class of switched systems with autonomous location transitions.

For the equivalent *state-space representations* $\Sigma^{state}(A_{q_i}, B, C_{q_i})$ (7.2) and (7.3), we have determined the common fixed structure. The general systems structure is represented by the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ in (7.35). We also have derived a *linear time-invariant implicit representation* for the initial linear switched system with autonomous location transitions, $\Sigma^{state}(A_{q_i}, B, C_{q_i})$. Note that the *implicit global representations* $\Sigma^{ig}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$ in (7.37) and (7.40) are time-dependent. Alternatively, the *implicit rectangular representation* $\Sigma^{ir}(E, A, B, C)$ in (7.35) is time-invariant.

As shown in some simple examples, the corresponding structure variation has a wide structure. For instance, it includes variable relative degree, variable gain and variable finite zeros.

In the particular structure (7.3) studied above, only the matrices A_{q_i} and C_{q_i} have a generic "switched" structure and additionally depend on index q_i. The independence (assumed above) of matrix B on the switchings $q_i \in \mathcal{Q}$ does not involve any restriction into the used formulation. The zeros and the unobservable subspace of the systems under consideration are indeed only characterized by the structure of the matrices A_{q_i} and C_{q_i}.

The main advantage of these implicit representations is reflected in the structural concept of solvability. Indeed, condition (7.30) for having at least one solution for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathcal{U}\right)$ and condition (7.31) for the existence of solution (for all $u(\cdot) \in \mathcal{C}^\infty\left(\mathbb{R}^+, \mathcal{U}\right)$ and for any initial condition) naturally lead to the *global implicit representation* (7.37).

When we restrict to proper systems, the matrices of the *implicit rectangular representation* (7.35), the *algebraic constraint* (7.36) and the *implicit rectangular representation* (7.37) have the particular form (7.40), which are precisely the structure of the switched system, (7.2), and (7.3) here considered.

In Sect. 7.4, we have tackled the most important concept studied in System Theory, *the reachability*. For the general case of implicit systems, represented by (7.10), with E and A not necessarily square, Frankowska [16] has been the first to give a functional interpretation of *reachability*. For this, she has used the Viability Theory [2]. More precisely, she has shown that *reachability* is equivalent to finding a smooth trajectory $x(\cdot)$, solution of (7.10), starting from the initial condition x_0 and reaching the desired x_1 in a given finite time t_1, namely, $x(0) = x_0$ and $x(t_1) = x_1$. Frankowska [16] has shown that *reachability* is geometrically characterized by the well-known *reachable space*, $\mathscr{R}^*_{\mathscr{X}_d}$. Of course, $\mathscr{R}^*_{\mathscr{X}_d}$ is contained in the viability kernel $\mathscr{V}^*_{\mathscr{X}_d}$, since this last guaranties the existence of at least one trajectory, solution of (7.10), leaving from x_0. This is also clear from $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d}$.

It has to be pointed out that the fundamental *reachability* Definition 7.11 requires no explicit control actions, and in general, the trajectories inside the *reachability*

subspace are handled by input actions and by internal degree of freedom actions; it might exist *reachable systems* without any effective external input (see, for instance, Example 7.2). In order to guarantee that the trajectories are caused by control inputs, we have introduced the *external reachability* concept; for this, we have combined the Frankowska's *reachability* notion [16] with the notion of *unicity of the descriptor variable solution* of Lebret [25].

In the case of *implicit descriptions* constituted by *minimal column indices*, there exists no external input for controlling the whole descriptor variable: this is due to the existence of completely free variables. In order to partly control implicit representations having an internal degree of freedom, we have introduced the *external output dynamics assignment* concept; for this we have used the characterization of Lebret and Loiseau [26], which enables us to make unobservable the degree of freedom by means of a $P.D.$ feedback, and insure that the closed-loop system gets the *external reachability* property.

In Sect. 7.5, we have proposed a control scheme based on *proportional and derivative feedbacks* of the descriptor variable, in order to obtain a *closed-loop system* which is *proper, linear and time-invariant*, whatever be the positions of the internal switches.

The Canonical Form of Lebret and Loiseau [26] has enabled us to characterize the internal structure of the *implicit descriptions*.

Following the typical geometric procedure of the *disturbance decoupling problem* [47], we have decoupled the variable structure by means of an ideal $P.D.$ feedback [8].

In [10], we have proposed an effective procedure to approximate the ideal static $P.D.$ feedback by means of a dynamic $P.$ feedback. Following the ideas of [33, 42], in [14] we have studied the stability aspects.

In Sect. 7.6, we have presented a MATLAB® numerical simulation. We have used a descriptor variable observer based on fault detection techniques [15]. This observer is composed of a Beard-Jones filter, which aim is to observe the existing degree of freedom in rectangular implicit representations. Notice that after the initial transient, this observer remains insensitive to the switchings events (see Fig. 7.4f, g); this is the case, because the observer is based on the fault detection of a continuous linear system. Since this observation is accomplished by a pole-zero cancellation, this technique is reserved to minimum phase systems, with respect to the output-degree-of-freedom transfer, namely, to *implicit rectangular representations* having Hurwitz output decoupling zeros. When unstable zeros are present, alternatives exist (see [15]).

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Chapter 7

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