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# Consensus and Flocking in a Class of Cucker-Smale Systems under Communication Failures

Benoît Bonnet\* and Émilien Flayac†

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## Abstract

In this paper, we study sufficient conditions for the emergence of exponential consensus and asymptotic flocking in a certain class of non-linear generalised Cucker-Smale systems subject to multiplicative communication failures. Our approach is based on the combination of strict Lyapunov design together with the formulation of a novel persistence condition for multi-agent systems. The latter can be interpreted as a lower bound on the algebraic connectivity of the time-average of the interaction graph generated by the communication weights, and provides quantitative decay of the variance functional along the solutions of the system.

**Keywords :** Multi-Agent Systems, Persistence of Excitation, Asymptotic Flocking, Lyapunov Methods.

**MSC Classification :** 34D05, 34D23, 91C20

## 1 Introduction

The study of emerging patterns in dynamical systems describing collective behaviour has been the object of an increasing attention in the past decades. There is by now a large literature devoted to the analysis of asymptotic *cluster formation* in the class of so-called *cooperative systems*, see e.g. [35]. These models are widely used, for example, to study crowd dynamics [11, 32], robot swarms [16], opinion dynamics [3, 22] and animal groups such as bird flocks or fish schools [2].

Since the seminal papers [12, 13], a great deal of interest has been manifested towards the analysis of the so-called *flocking behaviour* (see Definition 4 below) in second-order multi-agent systems, see e.g. [7, 8, 33]. The latter describes the appearance of consensus patterns (see Definition 1 below) in the velocity variable – otherwise known as *alignment* – in Cucker-Smale type systems of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i(t) - x_j(t)|) (v_j(t) - v_i(t)). \end{cases} \quad (\text{CS})$$

Here  $(x_1, \dots, x_N) \in (\mathbb{R}^d)^N$  and  $(v_1, \dots, v_N) \in (\mathbb{R}^d)^N$  stand respectively for the positions and velocities of the agents, and  $\phi(\cdot)$  is a non-linear positive interaction kernel. The functions  $\xi_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0, 1])$  represent *communication weights*, taking into account potential communication failures that can occur in the system (e.g. when  $\xi_{ij}(t) = 0$ , see Section 4 below). Alignment patterns are of high interest in many applications, in particular for controlling fleets of autonomous vehicles (see e.g. [4]), and have been thoroughly investigated in the *full-communication* case, i.e. when  $\xi_{ij}(\cdot) \equiv 1$ . When the interactions between agents are subject to possibly severe failures, it is then of high interest to identify conditions under which the formation of asymptotic flocking can still be guaranteed. For discrete-time first- and second-order systems, opinion formation models of this type have been widely studied in *graph-theoretic frameworks*, see for instance the seminal paper [30] and the reference monograph [29].

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\*CNRS, IMJ-PRG, UMR 7586, Sorbonne Université, 4 place Jussieu, 75252 Paris, France. *Email:* benoit.bonnet@imj-prg.fr

†University of Melbourne, Parkville VIC 3010, Melbourne, Australia. *Email:* emilien.flayac@unimelb.edu.au

Several variants of time-varying topologies have already been studied in the context of alignment formation for multi-agents system. In the early work [37], the authors investigated flocking formation for a time-continuous system with non-stationary interaction topologies, under the strong assumption that the corresponding time-dependent graphs are always connected and exhibit dwell times. This connectivity assumption was subsequently relaxed in [27], at the price of restricting the analysis to discrete-time systems in which the maximal position-spreading between agents needs to be a priori bounded. This led to a rather involved sufficient condition for flocking, in which both the aforementioned bound and the spreading radius of the initial state were appearing. In [34], the authors proved the convergence to flocking of a discrete-time version of (CS), in which the communication rates  $(\xi_{ij}(\cdot))$  are piecewise constant realisations of independent stationary Bernoulli processes. This convergence analysis was later improved in [21], where asymptotic flocking was obtained for the same type of discrete-time systems with weights  $(\xi_{ij}(\cdot))$  that are neither symmetric nor assumed to be independent. The latter result was building on the recent contribution [15] in which flocking formation was studied in the case of discrete-time systems with full communications and *directed* – i.e. possibly asymmetric – interaction topologies. We also mention the results of [14] on this topic, where flocking was studied for a discrete-time version of (CS) with an asymmetric and slightly more general right-hand side, under the additional structural assumption that the agents are all hierarchically directed towards a common leader (see also [36]). We stress that such models of communication failures – relying on time-varying interaction topologies – are substantially different from several known contributions in the literature such as [1, 18, 19]. Therein, the agents are fully communicating at all times, and the disturbances are modelled by additive white noises. However to the best of our knowledge, the analysis of flocking formation for general time-continuous systems of the form (CS) where the communication weights  $(\xi_{ij}(\cdot))$  are merely *measurable* and essentially bounded is still a completely open problem, even in the case of symmetric communications corresponding to *undirected* interaction topologies.

In order to establish such a convergence result for general time-continuous non-linear systems, the best identified setting is that of *Lyapunov analysis*. Indeed in [20], the authors proposed a simple proof of the emergence of asymptotic flocking in the full-communication setting, based on the derivation of strict dissipation inequalities along solutions of (CS) for the velocity variance functional (see Definition 2 below). However, this methodology cannot be directly transposed to the case where the weights  $(\xi_{ij}(\cdot))$  may vanish arbitrarily often, since the variance functional is not strictly dissipative anymore in this context. A natural idea to circumvent this difficulty is to try and formulate a suitable *persistence of excitation* condition on the communication weights. Persistence conditions are indeed quite standard in classical control theory (see e.g. [9, 10, 31]), and have proven their adaptability in stability analysis by allowing to build *strict* Lyapunov functions for perturbed non-stationary systems (see e.g. [24, 25, 28]). In addition to their practical interest (e.g. to study input-to-state stability), strict Lyapunov functions enable to recover quantitative convergence properties towards the equilibrium, which are crucial to ensure flocking formation in multi-agent systems as amply discussed below (see also [19, 20]).

The contributions of this paper are thus composed of two steps. The first one is the formulation of a suitable persistence condition for systems of the form (CS). We shall say that a collection of weights  $(\xi_{ij}(\cdot))$  satisfies the persistence condition (PE) (see Definition 3 below) if there exists a pair  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$  such that

$$B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}_\xi(s)ds\right)\mathbf{v}, \mathbf{v}\right) \geq \mu B(\mathbf{v}, \mathbf{v}), \quad (1)$$

for all  $\mathbf{v} \in (\mathbb{R}^d)^N$ . Here,  $B : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \rightarrow \mathbb{R}$  denotes the *variance bilinear form* (see Definition 2 below), and  $\mathbf{L}_\xi(\cdot)$  is the time-dependent graph-Laplacian associated to the interaction weights  $(\xi_{ij}(\cdot))$  of the system (see (3) below). In the context of cooperative dynamics, the persistence condition proposed in (1) has both a deep and simple meaning in terms of interaction topology. Indeed, it transcribes the fact that on average on a sliding time window of length  $\tau > 0$ , the communication graph of the multi-agent system is connected. It also imposes a uniform lower-bound  $\mu \in (0, 1]$  on the so-called *algebraic connectivity* of the time-averaged graph generated by the weights  $(\xi_{ij}(\cdot))$  (see Definition 5 below), which is the first non-zero eigenvalue of the time-averaged graph-Laplacians. In the way we formulate it, the persistence condition (1) further encodes two interesting ideas. First, it

only requires the system to be persistently excited with respect to the agents which have not reached flocking yet. Second, it solely involves the communication weights  $(\xi_{ij}(\cdot))$  and not the non-linear kernel  $\phi(\cdot)$ . This is in stark contrast compared to other contributions in the literature where the whole right-hand side of (CS) is assumed to be persistent, which boils down to implicitly assuming that the maximal distance between agents is a-priori bounded.

The second step is the explicit construction of time-varying Lyapunov functions for (CS) in the spirit of [28], obtained by combining the variance bilinear form and the persistence conditions (1). We show that these functionals are strictly-dissipative on a family of finite time intervals whose upper-bounds can go to infinity, which allows us to recover exponential convergence rates towards consensus for a first-order variant of (CS) (see Theorem 1 below) as well as a quantitative flocking formation result for the second-order system proper (see Theorem 2 below). While it is known that asymptotic consensus can be recovered in non-oriented first-order systems under mere infinite-time average connectivity assumptions – e.g. when  $\tau = +\infty$  and  $\mu$  is non-constant and vanishing at infinity – (see e.g. [23, 30] and other works in the literature), the corresponding convergence results are inherently non-quantitative. Additionally in [26], it is proven that quantitative integral connectivity conditions in the spirit of (1) are in fact *necessary* for the formation of exponential consensus in first-order multi-agent systems. This along with the seminal contributions of [20] point towards the idea that asymptotic flocking formation seems unlikely in the absence of a strictly dissipative underlying Lyapunov structure supported by a quantitative persistence assumption. Observe that both quantitative and non-quantitative variants of (1) have already been considered in the literature devoted to first-order time-varying interaction topologies (see e.g. [4, 5, 26, 30, 23, 36]). However to the best of our knowledge, this article is the first one to formulate a persistence condition in terms of positive-definiteness of the averaged graph-Laplacian associated to the communication weights with respect to the variance bilinear form, and to leverage this condition to perform a strict Lyapunov design for general time-continuous non-linear systems.

The structure of the article is the following. In Section 2, we introduce our Lyapunov approach by recovering a known result of consensus formation for persistently excited first-order dynamics. We then extend this result in Section 3 to prove the formation of flocking in a class of Cucker-Smale type systems with strongly interacting kernels in the sense of Hypothesis **(K)**, which is the main result of this paper. In Section 4, we illustrate our persistence condition on a general class of communication weights, and we conclude with some remarks and open perspectives in Section 5.

## 2 Exponential consensus under persistent excitation for first-order Cucker-Smale models

In this section, we introduce the main tools used throughout this article in the particular case of consensus formation. We study first-order cooperative systems of the form

$$\begin{cases} \dot{x}_i(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i(t) - x_j(t)|) (x_j(t) - x_i(t)), \\ x_i(0) = x_i^0, \end{cases} \quad (\text{CS1})$$

where  $(x_1^0, \dots, x_N^0) \in (\mathbb{R}^d)^N$  is a given initial datum. We assume that the interaction kernel  $\phi \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$  is strictly positive, and that the communication rates  $\xi_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0, 1])$  are symmetric, i.e.  $\xi_{ij}(t) = \xi_{ji}(t)$  for almost every  $t \geq 0$  and any  $i, j \in \{1, \dots, N\}$ .

From now on, the notation  $\mathbf{x} = (x_1, \dots, x_N)$  refers to the state of the system in  $(\mathbb{R}^d)^N$ , and we denote by  $\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N x_i \in \mathbb{R}^d$  its mean value. For systems of the form (CS1), we aim at studying the formation of *asymptotic consensus*, defined as follows.

**Definition 1.** A solution  $\mathbf{x}(\cdot)$  of (CS<sub>1</sub>) converges to consensus if for any  $i \in \{1, \dots, N\}$  it holds

$$\lim_{t \rightarrow +\infty} |x_i(t) - \bar{\mathbf{x}}(t)| = 0.$$

By the symmetry of the rates  $\xi_{ij}(\cdot)$ , the equations of (CS1) can be rewritten in matrix form as

$$\dot{\mathbf{x}}(t) = -\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (\text{CSM}_1)$$

where  $\mathbf{L} : \mathbb{R}_+ \times (\mathbb{R}^d)^N \rightarrow \mathcal{L}((\mathbb{R}^d)^N)$  is the so-called *graph-Laplacian* of the system, defined by

$$(\mathbf{L}(t, \mathbf{x})\mathbf{y})_i := \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) (y_i - y_j). \quad (2)$$

for almost every  $t \geq 0$  and any  $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ . Here,  $\mathcal{L}((\mathbb{R}^d)^N)$  denotes the vector-space of linear operators over  $(\mathbb{R}^d)^N$ . In the sequel, we will also use the graph-Laplacian  $\mathbf{L}_\xi : \mathbb{R}_+ \rightarrow \mathcal{L}((\mathbb{R}^d)^N)$  associated to the weights  $(\xi_{ij}(\cdot))$ , defined by

$$(\mathbf{L}_\xi(t)\mathbf{y})_i := \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) (y_i - y_j), \quad (3)$$

for almost every  $t \geq 0$  and any  $\mathbf{y} \in (\mathbb{R}^d)^N$ . We insist on the fact that since the communication rates  $\xi_{ij}(\cdot)$  are  $L^\infty$ -functions, both  $\mathbf{L}(\cdot, \cdot)$  and  $\mathbf{L}_\xi(\cdot)$  are defined only for almost every  $t \geq 0$ .

The structure displayed in (2) is fairly general and allows for a comprehensive study of both consensus and flocking problems via Lyapunov methods. With this goal in mind, we introduce the following bilinear form in the spirit of [6, 7].

**Definition 2.** *The variance bilinear form  $B : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \rightarrow \mathbb{R}$  is defined by*

$$B(\mathbf{x}, \mathbf{y}) := \frac{1}{N} \sum_{i=1}^N \langle x_i, y_i \rangle - \langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle, \quad (4)$$

for any  $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ . It is symmetric and positive semi-definite.

Observe that the evaluation  $B(\mathbf{x}, \mathbf{x})$  of the variance bilinear form is the distance of a given  $\mathbf{x} \in (\mathbb{R}^d)^N$  from the so-called *consensus manifold*  $\mathcal{C} = \{\mathbf{x} \in (\mathbb{R}^d)^N \text{ s.t. } x_1 = \dots = x_N\}$ . It then follows that  $B(\mathbf{x}, \mathbf{x}) = 0$  if and only if  $x_i = \bar{x}$  for any index  $i \in \{1, \dots, N\}$ , i.e. if  $\mathbf{x}$  is a consensus. We now list some properties of  $B(\cdot, \cdot)$  and  $\mathbf{L}(\cdot, \cdot)$ .

**Proposition 1.** *The graph-Laplacian  $\mathbf{L}(t, \mathbf{x})$  is positive-semi definite with respect to  $B(\cdot, \cdot)$ . Moreover, vectors of the form  $\mathbf{L}(t, \mathbf{x})\mathbf{y}$  have zero mean.*

*Proof.* By summing over  $i \in \{1, \dots, N\}$  the components in (2), one can check that  $\mathbf{L}(t, \mathbf{x})\mathbf{y}$  has zero mean. This together with the symmetry of the communication rates  $(\xi_{ij}(\cdot))$  implies that

$$B(\mathbf{L}(t, \mathbf{x})\mathbf{y}, \mathbf{y}) = \frac{1}{N^2} \sum_{i,j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) \langle y_i, y_i - y_j \rangle = \frac{1}{2N^2} \sum_{i,j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) |y_i - y_j|^2 \geq 0,$$

for all  $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ . □

We now introduce our concept of *persistence of excitation* for multi-agent systems, which is one of the main contributions of this article.

**Definition 3.** *The weights  $(\xi_{ij}(\cdot))$  are said to satisfy the persistence of excitation condition (PE) if there exist  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$  such that*

$$B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}_\xi(s) ds\right) \mathbf{x}, \mathbf{x}\right) \geq \mu B(\mathbf{x}, \mathbf{x}), \quad (\text{PE})$$

for almost every  $t \geq 0$  and all  $\mathbf{x} \in (\mathbb{R}^d)^N$ .

**Remark 1.** Condition (PE) only involves the communication weights  $(\xi_{ij}(\cdot))$  through  $\mathbf{L}_\xi(\cdot)$  and not the state of the system. Moreover, it is formulated using the bilinear form  $B(\cdot, \cdot)$ , illustrating the fact that one only needs the persistence to hold along directions which are orthogonal to the consensus manifold  $\mathcal{C}$ . Finally, (PE) can be interpreted as a lower bound on the so-called algebraic connectivity (see e.g. [27]) of the time-average of the graphs with weights  $(\xi_{ij}(\cdot))$ , as illustrated in Section 4 below.

In the following theorem, we prove that solutions of (CS1) exponentially converge to consensus when the persistence assumption (PE) holds. This result is not fully new, and can be derived from earlier works dealing with consensus in undirected graphs, see e.g. in [5, 30]. However, the corresponding proof strategy is original in itself, and allows for a progressive introduction of the concepts that will be necessary to prove our main result Theorem 2.

**Theorem 1 (Consensus).** *Suppose that  $\phi(\cdot) \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$  and that the weights  $(\xi_{ij}(\cdot))$  are such that (PE) holds with parameters  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$ . Then, any solution  $\mathbf{x}(\cdot)$  of (CS1) exponentially converges to consensus.*

*Proof.* First, recall that first-order cooperative systems have compactly supported trajectories as a consequence e.g. of [33, Lemma 1]. Hence, there exists  $R > 0$  depending only on  $\mathbf{x}_0 \in (\mathbb{R}^d)^N$  such that  $x_i(t) \in B(0, R)$  for all times  $t \geq 0$ . Besides, since  $\phi(\cdot)$  is positive and continuous, there exists a constant  $C_R > 0$  depending only on  $\mathbf{x}_0 \in (\mathbb{R}^d)^N$  such that

$$\max_{r \leq 2R} \phi(r) \leq C_R. \quad (5)$$

Let  $\|\mathbf{L}(t, \mathbf{x})\|_B$  be the operator norm of  $\mathbf{L}(t, \mathbf{x})$  with respect to  $B(\cdot, \cdot)$ , that is defined by

$$\|\mathbf{L}(t, \mathbf{x})\|_B = \sup_{\mathbf{y} \in (\mathbb{R}^d)^N} \sqrt{\frac{B(\mathbf{L}(t, \mathbf{x})\mathbf{y}, \mathbf{L}(t, \mathbf{x})\mathbf{y})}{B(\mathbf{y}, \mathbf{y})}},$$

and consider the constant

$$c := \sup_{(t, \mathbf{x}) \in \mathbb{R}_+ \times B(0, R)} \|\mathbf{L}(t, \mathbf{x})\|_B^{1/2},$$

which is finite by (5). We introduce the *standard deviation*  $X$  of an element  $\mathbf{x} \in (\mathbb{R}^d)^N$ , which writes

$$X := \sqrt{B(\mathbf{x}, \mathbf{x})},$$

and observe that a solution  $\mathbf{x}(\cdot)$  of (CS1) converges to consensus if and only if  $\lim_{t \rightarrow +\infty} X(t) = 0$ .

We consider the time-state dependent family of linear operators  $\psi_\tau : \mathbb{R}_+ \rightarrow \mathcal{L}((\mathbb{R}^d)^N)$ , given by

$$\psi_\tau(t) := (1 + c^2)\tau \text{Id} - \frac{1}{\tau} \int_t^{t+\tau} \int_t^s \mathbf{L}(\sigma, \mathbf{x}(\sigma)) d\sigma ds. \quad (6)$$

Observe that  $\psi_\tau(\cdot)$  is Lipschitz continuous and thus differentiable almost everywhere by Rademacher's theorem (see e.g. [17, Theorem 3.2]), with pointwise derivative

$$\dot{\psi}_\tau(t) = \mathbf{L}(t, \mathbf{x}(t)) - \frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s)) ds. \quad (7)$$

By construction, it further holds

$$\sqrt{\tau}X \leq \sqrt{B(\psi_\tau(t)\mathbf{x}, \mathbf{x})} \leq \sqrt{(1 + c^2)\tau}X. \quad (8)$$

We thus define the candidate Lyapunov function

$$\mathcal{X}_\tau(t) := \lambda X(t) + \sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}, \quad (9)$$

where  $\lambda > 0$  is a tuning parameter and  $\mathbf{x}(\cdot)$  solves (CSM<sub>1</sub>). Notice that by (8), one also has

$$(\lambda + \sqrt{\tau})X(t) \leq \mathcal{X}_\tau(t) \leq (\lambda + \sqrt{(1 + c^2)\tau})X(t). \quad (10)$$

This type of construction is inspired from [28] and appears frequently in strict Lyapunov design for persistent systems.

By Proposition 1, any solution  $\mathbf{x}(\cdot)$  of (CSM<sub>1</sub>) – and thus of (CS1) – has constant mean, i.e.  $\bar{\mathbf{x}}(\cdot) \equiv \bar{\mathbf{x}}_0$ . By invariance with respect to translation of (CS1), we can assume without loss of generality that  $\bar{\mathbf{x}}(\cdot) \equiv 0$ . We now want to prove that  $\mathcal{X}_\tau(\cdot)$  satisfies a strict-dissipation inequality of the form

$$\dot{\mathcal{X}}_\tau(t) \leq -\alpha \mathcal{X}_\tau(t), \quad (11)$$

for all times  $t \geq 0$  and some constant  $\alpha > 0$ . With this goal in mind, we compute the time-derivative

$$\dot{\mathcal{X}}_\tau(t) = -\frac{\lambda}{X(t)} B(\mathbf{L}(t, \mathbf{x}(t)), \mathbf{x}(t)) + \frac{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}{2\sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}} - \frac{B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \psi_\tau(t)\mathbf{x}(t))}{\sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}}.$$

By (7) and (8), it holds

$$\begin{aligned} \dot{\mathcal{X}}_\tau(t) \leq & -\frac{1}{2\sqrt{(1+c^2)\tau}X(t)} B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s))ds\right)\mathbf{x}(t), \mathbf{x}(t)\right) \\ & + \frac{1}{\sqrt{\tau}X(t)} \left(\frac{1}{2} - \sqrt{(1+c^2)\tau} - \sqrt{\tau}\lambda\right) B\left(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t)\right) \\ & + \frac{1}{\sqrt{\tau}X(t)} B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \int_t^s \mathbf{L}(\sigma, \mathbf{x}(\sigma))d\sigma ds\right)\mathbf{x}(t), \mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t)\right). \end{aligned} \quad (12)$$

We start by estimating the first line in (12). Since  $\phi(\cdot)$  is positive and continuous and  $x_i(t) \in B(0, R)$  for all times  $t \geq 0$ , there exists a constant  $C_0 > 0$  depending only on  $\mathbf{x}_0 \in (\mathbb{R}^d)^N$ , such that

$$\min_{1 \leq i, j \leq N} \phi(|x_i(t) - x_j(t)|) \geq C_0,$$

for all times  $t \geq 0$ . By (2), this implies that

$$B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{y}, \mathbf{y}) \geq \frac{1}{2N^2} \sum_{i, j=1}^N C_0 \xi_{ij}(t) |y_i - y_j|^2 = C_0 B(\mathbf{L}_\xi(t)\mathbf{y}, \mathbf{y}),$$

for any  $\mathbf{y} \in (\mathbb{R}^d)^N$ , which together with (PE) further yields

$$B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s))ds\right)\mathbf{x}(t), \mathbf{x}(t)\right) \geq B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}_\xi(s)ds\right)\mathbf{x}(t), \mathbf{x}(t)\right) \geq C_0 \mu X^2(t), \quad (13)$$

for all  $t \geq 0$ . For the second line of (12), one has

$$\begin{aligned} & B\left(\frac{1}{\tau}\left(\int_t^{t+\tau} \int_t^s \mathbf{L}(\sigma, \mathbf{x}(\sigma))d\sigma ds\right)\mathbf{x}(t), \mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t)\right) \\ & \leq \tau c^2 X(t) \sqrt{B\left(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t)\right)} \\ & \leq \tau c^2 X(t) \|\mathbf{L}(t, \mathbf{x}(t))\|_B^{1/2} \sqrt{B\left(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t)\right)} \\ & \leq \tau c^3 \left(\frac{\epsilon}{2} X(t)^2 + \frac{1}{2\epsilon} B\left(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t)\right)\right), \end{aligned} \quad (14)$$

for any  $\epsilon > 0$ , by definition of  $\|\cdot\|_B$  and by applying Cauchy-Schwartz's and Young's inequalities. Merging (12),(13) and (14) while observing that  $\mathbf{L}(\cdot, \cdot)$  is positive semi-definite, we obtain

$$\dot{\mathcal{X}}_\tau(t) \leq -\left(\frac{C_0 \mu}{2\sqrt{(1+c^2)\tau}} - \frac{c^3 \sqrt{\tau}}{2} \epsilon\right) X(t) + \frac{1}{X(t)} \left(\frac{1}{2\sqrt{\tau}} + \frac{c^3 \sqrt{\tau}}{2\epsilon} - \lambda\right) B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t)).$$

Therefore, choosing the parameters

$$\epsilon := \frac{C_0 \mu}{2c^3 \tau \sqrt{(1+c^2)}} \quad \text{and} \quad \lambda := \frac{1}{2\sqrt{\tau}} + \frac{c^3 \sqrt{\tau}}{2\epsilon},$$

and using (10), we recover

$$\dot{\mathcal{X}}_\tau(t) \leq -\frac{C_0 \mu}{4\sqrt{(1+c^2)\tau}} X(t) \leq -\frac{C_0 \mu}{4\sqrt{(1+c^2)\tau}(\lambda + \sqrt{(1+c^2)\tau})} \mathcal{X}_\tau(t),$$

so that (11) holds with a given constant  $\alpha > 0$  depending only on  $\mathbf{x}_0, \phi(\cdot), \tau$  and  $\mu$ . By an application of Grönwall's Lemma, we obtain that  $\lim_{t \rightarrow +\infty} \mathcal{X}_\tau(t) = 0$ , and thus  $\lim_{t \rightarrow +\infty} X(t) = 0$  by (10). By definition of  $X(\cdot)$ , this implies that  $\mathbf{x}(\cdot)$  exponentially converges to consensus with rate  $\alpha > 0$ .  $\square$

### 3 Flocking under persistent excitation for Cucker-Smale systems with strong interactions

In this section, we prove the main result of this article, which is the formation of asymptotic flocking for a class of Cucker-Smale systems subject to communication failures. These systems are of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i(t) - x_j(t)|) (v_j(t) - v_i(t)). \end{cases} \quad (\text{CS2})$$

Similarly to Section 2, the dynamics in (CS2) can be rewritten in matrix form using graph-Laplacians

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{v}(t), & \mathbf{x}(0) = \mathbf{x}_0, \\ \dot{\mathbf{v}}(t) = -\mathbf{L}(t, \mathbf{x}(t))\mathbf{v}(t), & \mathbf{v}(0) = \mathbf{v}_0. \end{cases} \quad (\text{CSM}_2)$$

We now recall the definition of *asymptotic flocking*.

**Definition 4.** A solution  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$  of (CS2) converges to flocking if for any  $i \in \{1, \dots, N\}$  it holds

$$\sup_{t \geq 0} |x_i(t) - \bar{\mathbf{x}}(t)| < +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} |v_i(t) - \bar{\mathbf{v}}(t)| = 0.$$

For (CS2), we assume that the interaction kernel  $\phi(\cdot) \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$  also satisfies the following *strong interaction* conditions.

**Hypotheses (K).** There exist two constants  $K, \sigma > 0$  along with a parameter  $\beta \in (0, \frac{1}{2})$  such that

$$\phi(r) \geq \frac{K}{(\sigma + r)^\beta}, \quad (15)$$

for any  $r \geq 0$ .

Observe that in particular  $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$ , so that up to replacing  $\phi(\cdot)$  by this lower estimate, we can assume without of generality that  $\phi(\cdot)$  is non-increasing.

**Remark 2.** Hypothesis (K) is a strengthened version of the usual fat tail condition which requires that  $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$ , see e.g. [20]. Remark that here, we require that the Cucker-Smale exponent  $\beta$  be less than  $\frac{1}{2}$ , whereas in the literature the expected critical exponent beyond which unconditional flocking may fail to occur is  $\beta = 1$ , see also Section 5 below.

**Remark 3.** When  $\phi(\cdot)$  is bounded from below by a positive constant, flocking always occurs for (CS2) in the full-communication setting, i.e. when  $\xi_{ij}(\cdot) \equiv 1$  (see e.g. [12, 20, 33]). In the case where the communication weights  $(\xi_{ij}(\cdot))$  satisfy (PE), this result still holds and can be recovered as a simple consequence of Theorem 1. On the other hand for kernels  $\phi(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}_+^*)$ , one can construct examples of initial conditions  $(\mathbf{x}_0, \mathbf{v}_0)$  for which asymptotic flocking fails (see e.g. [7]).



One can check that solutions of (CSM<sub>2</sub>) are such that

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{v}}(t) \quad \text{and} \quad \dot{\bar{\mathbf{v}}}(t) = 0.$$

By invoking again the invariance properties of multi-agent systems, we can assume without loss generality that  $\bar{\mathbf{x}}(\cdot) = \bar{\mathbf{v}}(\cdot) \equiv 0$ . We also define the standard deviation maps

$$X(t) := \sqrt{B(\mathbf{x}(t), \mathbf{x}(t))} \quad \text{and} \quad V(t) := \sqrt{B(\mathbf{v}(t), \mathbf{v}(t))}.$$

evaluated along solutions of (CSM<sub>2</sub>). As a consequence of the symmetry of the rates  $(\xi_{ij}(\cdot))$ , the system (CSM<sub>2</sub>) is *weakly dissipative* in the sense that

$$\dot{X}(t) \leq V(t), \quad \dot{V}(t) \leq 0. \quad (16)$$

In the seminal paper [20], the authors introduced a concise proof of the Cucker-Smale flocking based on the analysis of a system of *strictly dissipative inequalities*. More precisely, they showed that if the dissipative inequalities

$$\dot{X}(t) \leq V(t), \quad \dot{V}(t) \leq -\phi(2\sqrt{N}X(t))V(t), \quad (17)$$

hold with an interaction kernel  $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$ , then the system converges to flocking. Our aim is to adapt their strategy by building a strict Lyapunov function based on the persistence condition (PE). We prove the following main result of this paper.

**Theorem 2** (Main result - Flocking). *Let  $\phi(\cdot)$  be positive, non-increasing and satisfying **(K)**. Assume that the weights  $(\xi_{ij}(\cdot))$  are such that (PE) holds with parameters  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$ . Then any solution  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$  of (CS2) converges to flocking.*

The proof of this result relies on the construction of a strict and time-dependent Lyapunov function for (CS2), for which a system of inequalities akin to (17) holds only on a bounded time interval. This local-in-time strict dissipation allows us in turn to recover the asymptotic flocking of the system by a reparametrisation of the time variable. To the best of our knowledge, this combination of strict Lyapunov design and flocking analysis via locally dissipative inequalities is fully new in the context of multi-agent systems.

**Notation 1.** *In what follows, we will use the rescaled interaction kernel defined by*

$$\phi_\tau(r) := \phi(2\sqrt{N}(r + \tau V(0))) \quad (18)$$

for any  $r \geq 0$ , and denote by  $\Phi_\tau(\cdot)$  its uniquely determined primitive which vanishes at  $X(0)$ , i.e.

$$\Phi_\tau(X) := \int_{X(0)}^X \phi_\tau(r) dr. \quad (19)$$

We start the proof of Theorem 2 by a series of lemmas which will progressively highlight the role of the different assumptions made on the system.

**Lemma 1.** *Let  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$  solve (CSM<sub>2</sub>). If (PE) holds with  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$ , then*

$$B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s)) ds\right) \mathbf{w}, \mathbf{w}\right) \geq \mu \phi_\tau(X(t)) B(\mathbf{w}, \mathbf{w}), \quad (20)$$

for any  $\mathbf{w} \in (\mathbb{R}^d)^N$ , with  $\phi_\tau(\cdot)$  defined as in (18).

*Proof.* By the definition (2) of  $\mathbf{L} : \mathbb{R}_+ \times (\mathbb{R}^d)^N \rightarrow (\mathbb{R}^d)^N$ , one has

$$\begin{aligned} B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s)) ds\right) \mathbf{w}, \mathbf{w}\right) &= \frac{1}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau} \int_t^{t+\tau} \xi_{ij}(s) \phi(|x_i(s) - x_j(s)|) ds\right) |w_i - w_j|^2 \\ &\geq \frac{1}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau} \int_t^{t+\tau} \xi_{ij}(s) \phi(2\sqrt{N}X(s)) ds\right) |w_i - w_j|^2, \end{aligned} \quad (21)$$

since  $\phi(\cdot)$  is non-increasing. As a consequence of the weak dissipation (16), it further holds

$$X(s) = X(t) + \int_t^s \dot{X}(\sigma) d\sigma \leq X(t) + \tau V(0),$$

for all  $s \in [t, t + \tau]$ . By (21), and using again that  $\phi(\cdot)$  is non-increasing, one has

$$\begin{aligned} B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s)) ds\right) \mathbf{w}, \mathbf{w}\right) &\geq \frac{\phi(2\sqrt{N}(X(t) + \tau V(0)))}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau} \int_t^{t+\tau} \xi_{ij}(s) ds\right) |w_i - w_j|^2 \\ &= \phi_\tau(X(t)) B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}_\xi(s) ds\right) \mathbf{w}, \mathbf{w}\right) \\ &\geq \phi_\tau(X(t)) \mu B(\mathbf{w}, \mathbf{w}), \end{aligned}$$

where we used (PE) in the last inequality.  $\square$

We now consider for all times  $t \geq 0$  the candidate Lyapunov function

$$\mathcal{V}_\tau(t) := \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\mathbf{v}(t), \mathbf{v}(t))}, \quad (22)$$

where  $\psi_\tau(\cdot)$  is defined as in (6) and  $\lambda(\cdot)$  is a smooth tuning curve. We have the following lemma.

**Lemma 2.** *For any  $\epsilon_0 > 0$ , there exists a time horizon  $T_{\epsilon_0} > 0$  such that*

$$\dot{\mathcal{V}}_\tau(t) \leq -\frac{\mu \phi_\tau(X(t))}{2\sqrt{(1+c^2)\tau}} V(t). \quad (23)$$

for all times  $t \in [0, 2T_{\epsilon_0})$ .

*Proof.* By repeating the computations in the proof of Theorem 1, we can estimate the time-derivative

$$\begin{aligned} \dot{\mathcal{V}}_\tau(t) &\leq -\left(\frac{\mu \phi_\tau(X(t))}{2\sqrt{(1+c^2)\tau}} - \frac{c^3\sqrt{\tau}}{2}\epsilon(t) - \dot{\lambda}(t)\right) V(t) \\ &\quad + \frac{1}{V(t)} \left(\frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon(t)} - \lambda(t)\right) B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{v}(t), \mathbf{v}(t)). \end{aligned} \quad (24)$$

The two main differences with respect to the proof of Theorem 1 are the choice of *time-dependent families of parameters*  $(\lambda(\cdot), \epsilon(\cdot))$  and the use of the refined estimate (20) instead of (PE).

We start by fixing the curve

$$\lambda(t) := \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon(t)}. \quad (25)$$

for all times  $t \geq 0$  such that  $\lambda(\cdot)$  is well-defined. This choice implies that  $\dot{\lambda}(t) = -\frac{c^3\sqrt{\tau}}{2\epsilon^2(t)}\dot{\epsilon}(t)$ , and let us now define  $\epsilon(\cdot)$  as the unique solution of the ordinary differential equation

$$\dot{\epsilon}(t) = \epsilon^3(t), \quad \epsilon(0) = \epsilon_0,$$

for a given constant  $\epsilon_0 > 0$ . The corresponding solution is unique and can be written explicitly as

$$\epsilon(t) = \frac{\epsilon_0}{\sqrt{1 - 2\epsilon_0^2 t}}, \quad (26)$$

for any  $t \in [0, 1/2\epsilon_0^2)$ . Then, (24) can be reformulated as

$$\dot{\mathcal{V}}_\tau(t) \leq -\frac{\mu \phi_\tau(X(t))}{2\sqrt{(1+c^2)\tau}} V(t),$$

for all times  $t \in [0, \frac{1}{2\epsilon_0^2})$ , and (23) holds with  $T_{\epsilon_0} = 1/4\epsilon_0^2$ .  $\square$

Observe that (23) involves both  $V(\cdot)$  and  $\mathcal{V}_\tau(\cdot)$ . In order to prove Theorem 2 however, we need to derive an estimate which solely involves  $V(\cdot)$ . This is the object of the following lemma.

**Lemma 3.** *There exists a function  $\epsilon_0 \in \mathbb{R}_+^* \mapsto X_M(\epsilon_0)$  such that  $X(t) \leq X_M(\epsilon_0)$  for all  $t \in [0, T_{\epsilon_0}]$ . Moreover, for any  $\epsilon_0 > 0$  it holds*

$$V(T_{\epsilon_0}) \leq \left( \frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0} \right) V(0) \exp \left( - \frac{\mu \phi_\tau(X_M(\epsilon_0))}{4(\alpha_3 + \beta_3 \epsilon_0) \epsilon_0} \right), \quad (27)$$

where  $\{\alpha_k, \beta_k\}_{k=1}^3$  are positive constants which depend only on  $(c, \tau)$ .

*Proof.* Choose  $\epsilon_0 > 0$  and denote by  $(\lambda(\cdot), \epsilon(\cdot))$  the corresponding tuning functions given respectively by (25) and (26). Similarly to (8), it holds for any  $\mathbf{v} \in (\mathbb{R}^d)^N$  that

$$\sqrt{\tau} V \leq \sqrt{B(\psi_\tau(t)\mathbf{v}, \mathbf{v})} \leq \sqrt{(1 + c^2)\tau} V.$$

By definition of  $\mathcal{V}_\tau(\cdot)$  in (22), we then have

$$\left( \sqrt{\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3 \sqrt{2\tau}}{4\epsilon_0} \right) V(t) \leq \mathcal{V}_\tau(t) \leq \left( \sqrt{(1 + c^2)\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3 \sqrt{\tau}}{2\epsilon_0} \right) V(t),$$

for any  $t \in [0, T_{\epsilon_0}]$ , where we used the fact that  $\epsilon(t) \in [\epsilon_0, \sqrt{2}\epsilon_0]$  on this time interval. By simple identification of the coefficients, these estimates can be rewritten as

$$\left( \frac{\alpha_2}{\epsilon_0} + \beta_2 \right) V(t) \leq \mathcal{V}_\tau(t) \leq \left( \frac{\alpha_1}{\epsilon_0} + \beta_1 \right) V(t), \quad (28)$$

for some constants  $\{\alpha_k, \beta_k\}_{k=1}^2$  depending on  $(c, \tau)$ .

We can further integrate (23) on  $[0, t]$  to obtain

$$\mathcal{V}_\tau(t) \leq \mathcal{V}_\tau(0) - \frac{\mu}{2\sqrt{(1 + c^2)\tau}} \int_0^t \phi_\tau(X(s)) V(s) ds,$$

which in turn yields

$$V(t) \leq \left( \frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0} \right) V(0) - \frac{\mu \epsilon_0}{\alpha'_2 + \beta'_2 \epsilon_0} \int_0^t \phi_\tau(X(s)) V(s) ds, \quad (29)$$

where  $(\alpha'_2, \beta'_2) = 2\sqrt{(1 + c^2)\tau}(\alpha_2, \beta_2)$ . Recall now that  $\dot{X}(s) \leq V(s)$  by (16), so that applying the change of variable  $r = X(s)$  in (29) we recover

$$V(t) \leq \left( \frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0} \right) V(0) - \frac{\mu \epsilon_0}{\alpha'_2 + \beta'_2 \epsilon_0} \int_{X(0)}^{X(t)} \phi_\tau(r) dr = \left( \frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0} \right) V(0) - \frac{\mu \epsilon_0}{\alpha'_2 + \beta'_2 \epsilon_0} \Phi_\tau(X(t)). \quad (30)$$

for all times  $t \in [0, T_{\epsilon_0}]$ .

Since  $\phi_\tau \notin L^1(\mathbb{R}_+, \mathbb{R}_+^*)$ , its primitive  $\Phi_\tau(\cdot)$  is a strictly increasing map which image continuously spans  $\mathbb{R}_+$ . It is therefore invertible, and for any  $\epsilon_0 > 0$  there exists a radius  $X_M(\epsilon_0)$  such that

$$X_M(\epsilon_0) = \Phi_\tau^{-1} \left( \frac{2\sqrt{(1 + c^2)\tau}(\alpha_1 + \beta_1 \epsilon_0) V(0)}{\mu \epsilon_0} \right). \quad (31)$$

Since  $V(\cdot)$  is a non-negative quantity by definition, it necessarily follows by plugging (31) into (30) that  $X(t) \leq X_M(\epsilon_0)$  on  $[0, T_{\epsilon_0}]$ .

Going back to the dissipative differential inequality (23) combined with (28), we can again use the fact that  $\phi_\tau(\cdot)$  is non-increasing to obtain

$$\dot{\mathcal{V}}_\tau(t) \leq - \frac{\mu \epsilon_0 \phi_\tau(X_M(\epsilon_0))}{(\alpha_3 + \beta_3 \epsilon_0)} \mathcal{V}_\tau(t)$$

for almost every  $t \in [0, T_{\epsilon_0}]$ , where  $(\alpha_3, \beta_3) = 2\sqrt{(1 + c^2)\tau}(\alpha_1, \beta_1)$ . By an application of Grönwall's Lemma to  $\mathcal{V}_\tau(\cdot)$  along with (28), we finally recover the estimate

$$V(T_{\epsilon_0}) \leq \left( \frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0} \right) V(0) \exp \left( - \frac{\mu \phi_\tau(X_M(\epsilon_0))}{4(\alpha_3 + \beta_3 \epsilon_0) \epsilon_0} \right)$$

where we used the fact that  $T_{\epsilon_0} = 1/4\epsilon_0^2$ . □

Building on the dissipative inequality (27) obtained in Lemma 3, we now conclude the proof of our main result Theorem 2. To lighten the computations, most of the argument will be carried out in terms of asymptotic estimates.

*Proof (of Theorem 2).* Using the analytical expression (31) of  $X_M(\epsilon_0)$ , we have

$$\phi_\tau(X_M(\epsilon_0)) = \phi_\tau \circ \Phi_\tau^{-1} \left( A_1 + \frac{A_2}{\epsilon_0} \right),$$

where  $A_1, A_2 > 0$  are given constants depending on the datum of the problem. By integrating (15), one further has

$$\Phi(X) \geq \frac{K}{1-\beta} \left( (\sigma + X)^{1-\beta} - (\sigma + X(0))^{1-\beta} \right),$$

which can be reformulated into

$$X \leq \left( \frac{1-\beta}{K} \Phi(X) + (\sigma + X(0))^{1-\beta} \right)^{\frac{1}{1-\beta}} - \sigma, \quad (32)$$

for every  $X \geq X_0$ . Notice now that  $\Phi(X) = A_3 \Phi_\tau(X) + A_4$  for given constants  $A_3, A_4 > 0$  depending only on  $V(0), N$  and  $\tau$ , so that choosing  $X := X_M(\epsilon_0) = \Phi_\tau^{-1} \left( C_1 + \frac{C_2}{\epsilon_0} \right)$  and recalling that  $\phi_\tau(\cdot)$  is decreasing, we obtain as a consequence of (32) together with hypothesis **(K)** that

$$\phi_\tau(X_M(\epsilon_0)) \geq \phi_\tau \left( \left( C_1 + \frac{C_2}{\epsilon_0} \right)^{\frac{1}{1-\beta}} - \sigma \right) \geq K \left( C_1 + \frac{C_2}{\epsilon_0} \right)^{\frac{\beta}{\beta-1}}, \quad (33)$$

where  $C_1, C_2 > 0$  depend only on  $A_1, A_2, A_3, A_4, X(0), \sigma$  and  $\tau$ .

Plugging the expression (33) into (27) and recalling that  $T_{\epsilon_0} = 1/4\epsilon_0^2$ , we finally recover

$$V(T_{\epsilon_0}) \leq C_3 \exp \left( - C_4 \mu T_{\epsilon_0}^{\frac{1-2\beta}{2(1-\beta)}} \right), \quad (34)$$

for every  $\epsilon_0 > 0$ , where  $C_3, C_4 > 0$  are uniform constants depending only on the datum of the problem. Observe now that since  $\epsilon_0 \in \mathbb{R}_+^* \mapsto T_{\epsilon_0}$  continuously spans the whole of  $\mathbb{R}_+$ , we can reparametrise time using  $T := T_{\epsilon_0}$ . As we assumed in **(K)** that  $\beta \in (0, \frac{1}{2})$ , the estimate of (34) implies that

$$V(T) \xrightarrow{T \rightarrow +\infty} 0.$$

We now turn our attention to the uniform boundedness of the position radius  $X(\cdot)$ . The weak-dissipativity (16) of  $(\text{CSM}_1)$  expressed in terms of the new time variable  $T := T_{\epsilon_0}$  writes

$$\sup_{T \geq 0} X(T) \leq X(0) + \int_0^{+\infty} V(T) dT.$$

This implies that  $\sup_{T \geq 0} X(T) < +\infty$  as a consequence of (34) and of the fact that  $\beta \in (0, \frac{1}{2})$ , which concludes the proof of Theorem 2.  $\square$

## 4 Illustration of the persistence condition (PE)

In this section, we exhibit a general situation in which (PE) holds. We start by fixing a constant  $\mu \in (0, 1]$  and recalling known facts about interaction graphs and their Laplacians, for which we refer the reader e.g. to [29].

**Definition 5.** *The algebraic connectivity of a graph with weights  $(\xi_{ij})$  is the smallest non-zero eigenvalue of  $\mathbf{L}_\xi$  seen as an  $N \times N$  matrix, denoted by  $\lambda_2(\mathbf{L}_\xi)$ .*

**Lemma 4.** *If an interaction graph with weights  $(\xi_{ij})$  is such that  $\lambda_2(\mathbf{L}_\xi) \geq \mu$ , then*

$$B(\mathbf{L}_\xi \mathbf{v}, \mathbf{v}) \geq \mu B(\mathbf{v}, \mathbf{v}),$$

for any  $\mathbf{v} \in (\mathbb{R}^d)^N$ .

*Proof.* This follows from the definition of algebraic connectivity, along with the fact that

$$B(\mathbf{L}_\xi \mathbf{v}, \mathbf{v}) = \frac{1}{2N^2} \sum_{i,j=1}^N \xi_{ij} |v_i - v_j|^2,$$

for any  $\mathbf{v} \in (\mathbb{R}^d)^N$ .  $\square$

**Lemma 5.** Let  $\mathbf{L}_{\xi_1}, \mathbf{L}_{\xi_2}$  be the graph-Laplacian associated to two interaction graphs with weights  $(\xi_{ij}^1)$  and  $(\xi_{ij}^2)$  respectively. Then

$$\mathbf{L}_\xi := \mathbf{L}_{\xi_1 + \xi_2} = \mathbf{L}_{\xi_1} + \mathbf{L}_{\xi_2},$$

is the graph-Laplacian of the union of the two graphs, whose weights are given by  $\xi_{ij} = \xi_{ij}^1 + \xi_{ij}^2$  for any  $i, j \in \{1, \dots, N\}$ .

From now on, we fix  $\tau \in \mathbb{R}_+^*$ , an integer  $n \geq 1$ , and time-dependent communication rates  $(\xi_{ij}(\cdot))$  which are constant on all the intervals of the form  $[\frac{m\tau}{n}, \frac{(m+1)\tau}{n})$  for some  $m \geq 0$ .

**Proposition 2.** Suppose that for all  $m \geq 0$ , the time-average of the graphs  $\{\xi_{ij}(\frac{m+k}{n}\tau)\}_{k=0}^{n-1}$ , whose weights are given by

$$\xi_{ij}^m := \frac{1}{n} \sum_{k=0}^{n-1} \xi_{ij}(\frac{m+k}{n}\tau), \quad (35)$$

for any  $i, j \in \{1, \dots, N\}$  is connected with  $\lambda_2(\mathbf{L}_{\xi^m}) \geq \mu$ . Then (PE) holds.

*Proof.* For  $m \geq 0$  and  $t \in [\frac{m\tau}{n}, \frac{(m+1)\tau}{n})$ , we have

$$\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}_\xi(s) ds = \left(\frac{m+1}{n} - \frac{t}{\tau}\right) \mathbf{L}_\xi\left(\frac{m\tau}{n}\right) + \frac{1}{n} \sum_{k=1}^{n-1} \mathbf{L}_\xi\left(\frac{m+k}{n}\tau\right) + \left(\frac{t}{\tau} - \frac{m\tau}{n}\right) \mathbf{L}_\xi\left(\frac{(m+n)\tau}{n}\right). \quad (36)$$

Now, remark that  $\max\{\frac{m+1}{n} - \frac{t}{\tau}, \frac{t}{\tau} - \frac{m\tau}{n}\} \geq \frac{1}{2n}$ . Without loss of generality, assume that  $\frac{m+1}{n} - \frac{t}{\tau} \geq \frac{1}{2n}$ , so that by (36) it holds that

$$B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}_\xi(s) ds\right) \mathbf{v}, \mathbf{v}\right) \geq B\left(\left(\frac{1}{2n} \sum_{k=0}^{n-1} \mathbf{L}_\xi\left(\frac{m+k}{n}\tau\right)\right) \mathbf{v}, \mathbf{v}\right) = B\left(\mathbf{L}_{\xi^m/2} \mathbf{v}, \mathbf{v}\right) \geq \frac{\mu}{2} B(\mathbf{v}, \mathbf{v}),$$

for all  $\mathbf{v} \in (\mathbb{R}^d)^N$ , where the weights  $\xi_{ij}^m$  are defined as in (35).  $\square$

**Corollary 1.** Suppose that the piecewise constant weights  $(\xi_{ij}(\cdot))$  take their values in an arbitrary finite set  $I \subset [0, 1]$ . Then (PE) holds if and only if for all  $m \geq 0$ , the time-averaged graph whose weights  $\xi_{ij}^m$  are given by (35) is connected.

*Proof :* The direct implication of this statement is evident. For the converse, observe that since  $I \subset [0, 1]$  is a finite set, there exists a finite number of graphs with weights given by (35) which are connected. In particular, the quantity

$$\mu := \min \left\{ \lambda_2(\mathbf{L}_{\xi^m}) \text{ s.t. } (\xi_{ij}^m) \text{ are given by (35)} \right\},$$

is positive and independent from  $m \geq 0$ . Thus, (PE) holds with parameters  $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$ .  $\square$

We now illustrate these general results for piecewise constant communication rates on a simple example with  $N = 4$  agents. For  $\tau \in \mathbb{R}_+^*$  and  $t \geq 0$ , define the interactions weights as follows

$$\begin{aligned} \xi_{14}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 1 \bmod[6], \\ 0 & \text{otherwise,} \end{cases} & \xi_{34}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 3 \bmod[6], \\ 0 & \text{otherwise,} \end{cases} \\ \xi_{23}(t) = \xi_{24}(t) &= \begin{cases} 1 & \text{if } \lfloor t/\tau \rfloor = 5 \bmod[6], \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (37)$$

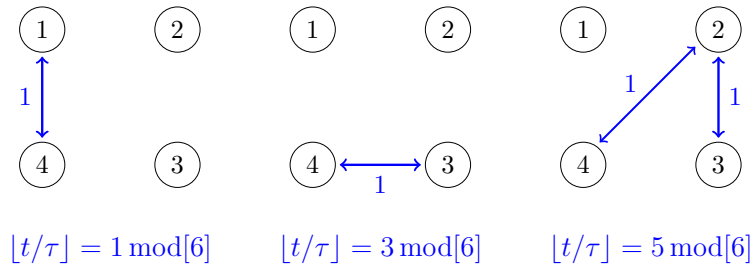


Figure 1: *Admissible connections between agents*

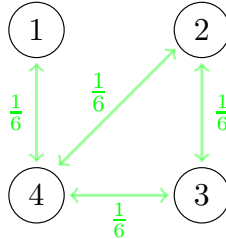


Figure 2: *Averaged interaction graph on a time window of the form  $[t, t + \tau]$*

where  $\lfloor \cdot \rfloor$  denotes the lower integer part of a real number, and we set all the other weights to 0. In this example, our signals are piecewise constant on intervals of the form  $[\frac{m\tau}{6}, \frac{(m+1)\tau}{6})$  for any  $m \geq 0$ .

The weights  $(\xi_{ij}(\cdot))$  defined in (37) are such that the persistence condition (PE) holds. This can be verified e.g. by computing the smallest positive eigenvalue of the averaged graph-Laplacian matrix  $\mathbf{L}_{\xi^m}$ , where  $\xi^m$  is defined as in (35) with  $t \in [\frac{m\tau}{6}, \frac{(m+1)\tau}{6})$ . In this example, the spectrum of  $\mathbf{L}_{\xi^m}$  for all  $m \geq 0$  is given explicitly by

$$\text{Sp}(\mathbf{L}_{\xi^m}) = \{0, \frac{1}{6}, \frac{1}{2}, \frac{2}{3}\},$$

so that (PE) holds with  $\tau \in \mathbb{R}_+^*$ ,  $\mu = \lambda_2(\mathbf{L}_{\xi^m}) = \frac{1}{6}$ .

## 5 Conclusion and perspectives

In this article, we proved two convergence results for multi-agent systems subject to general multiplicative communication failures. If communication rates satisfy a persistence of excitation condition, then one has both exponential convergence to consensus for first-order systems (Theorem 1), and quantitative convergence to flocking for Cucker-Smale systems under an additional strong interaction condition (Theorem 2). For the sake of conciseness and readability, we assumed that the initial time of the non-stationary dynamics was fixed and equal to 0. Yet, it could be checked by repeating our argument that both convergence results are *uniform with respect to the initial time*. In the future, we aim at improving our main result Theorem 2 in three directions.

First, we will investigate whether the rather surprising exponent range  $\beta \in (0, \frac{1}{2})$  – which is currently needed in order to ensure that asymptotic flocking occurs – has an intrinsic meaning, or if it just arises as a limit of our current choice of Lyapunov function. Answering this question might also pave the way for flocking results with weaker interactions, involving confinement conditions linking the initial state and velocity mean-deviations as well as the persistence parameters.

Then, we will study communication failures defined as the realisations of stochastic processes and try to see under which assumptions and in what sense the convergence towards consensus and flocking can occur (almost surely, in probability, etc...). In this setting, one of the main difficulty will most likely lie in the identification of proper stochastic generalisations of (PE).

Lastly, we will investigate whether our dissipative approach, applied here to the mean-square energies – which are  $L^2$ -functionals –, can be adapted to  $L^\infty$ -type Lyapunov functionals in the spirit of [20, 33]. The motivation behind this line of study is that  $L^2$ -type functionals do not allow for the study of flocking formation in the mean-field setting as the number of agents  $N$  goes to infinity, while

$L^\infty$ -type functionals do.

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